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An upper bound for the 3-primary homotopy exponent of the exceptional Lie group E_7

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Abstract

A new homotopy fibration is constructed at the prime 3 which shows that the quotient group E_7/F_4 is spherically resolved. This is then used to show that the 3-primary homotopy exponent of E_7 is bounded above by 3^{23}, which is at most four powers of 3 from being optimal.

1. Introduction

Let p be a prime. A *torsion Lie group* is a Lie group which has p-torsion in its integral cohomology. When p is odd the only torsion Lie groups which are connected, compact, and simple are F_4, E_6, E_7, and E_8 at the prime 3, and E_8 at the prime 5. The *homotopy exponent* of a space X is the least power of p which annihilates the p-torsion in $\pi_*(X)$. We write this as $\exp(X) = p^r$. In [Th1] upper bounds were calculated for the homotopy exponents of F_4 and E_6 at 3 which equalled the known lower bounds, thereby determining exact values for the homotopy exponents. In [Th2] an upper bound was calculated for the homotopy exponent of E_8 at 5 which differed from the known lower bound by one power of 5. The purpose of this paper is to deal with the next case by calculating an upper bound on the homotopy exponent of E_7 at 3. Specifically, we show:

Theorem 1.1. $\exp(E_7) \leq 3^{23}$.

A lower bound for the 3-primary homotopy exponent of E_7 was given by Davis [D2]. He calculated the v_1-periodic homotopy groups of E_7, which represent a certain subcollection of all the homotopy groups of E_7. His calculation showed that $\exp(E_7) \geq 3^{19}$. Thus Theorem 1.1 is at most four powers of 3 from being optimal. Davis has conjectured that the homotopy exponent of any connected, compact, simple Lie group equals the exponent of its v_1-periodic homotopy groups. Theorem 1.1 leaves open the possibility that there exists elements in $\pi_*(E_7)$ which are of order greater than 3^{19}. It would be very interesting to know if such elements exist.

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Theorem 1.1 is proved by closely examining the fibration $F_4 \longrightarrow E_7 \longrightarrow E_7/F_4$. A crucial step, which is of interest in its own right, is to show that E_7/F_4 is spherically resolved. This positively resolves a conjecture of Mimura. The lack of such a spherical resolution until now has often been a thorn in the side when studying the 3-primary homotopy theory of E_7. Cohomologically, with mod-3 coefficients, we have $H^*(E_7/F_4) \cong \Lambda(x_{19}, x_{27}, x_{35})$, where the right side is the exterior algebra on generators in dimensions $19, 27$, and 35 respectively. The action of the Steenrod algebra is trivial but there do exist nontrivial secondary operations. It is well known that the stable class $\alpha_2 \in \pi_{m+7}(S^m)$ is detected by a secondary operation Φ. Mimura, as reported in [D2], showed that $\Phi x_{19} = x_{27}$ and $\Phi x_{27} = x_{35}$. We prove:

Theorem 1.2. *There is a 3-primary homotopy fibration*

$$S^{19} \longrightarrow E_7/F_4 \longrightarrow B(27,35)$$

where $H^*(B(27,35)) \cong \Lambda(x_{27}, x_{35})$ *with* $\Phi x_{27} = x_{35}$ *and there is a homotopy fibration*

$$S^{27} \longrightarrow B(27,35) \longrightarrow S^{35}.$$

An alternative spherical resolution is given by:

Corollary 1.1. *The composition* $E_7/F_4 \longrightarrow B(27,35) \longrightarrow S^{35}$ *results in a homotopy fibration*

$$B(19,27) \longrightarrow E_7/F_4 \longrightarrow S^{35}$$

where $H^*(B(19,27)) \cong \Lambda(x_{19}, x_{27})$ *with* $\Phi x_{19} = x_{27}$ *and there is a homotopy fibration*

$$S^{19} \longrightarrow B(19,27) \longrightarrow S^{27}.$$

The strategy behind the proof of Theorem 1.2 is as follows. The standard first approach to constructing a map $E_7/F_4 \longrightarrow B(27,35)$ which is an injection in cohomology is to start with a map from some skeleton of E_7/F_4 to $B(27,35)$ and then extend over the remaining cells of E_7/F_4 one at a time. In trying to do so, however, one encounters potential obstructions to the extensions in the homotopy groups of $B(27,35)$ which are difficult to resolve. Instead, we take a different approach which is based on Cohen and Neisendorfer's [CN] construction of finite H-spaces. We produce a space B whose 62-skeleton satisfies: (i) $H^*(B_{62}) \cong H^*((E_7/F_4)_{62})$ and (ii) there exists a map $B_{62} \longrightarrow B(27,35)$ which is an injection in cohomology. Other methods from [CN] then allow us to produce a homotopy equivalence $(E_7/F_4)_{62} \simeq B_{62}$. It remains to extend $(E_7/F_4)_{62} \longrightarrow B(27,35)$ over the 81-cell of E_7/F_4. This is done using Davis' proof [D2] that the top cell of E_7/F_4 is stably spherical.

This paper is organized as follows. Section 2 records some simple but useful facts about the homology of E_7/F_4. Section 3 reviews the work of Cohen and Neisendorfer and does the work to set up Section 4, which proves Theorem 1.2.

Section 5 records a method for calculating an upper bound on the homotopy exponent of a spherically resolved space. This requires certain ingredient maps, which Section 6 establishes in the context of E_7. The exponent bound in Theorem 1.1 is then proved in Section 7.

2. The homology of E_7/F_4

This brief section simply records some observations about the homology of E_7/F_4 which the reader may find helpful to keep in mind. Dualizing $H^*(E_7/F_4) \cong \Lambda(x_{19}, x_{27}, x_{35})$ we obtain $H_*(E_7/F_4) \cong \Lambda(u, v, w)$ where u, v, w are in degrees $19, 27, 35$ respectively. Let Φ_* be the hom-dual of the secondary operation Φ. Then $\Phi_* w = v$ and $\Phi_* v = u$.

Let $A = (E_7/F_4)_{35}$, the 35-skeleton of E_7/F_4. Observe that A has three cells and $H_*(A) = \{u, v, w\}$. It is useful to observe that $H_*(E_7/F_4) \cong \Lambda(\widetilde{H}_*(A))$. Let $V = \widetilde{H}_*(A)$. In terms of monomials, we can write $\Lambda(V) \cong \oplus_{i=1}^{3} \Lambda_i(V)$, where $\Lambda_i(V)$ consists of the monomials of length i in $\Lambda(V)$. Observe that $\Lambda_1(V) = V$ has dimension 35, $\Lambda_2(V) = \{uv, uw, vw\}$ has "connectivity" > 35 and dimension 62, while $\Lambda_3(V) = \{uvw\}$ consists of a single element in degree 81. So $\Lambda_1(V)$, $\Lambda_2(V)$, and $\Lambda_3(V)$ occupy distinct degree ranges.

We will also often use the 62-skeleton of E_7/F_4. Observe that there is an isomorphism $H_*((E_7/F_4)_{62}) \cong \Lambda_1(V) \oplus \Lambda_2(V)$.

3. Cohen and Neisendorfer's construction of finite H-spaces

In this section we review the work of Cohen and Neisendorfer [CN] in detail. Their methods led to a construction of finite p-local H-spaces, which we state in Theorem 3.1. But the methods can do more, as they knew, and we present one generalization in Theorem 3.2. This is then applied in the context of E_7/F_4. In terms of the strategy to prove Theorem 1.2 outlined in the Introduction, the goal of this section is to produce a 3-local space F such that $H_*(F_{62}) \cong H_*((E_7/F_4)_{62})$ and there exists a map $F_{62} \longrightarrow B(27, 35)$ which is onto in homology.

We begin with Cohen and Neisendorfer's statement regarding H-spaces. Assume that homology is taken with mod-p coefficients.

Theorem 3.1. *Fix an odd prime p. Let A be a p-local complex of l odd dimensional cells, where $l < p - 1$. Then there is a homotopy fibration $B \longrightarrow R \longrightarrow \Sigma A$ satisfying*:

(a) $\Omega\Sigma A \simeq B \times \Omega R$;

(b) $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$;

(c) *the composite* $A \xrightarrow{E} \Omega\Sigma A \longrightarrow B$ *includes* $\widetilde{H}_*(A)$ *into* $H_*(B)$ *as the generating set of the exterior algebra.*

Further, all of these statements are functorial for maps $f : A \longrightarrow A'$ between spaces satisfying the hypotheses.

