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Towards a general theory of unprojection

By

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Abstract

Unprojection is an effort, initiated by Miles Reid, to develop an algebraic language for the study of birational geometry. [Ki] contains motivation and examples, and poses the problem of developing a general theory of unprojection. The main purpose of the present work is to suggest a general definition of unprojection, and to show that it indeed generalizes previous work done in the topic. In addition, in Section 6 we present an unprojection analysis of an example of Reid and K. Suzuki, and Section 7 contains more examples.

1. Introduction

Birational geometry is an old and important field of algebraic geometry. Since the late 1970s there has been spectacular progress, especially in the establishment of the Mori minimal model program for threefolds due to work of S. Mori and many others. The methods of the Mori minimal model program are often abstract and cohomological, while explicit birational geometry [EBG] initiated by Reid and A. Corti aims to study the objects (such as Fano 3-folds) and the maps between them (such as birational contractions) in more detail on specific situations. Unprojection plays an important role in this study and has found many applications, for example in the birational geometry of Fano 3-folds [CPR] and [CM], in the construction of weighted complete intersection K3 surfaces and Fano 3-folds [Al], and in the study of Mori flips [BrR].

[Ki] discusses more examples and applications of unprojection, and poses the problem of developing a general theory of unprojection. The cases that have been studied so far are the unprojection of type Kustin–Miller (or type I) [KM], [PR] and [P], the generic case of type II unprojection [P2], and the generic case of type III unprojection [P3], while [R] contains examples of type IV unprojection. The definitions of unprojection in [P2] and [P3] apply only to their respective generic cases, while the definitions in [KM] and [PR] need strong Gorenstein assumptions.

In Section 2 we propose a general definition of unprojection (Definition 2.1), while in Section 3 we use the well–known general machinery of homological algebra to write down explicitly some of the constructions needed.

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In Section 4 we study in some detail the important special case of unprojection of an ideal, while in Section 5 we prove that Definition 2.1 indeed generalizes those of [PR], [P2] and [P3].

Section 6 presents an analysis of a construction of Reid and Suzuki [RS] which is an unprojection but is not the unprojection of an ideal. Finally, Section 7 contains more examples of unprojection analysis of rings appearing in geometry.

A very interesting open question stated in Remark 7 is to study under which conditions good properties of the unprojection initial data are preserved by the unprojection ring. Another open question is whether unprojection can be used for an inductive treatment of families of rings arising in geometry such as homogeneous coordinate rings of Grassmannians and other homogeneous spaces, cf. [KM, Section 2].

2. General definition of unprojection

Assume $\mathcal{O}_X$ is a commutative ring with unit, M is an $\mathcal{O}_X$-module, and

$$\phi\colon \mathcal{O}_X \to M$$

is a homomorphism of $\mathcal{O}_X$-modules. We assume that there exists an $\mathcal{O}_X$-regular element $q \in \mathcal{O}_X$ such that $qM = 0$. We will define an $\mathcal{O}_X$-algebra $\operatorname{unpr}_{\mathcal{O}_X} \phi$ which we will call the unprojection algebra of ϕ.

Since a map $M \to \mathcal{O}_X/(q)$ is a homomorphism of $\mathcal{O}_X$-modules if and only if is a homomorphism of $\mathcal{O}_X/(q)$-modules, by Rees lemma ([BH, Lemma 3.1.16]) there are canonical isomorphisms

$$\operatorname{Ext}^1_{\mathcal{O}_X}(M, \mathcal{O}_X) \cong \operatorname{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X/(q)),$$

and

$$\operatorname{Ext}^1_{\mathcal{O}_X}(\operatorname{Ext}^1_{\mathcal{O}_X}(M, \mathcal{O}_X), \mathcal{O}_X) \cong \operatorname{Hom}_{\mathcal{O}_X}(\operatorname{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X/(q)), \mathcal{O}_X/(q)).$$

In Section 3 we explicitly write down, using the well–known general machinery of homological algebra, how an extension induces a homomorphism and also how a homomorphism induces an extension.

As a consequence, composing the natural double dual map

$$M \to \operatorname{Hom}_{\mathcal{O}_X}(\operatorname{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X/(q)), \mathcal{O}_X/(q))$$

with ϕ and taking the value of the composition at $1_{\mathcal{O}_X} \in \mathcal{O}_X$ we obtain an extension

$$0 \to \mathcal{O}_X \xrightarrow{\ i\ } Q \to \operatorname{Ext}^1_{\mathcal{O}_X}(M, \mathcal{O}_X) \to 0. \tag{2.1}$$

Again from general principles of homological algebra, if we choose another $\mathcal{O}_X$-regular element q' with $q'M = 0$ we would get an extension isomorphic to (2.1).

We define the $\mathcal{O}_X$-algebra R_1 with

$$R_1 = \frac{\operatorname{Sym}_{\mathcal{O}_X} Q}{(i(1_{\mathcal{O}_X}) - 1)},$$

where $\operatorname{Sym}_{\mathcal{O}_X} Q$ is the symmetric algebra of the $\mathcal{O}_X$-module Q. We also define the multiplicatively closed subset $T \subset \mathcal{O}_X$

$$T = \{t \in \mathcal{O}_X : t \text{ is both } \mathcal{O}_X \text{ and } M\text{-regular } \} \subset \mathcal{O}_X.$$

Therefore, the set T consists of the elements of $\mathcal{O}_X$ which multiplied by any nonzero element of $\mathcal{O}_X$ or M give a nonzero element of the respective $\mathcal{O}_X$-module. In particular T contains the invertible elements of $\mathcal{O}_X$.

