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The Chow ring of the moduli space of bundles on $\mathbb{P}^2$ with charge 1

By

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Abstract

For an algebraically closed field K with $\mathrm{ch}(K) \neq 2$, let $\mathcal{O}M(1, SO(n, K))$ denote the moduli space of holomorphic bundles on $\mathbb{P}^2$ with the structure group $SO(n, K)$ and half the first Pontryagin index being equal to 1, each of which is trivial on a fixed line l_∞ and has a fixed holomorphic trivialization there. In this paper we determine the Chow ring of $\mathcal{O}M(1, SO(n, K))$.

1. Introduction

Let G be a compact connected simple Lie group. The fact $\pi_3(G) \cong \mathbb{Z}$ leads to the classification of principal G-bundles P_k over S^4 by the integer k in $\mathbb{Z}$. Denote by $M(k, G)$ the moduli space based equivalence classes of G-instantons on P_k. Let $i_k : M(k, G) \to \Omega^3_k G$ be the inclusion. In [3] Boyer, Mann and Waggoner posed an idea of constructing homology classes in $H_*(M(k, G); \mathbb{Z}/p)$ from those in $H_*(M(1, G); \mathbb{Z}/p)$, where p is a prime. The crucial part of their idea is as follows: (i) First they gave a description of $M(1, G)$ in terms of a homogeneous space. They also showed that the map i_1 is a generalization of the well-known J-homomorphism; (ii) Next they observed that a homology class in $H_*(M(k, G); \mathbb{Z}/p)$, which is constructed from a class $\alpha \in H_*(M(1, G); \mathbb{Z}/p)$, can be shown to be non-trivial if one can show that $i_{1*}(\alpha) \neq 0$.

For $G = SU(n)$ or $Sp(n)$, an explicit topological type of $M(1, G)$ is obtained from the above description. Hence we can determine $H_*(M(1, G); \mathbb{Z}/p)$. Then in [3] they considered the case $G = SU(n)$ and proved that $i_{1*}(\alpha) \neq 0$, where $\alpha \in H_*(M(1, SU(n)); \mathbb{Z}/2)$ is an even dimensional generator.

For $G = SU(n)$, the structure $H_*(M(1, SO(n)); \mathbb{Z}/p)$ was studied in detail in [11]. At present, we know the following result ([10]): For $G = SU(n), Sp(n)$ or $SO(n)$, the homomorphism i_{1*}: $H_*(M(1, G); \mathbb{Z}/2) \to H_*(\Omega^3_1 G; \mathbb{Z}/2)$ is injective. (In [10], the group $\mathrm{Spin}(n)$ is used for $SO(n)$. But by the definition of instanton moduli spaces, we have $M(k, \mathrm{Spin}(n)) = M(k, SO(n))$).

Thus it is important to study the topology of $M(1, G)$ to understand

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$M(k, G)$. The purpose of this paper is to study an algebraic version of the results in [11]. The motivation for the study is as follows:

Once defined algebraically, algebraic cycles can be considered. The distinction between topological and algebraic cycles is an eminent object to be attacked. Hence our main concern of this note is to calculate the Chow ring of $M(1, G)$ explicitly (Theorem 4.1). The Chow ring of a classifying space is studied by B. Totaro [21] and has valuable applications. Our approach will be a first step to obtain a Chow ring of a threefold loop space $\Omega_k^3 G$ and a moduli space $M(k, G)$.

This paper is organized as follows. In Section 2 we define an algebraic version of $M(k, SO(n))$. (See Definition 2.1.) For the case $k = 1$, we prove a similar assertion to [3] which describes the space in terms of a homogeneous space. (See Proposition 2.2.) In Section 3 we collect some facts on the Chow ring of the related space Y_n and prove the combinatorial results to describe it explicitly. In Section 4 we determine the Chow ring of $M(1, G)$. It turns out that the additive structure has a four period. When we consider the ring structure, it has an eight period. The relation to the Bott periodicity theorem is recently given by Kishimoto [13].

We thank N. Yagita for turning our interest to the Chow ring and explaining the paper [19].

