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Asymptotic stability of small solitons for 2D nonlinear Schrödinger equations with potential

By

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Abstract

We consider asymptotic stability of a small solitary wave to supercritical 2-dimensional nonlinear Schrödinger equations

$$iu_t + \Delta u = Vu \pm |u|^{p-1}u \quad \text{for } (x,t) \in \mathbb{R}^2 \times \mathbb{R},$$

in the energy class. This problem was studied by Gustafson-Nakanishi-Tsai [14] in the n-dimensional case ($n \geq 3$) by using the endpoint Strichartz estimate. Since the endpoint Strichartz estimate fails in 2-dimensional case, we use a time-global local smoothing estimate of Kato type to prove the asymptotic stability of a solitary wave.

1. Introduction

In this paper, we consider asymptotic stability of solitary wave solutions to

$$\begin{cases} iu_t + \Delta u = Vu + f(u) & \text{for } (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $V(x)$ is a real potential, $f(u) = \alpha|u|^{p-1}u$ with $\alpha = \pm 1$.

Let

$$H(u) = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x)|u|^2 + \frac{2\alpha}{p+1}|u|^{p+1} \right) dx,$$
$$N(u) = \int_{\mathbb{R}^2} |u|^2 dx.$$

Then a solution to (1.1) satisfies

$$H(u(t)) = E(u_0), \quad N(u(t)) = N(u_0) \tag{1.2}$$

during the time interval of existence. Stability of solitary waves was first studied by Cazenave and Lions [8], Grillakis-Shatah-Strauss [13] and Weinstein [53]

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(see also Rose-Weinstein [37], Oh [29] and Shatah-Strauss [41]). In the case of integrable equations such as cubic NLS and KdV, the inverse scattering theory tells us that if the initial data decays rapidly as $x \to \pm\infty$, a solution decomposes into a sum of solitary waves and a radiation part as $t \to \infty$ (see [40]). Soffer and Weinstein [44], [45] considered

$$iu_t + \Delta u = Vu \pm |u|^{p-1}u \quad \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \tag{1.3}$$

where $n \geq 2$ and $1 < p < (n+2)/(n-2)$. They proved that if $-\Delta+V$ has exactly one eigenvalue with negative value E_* and initial data is well localized and close to a nonlinear bound state, a solution tends to a sum of a nonlinear bound state nearby and a radiation part which disperses to 0 as $t \to \infty$ (see also [21] for 2-dimensional case). This result was extended by Yau and Tsai [51], [55]–[57] and Soffer-Weinstein [46] to the case where $-\Delta + V$ have two bound states (see also [12], [52]). In the 1-dimensional case, Buslaev and Perelman [5], [6] and Buslaev and Sulem [7] studied the asymptotic stability of (1.1) with $V \equiv 0$. Using the Jost functions, they built a local energy decay estimate of solutions to the linearized equation and prove asymptotic stability of solitary waves for super critical nonlinearities. Their results are extended to the higher dimensional case by Cuccagna [11]. See also Perelman [32] and Rodnianski-Schlag-Soffer [35] which study asymptotic stability of multi-solitons, and Krieger and Schlag [22] which study large time behavior of solutions around unstable solitons.

However, all these results assume that initial data is well localized so that a solution decays like $t^{-3/2}$. Martel and Merle [23], [24] proved the asymptotic stability of solitary waves to generalized KdV equations using the monotonicity of L^2-mass, which is a variant of the local smoothing effect proved by Kato [16]. They elegantly use the fact that the dispersive remainder part of a solution $v(t,x)$ satisfies

$$\int_0^\infty \|v(t,\cdot)\|_{H^1_{loc}}^2 \, dt < \infty \tag{1.4}$$

to prove the asymptotic stability of solitary waves in H^1 (see also Pego and Weinstein [31] for KdV with exponentially localized initial data and Mizumachi [25] for polynomially localized initial data). Gustafson-Nakanishi-Tsai [14] has proved asymptotic stability of a small solitary wave of (1.3) in the energy class with $n \geq 3$. Their idea is to use the endpoint Strichartz estimate instead of (1.4), which tells us that $\|v\|_{L_t^2 W_x^{1,6}}$ remains small globally in time for super critical nonlinearities. However, dispersive wave decays more slowly in the lower dimensional case and the endpoint Strichartz estimate does not hold in the 2-dimensional case. Recently, Mizumachi [26] has proved the asymptotic stability of small solitons in 1D case by using dispersive estimates such as

$$\|\partial_x e^{it(-\partial_x^2+V)} P_c f\|_{L_x^\infty L_t^2} \leq C\|f\|_{H^{1/2}}. \tag{1.5}$$

In the present paper, we apply local smoothing estimates

$$\|\langle x\rangle^{-1-0}e^{it(-\Delta+V)}P_cf\|_{L_t^2(0,\infty;L_x^2(\mathbb{R}^2))}\le C\|f\|_{L^2(\mathbb{R}^2)}, \tag{1.6}$$

$$\begin{aligned}&\|\langle x\rangle^{-1-0}\int_0^t e^{i(t-s)(-\Delta+V)}P_cg(s)ds\|_{L_t^2(0,\infty;L_x^2(\mathbb{R}^2))}\\ &\le C\|\langle x\rangle^{1+0}g\|_{L_t^2(0,\infty;L_x^2(\mathbb{R}^2))}\end{aligned} \tag{1.7}$$

to obtain the asymptotic stability of small solitons in the 2-dimensional case.

Local smoothing estimates such as (1.6) have been studied by many authors. See, for example, Ben-Artzi and Klainerman [3], Constantin and Saut [10], Kato and Yajima [17] and Kenig-Ponce-Vega [19, 20], Sjolin [42], Ruiz-Vega [38], Sugimoto [48] and Watanabe [58]. Especially, Ben-Artzi and Klainerman [3] and Barceló-Ruiz-Vega [2] prove time-global local smoothing estimates in n-dimensional case with $n\ge 3$. In the 2-dimensional case, it is well-known that

$$\|e^{it\Delta}f\|_{L_x^\infty(\mathbb{R}^2;L_t^2(\mathbb{R}))}\lesssim\|f\|_{L^2(\mathbb{R}^2)}, \tag{1.8}$$

follows from a special case of Thomas-Stein theorem ([47]) (see, e.g., Planchon [33]). However, to the best of our knowledge, there seems to be a lack of literature in the 2-dimensional case with $V\not\equiv 0$. Another purpose of the present paper is to fill the gap.

Our strategy to prove (1.6) is to apply Plancherel's theorem to the inversion of the Laplace transform formula. The key is to prove

$$\|\langle x\rangle^{-1-0}R(\lambda\pm i0)f\|_{L_\lambda^2(0,\infty;L_x^2)}\le C\|f\|_{L^2}\quad\text{for every } f\in L^2(\mathbb{R}^2). \tag{1.9}$$

Roughly speaking, Eq. (1.9) can be translated into (1.6) by using the Fourier transform with respect to λ.

To obtain (1.9), we utilize that the free resolvent operator $R_0(\lambda)=(-\Delta-\lambda)^{-1}$ satisfies

$$\sup_x\|R_0(\lambda\pm i0)f\|_{L_\lambda^2(0,\infty)}\le C\|f\|_{L^2}\quad\text{for every } f\in L^2(\mathbb{R}^2), \tag{1.10}$$

and apply a resolvent expansion obtained by Jensen and Nenciu [15] as well as Schlag [39].