Definition 2.1. The unprojection algebra $\operatorname{unpr}_{\mathcal{O}_X} \phi$ is the $\mathcal{O}_X$-algebra

$$\operatorname{unpr}_{\mathcal{O}_X} \phi = R_1/J,$$

where

$$J = \{u \in R_1 : \text{ there exists } t \in T \text{ with } tu = 0 \in R_1\}.$$

Remark 1. An important case for the applications is when a morphism $\widetilde{\phi}\colon \widetilde{D} \to X$ of affine schemes having codimension one image in X is given. In this case we set

$$\phi\colon \mathcal{O}_X \to M = \widetilde{\phi}_* \mathcal{O}_{\widetilde{D}}$$

to be the homomorphism of $\mathcal{O}_X$-modules induced by the morphism $\widetilde{\phi}$, compare the Reid–Suzuki example in Section 6. A particular case, which we study in some detail in Section 4, is when $D \subset X$ is a codimension one subscheme and $\widetilde{\phi}\colon D \to X$ is the inclusion morphism.

3. The relation between extensions and homomorphisms

In the following $\mathcal{O}_X$ is a commutative ring, A, B are two $\mathcal{O}_X$-modules, and $q \in \mathcal{O}_X$ is an $\mathcal{O}_X$ and B-regular element such that $qA = 0$. By Rees lemma ([BH, Lemma 3.1.16])

$$\operatorname{Ext}^1_{\mathcal{O}_X}(A, B) \cong \operatorname{Hom}_{\mathcal{O}_X}(A, B/(q)).$$

We use the well–known general machinery of homological algebra to write down explicitly the correspondence between homomorphisms $A \to B/(q)$ and extensions $0 \to B \to Q \to A \to 0$ used in Section 2.

3.1. Construction of the homomorphism given an extension

Assume we are given an extension

$$0 \to B \xrightarrow{q_1} Q \xrightarrow{q_2} A \to 0. \tag{3.1}$$

We will define a homomorphism of $\mathcal{O}_X$-modules

$$g\colon A \to B/(q)$$

as follows.

Let $a \in A$. Choose lifting $\widetilde{a} \in Q$. Then $q\widetilde{a} \in \ker q_2$ (since $qA = 0$), so there exists unique $b \in B$ such that $q\widetilde{a} = b$ (equality in Q). We set

$$g(a) = b + (q) \in B/(q).$$

Assume $\widetilde{a}' \in Q$ is another lifting of a, and $b' \in B$ such that $q\widetilde{a}' = b'$ Since $\widetilde{a} - \widetilde{a}' \in \operatorname{Ker} q_2$ and (3.1) is exact, there exists $b_3 \in B$ with

$$\widetilde{a} - \widetilde{a}' = b_3$$

(equality in Q). As a consequence

$$b - b' = q(\widetilde{a} - \widetilde{a}') = qb_3,$$

hence

$$b + (q) = b' + (q),$$

(equality in $B/(q)$) and therefore g is well defined, independent of the choice of the lifting of a.

3.2. Construction of the extension given a homomorphism

Assume now we are given a homomorphism

$$g\colon A \to B/(q)$$

We will use g to define an extension

$$0 \to B \xrightarrow{q_1} Q \xrightarrow{q_2} A \to 0.$$

of $\mathcal{O}_X$-modules.

Fix a generating set $a_i, i \in I$ for A, denote by F the free $\mathcal{O}_X$-module with basis $e_i, i \in I$, and by

$$p_1\colon F \to A$$

the surjective $\mathcal{O}_X$-module homomorphism with $p_1(e_i) = a_i$.

Moreover, fix (arbitrary) set–theoretic lifting

$$\widetilde{g}\colon A \to B$$

of g.

Lemma 3.1. *Assume $r_i \in \mathcal{O}_X$ with $i \in I$ and all except a finite number of r_i equal to 0, such that $\sum_{i \in I} r_i a_i = 0$ in A. We then have*

$$\sum_{i \in I} r_i \widetilde{g}(a_i) \in (q) \subset B.$$

(The map $\widetilde{g}$ in general is not a homomorphism of $\mathcal{O}_X$-modules, so it may happen that $\sum_{i \in I} r_i \widetilde{g}(a_i) \neq 0 \in B$.)

Proof. Clear, since $\sum_{i\in I} r_i g(a_i) = 0 \in B/(q)$ and $\widetilde{g}$ is a lifting of g. $\square$

By Lemma 3.1, if $\sum_{i\in I} r_i a_i = 0$ in A, there exists unique (since q is $\mathcal{O}_X$-regular) $b \in B$ such that

$$qb = \sum_{i\in I} r_i \widetilde{g}(a_i)$$

in B. We define an $\mathcal{O}_X$-submodule $M \subset B \times F$ with

$$M = \left\{ \left(b, \sum_{i\in I} r_i e_i \right) \in B \times F \colon \sum_{i\in I} r_i a_i = 0 \in A \ \text{ and } \ qb = \sum_{i\in I} r_i \widetilde{g}(a_i) \in B \right\}$$

and we set

$$Q = (B \times F)/M.$$

Lemma 3.2. *We have an extension of $\mathcal{O}_X$-modules,*

$$0 \to B \xrightarrow{q_1} Q \xrightarrow{q_2} A \to 0.$$

where q_1 is the map

$$b \mapsto (b, 0) + M \in Q$$

and q_2 is the map

$$\left(b, \sum_{i\in I} r_i e_i \right) + M \mapsto \sum_{i\in I} r_i a_i \in A.$$

Proof. The map q_2 is well defined, since if $(b, \sum_{i\in I} r_i e_i) \in M$ we have $\sum_{i\in I} r_i a_i = 0 \in A$.

We also have that q_1 is injective. Indeed, assume $(b, 0) \in M$. This implies that $qb = 0 \in B$, and since q is B-regular we have $b = 0$.

It is clear that

$$q_2 \circ q_1 = 0.$$

Assume now $(b, \sum_{i\in I} r_i e_i) + M \in \ker q_2$. This implies that $\sum_{i\in I} r_i a_i = 0 \in A$. By Lemma 3.1 we have

$$\sum_{i\in I} r_i \widetilde{g}(a_i) \in (q) \subset B,$$

so there exists $b' \in B$ such that

$$qb' = qb - \sum_{i\in I} r_i \widetilde{g}(a_i)$$

(equality in B). As a result

$$\left(b - b', \sum_{i\in I} r_i e_i \right) \in M,$$

hence

$$\left(b, \sum_{i\in I} r_i e_i\right) + M = (b', 0) + M = q_1(b') \in \operatorname{Im} q_1.$$

It is clear that q_2 is surjective, which finishes the proof of the lemma. □

It is easy to check that the extension does not depend on the choices of the generating set $a_i, i \in I$ of A and of the lifting $\widetilde{g}$ of g.