2. An algebraic version of $M(1, SO(n))$

We fix a line $l_\infty \subset \mathbb{CP}^2$. Let $SO(n, \mathbb{C}) \to E \to \mathbb{CP}^2$ be a holomorphic principal bundle such that $\frac{1}{2}\langle c_2(E), [\mathbb{CP}^2]\rangle = k$, $E \mid l_\infty$ is trivial and its holomorphic trivialization is fixed. Let $\mathcal{O}M(k, SO(n, \mathbb{C}))$ be the moduli space of such bundles. Then according to Donaldson ([7]), there is a diffeomorphism

$$M(k, SO(n)) \simeq \mathcal{O}M(k, SO(n, \mathbb{C})),\ n \geq 5 \text{ and } k \geq 1. \tag{2.1}$$

In order to explain why E must be trivial on l_∞, we construct a map $M(k, SO(n)) \to \mathcal{O}M(k, SO(n, \mathbb{C}))$. We define a map $f : \mathbb{CP}^2 \to \mathbb{HP}^1$ by $f([z_0, z_1, z_2]) = [z_0 + z_1 j, z_2]$ and set $l_\infty := \{[z_0, z_1, z_2] \in \mathbb{CP}^2 : z_2 = 0\}$. Then $f(l_\infty) = [1, 0]$.

As in Section 1, let $SO(n) \to P_k \to S^4$ be a principal bundle. We identify S^4 with $\mathbb{HP}^1$. By complexification the structure group, the bundle $f^*(P_k)$ induces a (topological) principal bundle $SO(n, \mathbb{C}) \to Q_k \to \mathbb{CP}^2$. Since $f(l_\infty) = [1, 0]$, $Q_k \mid l_\infty$ is trivial.

A connection A on P_k naturally defines a connection on Q_k, which we denote by $f^*(A)$. It is known that if A is an instanton, then there is an unique complex structure on the bundle $Q_k \to \mathbb{CP}^2$ such that the connection compatible with the complex structure is $f^*(A)$. We denote by $E \to \mathbb{CP}^2$ the bundle $Q_k \to \mathbb{CP}^2$ with this complex structure. Since we can show that a frame for A induces a holomorphic frame for $E|l_\infty$, we have the desired map.

We use the right-hand side of (2.1) to define an algebraic version of $M(k, SO(n))$. Hereafter we denote by K an algebraically closed field with $\mathrm{ch}(K) \neq 2$.

Definition 2.1. Let $n \geq 5$. We fix a line $l_\infty \subset \mathbb{P}^2$. Let $\mathcal{O}M(k, SO(n, K))$ be the moduli space of holomorphic principal bundles $SO(n,K) \to E \to \mathbb{P}^2$ such that $\frac{1}{2}\langle c_2(E), [\mathbb{P}^2]\rangle = k$, $E|l_\infty$ is trivial and its holomorphic trivialization is fixed.

Recall that for a parabolic subgroup P of an algebraic group G, P is a semidirect product of a reductive group called a Levi factor and its unipotent radical P_u.

The main interest in this paper is the case $k = 1$. In this case, we have the following:

Proposition 2.2. *We set*

$$X_n = SO(n, K)/(SO(n-4, K) \times SL(2, K)) \cdot P_u,$$

where P_u denotes the unipotent radical of a parabolic subgroup P with a Levi factor $SO(n-4, k) \times GL(2, k)$. Then there is a biregular map

$$\mathcal{O}M(1, SO(n, K)) \simeq \mathbb{A}^2 \times X_n.$$

Remark 2.3. Consider the proposition for $K = \mathbb{C}$. We set $W_n = SO(n)/(SO(n-4) \times SU(2))$. Note that there is a diffeomorphism $X_n \simeq \mathbb{R} \times W_n$. The proposition and (2.1) tell us that $M(1, SO(n)) \simeq \mathbb{R}^5 \times W_n$. This is in agreement with [11, Proposition 1], since by the definition of instanton moduli spaces, we have $M(k, \mathrm{Spin}(n)) = M(k, SO(n))$.