Our plan of the present paper is as follows. In Section 2, we state our main result and linear dispersive estimates that will be used later. In Section 3, we prove our main result assuming the linear estimates introduced in Section 2. In Section 4, we prove (1.9) and obtain (1.6). To prove (1.9), we use an argument of the resolvent expansion as well as (1.10) which follows from $L^2(0,\infty;\sqrt{x}dx)$-boundedness of the Hankel transform and the $\mathcal{Y}_0$-transform (see Rooney [36]).

Finally, we introduce several notations. Let

$$\|f\|_{L_t^qL_x^p}=\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}|f(t,x)|^pdx\right)^{q/p}dt\right)^{1/q},$$

$$\|f\|_{L_x^sL_t^r}=\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}|f(t,x)|^rdt\right)^{s/r}dx\right)^{1/s}.$$

We denote by $L^{2,s}$ and $H^{m,s}$ Hilbert spaces whose norms are defined by

$$\|u\|_{L^{2,s}} = \|\langle x\rangle^s u\|_{L^2(\mathbb{R}^2)} \quad \text{and} \quad \|u\|_{H^{m,s}} = \|\langle x\rangle^s u\|_{H^m(\mathbb{R}^2)},$$

where $m \in \mathbb{N}$, $s \in \mathbb{R}$ and $\langle x\rangle = (1+|x|^2)^{1/2}$. Let

$$\langle f_1, f_2\rangle_x = \int_{\mathbb{R}^2} f_1(x) f_2(x) dx, \quad \langle g_1, g_2\rangle_{t,x} = \int_{-\infty}^{\infty}\int_{\mathbb{R}^2} g_1(t,x) g_2(t,x) dx dt.$$

We set $L^2_{rad} = L^2(0,\infty; rdr)$ whose norm is defined by

$$\|f\|_{L^2_{rad}} = \left(\int_0^\infty |f(r)|^2 r dr\right)^{1/2}.$$

For any Banach spaces X, Y, we denote by $B(X,Y)$ the space of bounded linear operators from X to Y. We abbreviate $B(X,X)$ as $B(X)$.

We define the Fourier transform of $f(x)$ as

$$\mathcal{F}_x f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx,$$

and the inverse Fourier transform of $g(\xi)$ as

$$\mathcal{F}_\xi^{-1} g(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(\xi) e^{ix\xi} d\xi.$$

We denote by $\mathcal{S}_t(\mathbb{R}) \otimes \mathcal{S}_x(\mathbb{R}^2)$ a set of functions $f(t,x) = \sum_{i=1}^N f_i(t) g_i(x)$ with $f_i \in \mathcal{S}(\mathbb{R})$, $g_i \in \mathcal{S}(\mathbb{R}^2)$ $(1 \le i \le N)$.

2. The main result and preliminaries

In the present paper, we assume that the linear potential $V(x)$ is a C^1-function on $\mathbb{R}^2$ satisfying the following.

(H1) There exists a $\sigma > 3$ such that $\sup_{x\in\mathbb{R}^2} (\langle x\rangle^\sigma |V(x)| + |\nabla V(x)|) < \infty$.

(H2) $L = -\Delta + V$ has exactly one negative eigenvalue E_*.

(H3) 0 is neither a resonance nor an eigenvalue of L (see Definition 4.1 in Section 4).

From (H1)–(H3), it follows that the spectrum of L consists of the continuous spectrum $\sigma_c(L) = [0,\infty)$ and a simple eigenvalue E_* (see [34]). Let ϕ_* be a normalized eigenfunction of L (satisfying $\|\phi_*\|_{L^2} = 1$) belonging to E_*, and let P_d and P_c be spectral projections of L defined by

$$P_d u = \langle u, \phi_*\rangle \phi_*, \quad P_c u = (I - P_d) u.$$

Suppose $E \in \mathbb{R}$ and that $e^{-iEt}\phi_E(x)$ is a solitary wave solution of (1.1). Then $\phi_E(x)$ is a solution to

$$\begin{cases} \Delta\phi_E + E\phi_E = V\phi_E + \alpha|\phi_E|^{p-1}\phi_E & \text{for } x \in \mathbb{R}^2, \\ \lim_{|x|\to\infty} \phi_E(x) = 0. \end{cases} \tag{2.1}$$

Using the bifurcation theory, we have the following.

Proposition 2.1. *Assume (H1)–(H3). Let δ be a small positive number. Suppose that $E \in (E_*, E_*+\delta)$ and $\alpha = 1$ or $E \in (E_*-\delta, E_*)$ and $\alpha = -1$. Then, there exists a positive solution ϕ_E to (2.1) such that for every $k \in \mathbb{N}$,*

1. $\phi_E \in H^{1,k}$,
2. *The function $E \mapsto \phi_E$ is C^1 in $H^{1,k}$ for every $k \in \mathbb{N}$, and as $E \to E_*$,*

$$\phi_E = |E - E_*|^{1/(p-1)} \left(\|\phi_*\|_{L^{p+1}}^{-(p+1)/(p-1)} \phi_* + O(E - E_*) \right) \quad \text{in } H^{1,k}.$$

Proposition 2.1 follows from a rather standard argument. See for example [28] and [44, pp.123–124].

Remark 1. Let $\phi_{1,E} = \|\phi_E\|_{L^2}^{-1}\phi_E$ and $\phi_{2,E} = \|\partial_E\phi_E\|_{L^2}^{-1}\partial_E\phi_E$. By Proposition 2.1,

$$\|\phi_{1,E} - \phi_*\|_{H^{1,k}(\mathbb{R}^2)} + \|\phi_{2,E} - \phi_*\|_{H^{1,k}(\mathbb{R}^2)} \lesssim |E - E_*|.$$

Now, we introduce our main result.

Theorem 2.1. *Assume (H1)–(H3). Let $p \geq 3$ and let ε_0 be a sufficiently small positive number. Suppose $\|u_0\|_{H^1} < \varepsilon_0$. Then there exist an $E_+ < 0$, a C^1 real-valued function $\theta(t)$ and $v_+ \in P_c H^1(\mathbb{R}^2)$ such that*

$$\lim_{t\to\infty} \dot\theta(t) = E_+, \tag{2.2}$$

$$\lim_{t\to\infty} \|u(t) - e^{i\theta(t)}\phi_{E_+} - e^{-itL}v_+\|_{H^1(\mathbb{R}^2)} = 0. \tag{2.3}$$

Remark 2. Let us decompose a solution to (1.1) into a solitary wave part and a radiation part:

$$u(t,x) = e^{-i\theta(t)}(\phi_{E(t)}(x) + v(t,x)). \tag{2.4}$$

If we take initial data in the energy class, the dispersive part of the solutions decays more slowly than they do for well localized initial data. Thus $\int_t^\infty \dot E(s)ds$ cannot be expected to be integrable as it is for localized initial data (see e.g. Soffer-Weinstein [44], [45] and Buslaev-Perelman [5]). In general, we need dispersive estimates for a time-dependent linearized equations to prove asymptotic stability of solitary waves in H^1. To avoid this difficulty, we assume the smallness of solitary waves so that a generalized kernel of the linearized operator is well approximated by a 1-dimensional subspace $\{\beta\phi_* \,|\, \beta \in \mathbb{C}\}$.