Note that Theorem 3.1 (a) implies B is an H-space. The functorial property implies that B is spherically resolved. For if the bottom cell of A is S^{2m+1} then the homotopy cofibration

$$S^{2m+1} \longrightarrow A \longrightarrow A'$$

results in a homotopy fibration

$$S^{2m+1} \longrightarrow B \longrightarrow B'.$$

We can then iterate with respect to A' and B'.

Ideally, we would like to apply Theorem 3.1 to $A = (E_7/F_4)_{35}$. Then the resulting H-space B would satisfy $H_*(B) \cong \Lambda(\widetilde{H}_*(A)) \cong H_*(E_7/F_4)$ and B would be spherically resolved. Showing E_7/F_4 and B are homotopy equivalent would then prove Theorem 1.2. However, as $p = 3$ and A has three cells, Theorem 3.1 does not apply. Instead, we go back to the methods that made Theorem 3.1 work and derive a more general statement in Theorem 3.2 which can be applied to A, although in a more limited way.

Everything that follows comes from [CN]. The idea is to study $\Omega\Sigma A$ and see if it is possible to produce a retract corresponding to B. The first step is to see what is possible in homology. The Bott-Samelson Theorem says that $H_*(\Omega\Sigma A) \cong T(\widetilde{H}_*(A))$, where $T(\ )$ is the free tensor algebra. It is well known that $T(\widetilde{H}_*(A))$ is isomorphic to the universal enveloping algebra of the free Lie algebra generated by $\widetilde{H}_*(A)$. So we need to consider how the universal enveloping algebra behaves.

In general, for a graded vector space V, let $L = L\langle V\rangle$ be the free Lie algebra generated by V. Let UL be the universal enveloping algebra. Let $L_{ab} = L_{ab}\langle V\rangle$ be the free abelian Lie algebra generated by V, that is, the bracket in L_{ab} is identically zero. Let $[L, L]$ be the kernel of the quotient map $L \longrightarrow L_{ab}$. The short exact sequence of Lie algebras

$$0 \longrightarrow [L, L] \longrightarrow L \longrightarrow L_{ab} \longrightarrow 0$$

results in a short exact sequence of Hopf algebras

$$0 \longrightarrow U[L, L] \longrightarrow UL \longrightarrow UL_{ab} \longrightarrow 0.$$

When the elements of V are all of odd dimension, an explicit Lie basis for $[L, L]$ is given by the following.

Lemma 3.1. *Suppose $V = \{u_1, \ldots, u_l\}$ where each u_i is of odd dimension and l is a positive integer. Let $L = L\langle V\rangle$. Then a Lie basis for $[L, L]$ is given by the elements*

$$[u_i, u_j], [u_{k_1}, [u_i, u_j]], [u_{k_2}, [u_{k_1}, [u_i, u_j]]], \ldots$$

where $1 \leq j \leq i \leq l$ and $1 \leq k_t < k_{t-1} < \cdots < k_2 < k_1 < i$. In particular, the basis elements have bracket lengths from 2 through $l+1$.

We now bring in the topology. Let A be a CW-complex consisting of l odd dimensional cells. Localize at p. Let $V = \widetilde{H}_*(A)$ and $L = L\langle V\rangle$. We would like to geometrically realize the Lie basis elements of $[L, L]$ in Lemma 3.1 as certain Whitehead products. By [SW] we can do so if the bracket length is not a power of p. This is a much stronger statement than we need, however. For our purposes it will suffice to geometrically realize the brackets of length $< p$.

To describe how this comes about, let $A^{(k)}$ be the k-fold smash of A with itself. Let

$$w_k : \Sigma A^{(k)} \longrightarrow \Sigma A$$

be the k-fold Whitehead product of the identity map on ΣA with itself. Observe that if σ is a permutation in the symmetric group Σ_k on k letters then there is a corresponding map $\sigma : \Sigma A^{(k)} \longrightarrow \Sigma A^{(k)}$ defined by permuting the smash factors. Define a map

$$\beta_k : \Sigma A^{(k)} \longrightarrow \Sigma A^{(k)}$$

inductively by letting $\beta_2 = 1 - (1, 2)$ and

$$\beta_k = (1 - (k, k-1, \ldots, 2, 1)) \circ (1 \wedge \beta_{k-1}).$$

In homology (ignoring the suspension coordinate),

$$(\beta_k)_*(x_1 \otimes \cdots \otimes x_k) = [x_1, [x_2, \ldots [x_{k-1}, x_k] \ldots]].$$

This map has the property that $(\beta_k)_* \circ (\beta_k)_* \simeq k \cdot (\beta_k)_*$. Thus if we restrict to $k < p$ and define $\overline{\beta}_k = \frac{1}{k}\beta_k$ then $(\overline{\beta}_k)_*$ is an idempotent. Let R_k be the mapping telescope of $\overline{\beta}_k$. Then $H_*(R_k) \cong \text{Im } (\overline{\beta}_k)_*$. In particular, the cells in R_k are in one-to-one correspondence with the Lie basis elements in $[L, L]$ of bracket length k, where the dimension of the cell is one more than the degree of the corresponding bracket. Note that if A has l cells, where $l < p - 2$, then Lemma 3.1 implies that $[L, L]$ has *no* basis elements of length k for $l + 2 \leq k \leq p - 1$, and so $H_*(R_k) = 0$, implying that R_k is homotopy equivalent to a point. Let S_k be the mapping telescope of $1 - \overline{\beta}_k$. As $(\overline{\beta}_k)_* + (1 - \overline{\beta}_k)_*$ is the identity map, the map

$$\Sigma A^{(k)} \longrightarrow R_k \vee S_k$$

is an isomorphism in homology and so is a homotopy equivalence. Let R be the wedge sum

$$R \simeq \bigvee_{k=2}^{p-1} R_k.$$

Define

$$R \longrightarrow \Sigma A$$

as the wedge sum of the composites

$$R_i \longrightarrow \Sigma A^{(i)} \xrightarrow{w_i} \Sigma A.$$

Observe that the cells of R are in one-to-one correspondence with the Lie basis elements of $[L, L]$ of bracket length k for $2 \leq k \leq p-1$. Define B by the homotopy fibration

$$B \longrightarrow R \longrightarrow \Sigma A.$$

Cohen and Neisendorfer [CN] proved the following result. Note that their explicit statement restricted to the case when $l < p-1$, but their argument held in the generality stated below.

Theorem 3.2. *Let A be a p-local CW complex consisting of l odd dimensional cells, where l is some positive integer. Let $V = \widetilde{H}_*(A)$ and let $L = L\langle V\rangle$. Then there is a homotopy fibration sequence*

$$\Omega R \longrightarrow \Omega\Sigma A \longrightarrow B \longrightarrow R \longrightarrow \Sigma A$$

with the following property. Let t be the least degree of the Lie basis elements in $[L, L]$ of length p. A homological model for the homotopy fibration $\Omega R \longrightarrow \Omega\Sigma A \longrightarrow B$ in degrees $< t$ is given by the short exact sequence of Hopf algebras

$$0 \longrightarrow U[L, L] \longrightarrow UL \longrightarrow UL_{ab} \longrightarrow 0.$$

In particular, if $l < p-1$ then no basis element of $[L, L]$ has length p and so $t = \infty$, implying that $H_*(B) \cong UL_{ab} \cong \Lambda(\widetilde{H}_*(A))$. Furthermore, all of these statements are functorial for maps $A \longrightarrow A'$ between spaces satisfying the hypotheses.

Example 3.1. Let $p = 3$. Let $A = (E_7/F_4)_{35}$. Let $V = \widetilde{H}_*(A)$, so $V = \{u, v, w\}$ where u, v, and w have degrees $19, 27$, and 35 respectively. Note that $H_*(E_7/F_4) \cong \Lambda(V)$. Let $L = L\langle V\rangle$. Using the Lie basis for $[L, L]$ in Lemma 3.1, observe that the basis element of length three or more of least degree is $[u, [v, u]]$, which has degree 65. Theorem 3.2 says there is a homotopy fibration sequence

$$\Omega R \longrightarrow \Omega\Sigma A \xrightarrow{\partial} B \longrightarrow R \longrightarrow \Sigma A$$

where the terms $\Omega R \longrightarrow \Omega\Sigma A \longrightarrow B$ are modelled homologically through dimension 64 by the short exact sequence of Hopf algebras $0 \longrightarrow U[L, L] \longrightarrow UL \longrightarrow UL_{ab} \longrightarrow 0$. Note that as $p = 3$ the wedge defining R consists only of the summand R_2, which corresponds to the length two brackets in $[L, L]$. The isomorphism $H_*(B) \cong UL_{ab} \cong \Lambda(u, v, w)$ through degree 64 accounts for every element in $\Lambda(u, v, w)$ except uvw. Note that we only need this isomorphism up to degree 62 as the length two monomial of highest degree is vw. In particular, there is an isomorphim $H_*(B_{62}) \cong H_*((E_7/F_4)_{62})$. Further, there are isomorphisms $UL \cong T(V)$ and $UL_{ab} \cong S(V)$, where $T(V)$ is the tensor algebra generated by V and $S(V)$ is the symmetric algebra generated by V. The map $UL \longrightarrow UL_{ab}$ is the abelianization $T(V) \longrightarrow S(V)$. Thus ∂_* is the abelianization of the tensor algebra through degree 62.