For the rest of this section, we prove Proposition 2.2. For that purpose, we need the following:

Lemma 2.4. *Let $\mathscr{C}_n$ be the space of $n \times 2$ matrices c with coefficients in K satisfying*:
(i) $c^T c = O$,
(ii) *rank* $c = 2$.
The group $SL(2, K)$ acts on $\mathscr{C}_n$ from the right by the multiplication of matrices. Then there is a biregular map

$$\mathcal{O}M(1, SO(n, K)) \simeq \mathbb{A}^2 \times (\mathscr{C}_n / SL(2, K)).$$

Proof. For $K = \mathbb{C}$, an algebraic description of $\mathcal{O}M(k, SO(n, \mathbb{C}))$ was proved in [20, Proposition 1.8] and restated in [15, Proposition 1]. About these propositions, we use the following three remarks.

First, in the former proposition, the rank condition as in the latter proposition item d) is forgotten. Hence we need to add the condition.

Second, in these propositions, the results are stated as an algebraic description of $M(k, SO(n))$. But strictly speaking, they are a description of $\mathcal{O}M(k, SO(n, \mathbb{C}))$, and using (2.1), the description is interpreted as that of $M(k, SO(n))$.

Third, the proof of the propositions is given in [20, pp. 180–183]. The crucial step is to use an $SO(n,\mathbb{C})$ monad over $\mathbb{CP}^2$ as in [16]. It is easy to see that the monad remains valid even if we generalize $\mathbb{C}$ to K. Hence, if we generalize $\mathbb{C}$ to K in these propositions, then we have a description of $\mathcal{OM}(k, SO(n,K))$.

Under this modification, we consider [15, Proposition 1] for $k = 1$. Then the items a) and b) in Proposition 1 in [15] tell us that $\gamma_1 = uI_2$ and $\gamma_2 = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}$, where $u, v \in K$. Moreover, the item c) tells us that c is an $n \times 2$ matrix with coefficients in K such that $c^T c = 0$. Finally the item d) is equivalent to the assertion that rank $c = 2$.

Note that $Sp(1,K) = SL(2,K)$ and take an element g from this. Then about [15, Proposition 1], we have $g\gamma_1 g^{-1} = \gamma_1$ and $(g^{-1})^T\gamma_2 g^{-1} = \gamma_2$. Hence Lemma 2.4 is clear from the proposition. $\square$

From Lemma 2.4, it suffices to prove $X_n \simeq \mathscr{C}_n/SL(2,K)$. We prove this for the case $n = 2m$. (The case $n = 2m+1$ can be proved similarly.) Recall that in [2], $SO(n,K)$ is defined as follows: Let $q(x)$ be a quadratic form on $\mathbb{A}^n$ defined by $q(x) = \sum_{i=1}^m x_i x_{m+i}$, and let $B(x,y)$ be the associated bilinear form. Then $SO(n,K)$ is defined by

$$SO(n,K) = \{\sigma \in \mathrm{Aut}(\mathbb{A}^n) : B(\sigma(x), \sigma(y)) = B(x,y) \text{ for } x, y \in \mathbb{A}^n\}.$$

We write $c = \begin{pmatrix} z_1 & w_1 \\ \vdots & \vdots \\ z_n & w_n \end{pmatrix} \in \mathscr{C}_n$ and set

$$x_j = z_j + \sqrt{-1}z_j,\ x_{m+j} = z_j - \sqrt{-1}z_j,\ y_j = w_j + \sqrt{-1}w_j \quad \text{and} \quad y_{m+j} = w_j - \sqrt{-1}w_j,$$

where $1 \le j \le m$. Then Lemma 2.4 (i) is transformed into

$$q(x) = q(y) = 0 \quad \text{and} \quad B(x,y) = 0.$$

Clearly $SO(n,K)$ acts on $\mathscr{C}_n$. It is easy to prove the following lemma. (See [2, V23.4].)

Lemma 2.5.

$$SO(n,K)/SO(n-4,K) \cdot P_u \simeq \mathscr{C}_n,$$

where P_u is the unipotent radical of a parabolic subgroup with a Levi factor $SO(n-4,K) \times GL(2,K)$.

Now Proposition 2.2 follows from the above lemma.

3. The ring of $CH^{\cdot}(Y_n)$

First we recall basic facts on the Chow ring. We suppose that an algebraic variety V is defined over K. Let $CH^{\cdot}(V)$ denote the Chow ring and $CH^i(V)$ the subgroup of $CH^{\cdot}(V)$ generated by the cycles of codimension i.