4. Unprojection of an ideal

In the following, $\mathcal{O}_X$ is a commutative ring with unit, and $I_D \subset \mathcal{O}_X$ an ideal containing an $\mathcal{O}_X$-regular element $q \in I_D$. We set $\mathcal{O}_D = \mathcal{O}_X/I_D$ and denote by

$$\phi\colon \mathcal{O}_X \to \mathcal{O}_D$$

the natural projection map.

Definition 4.1. The unprojection algebra $\operatorname{unpr}_{\mathcal{O}_X} I_D$ of the ideal $I_D \subset \mathcal{O}_X$ is the $\mathcal{O}_X$-algebra

$$\operatorname{unpr}_{\mathcal{O}_X} I_D = \operatorname{unpr}_{\mathcal{O}_X} \phi,$$

where $\operatorname{unpr}_{\mathcal{O}_X} \phi$ has been defined in Definition 2.1.

It is easy to see that the extension corresponding to the projection map ϕ is just the natural short exact sequence

$$0 \to \mathcal{O}_X \to \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_D, \mathcal{O}_X) \to 0 \tag{4.1}$$

obtained by applying the derived functor of $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ to the exact sequence $0 \to I_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ (cf. [PR, p. 563]).

In particular, $Q = \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$, and hence we have that

$$\operatorname{unpr}_{\mathcal{O}_X} I_D = R_1/J,$$

where

$$R = \operatorname{Sym}_{\mathcal{O}_X} \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X),$$

$i \in \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is the inclusion map $I_D \to \mathcal{O}_X$, $R_1 = R/(i-1)$,

$$J = \{u \in R_1 \colon \text{ there exists } t \in T \text{ with } tu = 0 \in R_1\},$$

and $T \subset \mathcal{O}_X$ is the multiplicatively closed subset

$$T = \{t \in \mathcal{O}_X \colon t \text{ is both } \mathcal{O}_X \text{ and } \mathcal{O}_D\text{-regular } \} \subset \mathcal{O}_X.$$

Remark 2. In general, the ring $\operatorname{unpr}_{\mathcal{O}_X} I_D$ is not graded - for an important exception see Remark 3 below. However, the natural grading of R induces an increasing filtration

$$F_0 \subset F_1 \subset F_2 \subset \dots$$

of $\operatorname{unpr}_{\mathcal{O}_X} I_D$, where F_k is the image of the direct sum of the first $k+1$ graded components of R under the natural projection map $R \to \operatorname{unpr}_{\mathcal{O}_X} I_D$.

Remark 3. Let $\mathcal{O}_X$ be $\mathbb{Z}$-graded and $I_D \subset \mathcal{O}_X$ a homogeneous ideal. We call an $\mathcal{O}_X$-module homomorphism $\widetilde{s}\colon I_D \to \mathcal{O}_X$ graded, if there exists $k \in \mathbb{Z}$, called the degree of $\widetilde{s}$, such that $\widetilde{s}$ sends homogeneous elements of I_D of degree a to homogeneous elements of $\mathcal{O}_X$ of degree $a+k$ for all $a \in \mathbb{Z}$.

Assume that $\operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is generated by graded homomorphisms, a sufficient condition for that is that $\mathcal{O}_X$ is Noetherian (cf. [BH] p. 33). Then, there is a unique natural grading on R extending the gradings of $\mathcal{O}_X$ and $\operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$. Under this grading, i is homogeneous of degree 0, hence the ideal $(i-1) \subset R$ is homogeneous and R_1 and $\operatorname{unpr}_{\mathcal{O}_X} I_D$ become graded rings.

4.1. Relation with $\mathcal{O}_X[I_D^{-1}]$

From now on, we assume that I_D contains an $\mathcal{O}_X$-regular element. We denote by $K(X)$ the total quotient ring of $\mathcal{O}_X$, i.e., the localization of $\mathcal{O}_X$ with respect to the multiplicatively closed subset of $\mathcal{O}_X$ consisting of all $\mathcal{O}_X$-regular elements, and we set

$$I_D^{-1} = \{f \in K(X) \colon f I_D \subset \mathcal{O}_X\}.$$

I_D^{-1} is an $\mathcal{O}_X$-submodule of $K(X)$, which under the above assumption that I_D contains an $\mathcal{O}_X$-regular element is naturally isomorphic to $\operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$. Indeed, the map

$$I_D^{-1} \to \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$$

sends $f \in I_D^{-1}$ to the multiplication map by f, while the inverse map sends $g \in \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ to $g(q)/q \in I_D^{-1}$, where $q \in I_D$ is any $\mathcal{O}_X$-regular element. In the following we will identify these two isomorphic $\mathcal{O}_X$-modules.

We denote by $\mathcal{O}_X[I_D^{-1}]$ the $\mathcal{O}_X$-subalgebra of $K(X)$ generated by I_D^{-1}. The inclusion $I_D^{-1} \subset \mathcal{O}_X[I_D^{-1}]$ and the universal property of $R = \operatorname{Sym}_{\mathcal{O}_X} I_D^{-1}$ induce a surjective homomorphism of $\mathcal{O}_X$-algebras

$$\rho\colon R \to \mathcal{O}_X[I_D^{-1}],$$

which restricted to the first graded part of R is the identity.

Lemma 4.1. *The natural inclusion $\mathcal{O}_X \subset R$ as degree 0 graded part induces injective maps $\mathcal{O}_X \to R_1$ and $\mathcal{O}_X \to \operatorname{unpr}_{\mathcal{O}_X} I_D$.*

Proof. The second claim follows from the first, since T contains only $\mathcal{O}_X$-regular elements. We will show that $\mathcal{O}_X \cap (i-1) = 0 \subset R$, this will prove the first claim.