Substituting (2.4) into (1.1), we obtain

$$iv_t = Lv + g_1 + g_2 + g_3 + g_4, \tag{2.5}$$

where

$$g_1(t) = -\dot\theta(t)v(t), \quad g_2(t) = (E(t) - \dot\theta(t))\phi_{E(t)} - i\dot E(t)\partial_E\phi_{E(t)},$$
$$g_3(t) = f(\phi_{E(t)} + v(t)) - f(\phi_{E(t)}) - \partial_\varepsilon f(\phi_{E(t)} + \varepsilon v(t))|_{\varepsilon=0},$$
$$g_4(t) = \partial_\varepsilon f(\phi_{E(t)} + \varepsilon v(t))|_{\varepsilon=0} = \alpha\phi_{E(t)}^{p-1}\left(\frac{p+1}{2}v(t) + \frac{p-1}{2}\overline{v(t)}\right).$$

To fix the decomposition (2.4), we assume

$$\left\langle \Re v(t), \phi_{E(t)} \right\rangle = \left\langle \Im v(t), \partial_E \phi_{E(t)} \right\rangle = 0. \tag{2.6}$$

By Proposition 2.1, we have

$$|E(0) - E_*|^{1/(p-1)} + \|v(0)\|_{H^1} \lesssim \|u_0\|_{H^1}. \tag{2.7}$$

Since $u \in C(\mathbb{R}; H^1(\mathbb{R}^2))$, it follows from the implicit function theorem that there exist a $T > 0$ and $E, \theta \in C^1([-T,T])$ such that (2.6) holds for $t \in [-T,T]$. See, for example, [14] for the proof.

Differentiating (2.6) with respect to t and substituting (2.5) into the resulting equation, we obtain

$$\mathcal{A}(t) \begin{pmatrix} \dot{E}(t) \\ \dot{\theta}(t) - E(t) \end{pmatrix} = \begin{pmatrix} \langle \Im g_3(t), \phi_{E(t)} \rangle \\ \langle \Re g_3(t), \partial_E \phi_{E(t)} \rangle \end{pmatrix}, \tag{2.8}$$

where

$$\mathcal{A}(t) = \begin{pmatrix} \langle \partial_E \phi_{E(t)}, \phi_{E(t)} \rangle - \langle \Re v(t), \partial_E \phi_{E(t)} \rangle & \langle \Im v(t), \phi_{E(t)} \rangle \\ \langle \Im v(t), \partial_E^2 \phi_{E(t)} \rangle & \langle \partial_E \phi_{E(t)}, \phi_{E(t)} \rangle + \langle \Re v(t), \partial_E \phi_{E(t)} \rangle \end{pmatrix}.$$

To prove our main result, we will use the Strichartz estimate and the local smoothing effect of Kato type that is global in time. The Strichartz estimate follows from L^∞-L^1 estimate for 2-dimensional Schrödinger equations with linear potential obtained by Schlag [39]. See, for example, [18]. We say that (q,r) is *admissible* if q and r satisfy $2 < q \le \infty$, $2 \le r < \infty$ and $1/q + 1/r = 1/2$. For any $p \in [1,\infty]$, we denote by p' a Hölder conjugate exponent of p.

Lemma 2.1 (Strichartz estimate)**.** *Assume* (*H*1)–(*H*3).

(*a*) *Suppose that (q,r) is admissible. Then there exists a positive number C such that for every $f \in L^2(\mathbb{R})$,*

$$\|e^{-itL} P_c f\|_{L_t^q L_x^r} \le C \|f\|_{L^2}.$$

Furthermore, it holds that

$$\left\| \int_{\mathbb{R}} e^{isL} P_c g(s,\cdot) ds \right\|_{L_x^2} \le C \|g\|_{L_t^{q'} L_x^{r'}}.$$

(*b*) *Suppose that (q_1,r_1) and (q_2,r_2) are admissible. Then there exists a positive number C such that for every $g(t,x) \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$,*

$$\left\| \int_0^t e^{-i(t-s)L} P_c g(s,\cdot) ds \right\|_{L_t^{q_1} L_x^{r_1}} \le C \|g\|_{L_t^{q_2'} L_x^{r_2'}}.$$

Since Lemma 2.1 (a) does not hold with $q = 2$, we use the following local smoothing estimates to show that dE/dt is integrable with respect to t.

Lemma 2.2. *Assume (H1)–(H3). Let $s > 1$. Then there exists a positive constant C such that*

$$\|e^{-itL}P_cf\|_{L_t^2 L_x^{2,-s}} \le C\|f\|_{L^2}, \tag{2.9}$$

for every $f \in \mathcal{S}(\mathbb{R}^2)$ and that

$$\left\| \int_{\mathbb{R}} e^{isL}P_c g(s,\cdot)ds \right\|_{L_x^2} \le C\|g\|_{L_t^2 L_x^{2,s}}, \tag{2.10}$$

for every $g(t,x) \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^2)$.

Lemma 2.3. *Let $s > 1$. Then there exists a positive constant C such that*

$$\left\| \int_0^t e^{-i(t-s)L}P_c g(s,\cdot)ds \right\|_{L_t^2 L_x^{2,-s}} \le C\|g\|_{L_t^2 L_x^{2,s}}. \tag{2.11}$$

for every $g(t,x) \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^2)$ and $t \in \mathbb{R}$.

Since the linear term g_4 in (2.5) may not belong to $L_t^{q'} L_x^{r'}$ for admissible (q,r) (because $(q_2, r_2) = (2,\infty)$ is not admissible), we cannot apply Lemma 2.1 (b) to g_4. Instead, we will use the following to deal with g_4.

Corollary 2.1. *Let (q,r) be admissible and let $s > 1$. Then there exists a positive number C such that*

$$\left\| \int_{\mathbb{R}} e^{-i(t-s)L}P_c g(s,\cdot)ds \right\|_{L_t^q L_x^r} \le C\|g\|_{L_t^2 L_x^{2,s}} \tag{2.12}$$

for every $g(t,x) \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^2)$ and $t \in \mathbb{R}$.

Using a lemma by Christ and Kiselev [9], we see that Corollary 2.1 immediately follows from Lemmas 2.1 and 2.2 (see [43]).

The proof of Lemmas 2.2, 2.3 and Corollary 2.1 will be given in Section 4.

3. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1. To eliminate g_1 in (2.5), we put

$$w(t) = e^{-i\theta(t)}v(t). \tag{3.1}$$

Then (2.5) is translated into the integral equation

$$w(t) = e^{-itL}w(0) - i\sum_{2\le j\le 4}\int_0^t e^{-i(t-s)L}e^{-i\theta(s)}g_j(s)ds. \tag{3.2}$$

All nonlinear terms in (3.2) can be estimated in terms of the following.

$$\mathbb{M}_1(T) = \sup_{0\le t\le T} |E(t) - E_*|, \quad \mathbb{M}_2(T) = \|\langle x\rangle^{-s} P_c w\|_{L^2_t(0,T;H^1_x)},$$
$$\mathbb{M}_3(T) = \|\langle x\rangle^{-s} P_d w\|_{L^2_t(0,T;H^1_x)},$$
$$\mathbb{M}_4(T) = \sup_{0\le t\le T} \|P_c w(t)\|_{H^1} + \|P_c w\|_{L^q_t(0,T;W^{1,2p}_x)},$$
$$\mathbb{M}_5(T) = \sup_{0\le t\le T} \|P_d w(t)\|_{H^1} + \|P_d w\|_{L^q_t(0,T;W^{1,2p}_x)}.$$

where $2/q = 1 - 1/p$.