Theorem 3.1 ([4]). (i) *Let V be a nonsingular variety, X a nonsingular closed subvariety of V, and $U = X - V$. Then there exists an exact sequence*

$$CH^{\cdot}(X) \xrightarrow{i_*} CH^{\cdot}(V) \xrightarrow{j^*} CH^{\cdot}(U) \to 0,$$

where $i : X \to V$ (resp. $j : U \to V$) is a closed immersion (resp. an open immersion).

(ii) *Let $\pi : E \to V$ be a fiber bundle with an affine space $\mathbb{A}^n$ as a fiber. Then the induced map $\pi^* : CH^{\cdot}(V) \to CH^{\cdot}(E)$ is an isomorphism.*

For the definitions of i_* and j^*, see also [9].

The Chow ring of the following projective variety is well-known.

Theorem 3.2 ([1], [6]). *Let G be a reductive algebraic group and P a maximal parabolic subgroup. Then*

(i) *a quotient G/P is a nonsingular projective variety.*

(ii) *$CH^{\cdot}(G/P)$ is generated by the Schubert varieties.*

(iii) *$CH^{\cdot}(G/P)$ is independent of the field K. Moreover, $CH^{\cdot}(G/P) \simeq H^{\cdot}(G/P, \mathbb{Z})$ for $K = \mathbb{C}$.*

Before describing the results, we need some notations and results. We set

$$Y_n = SO(n, K)/(SO(n-4, K) \times GL(2, K)) \cdot P_u.$$

Then we have a principal bundle

$$\mathbb{G}_m \to X_n \xrightarrow{\pi} Y_n. \tag{3.1}$$

In this section we determine the ring structure of $CH^{\cdot}(Y_n)$. The cohomologies $H^{\cdot}(Y_n) \otimes \mathbb{Z}/p$ is given in [11]. By Theorem 3.2 (ii), (iii), we obtain the following theorem:

Theorem 3.3 ([11]). *We have an isomorphism as modules*:

(1) *For $n = 2m$,*

$$CH^{\cdot}(Y_n) \otimes \mathbb{Z}/2 \simeq \mathbb{Z}/2[c_1, c_2]/(b_{m-1}, c_2 b_{m-2}) \otimes \Delta(v_{2m-4}, v_{2m-2})$$

where $|c_1| = 1, |c_2| = 2, |b_i| = i$, $|v_i| = \frac{i}{2}$ and $\Delta(x_1, x_2)$ is a graded algebra over $\mathbb{Z}/2$ with a $\mathbb{Z}/2$-basis $\{x_1, x_2, x_1x_2\}$.

(2) *For $n = 2m + 1$,*

$$CH^{\cdot}(Y_n) \otimes \mathbb{Z}/2 \simeq \mathbb{Z}/2[c_1, c_2]/(b_{m-1}, c_2 b_{m-2}) \otimes \Delta(v_{2m-2}, v_{2m}),$$

Theorem 3.4 ([11]). *Let p be an odd prime. Then we have a ring isomorphism*:

(1) *For $n = 2m$,*

$$CH^{\cdot}(Y_n) \otimes \mathbb{Z}/p \simeq \mathbb{Z}/p[c_1, c_2, \chi_{2m-4}]/(c_2\chi_{2m-4}, \chi_{2m-4}^2 - d_{m-2}, d_{m-1}),$$

where $\chi_{2m-4} \in CH^{m-2}(BSO_{2m-4})$ is the Euler class, and $|d_i| = 2i$.

(2) *For $n = 2m+1$,*

$$CH^{\cdot}(Y_n) \otimes \mathbb{Z}/p \simeq \mathbb{Z}/p[c_1, c_2]/(d_{m-1}, c_2^2 d_{m-2}).$$

Here b_i, d_i and v_i are defined by the following. In a polynomial ring $\mathbb{Z}[\alpha, \beta]$, we set $c_1 = \alpha + \beta$ and $c_2 = \alpha\beta$. Then b_k and d_k are defined by

$$b_k = (-1)^k \sum_{i=0}^{k} \alpha^i \beta^{k-i}$$

and

$$d_k = (-1)^k \sum_{i=0}^{k} \alpha^{2i} \beta^{2k-2i}.$$

The element $v_{2r} \in CH^r(Y_n)$ is defined by

(1) For $n = 2m$,

$$\begin{cases} 2v_{2m-4} = \chi_{2m-4} - b_{m-2} \\ 2v_{2m-2} = b_{m-1}. \end{cases} \tag{3.2}$$