Let $a \in \mathcal{O}_X \cap (i-1)$. There exist homogeneous elements $b_0, \dots, b_r$ of R, with degree of b_t equal to t, such that

$$a = (1-i)(b_0 + \cdots + b_r).$$

Comparing homogeneous degrees we get $0 = b_r i = a i^{r+1}$. Using the map ρ we get $0 = \rho(a i^{r+1}) = a$. □

Clearly $(i-1) \subset \ker \rho$, so there is an induced map

$$\rho^1 \colon R_1 \to \mathcal{O}_X[I_D^{-1}].$$

Lemma 4.2. *We have*

$$\ker \rho^1 = J_2,$$

where

$$J_2 = \{u \in R_1 \colon \textit{there exists } t \in \mathcal{O}_X \textit{ which is } \mathcal{O}_X\textit{-regular with } tu = 0 \in R_1\}.$$

Proof. It is clear that $J_2 \subset \ker \rho^1$. We will show the opposite inclusion. By the assumptions, there exists $q \in I_D$ which is $\mathcal{O}_X$-regular. Therefore, if $a \in I_D^{-1}$, there exists $z \in \mathcal{O}_X \subset R_1$ (depending on a) with

$$qa - z = 0 \in R_1.$$

Let $u \in \ker \rho^1$. Since R_1 is generated as a ring by I_D^{-1}, u is a polynomial in elements of I_D^{-1}, hence for a sufficiently large integer n there exists $z \in \mathcal{O}_X \subset R_1$ with

$$q^n u - z = 0 \in R_1.$$

Since $u \in \ker \rho^1$ we necessarily have $\rho^1(z) = 0$, hence $z = 0$. □

By the above, the map ρ^1 factors through the natural quotient maps

$$R_1 \to \operatorname{unpr}_{\mathcal{O}_X} I_D \to \mathcal{O}_X[I_D^{-1}]$$

The last map $\operatorname{unpr}_{\mathcal{O}_X} I_D \to \mathcal{O}_X[I_D^{-1}]$ is often (examples of Section 7, normal case of unprojection of type Kustin–Miller ([PR, Remark 1.3]), generic type II [P2], generic type III [P3]) but not always an isomorphism.

Example 4.1. Consider the Kustin–Miller unprojection pair (cf. [P, Section 4])

$$I_D = (x, y) \subset \mathcal{O}_X = k[x,y]/(x^2 - y^3)$$

and set

$$u = \frac{y^2}{x} = \frac{x}{y} \in I_D^{-1}.$$

Then, $u^2 - y$ is zero in $\mathcal{O}_X[I_D^{-1}] \subset K(X)$, but, using Theorem 5.1 below, $u^2 - y$ is nonzero in $\operatorname{unpr}_{\mathcal{O}_X} I_D$.

Remark 4. It is obvious that when ρ^1 is an isomorphism, we have that $\operatorname{unpr}_{\mathcal{O}_X} I_D$ is isomorphic to $\mathcal{O}_X[I_D^{-1}]$ and to R_1 (case of only linear relations). In the generic type II unprojection [P2], ρ^1 is not an isomorphism (there exist quadratic relations) but still $\operatorname{unpr}_{\mathcal{O}_X} I_D$ is isomorphic to $\mathcal{O}_X[I_D^{-1}]$.

Remark 5. In our experience, we have often found easier to first study the ring $\mathcal{O}_X[I_D^{-1}]$ and then use the results to study the unprojection ring $\operatorname{unpr}_{\mathcal{O}_X} I_D$.

Remark 6. Unlike the case where $\mathcal{O}_X$ is Gorenstein, the $\mathcal{O}_X$-module $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_D, \mathcal{O}_X)$ which appears in (4.1) in general depends both on $\mathcal{O}_D$ and $\mathcal{O}_X$. For an example, fix

$$\mathcal{O}_D = \frac{k[x_1, x_2, x_3]}{(x_1, x_2, x_3)},$$

where k is any field, and set

$$(\mathcal{O}_X)_n = \frac{k[x_1, x_2, x_3]}{(I_X)_n},$$

where $(I_X)_n$ is generated by the maximal minors of a general $n \times n+1$ matrix with linear entries in x_i.

Remark 7. It will be very interesting to study under which conditions good properties of the unprojection initial data are preserved by the unprojection ring. Compare [PR, Theorem 1.5], [P2, Theorem 2.15] and [P3, Theorem 3.5].

5. Compatibility with previously defined unprojections

5.1. Case of unprojection of type Kustin–Miller

In this subsection, we prove that Definition 4.1 generalizes Definition 1.2 of [PR].

Assume we are under the assumptions of [PR, Section 1]. That is, $\mathcal{O}_X$ is a local Gorenstein ring, and $I_D = (f_1, \dots, f_r) \subset \mathcal{O}_X$ is a codimension one ideal such that the quotient ring $\mathcal{O}_D = \mathcal{O}_X/I_D$ is also Gorenstein. We choose as in [PR] an injective map $\widetilde{s}\colon I_D \to \mathcal{O}_X$ generating the $\mathcal{O}_X$-module $\operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)/\mathcal{O}_X$, and we set $g_i = \widetilde{s}(f_i) \in \mathcal{O}_X$.

Let

$$\psi\colon \mathcal{O}_X[S] \to R$$

be the $\mathcal{O}_X$-algebra homomorphism with $\psi(S) = \widetilde{s} \in R_1$, where $\mathcal{O}_X[S]$ is the polynomial ring over $\mathcal{O}_X$ in one variable. The map ψ is not surjective, but induces two surjective maps

$$\psi^1\colon \mathcal{O}_X[S] \to R_1$$

and

$$\psi^2 \colon \mathcal{O}_X[S] \to \operatorname{unpr}_{\mathcal{O}_X} I_D.$$

The following lemma gives a presentation of $I_D^{-1} = \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ as $\mathcal{O}_X$-module.

Lemma 5.1. *As an $\mathcal{O}_X$-module*

$$I_D^{-1} \cong \frac{\mathcal{O}_X s_0 \oplus \mathcal{O}_X s_1}{(f_i s_1 - g_i s_0)},$$

where s_0 corresponds to the inclusion $i \colon I_D \to \mathcal{O}_X$, and s_1 corresponds to $\widetilde{s}$.