Proof of Theorem 2.1. By Proposition 2.1, Remark 1 and (2.6),

$$\langle \partial_E \phi_E, \phi_E\rangle = O(|E - E_*|^{2/(p-1)-1}), \quad |\langle v, \partial_E^i \phi_E\rangle| \lesssim |E - E_*|^{p/(p-1)-i}\|v\|_{L^2}.$$

Thus by (2.8), we have

$$|\dot\theta(t) - E(t)| \lesssim \|\phi_{2,E(t)} v^2\|_{L^1} + \|\phi_{2,E(t)} f(v)\|_{L^1}, \tag{3.3}$$
$$|\dot E(t)| \lesssim \|\phi_{1,E(t)} v^2\|_{L^1} + \|\phi_{1,E(t)} f(v)\|_{L^1}. \tag{3.4}$$

Suppose that the decomposition (2.4) with (2.6) persists for $0 \le t \le T$ and that $\mathbb{M}_i(T)$ $(1 \le i \le 5)$ are bounded. Eqs. (3.3)–(3.4) imply that

$$\begin{aligned}
&\|\dot\theta - E\|_{L^1(0,T)} + \|\dot E\|_{L^1(0,T)}\\
&\quad\le C(\mathbb{M})(\|\phi_{1,E(t)} v^2\|_{L^1(0,T;L^1_x)} + \|\phi_{2,E(t)} v^2\|_{L^1(0,T;L^1_x)})\\
&\qquad + C(\mathbb{M})(\|\phi_{1,E(t)} f(v)\|_{L^1(0,T;L^1_x)} + \|\phi_{2,E(t)} f(v)\|_{L^1(0,T;L^1_x)})\\
&\quad\le C(\mathbb{M}) \left(\sum_{i=1,2} \left\|\langle x\rangle^{2s}\phi_{i,E(t)}\right\|_{L^\infty(0,T;L^\infty_x)}\right) \|v\|^2_{L^2_t(0,T;H^{1,-s}_x)}\\
&\quad\le C(\mathbb{M})(\mathbb{M}_2(T) + \mathbb{M}_3(T))^2,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\|\dot\theta - E\|_{L^\infty(0,T)} + \|\dot E\|_{L^\infty(0,T)} &\lesssim \sup_{0\le t\le T} (\|v\|^2_{H^1} + \|v\|^p_{H^1})\\
&\le C(\mathbb{M})(\mathbb{M}_4(T) + \mathbb{M}_5(T))^2.
\end{aligned} \tag{3.6}$$

Hereafter we denote by $C(\mathbb{M})$ various functions of $\mathbb{M}_1, \ldots, \mathbb{M}_5$ that are bounded in a finite neighborhood of 0. By (2.7) and (3.5),

$$\mathbb{M}_1(T) \lesssim \|u_0\|_{H^1} + C(\mathbb{M})(\mathbb{M}_2 + \mathbb{M}_3)^2. \tag{3.7}$$

From Remark 1 and (2.6), it follows that

$$\begin{aligned}
|\langle w(t), \phi_*\rangle| &\le \|v\|_{L^{2,-s}_x} \sum_{i=1,2} \|\langle x\rangle^s(\phi_{i,E} - \phi_*)\|_{L^2}\\
&\lesssim |E(t) - E_*|\|w\|_{L^{2,-s}_x},
\end{aligned}$$

and that

$$\mathbb{M}_3(T) \le C(\mathbb{M})\mathbb{M}_1(T)(\mathbb{M}_2(T) + \mathbb{M}_3(T)). \tag{3.8}$$

Similarly, we have

$$\mathbb{M}_5(T) \le C(\mathbb{M})\mathbb{M}_1(T)(\mathbb{M}_4(T) + \mathbb{M}_5(T)). \tag{3.9}$$

Next, we will estimate $\mathbb{M}_2(T)$. By (3.2),

$$\mathbb{M}_2(T) \le I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= \|e^{-itL} P_c w(0)\|_{L_t^2(0,T;H_x^{1,-s})}, \\
I_2 &= \left\| \int_0^t e^{-i(t-s)L} P_c g_2(s) ds \right\|_{L_t^2(0,T;H_x^{1,-s})}, \\
I_3 &= \left\| \int_0^t e^{-i(t-s)L} P_c f(v(s)) ds \right\|_{L_t^2(0,T;H_x^{1,-s})}, \\
I_4 &= \left\| \int_0^t e^{-i(t-s)L} P_c \tilde{g}(s) ds \right\|_{L_t^2(0,T;H_x^{1,-s})},
\end{aligned}$$

and $\tilde{g}(s) = g_3(s) + g_4(s) - f(v(s))$. Lemma 2.2 yields

$$I_1 \lesssim \|w(0)\|_{H^1}.$$

By Lemma 2.3, (3.5) and (3.6),

$$\begin{aligned}
I_2 &\lesssim \|P_c g_2\|_{L_t^2(0,T;H_x^{1,s})} \\
&\le \left\|P_c \phi_{E(t)}\right\|_{L^\infty(0,T;H_x^{1,s})} \|\dot{\theta} - E\|_{L^2(0,T)} + \left\|P_c \partial_E \phi_{E(t)}\right\|_{L^\infty(0,T;H_x^{1,s})} \|\dot{E}\|_{L^2(0,T)} \\
&\le C(\mathbb{M})\mathbb{M}_1(T)^{1/(p-1)}(\mathbb{M}_2(T) + \mathbb{M}_3(T) + \mathbb{M}_4(T) + \mathbb{M}_5(T))^2.
\end{aligned}$$

Note that $\|P_c \partial_E \phi_E\|_{H^1} \lesssim |E - E_*|^{1/(p-1)}$ follows from Proposition 2.1. By Minkowski's inequality and Lemma 2.2,

$$\begin{aligned}
I_3 &\lesssim \int_0^T \|e^{-i(t-\tau)L} P_c f(v(\tau))\|_{L_t^2(0,T;H_x^{1,-s})} d\tau \\
&\lesssim \int_0^T \|f(v(s))\|_{H_x^1} ds \\
&\lesssim \|v\|^q_{L^q(0,T;W_x^{1,2p})} \|v\|^{p-q}_{L_t^\infty(0,T;H_x^1)},
\end{aligned}$$

where $2/q + 1/p = 1$. Note that $p \ge q > 2$ if $p \ge 3$. Thus we have

$$I_3 \le C(\mathbb{M})(\mathbb{M}_4(T) + \mathbb{M}_5(T))^p.$$

Since $\tilde g = O(\phi_E^{p-1}|v| + \phi_E|v|^{p-1})$, Lemma 2.3 yields that

$$
\begin{aligned}
I_4 &\lesssim \|\tilde g\|_{L_t^2(0,T;H_x^{1,s})} \\
&\lesssim \|\langle x\rangle^{2s}\phi_{E(t)}^{p-1}\|_{L_t^\infty(0,T;W_x^{1,\infty})}\|v\|_{L_t^2(0,T;H_x^{1,-s})} \\
&\quad + \|\langle x\rangle^{s}\phi_{E(t)}\|_{L_t^\infty(0,T;W_x^{1,\infty})}\||v|^{p-1}\|_{L_t^2(0,T;H_x^1)}.
\end{aligned}
\tag{3.10}
$$