(2) For $n = 2m+1$,

$$\begin{cases} 2v_{2m-2} = b_{m-1} \\ 2v_{2m} = c_2 b_{m-2}. \end{cases} \tag{3.3}$$

Lemma 3.5. *We have*

$$b_k = (-1)^k \sum_{\mu=0}^{[\frac{k}{2}]} (-1)^\mu \binom{k-\mu}{\mu} c_1^{k-2\mu} c_2^\mu$$

and

$$d_k = (-1)^k \sum_{\mu=0}^{k} (-1)^\mu \binom{2k-\mu+1}{\mu} c_1^{2k-2\mu} c_2^\mu.$$

Proof. From (3.2), we have a relation

$$b_{k+1} = -c_1 b_k - c_2 b_{k-1}.$$

By induction,

$$\begin{aligned}
b_{k+1} &= (-1)^{k+1}\sum_{\mu=0}^{\left[\frac{k}{2}\right]}(-1)^{\mu}\binom{k-\mu}{\mu}c_1^{k+1-2\mu}c_2^{\mu} \\
&\quad - (-1)^{k-1}\sum_{\nu=0}^{\left[\frac{k-1}{2}\right]}(-1)^{\nu}\binom{k-1-\nu}{\nu}c_1^{k-1-2\nu}c_2^{\nu+1} \\
&= (-1)^{k+1}\sum_{\mu=0}^{\left[\frac{k-1}{2}\right]+1}(-1)^{\mu}\left(\binom{k-\mu}{\mu}+\binom{k-\mu}{\mu-1}\right)c_1^{k+1-2\mu}c_2^{\mu} \\
&= (-1)^{k+1}\sum_{\mu=0}^{\left[\frac{k-1}{2}\right]}(-1)^{\mu}\binom{k+1-\mu}{\mu}c_1^{k+1-2\mu}c_2^{\mu}.
\end{aligned}$$

For d_n, a relation $d_{n+1} = (-c_1^2 + 2c_2)d_n - c_2^2 d_{n-1}$ holds. The formula is proved in a similar way. $\square$

Lemma 3.6.

$$\sum_{\mu=0}^{h}(-1)^{\mu}c_2^{h-\mu}b_{2\mu} = d_h.$$

Proof. In $\mathbb{Z}[\alpha,\beta]$, we see that

$$\sum_{i=0}^{2k}\alpha^i\beta^{2k-i} - \sum_{j=0}^{2k}\alpha^{2j}\beta^{2k-2j} = \alpha\beta\left(\sum_{i=0}^{k-1}\alpha^{2j}\beta^{2k-2-2i}\right).$$

Hence, we have

$$d_k = (-1)^k b_{2k} + c_2 d_{k-1}.$$

Then the assertion follows by induction. $\square$

Lemma 3.7. *We set $f_n(x) = (1+x)^n - (1+x^n)$ and write $f_n(x)$ as*

$$f_n(x) = \sum_{\mu=1}^{\left[\frac{n}{2}\right]} a_{n,\mu}x^{\mu}(1+x)^{n-2\mu}.$$

Then we have

$$a_{n,\mu} = (-1)^{\mu+1}\frac{n}{\mu}\binom{n-1-\mu}{\mu-1}.$$

Especially, the term $a_{n,\mu}x^{\mu}(1+x)^{n-2\mu}$ in $f_n(x)$ for $\mu = [\frac{n}{2}]$ is given by

$$\begin{cases} (-1)^{s+1}2x^s & \text{for } n = 2s \\ (-1)^{s+1}(2s+1)x^s(1+x) & \text{for } n = 2s+1. \end{cases}$$

Proof. We have

$$f_n(x) = (1+x)f_{n-1}(x) + x(1+x)^{n-2} - xf_{n-2}(x).$$

Comparing the coefficients of the both sides, we get

$$a_{n,1} = a_{n-1,1} + 1, \ n \geq 3$$

$$a_{n,\mu} = a_{n-1,\mu} - a_{n-2,\mu-1}, \ \mu = 2, 3, \cdots, \left[\frac{n-1}{2}\right].$$

By induction, we see that

$$\begin{aligned} a_{n,\mu} &= (-1)^{\mu+1}\frac{n-1}{\mu}\binom{n-2-\mu}{\mu-1} - (-1)^{\mu}\frac{n-2}{\mu-1}\binom{n-2-\mu}{\mu-2} \\ &= (-1)^{\mu+1}\frac{n}{\mu}\frac{n-\mu-1}{\mu-1}\binom{n-2-\mu}{\mu-2} = (-1)^{\mu+1}\frac{n}{\mu}\binom{n-1-\mu}{\mu-1}. \end{aligned}$$

□

The following lemma is proved in the same way as in [11, Lemma 3.8].