Proof. Assume $l_1, l_2 \in \mathcal{O}_X$ with

$$l_1 i - l_2 \widetilde{s} = 0 \in I_D^{-1}.$$

Then

$$l_2 \widetilde{s} = 0 \in \operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)/\mathcal{O}_X \cong \mathcal{O}_D,$$

hence $l_2 \in I_D$. Fix $q \in I_D$ an $\mathcal{O}_X$-regular element, such an element exists since $\mathcal{O}_X$ is Cohen–Macaulay and I_D has codimension one. Then

$$l_1 q = (l_1 i)(q) = l_2 \widetilde{s}(q) = \widetilde{s}(l_2 q) = q \widetilde{s}(l_2),$$

therefore $l_1 = \widetilde{s}(l_2)$. □

Using elementary properties of the symmetric algebra of a module (cf. [Ei, p. 570 Prop. A2.2]) we have the following corollary.

Corollary 5.1. *We have that*

$$\ker \psi^1 = (S f_i - g_i).$$

Theorem 5.1. *We have that*

$$\ker \psi^2 = (S f_i - g_i).$$

As a consequence, Definition 4.1 *of unprojection generalizes Definition* 1.2 *of [PR] and*

$$\operatorname{unpr}_{\mathcal{O}_X} I_D \cong R_1.$$

Proof. Let $h = h(S) = a_n S^n + \cdots + a_0 \in \ker \psi^2$, with $a_i \in \mathcal{O}_X$ and $a_n \neq 0$. We prove by induction on the degree n of h that $h \in (S f_i - g_i)$.

Assume $n = 0$. Then $h \in \mathcal{O}_X$ and $th = 0 \in \mathcal{O}_X$ for $t \in T$ implies that $h = 0$, since T contains only $\mathcal{O}_X$-regular elements.

Assume $n \geq 1$, and that the result is true for all polynomials of degree strictly less than n. Using Corollary 5.1, and the definition of $\operatorname{unpr}_{\mathcal{O}_X} I_D$, there exist $l_i(S) \in \mathcal{O}_X[S]$ and $t \in T$, with

$$th(S) = \sum l_i(S)(S f_i - g_i) \in \mathcal{O}_X[S].$$

Write

$$l_i = m_{k_i} S^{k_i} + \text{ lower terms},$$

with $m_{k_i} \in \mathcal{O}_X$ nonzero.

We remark that for $u_i \in \mathcal{O}_X$, $\sum u_i f_i = 0$ implies $\widetilde{s}(\sum u_i f_i) = 0$, so $\sum u_i g_i = 0$. Therefore, $ta_n \in (f_i) = I_D$. By definition, T contains only $\mathcal{O}_D$-regular elements, hence $a_n \in (f_i)$. As a consequence, there exists $h' \in \mathcal{O}_X[S]$ with degree strictly less than n, with $h - h' \in (Sf_i - g_i)$. By the inductive hypothesis $h' \in (Sf_i - g_i)$, so also $h \in (Sf_i - g_i)$ which proves the theorem. □

5.2. Case of generic type II unprojection

In this subsection, we prove that Definition 4.1 generalizes Definition 2.2 of [P2].

We use the notations of [P2, Section 2]. In addition, we define $J_L \subset \mathcal{O}_X[T_0, \dots, T_k]$ to be the ideal generated by all affine linear polynomials $f^a_{i,j,p}$ and $f^b_{j,p}$.

Lemma 5.2. *Using the notations of [P2, Section 2], we have that*

$$a_{11} g^a_{ij} \in J_L$$

for all i, j with $i + j \le k$, and that

$$a_{11} g^b_{ij} \in J_L$$

for all i, j with $i + j \ge k + 1$.

Proof. We prove the first inclusion, the second follows by similar arguments. For $r = 1, 2, \dots, i$ we define elements $y_r, u_r \in \mathcal{O}_X[T_0, \dots, T_k]$ by

$$y_r = (T_j(a_{r,1}T_{i-r+1} + a_{r+1,1}T_{i-r}) - T_0(a_{r,1}T_{i+j-r+1} + a_{r+1,1}T_{i+j-r}))(-1)^{r+1}$$

and

$$u_r = (T_j f^a_{r,1,i-r} - T_0 f^a_{r,1,i+j-r})(-1)^{r+1}.$$

From the identity

$$a_{11}(T_i T_j - T_0 T_{i+j}) = y_1 + y_2 + \dots + y_i,$$

it follows that the element

$$a_{11} g^a_{ij} - u_1 - \dots - u_i$$

is affine linear in T_p and in the kernel of ϕ. As a consequence, it is an element of J_L. □

From the above Lemma 5.2 and Proposition 2.6 of [P2], it follows that Definition 4.1 generalizes Definition 2.2 of [P2].

5.3. Case of generic type III unprojection

In this subsection, we prove that Definition 4.1 generalizes Definition 3.3 of [P3].

We use the notations of [P3, Section 3].

From [P3, Theorem 3.5], it follows that the natural map $R \to \mathcal{O}_X[I_D^{-1}]$ induces an isomorphism $R/(i-1) \cong \mathcal{O}_X[I_D^{-1}]$. Therefore, using Remark 4, we have that

$$R_1 \cong \operatorname{unpr}_{\mathcal{O}_X} I_D \cong \mathcal{O}_X[I_D^{-1}].$$

6. The Reid–Suzuki example

The aim of this section is to analyze using unprojection a construction from [RS, p. 235], which provides an example of unprojection which is not the unprojection of an ideal.

We fix a field K, the projective line $\mathbb{P}^1$ with homogeneous coordinates v, w over K, and three points $P_1 = [1,0] = (w = 0), P_2 = [0,1] = (v = 0)$ and $Q = [1,1] = (v - w = 0)$ of $\mathbb{P}^1$.