We can push Example 3.1 further by using the functoriality of Theorem 3.2.

Lemma 3.2. *There is a map*

$$B_{62} \longrightarrow B(27,35)$$

which is an epimorphism in homology, where $B(27,35)$ is a space satisfying: (i) $H_*(B(27,35)) \cong \Lambda(y_{27}, y_{35})$ with $\Phi_* y_{35} = y_{27}$ and (ii) there is a homotopy fibration $S^{27} \longrightarrow B(27,35) \longrightarrow S^{35}$.

Proof. The content of Example 3.1 is assumed throughout. Also note that the homological description of E_7/F_4 in Section 2 implies there are (dual) secondary operations $\Phi_* w = v$ and $\Phi_* v = u$ in $V = \widetilde{H}_*(A)$. Including S^{19} into A gives a homotopy cofibration

$$S^{19} \longrightarrow A \longrightarrow A'$$

and then including S^{27} into A' gives a homotopy cofibration

$$S^{27} \longrightarrow A' \longrightarrow S^{35}.$$

The definition of A' implies that $\Phi_* w = v$ in $\widetilde{H}_*(A')$. Let $V' = \widetilde{H}_*(A') = \{v, w\}$ and let $V'' = \widetilde{H}_*(S^{35}) = \{w\}$. Let $L' = L\langle V'\rangle$ and $L'' = L\langle V''\rangle$. Observe that the Lie basis elements of length three or more in $[L', L']$ are in degrees 89 and higher, while the Lie basis elements of length three or more in $[L'', L'']$ are in degrees 105 and higher. Theorem 3.2 shows there are homotopy fibrations $B' \longrightarrow R' \longrightarrow \Sigma A'$ and $B'' \longrightarrow R'' \longrightarrow S^{35}$ whose associated homotopy fibrations $\Omega R' \longrightarrow \Omega\Sigma A' \longrightarrow B'$ and $\Omega R'' \longrightarrow \Omega\Sigma S^{35} \longrightarrow B''$ are modelled homologically through dimensions 88 and 104 respectively by the short exact sequences of Hopf algebras $0 \longrightarrow U[L', L'] \longrightarrow UL' \longrightarrow UL'_{ab} \longrightarrow 0$ and $0 \longrightarrow U[L'', L''] \longrightarrow UL'' \longrightarrow UL''_{ab} \longrightarrow 0$. The functoriality of Theorem 3.2 then shows there is a homotopy fibration diagram (in which the rows are fibrations)

$$\begin{array}{ccccc}
\Omega R & \longrightarrow & \Omega\Sigma A & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
\Omega R' & \longrightarrow & \Omega\Sigma A' & \longrightarrow & B' \\
\downarrow & & \downarrow & & \downarrow \\
\Omega R'' & \longrightarrow & \Omega\Sigma S^{35} & \longrightarrow & B''
\end{array} \tag{3.1}$$

where the diagram as a whole is modelled homologically through degree 62 by the diagram of Hopf algebras (in which the rows are short exact sequences)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U[L,L] & \longrightarrow & UL & \longrightarrow & UL_{ab} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U[L',L'] & \longrightarrow & UL' & \longrightarrow & UL'_{ab} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U[L'',L''] & \longrightarrow & UL'' & \longrightarrow & UL''_{ab} & \longrightarrow & 0.
\end{array}$$

Here, the vertical arrows are determined by the projections

$$V = \{u, v, w\} \longrightarrow V' = \{v, w\} \longrightarrow V'' = \{w\}$$

of the generating sets of the underlying Lie algebras.

Restrict (3.1) to 62-skeletons. In particular, consider the sequence $B_{62} \longrightarrow (B')_{62} \longrightarrow (B'')_{62}$. Observe that $UL'_{ab} \cong \Lambda(v, w)$ has dimension 62. So $H_*((B')_{62}) \cong UL'_{ab} \cong \Lambda(v, w)$. Further, $\Phi_* w = v$ since the same is true in $H_*(\Omega\Sigma A) \cong UL$. Define the space $B(27, 35)$ as $(B')_{62}$. Then the only assertion in the statement of the Lemma which remains to be proved is the existence of the homotopy fibration $S^{27} \longrightarrow B(27, 35) \longrightarrow S^{35}$. Observe that $UL''_{ab} \cong \Lambda(w)$ has dimension 35. So $H_*((B'')_{62}) \cong UL''_{ab} \cong \Lambda(w)$. Thus $(B'')_{62}$ is homotopy equivalent to S^{35}. The map $(B')_{62} \longrightarrow (B'')_{62}$ then becomes a map $B(27, 35) \longrightarrow S^{35}$ which is onto in homology. Its homotopy fiber is immediately seen to have homology isomorphic to $\Lambda(v)$ by the Serre spectral sequence, and so is homotopy equivalent to S^{27}. $\square$

Another aspect of Cohen and Neisendorfer's [CN] work we will use is the construction of certain retractions. Again, let A be a p-local CW-complex such that $\widetilde{H}_*(A)$ is concentrated in odd degrees. Consider $H_*(\Omega\Sigma A) \cong T(\widetilde{H}_*(A))$. Let $V = \widetilde{H}_*(A)$. Let $T_k(V)$ be the submodule of $T(V)$ consisting of the tensors of length k. Let $S_k(V)$ be the submodule of $T_k(V)$ consisting of the symmetric tensors of length k. Let $V^{\otimes k}$ be the k-fold tensor product of V with itself. The symmetric group Σ_k on k letters acts on $V^{\otimes k}$ by permuting the tensor factors. If $k < p$ then a standard calculation shows that the map $s_k : V^{\otimes k} \longrightarrow V^{\otimes k}$ defined by $s_k = \frac{1}{k!}\Sigma_{\sigma \in \Sigma_k} \sigma$ is an idempotent. The image of s_k is isomorphic to $S_k(V)$. All this algebra can be realized geometrically. The symmetric group Σ_k acts on $A^{(k)}$ by permuting the smash factors. Suspending so that we can add and restricting to $k < p$, the algebraic map s_k has a geometric analogue $\bar{s}_k : \Sigma A^{(k)} \longrightarrow \Sigma A^{(k)}$. If A is a co-H space then $[\Sigma A^{(k)}, \Sigma A^{(k)}]$ is an abelian group and we can perform the arithmetic to show that $\bar{s}_k$ is an idempotent. If A is not a co-H space then we only know in general that $\bar{s}_k$ is a self-map, but its image in homology is an idempotent as $(\bar{s}_k)_*$ is the suspension of s_k. In either case, the mapping telescope $\mathrm{Tel}(\bar{s}_k)$ of $\bar{s}_k$ has the property that $H_*(\mathrm{Tel}(\bar{s}_k)) \cong \Sigma S_k(V)$. Consequently, if $k < p$ then $\bigvee_{i=1}^k \mathrm{Tel}(\bar{s}_k)$ is a retract of $\bigvee_{i=1}^k \Sigma A^{(k)}$ and has the property that $H_*(\bigvee_{i=1}^k \mathrm{Tel}(\bar{s}_k)) \cong \oplus_{i=1}^k \Sigma S_k(V)$.