Lemma 3.8. *For a prime p, we abbreviate $CH^{\cdot}(Y_n) \otimes \mathbb{Z}_{(p)}$ as $CH^{\cdot}(Y_n)_{(p)}$. If p is odd, we have the following isomorphism of modules*:

(i) *For $n = 2m$,*

$$CH^{\cdot}(Y_n)_{(p)} \simeq \mathbb{Z}_{(p)}[c_1]/(c_1^{2(m-1)})\{1, \chi_{2m-4}\} \oplus \bigoplus_{i=1}^{m-2} \mathbb{Z}_{(p)}[c_1] /(c_1^{2(m-1-i)})\{c_2^{2i-1}, c_2^{2i}\}.$$

(ii) *For $n = 2m+1$,*

$$CH^{\cdot}(Y_n)_{(p)} \simeq \bigoplus_{i=0}^{m-2} \mathbb{Z}_{(p)}[c_1]/(c_1^{2(m-1-i)})\{c_2^{2i}, c_2^{2i+1}\}.$$

By using above lemmas and the integral basis, we can determine the ring structure of $CH^{\cdot}(Y_n)$. However it is too complicated to describe all the results here. The explicit results are written down in [12]. The above lemmas are also used in the proof of Theorem 4.1 in the next section.

4. The Chow ring of X_n

Let $\tilde{X}_n = X_n \times_{\mathbb{G}_m} \mathbb{A}^1$ be the associated bundle of (3.1) and $s : Y_n \to \tilde{X}_n$ the 0-section. Since $s^* : CH^{\cdot}(\tilde{X}_n) \xrightarrow{\sim} CH^{\cdot}(Y_n)$ by Theorem 3.1 (ii), the first assertion of the same theorem for $V = \tilde{X}_n$ and $X = s(Y_n)$ gives an exact sequence

$$CH^{\cdot}(Y_n) \xrightarrow{\cdot c_1} CH^{\cdot}(Y_n) \xrightarrow{\pi^*} CH^{\cdot}(X_n) \to 0. \tag{4.1}$$

Theorem 4.1. *Let $T(X_n)$ and $F(X_n)$ be the torsion part and the free part of $CH^{\cdot}(X_n)$, respectively. Then we have a ring isomorphism*:
(i) *For $n = 4t$,*

$$F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t)\{1, v_{4t-4}\}$$
$$T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^t)\{v_{4t-2}, v_{4t-4}v_{4t-2}\},$$
$$c_2^t = (-1)^t 2c_2 v_{4t-4},\ v_{4t-4}^2 = (-1)^t t c_2^{t-1} v_{4t-4}.$$

(ii) *For $n = 4t+1$,*

$$F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t) \oplus \mathbb{Z}[c_2]/(c_2^{t-1})\{v_{4t}\}$$
$$T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^t)\{v_{4t-2}, v_{4t-2}v_{4t}\} \oplus \mathbb{Z}/(2t)\{c_2^{t-1}v_{4t}\},$$
$$c_2^t = (-1)^t 2v_{4t},\ v_{4t-2}^2 = (-1)^{t+1} t c_2^{t-1} v_{4t}.$$

(iii) *For $n = 4t+2$,*

$$F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t)\{1, v_{4t}\} \oplus \mathbb{Z}\{v_{4t-2}\}$$
$$T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^{t-1})\{c_2 v_{4t-2}, c_2 v_{4t-2} v_{4t}\} \oplus \mathbb{Z}/4\{v_{4t-2}v_{4t}\},$$
$$c_2^t = (-1)^t 2v_{4t},\ c_2^t v_{4t-2} = 2v_{4t-2}v_{4t},\ v_{4t-2}^2 = (-1)^t t c_2^{t-1} v_{4t}.$$

(iv) *For $n = 4t+3$,*

$$F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t)\{1, v_{4t}\}$$
$$T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^t)\{v_{4t+2}, v_{4t}v_{4t+2}\} \oplus \mathbb{Z}/(2t+1)\{c_2^t v_{4t}\},$$
$$c_2^t = (-1)^t 2v_{4t},\ v_{4t}^2 = (-1)^{t+1} t c_2^t v_{4t}.$$

The other relations (e.g., $v_{4t-2}^2 = 0$ in (i)) are read from the additive presentations.