Since

$$\||v|^{p-1}\|_{H^1} \le \|v\|_{W^{1,2(p-1)/(p-2)}}\||v|^{p-2}\|_{L^{2(p-1)}} \lesssim \|v\|^{p-1}_{W^{1,2(p-1)/(p-2)}},$$

it follows from (3.10), Proposition 2.1 and the interpolation theorem that

$$
\begin{aligned}
I_4 &\le C(\mathbb{M})\left(\|v\|_{L_t^2(0,T;H_x^{1,-s})} + \|v\|^{p-1}_{L_t^{2(p-1)}\left(0,T;W_x^{1,2(p-1)/(p-2)}\right)}\right) \\
&\le C(\mathbb{M})\{\mathbb{M}_1(T)(\mathbb{M}_2(T)+\mathbb{M}_3(T)) + (\mathbb{M}_4(T)+\mathbb{M}_5(T))^{p-1}\}.
\end{aligned}
$$

Combining the above, we see that

$$\mathbb{M}_2(T) \le C(\mathbb{M}) \sum_{1\le i\le 5} \mathbb{M}_i(T)^2. \tag{3.11}$$

Finally, we will estimate $\mathbb{M}_4(T)$. In view of (3.2),

$$\mathbb{M}_4(T) \le J_1 + J_2 + J_3,$$

where

$$
\begin{aligned}
J_1 &= \left\|e^{-itL}P_c w(0)\right\|_{L^\infty(0,T;H_x^1)\cap L^q(0,T;W_x^{1,2p})} \\
J_2 &= \left\|\int_0^t e^{-i(t-s)L}P_c g_2(s)ds\right\|_{L^\infty(0,T;H_x^1)\cap L^q(0,T;W_x^{1,2p})}, \\
J_2 &= \left\|\int_0^t e^{-i(t-s)L}P_c (g_3(s)+g_4(s))ds\right\|_{L^\infty(0,T;H_x^1)\cap L^q(0,T;W_x^{1,2p})}.
\end{aligned}
$$

Using the Strichartz estimate (Lemma 2.1), we have

$$
\begin{aligned}
J_1 &\lesssim \|w(0)\|_{H^1}, \\
J_2 &\lesssim \|P_c g_2(s)\|_{L_t^1(0,T;H_x^1)}ds \\
&\lesssim \|\dot\theta - E\|_{L^1(0,T)} \sup_{t\in[0,T]}\|P_c\phi_{E(t)}\|_{H_x^1} + \|\dot E\|_{L^1(0,T)}\sup_{t\in[0,T]}\|P_c\partial_E\phi_{E(t)}\|_{H_x^1}.
\end{aligned}
$$

Hence by (3.5),

$$J_2 \le C(\mathbb{M})(\mathbb{M}_2(T)^2 + \mathbb{M}_3(T)^2).$$

Using the Strichartz estimate and Corollary 2.1, we have

$$J_3 \lesssim \|g_3 + g_4\|_{L_t^1(0,T;H_x^1)+L_t^2(0,T;H_x^{1,s})}.$$

Since $g_3(t)+g_4(t)=O(\phi_{E(t)}^{p-1}|v|+|v|^p)$,

$$\begin{aligned}&\|g_3+g_4\|_{L_t^1(0,T;H_x^1)+L_t^2(0,T;H_x^{1,s})}\\ &\quad\lesssim \|\phi_{E(t)}^{p-1}v\|_{L_t^2(0,T;H_x^{1,s})}+\|f(v)\|_{L_t^1(0,T;H_x^1)}\\ &\quad\lesssim \|\langle x\rangle^{2s}\phi_{E(t)}^{p-1}\|_{L_t^\infty(0,T;W_x^{1,\infty})}\|v\|_{L_t^2(0,T;H_x^{1,-s})}+\|v\|^q_{L_t^q(0,T;W_x^{1,2p})}\|v\|^{p-q}_{L_t^\infty(0,T;H_x^1)}.\end{aligned}$$

Thus we have

$$J_3\le C(\mathbb{M})\{\mathbb{M}_1(T)(\mathbb{M}_4(T)+\mathbb{M}_5(T))+(\mathbb{M}_4(T)+\mathbb{M}_5(T))^p\}.$$

Combining the above, we have

$$\mathbb{M}_4(T)\le C(\mathbb{M})\sum_{1\le i\le 5}\mathbb{M}_i(T)^2. \tag{3.12}$$

It follows from (3.7)–(3.9), (3.11) and (3.12) that if ε_0 is sufficiently small,

$$\sum_{1\le i\le 5}\mathbb{M}_i(T)\lesssim \|u_0\|_{H^1}. \tag{3.13}$$

Thus by continuation argument, we may let $T\to\infty$. By (3.5), there exists an $E_+<0$ satisfying $\lim_{t\to\infty}E(t)=E_+$ and $|E_+-E_*|\lesssim\|u_0\|_{H^1}$. In view of (3.13), we see that

$$w_1:=-i\lim_{t\to\infty}\sum_{2\le j\le 4}\int_0^t e^{isL}P_ce^{-i\theta(s)}g_j(s)ds$$

exists in H^1 and that

$$\begin{aligned}\|w_1\|_{H^1}&\lesssim\|g_2(s)\|_{L_t^1H_x^1}+\|g_3+g_4\|_{L_t^2H_x^{1,s}+L_t^1H_x^1}\\ &\lesssim\|u_0\|_{H^1},\\ \lim_{t\to\infty}&\|P_cw(t)-e^{-itL}(P_cw(0)+w_1)\|_{H^1}=0.\end{aligned}$$

By [39], we have $\|e^{-itL}P_cf\|_{L^4}\lesssim t^{-1/2}\|f\|_{L^{4/3}}$. Since $L^{4/3}(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, it follows that $\|e^{-itL}(P_cw(0)+w_1)\|_{L^4}\to 0$ as $t\to\infty$, and that

$$\begin{aligned}&\|P_cw(t)\|_{L^4}\\ &\quad\le\|P_cw(t)-e^{-itL}(P_cw(0)+w_1)\|_{H^1}+\|P_ce^{-itL}(P_cw(0)+w_1)\|_{L^4}\\ &\quad\to 0\quad\text{as } t\to\infty.\end{aligned} \tag{3.14}$$

Analogously to (3.8), we have

$$\|P_dw(t)\|_{H^1}\lesssim\|P_dw(t)\|_{L^4}\lesssim|E(t)-E_*|\|P_cw(t)\|_{L^4}. \tag{3.15}$$

Combining (3.14) and (3.15), we have $\lim_{t\to\infty}\|P_d w(t)\|_{H^1}=0$. Thus by (2.4) and (3.1),

$$\lim_{t\to\infty}\left\|u(t)-e^{-i\theta(t)}\phi_{E(t)}-e^{-itL}P_c(w(0)+w_1)\right\|_{H^1}=0.$$

Thus we complete the proof of Theorem 2.1. □

4. Dispersive estimates

Let $R(\lambda)=(L-\lambda)^{-1}$ and $dE_{ac}(\lambda)$ be the absolute continuous part of the spectrum measure. By the spectral decomposition theorem, we have

$$\begin{aligned}P_ce^{-itL}f&=\int_{-\infty}^{\infty}e^{-it\lambda}dE_{ac}(\lambda)f\\&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{-it\lambda}P_c(R(\lambda+i0)-R(\lambda-i0))fd\lambda.\end{aligned}\tag{4.1}$$

We will prove Lemma 2.2 by using Plancherel's theorem and the following estimate on the resolvent $R(\lambda)$.