Using this in tandem with the James decomposition [J] $\Sigma\Omega\Sigma A \simeq \bigvee_{i=1}^\infty \Sigma A^{(i)}$ we obtain:

Lemma 3.3. *If $k < p$ then $\bigvee_{i=1}^k$ Tel$(\bar{s}_k)$ is a retract of $\Sigma\Omega\Sigma A$ and has the property that*

$$H_*\left(\bigvee_{i=1}^k \mathit{Tel}(\bar{s}_k)\right) \cong \bigoplus_{i=1}^k \Sigma S_k(V).$$

In the context of E_7/F_4 at 3 we use Lemma 3.3 as follows. Let A be the 35-skeleton of E_7/F_4, so $\widetilde{H}_*(A) \cong \{u, v, w\}$ where u, v, w have degrees $19, 27,$

and 35 respectively. Let $V = \{u, v, w\}$. Then $H_*(E_7/F_4) \cong \Lambda(V)$. As u, v, w are of odd degree, $\Lambda(V)$ is isomorphic to the symmetric algebra $S(V)$, so we can write $H_*(E_7/F_4) \cong \oplus_{i=1}^{3} S_i(V)$, where $S_i(V)$ consists of the tensors of length i in $S(V)$. Observe that $S_2(V)$ has dimension 62 while $S_3(V)$ consists of the single element uvw in degree 81. Thus $H_*((E_7/F_4)_{62}) \cong S_1(V) \oplus S_2(V)$. We do not work with $(E_7/F_4)_{62}$ directly because we do not yet know enough about it. Instead, we work with the 62-skeleton of the space B constructed in Example 3.1. Recall that there is a homotopy fibration sequence $\Omega\Sigma A \xrightarrow{\partial} B \longrightarrow R \longrightarrow \Sigma A$ and $H_*(B_{62}) \cong S_1(V) \oplus S_2(V)$. Further, in homology the map ∂_* is the abelianization $T(V) \longrightarrow S(V)$ through degree 62.

Lemma 3.4. *There is a map $B_{62} \longrightarrow \Omega\Sigma A$ such that the composite $B_{62} \longrightarrow \Omega\Sigma A \xrightarrow{\partial} B$ is a monomorphism in homology.*

Proof. By Lemma 3.3 with $p = 3$, there is a map $\mathrm{Tel}(\bar{s}_1) \vee \mathrm{Tel}(\bar{s}_2) \longrightarrow \Sigma\Omega\Sigma A$ whose image in homology is $\Sigma S_1(V) \oplus \Sigma S_2(V)$. Since ∂_* is the abelianization $T(V) \longrightarrow S(V)$ through degree 62, the composite $\mathrm{Tel}(\bar{s}_1) \vee \mathrm{Tel}(\bar{s}_2) \longrightarrow \Sigma\Omega\Sigma A \xrightarrow{\Sigma\partial} \Sigma B$ is an isomorphism in homology through degree 63. Thus $\mathrm{Tel}(\bar{s}_1) \vee \mathrm{Tel}(\bar{s}_2) \longrightarrow \Sigma B_{62}$ is a homotopy equivalence. This proves a suspended version of the assertion: there is a map $\Sigma B_{62} \longrightarrow \Sigma\Omega\Sigma A$ such that the composite $\Sigma B_{62} \longrightarrow \Sigma\Omega\Sigma A \xrightarrow{\Sigma\partial} \Sigma B$ is a monomorphism in homology.

It remains to desuspend. Consider the composite $\theta : B_{62} \longrightarrow \Omega\Sigma\Omega\Sigma A \xrightarrow{\Omega ev} \Omega\Sigma A$ where the left map is the adjoint of $\Sigma B_{62} \longrightarrow \Sigma\Omega\Sigma A$ and ev is the evaluation map. For convenience, rewrite the elements of $V = \{u, v, w\}$ as $\{u_1, u_2, u_3\}$. To prove the assertion we will show that, in homology, the composite $B_{62} \xrightarrow{\theta} \Omega\Sigma A \xrightarrow{\partial} B$ acts as the identity map on both direct summands $S_1(V) = V$ and $S_2(V)$ of $H_*(B_{62})$. Observe that in homology the adjunction, the evaluation, and ∂ all act as the identity on the generators in V, and therefore so do θ_* and $(\partial \circ \theta)_*$. Next, consider how θ_* acts on $S_2(V)$. We have $\theta_*(u_i) = u_i$ for $1 \leq i \leq 3$. Since θ_* is a coalgebra map, it commutes with the reduced diagonal. A standard calculation then shows that $\theta_*(u_i u_j) = u_i u_j + \lambda$ where λ is primitive. The module of primitives in $H_*(\Omega\Sigma A) \cong T(V)$ consists of the free Lie algebra $L\langle V\rangle$ and the 3^{rd}-powers of even degree elements. In degrees ≤ 62, there are no such 3^{rd}-powers. As well, by comparing degrees, λ cannot be an element of V, so we have $\lambda \in [L, L] \subseteq L\langle V\rangle$. As ∂_* is the abelianization $T(V) \longrightarrow S(V)$ through degree 62, it therefore sends λ to zero and so $\partial_* \circ \theta_*(u_i u_j) = u_i u_j$. Thus $\partial_* \circ \theta_*$ acts as the identity map on the submodule $S_2(V) \subseteq H_*(B_{62})$. This finishes the proof. □

4. The spherical resolution of E_7/F_4

In this Section we prove Theorem 1.2. The proof considers E_7/F_4 in two pieces. First, we prove a statement about $(E_7/F_4)_{62}$ in Corollary 4.1 and then we prove a property of the one remaining cell in dimension 81 in Lemma 4.1.

Proposition 4.1 is the crucial step. We know that B_{62} and $(E_7/F_4)_{62}$ have the same homology. We also know that B_{62} is "spherically resolved," in the sense of Lemma 3.2, which is the property we want for E_7/F_4. We now connect the two by showing that B_{62} and $(E_7/F_4)_{62}$ are homotopy equivalent. To do so, we consider the stablilization $Q(E_7/F_4)$ of E_7/F_4. Here, for any space X, there is a sequence of iterated suspensions

$$E^\infty : X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\Omega\Sigma E} \Omega^2\Sigma^2 X \longrightarrow \cdots \longrightarrow \Omega^\infty\Sigma^\infty X = Q(X).$$

Let $\overline{E}$ be the truncation

$$\overline{E} : \Omega\Sigma X \xrightarrow{\Omega\Sigma E} \Omega^2\Sigma^2 X \longrightarrow \cdots \longrightarrow Q(X).$$

Consider the composite

$$\theta : B_{62} \longrightarrow \Omega\Sigma A \xrightarrow{\Omega\Sigma i} \Omega\Sigma E_7/F_4 \xrightarrow{\overline{E}} Q(E_7/F_4)$$

where the left map comes from Lemma 3.4 and i is the skeletal inclusion.

Proposition 4.1. *There is a lift*

$$\begin{array}{ccc} & & B_{62} \\ & \overset{\lambda}{\swarrow} & \downarrow{\scriptstyle\theta} \\ E_7/F_4 & \xrightarrow{E^\infty} & Q(E_7/F_4) \end{array}$$

for some map λ.

Deferring the proof of Proposition 4.1 for the moment, let us first state the consequence of interest.

Corollary 4.1. *The restriction of* λ *to the* 62*-skeleton of* E_7/F_4 *results in a homotopy equivalence* $B_{62} \xrightarrow{\simeq} (E_7/F_4)_{62}$.

Proof. Recall that $H_*(B_{62})$ is isomorphic to the submodule of the exterior algebra $\Lambda(u,v,w)$ consisting of the length 1 and 2 monomials. Consider how the three maps comprising the definition of θ act on $H_*(B_{62})$. By Lemma 3.4, $B_{62} \longrightarrow \Omega\Sigma A$ is a monomorphism in homology which sends $H_*(B_{62}) \cong S_1(V) \oplus S_2(V)$ onto the submodule $S_1 \oplus S_2(V) \subseteq T(V) \cong H_*(\Omega\Sigma A)$. The definition of $\overline{E}$ shows it is a loop map, so $\overline{E} \circ \Omega\Sigma i$ is a loop map, and so $(\overline{E} \circ \Omega\Sigma i)_*$ is multiplicative. As $(\overline{E} \circ \Omega\Sigma i)_*$ acts as the identity on $V = S_1(V)$, it multiplicatively acts as the identity on $S_2(V)$. Hence θ_* is a monomorphism. The factorization of θ in Proposition 4.1 then implies that λ_* is a monomorphism. As $H_*(B_{62}) \cong H_*((E_7/F_4)_{62})$, we see that λ_* is in fact an isomorphism in degrees ≤ 62, and so $B_{62} \longrightarrow (E_7/F_4)_{62}$ is a homotopy equivalence. □

Proof of Proposition 4.1. We need to consider the homology of $Q(E_7/F_4)$ up to dimension 63. For a simply connected space X, [DL] or [CLM] showed that $H_*(Q(X))$ (homology with mod-p coefficients) is isomorphic to the free symmetric algebra generated by the independent admissible sequences of Dyer-Lashof operations acting on $\widetilde{H}_*(X)$. Further, the map $X \xrightarrow{E^\infty} Q(X)$ induces the inclusion of the generating set in homology. We write the Dyer-Lashof operations using lower notation,

$$Q_{s(p-1)} : H_t(X) \longrightarrow H_{pt+s(p-1)}(X),$$

and note that for odd primes $s+t$ is required to be even. In our case, $p=3$ and $H_*(E_7/F_4) \cong \Lambda(u,v,w)$ where u,v,w have degrees $19, 27, 35$ respectively. Excluding the elements coming from $H_*(E_7/F_4)$, a basis for $H_*(Q(E_7/F_4))$ through dimension 63 is as follows:

(4.1)

Dimension	Basis
58	$\beta Q_2 u$
59	$Q_2 u$
62	$\beta Q_6 u$
63	$Q_6 u.$

Write $\mathcal{P}^t_*$ for the hom-dual of the Steenrod operation $\mathcal{P}^t$. Then there are Nishida relations

$$\mathcal{P}^1_* Q_6 u = Q_2 u$$
$$\mathcal{P}^1_* \beta Q_6 u = 2\beta Q_2 u.$$

Define C by the homotopy cofibration

$$E_7/F_4 \xrightarrow{E^\infty} Q(E_7/F_4) \longrightarrow C. \tag{4.2}$$

Since $(E^\infty)_*$ is an inclusion, a basis for $H_*(C)$ through dimension 63 is given by the images of the basis elements in (4.1). That is,

$$H_*(C) \cong \{y_{58}, y_{59}, y_{62}, y_{63}\}$$

for $* \leq 63$ with $\beta y_{59} = y_{58}$, $\beta y_{63} = y_{62}$, $\mathcal{P}^1_* y_{62} = 2y_{58}$, and $\mathcal{P}^1_* y_{63} = y_{59}$. In particular, C is 57-connected. As E_7/F_4 is 18-connected, the Serre exact sequence implies the homotopy cofibration defining C is also a homotopy fibration through dimension 75. Thus, as B_{62} is 62-dimensional, we can regard (4.2) as a homotopy fibration, which implies that the asserted lift λ will exist once we show that the composite $\varphi : B_{62} \xrightarrow{\theta} Q(E_7/F_4) \longrightarrow C$ is null homotopic.

Since C is 57-connected, φ collapses out the 57-skeleton of B_{62}, resulting in a factorization

$$\begin{array}{ccc} B_{62} & \xrightarrow{q} & S^{62} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\phi} \\ C & = & C \end{array}$$

for some map ϕ, where q is the pinch map onto the top cell. We will show that ϕ is null homotopic, implying that φ is null homotopic.

It suffices to restrict to the 62-skeleton of C. The homological description of C implies that there is a homotopy cofibration

$$P^{59}(3) \longrightarrow C_{62} \longrightarrow S^{62}.$$

Let $x \in H_{62}(S^{62})$ be a generator. If $\phi_* x = y_{58}$ then the relation $\mathcal{P}^1_* y_{62} = 2y_{58}$ implies that $\mathcal{P}^1_* x \neq 0$, a contradiction. Thus $\phi_* x = 0$. Hence the composite $S^{62} \xrightarrow{\phi} C \longrightarrow S^{62}$ is zero in homology and so is null homotopic. This results in a lift

$$\begin{array}{ccccc} & & S^{62} & & \\ & \overset{\gamma}{\swarrow} & \downarrow{\scriptstyle \phi} & & \\ P^{59}(3) & \longrightarrow & C_{62} & \longrightarrow & S^{62} \end{array}$$

for some map γ. Here, we have used the fact that the Serre exact sequence implies the homotopy cofibration along the bottom row of the diagram is also a homotopy fibration through a dimension larger than 62. To deal with γ, use the homotopy cofibration $S^{58} \longrightarrow P^{59}(3) \longrightarrow S^{59}$. A simple calculation shows that $\pi_{62}(P^{59}(3)) = \mathbb{Z}/3\mathbb{Z}$, where the generator $\bar{\alpha}_1$ is a lift of the stable class α_1 which generates $\pi_{62}(S^{59}) = \mathbb{Z}/3\mathbb{Z}$. As α_1 is detected by the Steenrod operation $\mathcal{P}^1$, so is $\bar{\alpha}_1$. But in $H_*(C)$ we have $\mathcal{P}^1_* y_{63} = y_{59}$, implying that $S^{59} \xrightarrow{\bar{\alpha}_1} P^{59}(3)$ has been coned off in C. Hence the composite $S^{59} \xrightarrow{\gamma} P^{59}(3) \longrightarrow C$ – that is, ϕ – is null homotopic, as required. □

We now turn to the 81-dimensional cell of E_7/F_4. The statement in Lemma 4.1 was proved in [D2]. The proof is included for the sake of completeness.

Lemma 4.1. *The* 81*-dimensional cell in* E_7/F_4 *is stably spherical.*

Proof. We prove the dual statement that the bottom cell of the Spanier-Whitehead dual of E_7/F_4 stably splits off. Since E_7/F_4 is a manifold, its Spanier-Whitehead dual is the Thom space of the stable normal bundle ν over E_7/F_4. The bottom cell of this Thom space stably splits off if and only if $J(f) = 0$, where $f : E_7/F_4 \longrightarrow BO$ classifies ν, and J is the stable J-homomorphism. For any space X, the standard map $KO(X) \longrightarrow J(X)$ is an epimorphism. But in [D2] it is shown that $KO(E_7/F_4) = 0$ and so $J(f) = 0$. □

Proof of Theorem 1.2. By Corollary 4.1, there is a homotopy equivalence $(E_7/F_4)_{62} \simeq B_{62}$. Using this in combination with Lemma 3.2, we obtain a map $f : (E_7/F_4)_{62} \longrightarrow B(27, 35)$ which is onto in homology. Now consider the homotopy cofibration

$$S^{80} \longrightarrow (E_7/F_4)_{62} \longrightarrow E_7/F_4$$

which attaches the top cell to E_7/F_4. Let g be the composite

$$g : S^{80} \longrightarrow (E_7/F_4)_{62} \xrightarrow{f} B(27, 35).$$

If g is null homotopic then f extends to a map

$$h : E_7/F_4 \longrightarrow B(27,35)$$

which is onto in homology because f is. A Serre spectral sequence calculation then immediately shows that the homotopy fiber M of h satisfies $H_*(M) \cong H_*(S^{19})$ and so M is homotopy equivalent to S^{19}, completing the proof.

It remains to show that g is null homotopic. Consider the homotopy fibration

$$S^{27} \longrightarrow B(27,35) \longrightarrow S^{35}$$

from Lemma 3.2. Composing g to S^{35} gives an element in $\pi_{80}(S^{35})$. This is in the stable range. On the other hand, Lemma 4.1 says that the attaching map $S^{80} \longrightarrow (E_7/F_4)_{62}$ is stably trivial, and so g is stably trivial. Thus g composed to S^{35} is null homotopic. This means that g lifts to a map $S^{80} \longrightarrow S^{27}$. But by [To2], $\pi_{80}(S^{27}) = 0$. Hence g is null homotopic. □

Proof of Corollary 1.1. The composition $E_7/F_4 \longrightarrow B(27,35) \longrightarrow S^{35}$ results in a homotopy pullback

$$\begin{array}{ccccc}
S^{19} & \longrightarrow & B(19,27) & \longrightarrow & S^{27} \\
\| & & \downarrow & & \downarrow \\
S^{19} & \longrightarrow & E_7/F_4 & \longrightarrow & B(27,35) \\
 & & \downarrow & & \downarrow \\
 & & S^{35} & = & S^{35}
\end{array}$$

which defines the space $B(19,27)$. The asserted homotopy fibration is the top row of the pullback. A Serre spectral sequence calculation applied to the homotopy fibration $B(19,27) \longrightarrow E_7/F_4 \longrightarrow S^{35}$ immediately shows that $H^*(B(19,27)) \cong \Lambda(x_{19}, x_{27})$ and $H^*(E_7/F_4)$ surjects onto $H^*(B(19,27))$. The existence of the secondary operation $\Phi x_{19} = x_{27}$ in $H^*(B(19,27))$ follows. □

5. A method for computing upper bounds on exponents

In this section we outline a general method for calculating an upper bound on the homotopy exponent of spaces which arise as the total space in certain homotopy fibrations. The method is also described and applied in [Th1, Th2].

If B is an H-space, the identity map can be multiplied by p^r to give a map $B \xrightarrow{p^r} B$. Let $B\{p^r\}$ be the homotopy fiber of this map. By [N2], if p is odd then the homotopy exponent of $B\{p^r\}$ is p^r.