Before we begin a proof, we need some preparations. Hereafter, we focus on the case $n = 4t+1$ mainly.

Lemma 4.2. *We have a ring isomorphism for an odd prime p*:
(1) *For $n = 2m$,*

$$CH^{\cdot}(X_n)_{(p)} \simeq \mathbb{Z}_{(p)}\{\chi_{2m-4}\} \oplus \mathbb{Z}_{(p)}\{c_2^i \mid 0 \le i \le m-2\}.$$

(2) *For $n = 2m+1$,*

$$CH^{\cdot}(X_n)_{(p)} \simeq \mathbb{Z}_{(p)}\{c_2^i \mid 0 \le i \le m-2\} \oplus \mathbb{Z}_{(p)}/m\mathbb{Z}_{(p)}\{c_2^{m-1}\}.$$

Proof. We localize the exact sequence (4.1) at an odd prime p, and consider the case $n = 2m+1$. From Lemma 3.8(ii), a c_1-torsion element with the minimal degree is c_1^{2m-3}. Using $d_{m-1} = 0$ in $CH^{\cdot}(Y_n)_{(p)}$ by Theorem 3.4(2) and Lemma 3.5, we have

$$c_1 c_1^{2m-3} = c_1^{2m-2} \equiv (-1)^m m c_2^{m-1} \ mod\ Im(c_1). \tag{4.2}$$

Hence $mc_2^{m-1} \in Im(c_1)$.

The next c_1-torsion element with respect to the degree is $c_1^{2m-3}c_2$. Since $c_2^2 d_{m-2} = 0$ in $CH^{\cdot}(Y_n)_{(p)}$ by Theorem 3.4(2), we see that $(m-1)c_2^m \in Im(c_1)$ similarly. From (4.2), we note $mc_2^m \in Im(c_1)$. It implies that $c_2^m \in Im(c_1)$. Using a relation $d_{m-1}+(c_1^2-2c_2)d_{m-2} = -c_2^2 d_{m-3}$ repeatedly, we have $c_2^{2i} d_{m-1-i} = 0$. Repeating the above argument, we see that $c_2^{m+i} \in Im(c_1)$. We have proved the assertion from the short exact sequence (4.1). $\square$

Lemma 4.3. *For $n = 4t+1$, there are relations in $CH^{\cdot}(Y_n)$:*
(1)

$$c_1^{2t-2}c_2 = \sum_{\mu=1}^{t-1}(-1)^{1+\mu}\binom{2t-2-\mu}{\mu} c_1^{2(t-1-\mu)} c_2^{\mu+1} + 2v_{4t}.$$

(2)

$$c_1^{2t-2}c_2 v_{4t-2} = \left\{\sum_{\mu=1}^{t-1}(-1)^{1+\mu}\binom{2t-2-\mu}{\mu} c_1^{2(t-1-\mu)} c_2^{1+\mu}\right\} v_{4t-2} + 2v_{4t-2}v_{4t}.$$

(3)

$$v_{4t-2}^2 = (-1)^{t+1} d_{t-1} v_{4t}.$$

Proof. (1). By (3.4) and Lemma 3.5, we have

$$2v_{4t} = c_2 b_{2t-2} = \sum_{\mu=0}^{t-1}(-1)^{\mu}\binom{2t-2-\mu}{\mu} c_1^{2(t-1)-2\mu} c_2^{\mu}.$$

Hence we get the relation.

(2). The formula (2) follows from (1) immediately.