Lemma 4.1. *Let $s>1$. Then there exists a positive constant C such that*

$$\|R(\lambda\pm i0)P_cf\|_{L^2_\lambda(0,\infty;L^{2,-s}_x)}\le C\|f\|_{L^2}$$

for every $f\in L^2(\mathbb{R}^2)$.

First, we prove Lemma 2.2 assuming Lemma 4.1.

Proof of Lemma 2.2. By the inversion of the Laplace formula (see [30]), we have

$$\begin{aligned}e^{-itL}P_cf&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\lambda e^{-it\lambda}(R(\lambda+i0)-R(\lambda-i0))P_cf\\&=\frac{(it)^{-j}}{2\pi i}\int_{-\infty}^{\infty}d\lambda e^{-it\lambda}\partial_\lambda^j(R(\lambda+i0)-R(\lambda-i0))P_cf\quad\text{in }\mathcal{S}'_x(\mathbb{R}^2)\end{aligned}$$

for any $t\ne 0$ and $f\in\mathcal{S}_x(\mathbb{R}^2)$. Since

$$\|\partial_\lambda^jR(\lambda\pm i0)P_c\|_{B(L^{2,j+1/2+0},L^{2,-(j+1/2)-0})}\lesssim\langle\lambda\rangle^{-(j+1)/2},$$

the above integral absolutely converges in $L_x^{2,-(j+1/2)-0}$ for $j\ge 2$.

Suppose $g(t,x)=g_1(t)g_2(x)$, $g_1\in C_0^\infty(\mathbb{R}\setminus\{0\})$ and $g_2\in\mathcal{S}(\mathbb{R}^2)$. Making use of Fubini's theorem and integration by parts, we have for $j\ge 2$,

$$\begin{aligned}&\langle e^{-itL}P_cf,g\rangle_{t,x}\\&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dt(it)^{-j}g_1(t)\int_{-\infty}^{\infty}d\lambda e^{-it\lambda}\partial_\lambda^j\left\langle(R(\lambda+i0)-R(\lambda-i0))P_cf,g_2\right\rangle_x\\&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\lambda\partial_\lambda^j\left\langle(R(\lambda+i0)-R(\lambda-i0))P_cf,g_2\right\rangle_x\int_{-\infty}^{\infty}dt(it)^{-j}g_1(t)e^{-it\lambda}\\&=\frac{1}{\sqrt{2\pi}i}\int_{-\infty}^{\infty}d\lambda(\mathcal{F}_tg_1)(\lambda)\left\langle(R(\lambda+i0)-R(\lambda-i0))P_cf,g_2\right\rangle_x.\end{aligned}$$

Hence it follows from the above that

$$\left\langle e^{-itL}P_cf, g\right\rangle = \frac{1}{\sqrt{2\pi}i}\int_{-\infty}^{\infty} d\lambda \left\langle (R(\lambda+i0)-R(\lambda-i0))P_cf, \mathcal{F}_tg(\lambda,\cdot)\right\rangle_x$$

for every $g \in C_0^\infty(\mathbb{R}_t \setminus \{0\}) \otimes \mathcal{S}(\mathbb{R}_x^2)$. Using Plancherel's theorem, we have

$$\begin{aligned}
&\left|\langle e^{-itL}P_cf, g\rangle_{t,x}\right| \\
&\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\lambda \|(R(\lambda+i0)-R(\lambda-i0))P_cf\|_{L_x^{2,-s}}\|\mathcal{F}_tg(\lambda,\cdot)\|_{L_x^{2,s}} \\
&\le (2\pi)^{-1/2}\|(R(\lambda+i0)-R(\lambda-i0))P_cf\|_{L_\lambda^2(0,\infty;L_x^{2,-s})}\|g\|_{L_t^2L_x^{2,s}}.
\end{aligned} \tag{4.2}$$

Since $C_0^\infty(\mathbb{R}_t\setminus\{0\})\otimes\mathcal{S}(\mathbb{R}_x^2)$ is dense in $L_t^2L_x^{2,s}$, Lemma 2.2 immediately follows from (4.2). □

Now, we turn to prove Lemma 4.1. First, we will investigate the free resolvent operator $R_0(\lambda)$ in $\mathbb{R}^2$.

Lemma 4.2. *There exists a positive constant C such that*

$$\sup_x \|R_0(\lambda \pm i0)f\|_{L_\lambda^2(0,\infty)} \le C\|f\|_{L^2}$$

for every $f \in L^2(\mathbb{R}^2)$.

Remark 3. Obviously, the estimate $\|R_0(\lambda\pm i0)\|_{B(L^{2,s},L^{2,-s})} \lesssim \langle\lambda\rangle^{-1/2}$ does not suffice to prove Lemma 4.2. We will use the boundedness of the Hankel transform in L^2_{rad}.

Proof of Lemma 4.2. For any $k \ge 0$, we have

$$R_0(k^2 \pm i0)f(x) = \frac{\pm i}{4}\int_{\mathbb{R}^2} H_0(k|x-y|)f(y)dy,$$

where $H_0^\pm$ are the Hankel functions of order 0 and

$$H_0^\pm(z) = J_0(z) \pm Y_0(z).$$

Let $(\tau_xf)(y) := f(x-y)$ and decompose $\tau_xf \in L^2(\mathbb{R}^2)$ into a Fourier series as

$$\tau_xf = \sum_{m\in\mathbb{Z}} f_{x,m}(r)e^{im\theta} \in \bigoplus_{m\in\mathbb{Z}} e^{im\theta}L^2_{rad}.$$

Then

$$\begin{aligned}
R_0(k^2 \pm i0)f(x) &= \frac{\pm i}{4}\int_{\mathbb{R}^2} H_0^\pm(k|y|)\tau_xf(y)dy \\
&= \frac{\pm\pi i}{2}\int_0^\infty H_0^\pm(kr)f_{x,0}(r)rdr.
\end{aligned}$$

Titchmarsh [49] and Rooney [36] tell us that the operators T_1 and T_2 defined by

$$T_1 f(x) = \int_0^\infty J_0(xy)f(y)dy, \quad T_2 f(x) = \int_0^\infty Y_0(xy)f(y)dy,$$

are bounded on L^2_{rad}. Thus we have

$$\sup_x \left(\int_0^\infty |R_0(k^2 \pm i0)f|^2 kdk \right)^{1/2} \lesssim \|f_{x,0}\|_{L^2_{rad}}.$$

Since

$$\|f\|_{L^2} = \|\tau_x f\|_{L^2} = \left(2\pi \sum_{m\in\mathbb{Z}} \int_0^\infty |f_{x,m}(r)|^2 rdr \right)^{1/2},$$

it follows that

$$\sup_x \|R_0(\lambda \pm i0)f\|_{L^2_\lambda(0,\infty)} \lesssim \|f\|_{L^2}.$$

Thus we complete the proof of Lemma 4.2. □

We will prove Lemma 4.1 by using Lemma 4.2 and the resolvent expansion obtained by Schlag [39] based on Jensen and Nenciu [15].