Lemma 5.1. *Suppose there is a homotopy fibration*

$$F \xrightarrow{f} E \xrightarrow{g} B$$

where E and B are simply connected H-spaces. Suppose as well that there is a map $B \xrightarrow{i} E$ such that $g \circ i \simeq p^r$. Then there is a homotopy fibration

$$\Omega F \times \Omega B \xrightarrow{\Omega f \cdot (-\Omega i)} \Omega E \longrightarrow B\{p^r\}.$$

Consequently, $\exp(E) \leq p^r \cdot \max(\exp(F), \exp(B))$.

Proof. The homotopy $g \circ i \simeq p^r$ results in a homotopy pullback

$$\begin{array}{ccccc} B\{p^r\} & \longrightarrow & B & \xrightarrow{p^r} & B \\ \downarrow & & \downarrow{\scriptstyle i} & & \| \\ F & \xrightarrow{f} & E & \xrightarrow{g} & B. \end{array}$$

Since E is an H-space we can multiply the maps f and $-i$. The pullback in the diagram above then results in a homotopy fibration

$$B\{p^r\} \longrightarrow F \times B \xrightarrow{f \cdot (-i)} E$$

which is analogous to a Mayer-Vietoris sequence. Continuing the homotopy fibration sequence two steps to the left gives the fibration stated in the lemma. The exponent bound immediately follows. □

Lemma 5.1 is typically applied when $B = S^{2n+1}$ or $B = \Omega S^{2n+1}$. In what follows, we will also need a modified version of Lemma 5.1 which allows for the possibility that E is not an H-space. We state it with $B = \Omega^k S^{2n+1}$ as this is how it will occur in practise.

Lemma 5.2. *Let $0 \leq k < 2n-1$. Suppose there is a homotopy fibration*

$$F \xrightarrow{f} E \xrightarrow{g} \Omega^k S^{2n+1}$$

in which each space is 2-connected. Suppose there is a map $\Omega^k S^{2n+1} \xrightarrow{i} E$ such that $g \circ i \simeq p^r$. Then there is a homotopy fibration

$$\Omega^2 F \times \Omega^{k+2} S^{2n+1} \xrightarrow{\epsilon} \Omega^2 E \longrightarrow \Omega^{k+1} S^{2n+1}\{p^r\}$$

where $\epsilon = \Omega^2 f \cdot (-\Omega^{k+2} i)$. Consequently, $\exp(E) \leq p^r \cdot \max(\exp(F), \exp(S^{2n+1}))$.

Proof. Loop to obtain a homotopy fibration $\Omega F \xrightarrow{\Omega f} \Omega E \xrightarrow{\Omega g} \Omega^{k+1} S^{2n+1}$ and apply Lemma 5.1. □

We now consider two examples of Lemma 5.1 which later play a role in our exponent calculations. Recall from [CMN, N1] that at odd primes we have $\exp(S^{2n+1}) = p^n$.

Example 5.1. Let $q = 2(p-1)$. Let $\alpha_1 \in \pi^S_{q-1}(S^0)$ be a generator of the stable stem. Following Mimura and Toda [MT], for $m \geq 1$ define a space $B = B(2m+1, 2m+q+1)$ as the homotopy pullback

$$\begin{array}{ccccc}
S^{2m+1} & \longrightarrow & B(2m+1,2m+q+1) & \xrightarrow{q} & S^{2m+q+1} \\
\| & & \downarrow & & \downarrow \alpha_1 \\
S^{2m+1} & \longrightarrow & S^{4m+3} & \xrightarrow{w} & S^{2m+2}
\end{array}$$

where w is the Whitehead product of the identity map on S^{2m+2} with itself. Since α_1 has order p there is a characteristic map $i : S^{2m+q+1} \longrightarrow B$ satisfying $q \circ i \simeq p$. As B is not immediately known to be an H-space, we use Lemma 5.2 and obtain $\exp(B) \leq p \cdot \exp(S^{2m+q+1}) = p^{m+p}$.

Example 5.2. Let $\alpha_2 \in \pi^S_{2q-1}(S^0)$ be a generator of the stable stem. As in Example 5.1, for $m \geq 1$ there is a homotopy pullback

$$\begin{array}{ccccc}
S^{2m+1} & \longrightarrow & B(2m+1,2m+2q+1) & \xrightarrow{q} & S^{2m+2q+1} \\
\| & & \downarrow & & \downarrow \alpha_2 \\
S^{2m+1} & \longrightarrow & S^{4m+3} & \xrightarrow{w} & S^{2m+2}
\end{array}$$

which defines the space $B = B(2m+1, 2m+2q+1)$. Since α_2 has order p there is a characteristic map $j : S^{2m+2q+1} \longrightarrow B$ satisfying $q \circ j \simeq p$. Again, as B is not immediately known to be an H-space, we use Lemma 5.2 and obtain $\exp(B) \leq p \cdot \exp(S^{2m+2q+1}) = p^{m+2p-1}$.

6. Characteristic maps

In this section we wish to produce "characteristic" maps $S^{19} \longrightarrow E_7$, $S^{27} \longrightarrow E_7$, and $S^{35} \longrightarrow E_7$ which will allow us, in the right context, to apply Lemma 5.1 in order to calculate an upper bound on the 3-primary exponent of E_7. This is ultimately done in Proposition 6.2.

We first consider E_7/F_4. The spherical resolution in Theorem 1.2 implies there are homotopy fibrations

$$S^{19} \longrightarrow E_7/F_4 \longrightarrow B(27,35)$$
$$S^{27} \longrightarrow B(27,35) \longrightarrow S^{35}$$

as well as homotopy fibrations

$$B(19,27) \longrightarrow E_7/F_4 \longrightarrow S^{35}$$
$$S^{19} \longrightarrow B(19,27) \longrightarrow S^{27}.$$

Lemma 6.1. *The following hold*:

(a) *there is a map* $b_1 : S^{27} \longrightarrow B(19,27)$ *such that the composite* $S^{27} \xrightarrow{b_1} B(19,27) \longrightarrow S^{27}$ *is of degree* 3; *and*

(b) *there is a map* $b_2 : S^{35} \longrightarrow E_7/F_4$ *such that the composite* $S^{35} \xrightarrow{b_2} E_7/F_4 \longrightarrow S^{35}$ *is of degree* 3^2.

Proof. Recall that the 35-skeleton of E_7/F_4 consists of cells in dimensions $19, 27$, and 35. The 19 and 27-cells are attached by an α_2 as are the 27 and 35-cells. In particular, as the bottom two cells in $B(19,27)$ are attached by an α_2, the existence of the map b_1 in part (a) follows from Example 5.2.

Similarly, as the bottom two cells in $B(27,35)$ are attached by an α_2, Example 5.2 implies there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{35} & \xrightarrow{3} & S^{35} \\ \downarrow{\scriptstyle b} & & \| \\ B(27,35) & \longrightarrow & S^{35} \end{array}$$

for some map b. Let $\tilde{b} : S^{34} \longrightarrow \Omega B(27,35)$ be the adjoint of b. Consider the homotopy fibration sequence

$$\Omega E_7/F_4 \longrightarrow \Omega B(27,35) \xrightarrow{\partial} S^{19} \longrightarrow E_7/F_4 \longrightarrow B(27,35).$$

By [To1], 3-locally we have $\pi_{34}(S^{19}) = \mathbb{Z}/3\mathbb{Z}$, generated by α_4, implying that the composite $S^{34} \xrightarrow{\tilde{b}} \Omega B(27,35) \xrightarrow{\partial} S^{19}$ has order at most p. Thus $\tilde{b} \circ 3$ lifts to $\Omega E_7/F_4$, resulting in a homotopy commutative diagram

$$\begin{array}{ccc} S^{34} & \xrightarrow{3} & S^{34} \\ \downarrow{\scriptstyle \tilde{b}_2} & & \downarrow{\scriptstyle \tilde{b}} \\ \Omega E_7/F_4 & \longrightarrow & \Omega B(27,35) \end{array}$$

for some map $\tilde{b}_2$. Taking the adjoint of this square and combining it with the square above we obtain a homotopy commutative diagram

$$\begin{array}{ccccc} S^{35} & \xrightarrow{3} & S^{35} & \xrightarrow{3} & S^{35} \\ \downarrow{\scriptstyle b_2} & & \downarrow{\scriptstyle b} & & \| \\ E_7/F_4 & \longrightarrow & B(27,35) & \longrightarrow & S^{35} \end{array}$$

which proves part (b). □

Now consider the homotopy fibration $E_7 \longrightarrow E_7/F_4 \longrightarrow BF_4$. Rationally, F_4 has homotopy groups in dimensions $3, 11, 15, 23$, so $\pi_t(BF_4)$ is torsion for $t \in \{19, 27, 35\}$. Therefore when the inclusion $i : S^{19} \longrightarrow E_7/F_4$ is multiplied by some power of 3 it will lift to E_7. Similarly, when the maps $S^{27} \xrightarrow{b_1} B(19,27) \longrightarrow E_7/F_4$ and $S^{35} \xrightarrow{b_2} E_7/F_4$ coming from Lemma 6.1 are multiplied by some other powers of 3, they too will lift to E_7. We wish to minimize the number of powers of 3 necessary to produce each lift. This is the intention of Proposition 6.2.