Remark 8. Recall that for a $\mathbb{Q}$-divisor D of $\mathbb{P}^1$, $H^0(\mathbb{P}^1, D)$ is the K-vector space

$$H^0(\mathbb{P}^1, D) = \{f \in K(\mathbb{P}^1)^* \colon \operatorname{div}(f) + D \text{ is effective } \mathbb{Q}\text{-divisor } \} \cup \{0\}.$$

Consider now the two $\mathbb{Q}$-divisors

$$D_1 = \frac{3}{5}P_1 + \frac{4}{5}P_2 - Q, \qquad D_2 = \frac{3}{5}P_1 + \frac{4}{5}P_2$$

of $\mathbb{P}^1$, and the corresponding graded rings

$$\mathcal{O}_X = \bigoplus_{k \geq 0} H^0(\mathbb{P}^1, kD_1),$$
$$\mathcal{O}_Y = \bigoplus_{k \geq 0} H^0(\mathbb{P}^1, kD_2).$$

By [Wa] $\mathcal{O}_X$ and $\mathcal{O}_Y$ are both Cohen–Macaulay but not Gorenstein.

6.1. Explicit calculations for the rings $\mathcal{O}_X$ and $\mathcal{O}_Y$

Define $y, z, t, u_1, u_2 \in K(\mathbb{P}^1)$ with

$$y = \frac{(v-w)^2}{vw}, \quad z = \frac{(v-w)^3}{v^2 w}, \quad t = \frac{(v-w)^4}{v^3 w},$$
$$u_1 = \frac{(v-w)^5}{v^4 w}, \quad u_2 = \frac{(v-w)^5}{v^2 w^3}.$$

We consider these elements as homogeneous elements of $\mathcal{O}_X$ of degrees $2, 3, 4, 5, 5$ respectively. It is easy to see that they generate the K-algebra $\mathcal{O}_X$, and

$$\mathcal{O}_X \cong K[Y, Z, T, U_1, U_2]/I, \tag{6.1}$$

with

$I = (Z^2 - YT, h_1 = ZT - YU_1, h_2 = Y^4 - ZU_2, T^2 - ZU_1, Y^3Z - TU_2, Y^3T - U_1U_2)$

where Y, Z, T, U_1, U_2 are indeterminants of degrees $2, 3, 4, 5, 5$ respectively.

Remark 9. Macaulay 2 code to produce $\mathcal{O}_X$

```
kk = ZZ/101
S = kk[v,w]
R = kk[y,z,t,u1,u2, Degrees => {2,3,4,5,5}]
M = matrix {{ v*w, v*w^2,v*w^3, v*w^4, v^3*w^2}}
I = kernel (map (S, R, M))
-- we have \Oh_X = S/I
-- codim I --answer 3 -- res I --answer r, r^6, r^8 , r^3,
-- \Oh_X is CM, but not Gorenstein
```

For the ring $\mathcal{O}_Y$, we take coordinates $s_0, s_1, y, z, t, u_1, u_2$ where y, z, t, u_1, u_2 are as above and $s_0, s_1 \in K(\mathbb{P}^1)$ with

$$s_0 = 1, \quad s_1 = \frac{v - w}{w}. \tag{6.2}$$

We consider s_0 and s_1 as homogeneous elements of $\mathcal{O}_Y$ of degrees 1 and 2 respectively, and we also notice that

$$s_0 = \frac{zy}{y^2 - t}, \quad s_1 = \frac{y^3}{y^2 - t}.$$

Using a computer algebra system such as Singular [GPS01] or Macaulay 2 [GS93-08], we easily get

$$\mathcal{O}_Y \cong K[S_0, S_1, Y, Z, T, U_1, U_2]/J, \tag{6.3}$$

where $S_0, S_1, Y, Z, T, U_1, U_2$ are indeterminants of degrees $1, 2, 2, 3, 4, 5, 5$ respectively, and

$$J = J_1 + J_2 + J_3$$

with $J_1 = I$,

$$\begin{aligned} J_2 = (&-ZS_0 + YS_1 - Y^2, -TS_0 + ZS_1 - YZ, (Y^2 - T)S_0 - YZ, \\ &h_3 = -U_1S_0 + TS_1 - YT, (YZ - U_1)S_0 - YT, \\ &h_4 = (U_1 - U_2)S_0 + Y^3 + YT, h_5 = YTS_0 - U_1S_1, \\ &Z(U_1 - U_2)S_0 + (Y^2U_1 + TU_2), T(U_1 - U_2)S_0 + (YZU_1 + U_1U_2), \\ &h_6 = -Y(Y^2 + T)S_0 + (U_1 + U_2)S_1 - YU_2, \\ &h_7 = 2YU_2S_0 - Y(Y^2 + T)S_1 - ZU_2), \end{aligned}$$

(the linear relations between s_0 and s_1) and

$$J_3 = (h_8 = f_4 = YS_0^2 - S_1^2 + ZS_0 + Y^2)$$

(the quadratic relation).

Remark 10. Macaulay 2 code to produce $\mathcal{O}_Y$

```
kk = ZZ/101
S = kk[v,w]
newR = kk[y,z,t,u1,u2,s0,s1, Degrees =>{2,3,4,5,5,1,2}]
newM = matrix {{ v*w*(v-w)^2, v*w^2*(v-w)^3, v*w^3*(v-w)^4,
      v*w^4*(v-w)^5, v^3*w^2*(v-w)^5, v*w, (v-w)*v^2*w}}
newI = kernel (phi2= map (S, newR, newM))
--  we have \Oh_Y = newR / newI
-- codim newI  --answer 5
-- res newI --answer  r, r^15, r^40, r^45, r^24, r^5
-- \Oh_Y  is CM  but not Gorenstein
```

Define the prime ideal $I_D \subset \mathcal{O}_X$, with

$$I_D = (z^2 - y^3, t - y^2, u_1 - yz, u_2 - yz)$$

It is easy to see that I_D is the ideal of the point $[1, 1, 1, 1, 1] \in X = \operatorname{Proj} \mathcal{O}_X \subset \mathbb{P}(2, 3, 4, 5, 5)$.