(3). We note that a homogeneous polynomial algebra $\mathbb{Z}[\alpha,\beta]$ is identified with an inhomogeneous ring $\mathbb{Z}[x]$ by $x = \frac{\beta}{\alpha}$. The we have from (3.2)

$$\begin{cases} c_1 = 1+x \\ c_2 = x \end{cases}$$

and

$$\begin{cases} b_k = (-1)^k \frac{1-x^{2k+1}}{1-x} \\ d_k = (-1)^k \frac{1-x^{2k+2}}{1-x^2} \end{cases}$$

Under the identification, we see that

$$\begin{aligned}(2v_{4t-2})^2 + d_{2t-1} &= (b_{2t-1})^2 + d_{2t-1}\\ &= \frac{(1+x)}{(1+x)} \cdot \frac{(1-x^{2t})^2}{(1-x)^2} - \frac{(1-x)}{(1-x)} \cdot \frac{(1-x^{4t})}{(1-x^2)}\\ &= 2\frac{(1-x^{2t})}{1-x^2} \cdot \frac{x(1-x^{2t-1})}{1-x}\\ &= 2(-1)^{t+1} d_{t-1} \cdot (c_2 b_{2t-2})\\ &= 4(-1)^{t+1} d_{t-1} \cdot v_{4t}.\end{aligned}$$

Since $d_{2t-1} = 0$ in $CH^{\cdot}(Y_{4t+1})$ by Theorem 3.4 (2), we obtain the relation

$$v_{4t-2}^2 = (-1)^{t+1} d_{t-1} v_{4t}.$$

□

Proof of Theorem 4.1. For $CH^{\cdot}(Y_n)_{(2)}$, we can calculate the c_1-image in a similar but more complicated way as in Lemma 4.1 by using the table [12, 5.4]. While we do not repeat this argument, we obtain the additive presentation of $CH^{\cdot}(X_n)$. Then a ring structure is given as follows: We consider the case (ii). From Lemma 3.8(ii) and Lemma 4.3(1), we have

$$c_1^{2t-2} c_2 \equiv (-1)^t c_2^t + 2v_{4t} \ mod\ Imc_1.$$

Hence we obtain $c_2^t = (-1)^t 2v_{4t}$. The relation $v_{4t-2}^2 = (-1)^{t+1} t c_2^{t-1} v_{4t}$ follows from Lemmas 3.5, 3.8(ii) and 4.3(3). Using the presentation of Theorem (4.1)(ii) and Lemma 4.3(2), we have

$$c_1^{2t-2} c_2 v_{4t-2} \equiv (-1)^t c_2^t v_{4t-2} + 2v_{4t-2} v_{4t} \ mod\ Imc_1$$

$$and \quad c_2^t v_{4t-2} = 0.$$

Similarly we see that

$$c_2^t v_{4t-2} v_{4t} = v_{4t}^2 = 0.$$

□

Next we consider the cycle map. The cohomology groups mean an étale cohomology [8], [14]. All varieties are defined over K', which is a subfield of an algebraically closed field K. Let l be a prime with $(l, \mathrm{ch}(K)) = 1$. We denote a locally constant sheaf $\mu_l^{\otimes i}$ by $\mathbb{Z}/l(i)$.

Corollary 4.4. *The homomorphism* $cl : CH^i(X_n) \to H^{2i}(X_n, \mathbb{Z}/l(i))$ *is injective.*

Proof. Since $(\tilde{X}_n, Y_n)$ is a smooth pair, we have the Gysin sequence as in [5, Appendice 1.3.3] and [14, VI Remark 5.4]. Since the cycle map and the Gysin map are commutative, we have the following commutative diagram, where each row is exact:

$$\begin{array}{ccccccc}
CH^i(Y_n) & \xrightarrow{\cdot c_1} & CH^{i+1}(Y_n) & \xrightarrow{\pi^*} & CH^{i+1}(X_n) & \longrightarrow & 0 \\
\downarrow{\scriptstyle cl} & & \downarrow{\scriptstyle cl} & & \downarrow{\scriptstyle cl} & & \\
H^{2i}(Y_n, \mathbb{Z}/l(i)) & \xrightarrow{\cdot c_1} & H^{2(i+1)}(Y_n, \mathbb{Z}/l(i+1)) & \xrightarrow{\pi^*} & H^{2(i+1)}(X_n, \mathbb{Z}/l(i+1)) & &
\end{array}$$

This corollary follows from this diagram. □

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