To prepare the way, we need to more closely examine the homotopy theory of F_4. Harper [H], and subsequently Kono and Wilkerson in unpublished work using different methods, showed that there is a 3-primary homotopy decomposition

$$F_4 \simeq K_3 \times B(11,15) \tag{6.1}$$

where the spaces K_3 and $B(11,15)$ are described cohomologically as follows. First,

$$H^*(K_3) \cong \mathbb{Z}/3\mathbb{Z}[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7)$$

and the action of the Steenrod algebra is given by $\mathcal{P}^1 x_3 = x_7$ and $\beta x_7 = x_8$. Second,

$$H^*(B(11,15)) \cong \Lambda(x_{11}, x_{15})$$

and $\mathcal{P}^1 x_{11} = x_{15}$. Further, the space $B(11,15)$ is spherically resolved; there is a homotopy fibration

$$S^{11} \longrightarrow B(11,15) \longrightarrow S^{19}. \tag{6.2}$$

As described in [Th1], a consequence of [D2] is the existence of a homotopy fibration of H-maps and H-spaces

$$A \longrightarrow \Omega K_3\langle 3\rangle \longrightarrow \Omega S^{23} \tag{6.3}$$

where $K_3\langle 3\rangle$ is the three-connected cover of K_3, and the 34-skeleton of A is the mod-3 Moore space $P^{18}(3)$ of dimension 18. We will need the following two properties of (6.3) which were proved in [Th1].

Lemma 6.2. *The following hold*:

(a) *there is a map* $\Omega S^{23} \longrightarrow \Omega K_3\langle 3\rangle$ *such that the composite* $\Omega S^{23} \longrightarrow \Omega K_3\langle 3\rangle \longrightarrow \Omega S^{23}$ *is homotopic to multiplication by* 3;

(b) $\exp(A) = 3$.

In what follows, we also need to know the orders of select homotopy groups of K_3 and $B(11,15)$. These are stated in terms of loop spaces for later convenience. The proofs of both Lemmas 6.3 and 6.4 freely use Toda's [To1] calculations of the low dimension homotopy groups of spheres, and share the same notation.

Lemma 6.3. *The following hold*:

(a) $3 \cdot \pi_{17}(\Omega K_3\langle 3\rangle) = 0$;

(b) $3^2 \cdot \pi_{25}(\Omega K_3\langle 3\rangle) = 0$;

(c) $3^3 \cdot \pi_{33}(\Omega K_3\langle 3\rangle) = 0$.

Proof. All three cases use the homotopy fibration $A \longrightarrow \Omega K_3\langle 3\rangle \longrightarrow \Omega S^{23}$. On the level of homotopy groups, this homotopy fibration implies that for any $m \geq 1$ the order of $\pi_m(\Omega K_3\langle 3\rangle)$ is bounded above by the product of the orders of $\pi_m(\Omega S^{23})$ and $\pi_m(A)$. As we only care about $m \leq 33$, we may replace A by its 34-skeleton, which is $P^{18}(3)$. In dimensions 33 and below, the homotopy cofibration $S^{17} \longrightarrow P^{18}(3) \longrightarrow S^{18}$ is in the stable range and so is also a homotopy fibration. So if $m \leq 33$ then the order of $\pi_m(P^{18}(3))$ is bounded above by the product of the orders of $\pi_m(S^{17})$ and $\pi_m(S^{18})$. Hence for $m \in \{17, 25, 33\}$, the order of $\pi_m(\Omega K_3\langle 3\rangle)$ is bounded above by the product of the orders of $\pi_m(\Omega S^{23})$, $\pi_m(S^{17})$, and $\pi_m(S^{18})$.

The $m = 17$ case is a bit special, in that $\pi_{17}(S^{17}) = \mathbb{Z}_{(3)}$. But as $P^{18}(3)$ is the homotopy cofiber of the degree 3 map on S^{17}, we have $\pi_{17}(P^{18}(3)) = \mathbb{Z}/3\mathbb{Z}$. On the other hand, by connectivity, $\pi_m(\Omega S^{23}) = 0$. Part (a) then follows.

When $m = 25$, we have $\pi_{25}(\Omega S^{23}) = \mathbb{Z}/3\mathbb{Z}$, generated by the stable element α_1, $\pi_{25}(S^{17}) = 0$, and $\pi_{25}(S^{18}) = \mathbb{Z}/3\mathbb{Z}$, generated by the stable class α_2. Part (b) now follows.

When $m = 33$, we have $\pi_{33}(\Omega S^{23}) = \mathbb{Z}/3^2\mathbb{Z}$, generated by the stable element α', $\pi_{33}(S^{17}) = 0$, and $\pi_{33}(S^{18}) = \mathbb{Z}/3\mathbb{Z}$, generated by the stable class α_5. Part (c) now follows. □

Lemma 6.4. *The following hold*:

(a) $3^2 \cdot \pi_{17}(\Omega B(11, 15)) = 0$;

(b) $3^3 \cdot \pi_{25}(\Omega B(11, 15)) = 0$;

(c) $3^3 \cdot \pi_{33}(\Omega B(11, 15)) = 0$.

Proof. All three cases use the homotopy fibration $\Omega S^{11} \longrightarrow \Omega B(11, 15) \longrightarrow \Omega S^{15}$ As in Lemma 6.3, this homotopy fibration implies that for any $m \geq 1$ the order of $\pi_m(\Omega B(11, 15))$ is bounded above by the product of the orders of $\pi_m(\Omega S^{11})$ and $\pi_m(\Omega S^{15})$.

When $m = 17$, we have $\pi_{17}(\Omega S^{11}) = \mathbb{Z}/3\mathbb{Z}$, generated by the stable class α_2, while $\pi_{17}(\Omega S^{15}) = \mathbb{Z}/3\mathbb{Z}$, generated by the stable class α_1. Part (a) follows.

When $m = 25$, we have $\pi_{25}(\Omega S^{11}) = \mathbb{Z}/3\mathbb{Z}$, generated by the stable class α_4, while $\pi_{25}(\Omega S^{15}) = \mathbb{Z}/3^2\mathbb{Z}$, generated by the stable class α'_3. Part (b) follows.

When $m = 33$, we have $\pi_{33}(\Omega S^{11}) = \mathbb{Z}/3^2\mathbb{Z}$, generated by the stable class α'_6, while $\pi_{33}(\Omega S^{15}) = \mathbb{Z}/3\mathbb{Z}$, generated by the stable class α_5. Part (c) follows. □

Next, we define two new spaces as homotopy pullbacks. First, the composite $E_7 \longrightarrow E_7/F_4 \longrightarrow S^{35}$ results in a homotopy pullback

$$\begin{CD} X @>>> E_7 @>>> S^{35} \\ @VVV @VVV @| \\ B(19,27) @>>> E_7/F_4 @>>> S^{35} \\ @VVV @VVV \\ BF_4 @= BF_4 \end{CD}$$

which defines the space X. Second, the composite $X \longrightarrow B(19,27) \longrightarrow S^{27}$ results in a homotopy pullback

$$\begin{CD} Y @>>> X @>>> S^{27} \\ @VVV @VVV @| \\ S^{19} @>>> B(19,27) @>>> S^{27} \\ @VVV @VVV \\ BF_4 @= BF_4 \end{CD}$$

which defines the space Y.

Proposition 6.1. *There are homotopy commutative diagrams*

$$\begin{CD} S^{19} @>3^2>> S^{19} \\ @Vc_1VV @| \\ Y @>>> S^{19} \end{CD} \qquad \begin{CD} S^{27} @>3^3>> S^{27} \\ @Vc_2VV @VVb_1V \\ X @>>> B(19,27) \end{CD} \qquad \begin{CD} S^{35} @>3^3>> S^{35} \\ @Vc_3VV @VVb_2V \\ E_7 @>>> E_7/F_4 \end{CD}$$

for some maps $c_1, c_2,$ and c_3.