We have

$$\mathcal{O}_Y \subset \mathcal{O}_X[I_D^{-1}],$$

however the inclusion is strict. Indeed, using the following Macaulay 2 code continuing the code in Remark 9

```
ID = ideal(t-y^2, z^2-y^3,u1-y*z, u2-y*z)
betti presentation Hom(ID, R^1/I)
presentation Hom(ID, R^1/I)
```

we get that $\operatorname{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is generated by

$$q_0 = \frac{z}{y^2 - t}, \quad q_1 = \frac{y^2}{y^2 - t}, \quad q_2 = 1$$

the first of homogenous degree -1, the other two of homogeneous degrees 0. Clearly

$$s_0 = yq_0 \quad \text{and} \quad s_1 = yq_1.$$

Another calculation shows that

$$\begin{aligned}(u_1 + u_2)h_8 = &(s_1 - y)h_1 + (-2s_0)h_2 + (-z)h_3 + (s_0 y)h_4 \\ &+ (-2y)h_5 + (-s_1 - y)h_6 + (s_0)h_7\end{aligned}$$

and it is easy to see that $u_1 + u_2$ is both $\mathcal{O}_X$ and $\mathcal{O}_D = \mathcal{O}_X/I_D$-regular element. As a consequence, the quadratic relation h_8 multiplied by a regular element is inside the ideal generated by the linear relations (cf. Definition 2.1).

6.2. The $\mathcal{O}_X$-algebra $\mathcal{O}_Y$ as unprojection

Define $\widetilde{D} = \operatorname{Proj} K[a] \cong \mathbb{P}(1)$ where a is an indeterminate of degree 1 and the morphism of schemes

$$\widetilde{\phi}\colon \widetilde{D} \to X \subset \mathbb{P}(2,3,4,5,5)$$

with

$$\widetilde{\phi}([a]) = [a^2, a^3, a^4, a^5, a^5]$$

induced by the graded homomorphism of graded K-algebras

$$\phi\colon \mathcal{O}_X \to K[a]$$

specified by

$$\phi(y) = a^2,\ \phi(z) = a^3,\ \phi(t) = a^4,\ \phi(u_1) = \phi(u_2) = a^5.$$

It is clear that the scheme–theoretic image of $\widetilde{\phi}$ is $D \subset X$, and also that $K[a]$ needs two generators, corresponding to 1 and a, when viewed as $\mathcal{O}_X$-module via ϕ.

The interpretation of the calculations in Subsection 6.1 is that we have

$$\mathcal{O}_Y \cong \operatorname{unpr}_{\mathcal{O}_X} \phi,$$

while $\mathcal{O}_Y$ is not isomorphic to $\operatorname{unpr}_{\mathcal{O}_X} I_D$.

7. More examples

We discuss below more examples of unprojections related to geometry. In all cases we start from a triple of embedded schemes

$$D \subset X \subset \mathbb{P}$$

and construct by unprojection a new embedded scheme

$$Y \subset \mathbb{P}'.$$

$\mathbb{P}$ and $\mathbb{P}'$ are usual projective spaces, in each case clear from the construction. By I_X, I_Y and I_D we will denote the homogeneous ideals of the corresponding schemes, and by $\mathcal{O}_X, \mathcal{O}_Y$ etc. the corresponding homogeneous coordinate rings. By abuse of notation, we will also denote by I_D the homogeneous ideal of $\mathcal{O}_X$ corresponding to $D \subset X$. In all cases

$$\mathcal{O}_Y = \operatorname{unpr}_{\mathcal{O}_X} I_D$$

and moreover the map ρ^1 defined in Section 4.1 is an isomorphism, hence by Remark 4

$$\mathcal{O}_Y \cong R_1 \cong \mathcal{O}_X[I_D^{-1}].$$

If the minimal resolution of $\mathcal{O}_X$ as $\mathcal{O}_{\mathbb{P}}$-module taking no accounts of the twists is

$$\mathcal{O}_{\mathbb{P}}^{h_0} \leftarrow \mathcal{O}_{\mathbb{P}}^{h_1} \leftarrow \cdots \leftarrow \mathcal{O}_{\mathbb{P}}^{h_d} \leftarrow 0$$

we say that the Betti vector of X is $(h_0, h_1, \dots, h_d)$, and similarly for $\mathcal{O}_D$ and $\mathcal{O}_Y$.

We have checked the calculations of all examples using the computer algebra programs Macaulay 2 [GS93-08] and Singular [GPS01] over the field $\mathbb{Q}$ and the finite field $\mathbb{Z}_{101}$. We believe that with a little extra effort one should be able to prove the results with coefficients over any field, and even more generally.

7.1. Rolling factors format

According to [Ki, Example 10.8], the rolling factors format first appeared - in a disguised codimension four form - in D. Dicks Ph.D. thesis [D]. It has also appeared in [R1], [S] and [BCP].

Let M be the 2×2 matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}$$

and denote by I_M the ideal generated by the 2×2 minors of M. Let

$$\begin{aligned} f_1 &= a_1x_1 + \cdots + a_4x_4 \\ f_2 &= b_1x_1 + \cdots + b_4x_4 \\ g_1 &= a_1y_1 + \cdots + a_4y_4 \\ g_2 &= b_1y_1 + \cdots + b_4y_4 \end{aligned}$$

and set

$$I_Y = I_M + (f_1, f_2, g_1, g_2).$$

The ideal I_Y defines a codimension five projectively Gorenstein subscheme $Y \subset \mathbb{P}^{15}$ with Betti vector $(1, 10, 19, 19, 10, 1)$.

Let N be the submatrix of M obtained by deleting the first column of M, and denote by I_N the ideal generated by the 2×2 minors of N.

Define $X \subset \mathbb{P}^{13}$ (with coordinates of $\mathbb{P}^{13}$ those of $\mathbb{P}^{15}$ minus x_1, y_1) by

$$I_X = I_N + (b_1f_1 - a_1f_2, b_1g_1 - a_1g_2).$$

X is a codimension three projectively Gorenstein subscheme with Betti vector $(1, 5, 5, 1)$.