Before we prove Lemma 4.1, let us introduce a definition of the non-resonance condition given by Jensen and Nenciu [15].

Definition 4.1. Let $v(x) = |V(x)|^{1/2}$ and let P and Q be orthogonal projections defined by

$$Pf = \frac{\langle f, v\rangle v}{\|V\|_{L^1}}, \quad Q = I - P.$$

We say that 0 is not a resonance of L if $D_0 := Q(U + vG_0v)Q$ is invertible on $QL^2(\mathbb{R}^2)$.

Proof of Lemma 4.1. For every $f \in \mathcal{S}(\mathbb{R}^2)$, we have

$$R(\lambda \pm i0)f = R_0(\lambda \pm i0)f - R_0(\lambda \pm i0)VR(\lambda \pm i0)f. \tag{4.3}$$

By Lemma 4.2, there exists a $C > 0$ such that for every $f \in L^2(\mathbb{R}^2)$,

$$\begin{aligned} \|R_0(\lambda \pm i0)f\|_{L^{2,-s}_x L^2_\lambda(0,\infty)} &\le \|\langle x\rangle^{-s}\|_{L^2} \|R_0(\lambda \pm i0)f\|_{L^\infty_x L^2_\lambda(0,\infty)} \\ &\le C\|f\|_{L^2}. \end{aligned} \tag{4.4}$$

Next, we deal with the low energy part of the second term of (4.3). As [15, 39], we put $U(x) = 1$ for $x \in V^{-1}([0,\infty))$, $U(x) = -1$ for $x \in V^{-1}((-\infty,0))$, and $M^\pm(\lambda) := U + vR_0(\lambda \pm i0)v$. Then

$$R_0(\lambda \pm i0)VR(\lambda \pm i0)f = R_0(\lambda \pm i0)vM^\pm(\lambda)^{-1}vR_0(\lambda \pm i0)f.$$

Schlag [39, Lemma 9] tells us that

$$M^{\pm}(\lambda)^{-1} = h_{\pm}(\lambda)^{-1}S + QD_0Q + E^{\pm}(\lambda) \quad \text{in } B(L^2(\mathbb{R}^2)), \tag{4.5}$$

where S is a finite rank operator, $\|E^{\pm}(\lambda)\|_{B(L^2)} = O(\lambda^{1/4})$ as $\lambda \to 0$, and

$$h_+(\lambda) = a\log\lambda + z, \quad h_-(\lambda) = \overline{h_+(\lambda)}, \tag{4.6}$$

and $a \in \mathbb{R}$ and $z \in \mathbb{C}$ are constants with $a \neq 0$ and $\Im z \neq 0$.

Let λ_1 be a sufficiently small positive number. From [39, Lemma 5], it follows that for $0 < \lambda \leq \lambda_1$,

$$R_0(\lambda \pm i0) = c_{\pm}(\lambda)P_0 + G_0 + E_0^{\pm}(\lambda) \quad \text{in } B(L^{2,s}, L^{2,-s}), \tag{4.7}$$

and

$$\|E_0^{\pm}(\lambda)\|_{B(L^{2,s},L^{2,-s})} = O(\lambda^{1/4}), \tag{4.8}$$

where $P_0 f = \langle f, 1\rangle_x$, $G_0 = (-\Delta)^{-1}$, γ is the Euler number and

$$c_{\pm}(\lambda) = \pm\frac{i}{4} - \frac{\gamma}{2\pi} - \frac{1}{4\pi}\log\left(\frac{\lambda}{4}\right). \tag{4.9}$$

Thus $\widetilde{R}_0^{\pm}(\lambda) = R_0(\lambda \pm i0) - c_{\pm}(\lambda)P_0$ satisfies

$$\sup_{0<\lambda<\lambda_1} \|\widetilde{R}_0^{\pm}(\lambda)\|_{B(L^{2,s},L^{2,-s})} < \infty. \tag{4.10}$$

Let $\chi(\lambda)$ be a characteristic function on $[0, \lambda_1]$. Using Lemma 4.2, (4.5), (4.10) and the fact that $v(x) \lesssim \langle x\rangle^{-\sigma/2}$ with $\sigma > 3$, we have

$$\begin{aligned}
&\|\chi(\lambda)\widetilde{R}_0^{\pm}(\lambda)vM^{\pm}(\lambda)^{-1}vR_0(\lambda \pm i0)f\|_{L^2_\lambda(0,\infty;L^{2,-s}_x)} \\
&\quad\leq \sup_{0<\lambda<\lambda_1} \|\widetilde{R}_0^{\pm}(\lambda)\|_{B(L^{2,s},L^{2,-s})} \left\| \|\chi(\lambda)vM^{\pm}(\lambda)^{-1}vR_0(\lambda \pm i0)f\|_{L^{2,s}_x} \right\|_{L^2_\lambda(0,\infty)} \\
&\quad\lesssim \|\chi(\lambda)vR_0(\lambda \pm i0)f\|_{L^2_{x,\lambda}} \\
&\quad\lesssim \|v\|_{L^2_x} \sup_x \|R_0(\lambda \pm i0)f\|_{L^2_\lambda(0,\infty)} \lesssim \|f\|_{L^2}
\end{aligned}$$

for any $s \in (1, 3/2)$. Since $P_0 vQ = 0$, it follows from (4.5) that

$$c^{\pm}(\lambda)P_0 vM^{\pm}(\lambda)^{-1}vR_0(\lambda \pm i0) = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= c^{\pm}(\lambda)h_{\pm}(\lambda)^{-1}P_0 vSvR_0(\lambda \pm i0), \\
I_2 &= c^{\pm}(\lambda)P_0 vE^{\pm}(\lambda)vR_0(\lambda \pm i0).
\end{aligned}$$

By (4.6), (4.9), $\sup_{0<\lambda\le\lambda_1}|c_\pm(\lambda)/h_\pm(\lambda)|<\infty$. Hence from Lemma 4.1,

$$\begin{aligned}\|I_1 f\|_{L^2_\lambda(0,\infty;L^{2,-s}_x)} &\le \|\langle x\rangle^{-s}\|_{L^2}\|vSvR_0(\lambda\pm i0)f\|_{L^2_\lambda(0,\infty;L^1_x)}\\ &\lesssim \|v\|_{L^2}\|vR_0(\lambda\pm i0)f\|_{L^2_\lambda(0,\infty;L^2_x)}\\ &\lesssim \|v\|_{L^2}^2\sup_x\|R_0(\lambda\pm i0)f\|_{L^2_\lambda(0,\infty)}\\ &\lesssim \|f\|_{L^2}.\end{aligned}$$