Consider the subscheme $D \subset X$ with

$$I_D = I_X + (a_1, b_1).$$

The ideal I_D is codimension four projectively Cohen–Macaulay with Betti vector $(1, 5, 9, 7, 2)$ (actually I_D is a hyperplane section of I_N), and we have

$$\mathcal{O}_Y = \operatorname{unpr}_{\mathcal{O}_X} I_D \cong R_1 \cong \mathcal{O}_X[I_D^{-1}].$$

7.2. Veronese surface

Let $Y \subset \mathbb{P}^5$ be the Veronese surface. It is well known that if

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ \text{sym} & & a_{33} \end{pmatrix}$$

is the generic 3×3 symmetric matrix on the coordinates of $\mathbb{P}^5$, then I_Y is generated by the six 2×2 minors of M. In addition, Y has codimension three and Betti vector $(1, 6, 8, 3)$. Therefore, it is projectively Cohen–Macaulay, but not projectively Gorenstein.

Let N be the submatrix of M obtained by deleting the first row of M.

Define $X \subset \mathbb{P}^4$ (with coordinates of $\mathbb{P}^4$ those of $\mathbb{P}^5$ minus a_{11}) with I_X generated by the three 2×2 minors of N. X has codimension two and Betti vector $(1, 3, 2)$. Consider the codimension three complete intersection $D \subset \mathbb{P}^4$, with

$$I_D = (a_{22}, a_{23}, a_{33}).$$

We have $D \subset X$ and

$$\mathcal{O}_Y = \operatorname{unpr}_{\mathcal{O}_X} I_D \cong R_1 \cong \mathcal{O}_X[I_D^{-1}].$$

7.3. Grass(2,6)

Let $Y \subset \mathbb{P}^{14}$ be the Grassmannian Grass$(2, 6)$ of dimension 2 vector subspaces of a 6 dimensional vector space in its natural Plücker embedding. It is well known that if $M = [a_{ij}]$ is the generic 6×6 skew–symmetric matrix on the coordinates of P^{14}, then I_Y is the ideal generated by the 15 4×4 Pfaffians of M (that is, the Pfaffians of all 4×4 skew–symmetric submatrices of M obtained by deleting from M two rows and the corresponding two columns). In addition, Y is codimension six and projectively Gorenstein, with Betti vector $(1, 15, 35, 42, 35, 15, 1)$.

Let N be the 5×5 submatrix of M obtained by deleting the first row and the first column of M, and let $\operatorname{Pf}(N)$ be the ideal generated by the 5 4×4 Pfaffians of N.

Define $X \subset \mathbb{P}^{12}$ (with coordinates of $\mathbb{P}^{12}$ those of $\mathbb{P}^{14}$ minus a_{12}, a_{13}) by

$$I_X = \operatorname{Pf}(N) + (a_{14}a_{56} - a_{15}a_{46} + a_{16}a_{45}),$$

and $D \subset X$ with

$$I_D = I_X + (a_{45}, a_{46}, a_{56}).$$

X is codimension four projectively Gorenstein with Betti vector $(1, 6, 10, 6, 1)$, and D is codimension five projectively Cohen–Macaulay with Betti vector $(1, 6, 14, 16, 9, 2)$. We have

$$\mathcal{O}_Y = \operatorname{unpr}_{\mathcal{O}_X} I_D \cong R_1 \cong \mathcal{O}_X[I_D^{-1}].$$

7.4. Spinor Variety

Denote by $Y \subset P^{15}$ the ten-dimensional spinor variety Spin(5, 10) in its canonical embedding. It has Betti vector $(1, 10, 16, 16, 10, 1)$, hence it is projectively Gorenstein.

According to [Ki, Problem 8.7],

$$I_Y = (\mathrm{pos}_1, \dots, \mathrm{pos}_5, \mathrm{neg}_1, \dots, \mathrm{neg}_5),$$

where

$$\begin{aligned}
\mathrm{pos}_1 &= zy_1 - x_{23}x_{45} + x_{24}x_{35} - x_{25}x_{34} \\
\mathrm{pos}_2 &= zy_2 - x_{13}x_{45} + x_{14}x_{35} - x_{15}x_{34} \\
\mathrm{pos}_3 &= zy_3 - x_{12}x_{45} + x_{14}x_{25} - x_{15}x_{24} \\
\mathrm{pos}_4 &= zy_4 - x_{12}x_{35} + x_{13}x_{25} - x_{15}x_{23} \\
\mathrm{pos}_5 &= zy_5 - x_{12}x_{34} + x_{13}x_{24} - x_{14}x_{23} \\
\mathrm{neg}_1 &= x_{12}y_2 - x_{13}y_3 + x_{14}y_4 - x_{15}y_5 \\
\mathrm{neg}_2 &= x_{12}y_1 - x_{23}y_3 + x_{24}y_4 - x_{25}y_5 \\
\mathrm{neg}_3 &= x_{13}y_1 - x_{23}y_2 + x_{34}y_4 - x_{35}y_5 \\
\mathrm{neg}_4 &= x_{14}y_1 - x_{24}y_2 + x_{34}y_3 - x_{45}y_5 \\
\mathrm{neg}_5 &= x_{15}y_1 - x_{25}y_2 + x_{35}y_3 - x_{45}y_4.
\end{aligned}$$

Define $X \subset \mathbb{P}^{14}$ (with coordinates of $\mathbb{P}^{14}$ those of $\mathbb{P}^{15}$ minus z), by

$$I_X = (\mathrm{neg}_1, \dots, \mathrm{neg}_5).$$

X is a codimension four almost complete intersection (hence not projectively Gorenstein) with Betti vector $(1, 5, 12, 10, 2)$, as a consequence it is projectively Cohen–Macaulay.

Consider the codimension five complete intersection $D \subset \mathbb{P}^{14}$ with

$$I_D = (y_1, y_2, \dots, y_5).$$

We have $D \subset X$, and

$$\mathcal{O}_Y = \mathrm{unpr}_{\mathcal{O}_X} I_D \cong R_1 \cong \mathcal{O}_X[I_D^{-1}].$$

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