Using Schwarz's inequality and (4.8), we have

$$\begin{aligned}\|P_0vE^\pm(\lambda)vR_0(\lambda\pm i0)f\|_{L^{2,-s}_x} &\lesssim \|v\|_{L^2}\|E^\pm(\lambda)vR_0(\lambda\pm i0)f\|_{L^2}\\ &\lesssim |\lambda|^{1/4}\|vR_0(\lambda\pm i0)f\|_{L^2}.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\|\chi(\lambda)I_2\|_{L^2(0,\infty;L^{2,-s}_x)} &\lesssim \sup_{\lambda>0}\left(\chi(\lambda)|\lambda|^{1/4}|c_\pm(\lambda)|\right)\|vR_0(\lambda\pm i0)f\|_{L^2_{x,\lambda}}\\ &\lesssim \sup_x\|R_0(\lambda\pm i0)f\|_{L^2_\lambda}\\ &\lesssim \|f\|_{L^2}.\end{aligned}$$

Combining the above, we obtain

$$\|\chi(\lambda)R_0(\lambda\pm i0)VR(\lambda\pm i0)f\|_{L^2_\lambda(0,\infty;L^{2,-s}_x)}\lesssim\|f\|_{L^2}. \tag{4.11}$$

Next, we consider the high energy part. The assumptions (H2) and (H3) imply that

$$\sup_{\lambda\ge\lambda_1}\|R(\lambda\pm i0)P_c\|_{B(L^{2,s},L^{2,-s})}\lesssim\langle\lambda_1\rangle^{-1/2}, \tag{4.12}$$

See [1, Appendix A] and [27] for the proof. Let $\tilde\chi(\lambda)=1-\chi(\lambda)$. By (4.12) and Fubini's theorem,

$$\begin{aligned}&\|\tilde\chi(\lambda)P_cR(\lambda\pm i0)VR_0(\lambda\pm i0)f\|_{L^2_\lambda(0,\infty;L^{2,-s}_x)}\\ &\quad\lesssim \left\|\|VR_0(\lambda\pm i0)f\|_{L^{2,s}_x}\right\|_{L^2_\lambda(0,\infty)}\\ &\quad\le \|V\|_{L^{2,s}}\sup_x\|R_0(\lambda\pm i0)f\|_{L^2_\lambda(0,\infty)}\lesssim\|f\|_{L^2}.\end{aligned} \tag{4.13}$$

Combining (4.3), (4.4), (4.11) and (4.13), we obtain

$$\|R(\lambda\pm i0)P_cf\|_{L^2_\lambda(0,\infty;L^{2,-s}_x)}\le C\|f\|_{L^2}.$$

Thus we complete the proof of Lemma 4.1. □

Next, we will prove Lemma 2.3. For the purpose, we need the following.

Lemma 4.3. *Assume (H1)–(H3). Let $g(t,x) \in \mathcal{S}_t(\mathbb{R}) \otimes \mathcal{S}_x(\mathbb{R}^2)$ and*

$$U(t,x) = \frac{1}{\sqrt{2\pi} i} \int_{-\infty}^{\infty} d\lambda e^{-it\lambda} \{R(\lambda - i0) + R(\lambda + i0)\} P_c(\mathcal{F}_t^{-1} g)(\lambda, \cdot).$$

Then,

$$\begin{aligned} U(t,x) = 2 \int_0^t ds e^{-i(t-s)L} P_c g(s,\cdot) + \int_{-\infty}^0 ds e^{-i(t-s)L} P_c g(s,\cdot) \\ - \int_0^\infty ds e^{-i(t-s)L} P_c g(s,\cdot). \end{aligned}$$

Proof. Since Lemma 4.3 can be proved in the same as that of Lemma 11 in [26], we omit the proof. □

Proof of Lemma 2.3. Suppose that $g(t,x)$ and $h(t,x)$ belong to $\mathcal{S}_t(\mathbb{R}) \otimes \mathcal{S}_x(\mathbb{R}^2)$. It follows from Fubini's theorem that

$$\begin{aligned} &\langle U, h\rangle_{t,x} \\ &= \frac{1}{\sqrt{2\pi} i} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} dt e^{-it\lambda} \left\langle (R(\lambda+i0) + R(\lambda - i0)) P_c \mathcal{F}_t^{-1} g(\lambda,\cdot), h(t,\cdot) \right\rangle_x \\ &= i^{-1} \int_{-\infty}^{\infty} d\lambda \left\langle (R(\lambda+i0) + R(\lambda-i0)) P_c \mathcal{F}_t^{-1} g(\lambda,\cdot), \mathcal{F}_t h(\lambda,\cdot) \right\rangle_x . \end{aligned}$$

Using Plancherel's theorem and (4.12), we obtain

$$\begin{aligned} &|\langle U, h\rangle_{t,x}| \\ &\le \left\| (R(\lambda+i0) + R(\lambda-i0)) P_c \mathcal{F}_t^{-1} g(\lambda,\cdot) \right\|_{L^2_\lambda L^{2,-s}_x} \|\mathcal{F}_t h(\lambda,\cdot)\|_{L^2_\lambda L^{2,s}_x} \\ &\le \sup_{\lambda\in\mathbb{R}} \|(R(\lambda+i0) + R(\lambda-i0)) P_c\|_{B(L^{2,s}, L^{2,-s})} \|g\|_{L^2_t L^{2,-s}_x} \|h\|_{L^2_t L^{2,s}_x} \end{aligned}$$

Since $\mathcal{S}_t(\mathbb{R}) \otimes \mathcal{S}_x(\mathbb{R}^2)$ is dense in $L^2_t L^{2,s}_x$ and $L^2_t L^{2,-s}_x$, we see that

$$\|U\|_{L^2_t L^{2,-s}_x} \lesssim \|g\|_{L^2_t L^{2,s}_x} \tag{4.14}$$

holds for every $g \in L^2_t L^{2,s}_x$.

On the other hand, Lemma 2.2 implies

$$\left\| \int_I e^{-i(t-s)L} Q g(s) ds \right\|_{L^2_t L^{2,-s}_x} \lesssim \left\| \int_I e^{isL} g(s) ds \right\|_{L^2} \lesssim \|g\|_{L^2_t L^{2,s}_x}$$

for every $g \in L^2_t L^{2,s}_x$ and $I \subset \mathbb{R}$. Combining the above with (4.14) and Lemma 4.3, we obtain Lemma 2.3. Thus we complete the proof. □

Finally, we prove Corollary 2.1.

Proof of Corollary 2.1. Let (q,r) be admissible and let T be an operator defined by

$$Tg(t) = \int_{\mathbb{R}} ds e^{-i(t-s)L} P_c g(s).$$

Lemmas 2.1 and 2.2 yield $f := \int_{\mathbb{R}} e^{isL} P_c g(s)ds \in L^2(\mathbb{R})$ and that there exists a $C > 0$ such that

$$\|Tg(t)\|_{L_t^q L_x^r} \le C\|g\|_{L_t^2 L_x^{2,s}} \tag{4.15}$$

for every $g \in L_t^2 L_x^{2,s}$. Since $q > 2$, it follows from Lemma 3.1 in [43] and (4.15) that

$$\left\| \int_{s<t} ds e^{-i(t-s)L} P_c g(s) \right\|_{L_t^q L_x^r} \lesssim \|g\|_{L_t^2 L_x^{2,s}}. \tag{4.16}$$

Thus we prove Corollary 2.1. □

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