Proof. Consider the homotopy fibration $Y \longrightarrow S^{19} \longrightarrow BF_4$. To show the existence of a map c_1 which satisfies the asserted homotopy commutative diagram, it suffices to show that the map $S^{19} \longrightarrow BF_4$ has order at most 3^2. Equivalently, we need to show that the double adjoint, $t : S^{17} \longrightarrow \Omega F_4$ has order at most 3^2. The homotopy decomposition $\Omega F_4 \simeq \Omega K_3 \times \Omega B(11,15)$ then implies that it suffices to show that the projections of t to both ΩK_3 and $\Omega B(11,15)$ have order at most 3^2. But $\pi_{17}(\Omega K_3) = \pi_{17}(\Omega K_3\langle 3\rangle)$ and $\pi_{17}(\Omega B(11,15))$ are both annihilated by 3^2 by Lemmas 6.3 (a) and 6.4 (a) respectively.

Next, consider the homotopy fibration $X \longrightarrow B(19,27) \longrightarrow BF_4$. To show the existence of a map c_2 which satisfies the asserted homotopy commutative diagram, it suffices to show that the composite $S^{27} \xrightarrow{b_1} B(19,27) \longrightarrow BF_4$ has order at most 3^3. Arguing as in the first case, this follows since $\pi_{25}(\Omega K_3) = \pi_{25}(\Omega K_3\langle 3\rangle)$ and $\pi_{25}(\Omega B(11,15))$ are both annihilated by 3^3 by Lemmas 6.3 (a) and 6.4 (a) respectively.

Finally, consider the homotopy fibration $E_7 \longrightarrow E_7/F_4 \longrightarrow BF_4$. To show the existence of a map c_3 which satisfies the asserted homotopy commutative diagram, it suffices to show that the composite $S^{33} \xrightarrow{b_2} E_7/F_4 \longrightarrow BF_4$ has order at most 3^3. Arguing as before, this follows since $\pi_{33}(\Omega K_3) = \pi_{33}(\Omega K_3\langle 3\rangle)$ and $\pi_{33}(\Omega B(11,15))$ are both annihilated by 3^3 by Lemmas 6.3 (a) and 6.4 (a) respectively. □

Combining the homotopy commutative diagrams of Lemma 6.1 and Proposition 6.1 we obtain:

Proposition 6.2. *There are homotopy commutative diagrams*

$$\begin{array}{ccc} S^{19} & \xrightarrow{3^2} & S^{19} \\ \downarrow{\scriptstyle c_1} & & \| \\ Y & \longrightarrow & S^{19} \end{array} \qquad \begin{array}{ccc} S^{27} & \xrightarrow{3^4} & S^{27} \\ \downarrow{\scriptstyle c_2} & & \| \\ X & \longrightarrow & S^{27} \end{array} \qquad \begin{array}{ccc} S^{35} & \xrightarrow{3^5} & S^{35} \\ \downarrow{\scriptstyle c_3} & & \| \\ E_7 & \longrightarrow & S^{35}. \end{array}$$

7. An upper bound for the homotopy exponent of E_7

In this section we prove Theorem 1.1. We are almost immediately ready to apply Lemma 5.1 to the maps in Proposition 6.2 in order to compute the exponent bound of E_7. But first we have to deal with a complication. In what follows, we need to show that $\exp(Y) \leq 3^{13}$. However, the usual argument gives a weaker bound, as follows. From here on, we repeatedly use the fact from [CMN, N1] that 3-locally we have $\exp(S^{2n+1}) = 3^n$. Consider the homotopy fibration $F_4 \longrightarrow Y \longrightarrow S^{19}$. By Corollary 6.1, the composite $S^{19} \xrightarrow{c_1} Y \longrightarrow S^{19}$ has degree 3^2. Since Y is not necessarily an H-space, we loop and use Lemma 5.2 to obtain a homotopy fibration $\Omega^2 F_4 \times \Omega^2 S^{19} \longrightarrow \Omega^2 X \longrightarrow \Omega S^{19}\{3^2\}$. Thus $\exp(Y) \leq 3^2 \cdot \max(\exp(F_4), \exp(S^{19}))$. Now $\exp(S^{19}) = 3^9$, and by [Th1] $\exp(F_4) = 3^{12}$. So $\exp(Y) \leq 3^{14}$. A more delicate argument is needed to reduce this bound by another power of 3. This is done in the following Lemma.

Lemma 7.1. $\exp(Y) \leq 3^{13}$.

Proof. *Step* 1: We begin by defining some spaces and maps. Using the three-connected cover of F_4, there is a homotopy fibration sequence $\Omega S^{19} \xrightarrow{\delta} F_4\langle 3\rangle \longrightarrow Y\langle 3\rangle \longrightarrow S^{19}$. Recall from (6.1) and (6.3) that $F_4\langle 3\rangle \simeq K_3\langle 3\rangle \times B(11,15)$ and that there is a homotopy fibration of H-spaces and H-maps $A \longrightarrow \Omega K_3\langle 3\rangle \longrightarrow \Omega S^{23}$. Let f be the composite

$$f : \Omega F_4\langle 3\rangle \xrightarrow{\pi} \Omega K_3\langle 3\rangle \longrightarrow \Omega S^{23}$$

where π is the projection. Note that f is an H-map as it is a composite of H-maps. Consider the composite $\Omega^2 S^{19} \xrightarrow{\Omega\delta} \Omega F_4\langle 3\rangle \xrightarrow{f} \Omega S^{23}$. By [CMN, N1], multiplication by 3 on $\Omega^2 S^{19}$ factors as a composite $\Omega^2 S^{19} \longrightarrow S^{17} \xrightarrow{E^2} \Omega^2 S^{19}$ where E^2 is the double suspension. Thus $f\circ\Omega\delta\circ 3$ is null homotopic as it factors through a map $S^{17} \longrightarrow \Omega S^{23}$. Since f is an H-map, $f\circ\Omega\delta\circ 3 \simeq 3\circ f\circ\Omega\delta$, and so the composite $\Omega^2 S^{19} \xrightarrow{\Omega\delta} \Omega F_4\langle 3\rangle \xrightarrow{3\circ f} \Omega S^{23}$ is null homotopic. Define Q by the homotopy fibration

$$Q \longrightarrow \Omega F_4\langle 3\rangle \xrightarrow{3\circ f} \Omega S^{23}.$$

Then $\Omega\delta$ lifts to Q and we obtain a homotopy pullback

$$\begin{array}{ccccc}
R & \longrightarrow & \Omega^2 Y\langle 3\rangle & \xrightarrow{b} & \Omega^2 S^{23} \\
\| & & \downarrow & & \downarrow \\
R & \longrightarrow & \Omega^2 S^{19} & \longrightarrow & Q \\
 & & \downarrow{\scriptstyle \Omega\delta} & & \downarrow \\
 & & \Omega F_4\langle 3\rangle & = & \Omega F_4\langle 3\rangle
\end{array}$$

which defines the space R and the map b.

Step 2: We next find an upper bound for $\exp(Q)$. The definition of Q implies there is a homotopy pullback of H-spaces and H-maps

$$\begin{array}{ccccc}
A\times\Omega B(11,15) & \longrightarrow & Q & \longrightarrow & \Omega S^{23}\{3\} \\
\| & & \downarrow & & \downarrow \\
A\times\Omega B(11,15) & \longrightarrow & \Omega F_4\langle 3\rangle & \xrightarrow{f} & \Omega S^{23} \\
 & & \downarrow{\scriptstyle 3\circ f} & & \downarrow{\scriptstyle 3} \\
 & & \Omega S^{23} & = & \Omega S^{23}.
\end{array}$$

Since $\Omega B(11,15)$ is a retract of $\Omega F_4\langle 3\rangle$, it is also a retract of Q. As Q is an H-space we then obtain a homotopy decomposition $Q\simeq \Omega B(11,15)\times Q'$ where there is a homotopy fibration $A\longrightarrow Q'\longrightarrow \Omega S^{23}\{3\}$. By Lemma 6.2, $\exp(A)=3$, while by [N2], $\exp(S^{23}\{3\})=3$. Thus $\exp(Q')\le 3^2$. By Example 5.1, $\exp(B(11,15))\le 3^8$. The decomposition of Q then implies that $\exp(Q)\le 3^8$.

Step 3: Now we find an upper bound for $\exp(R)$. Consider the homotopy fibration $R\longrightarrow \Omega^2 S^{19}\longrightarrow Q$ from Step 1. We claim that the composite $g:\Omega^2 S^{19}\xrightarrow{3^2}\Omega^2 S^{19}\longrightarrow Q$ is null homotopic. It is equivalent to check that g composes trivially to each factor of $Q\simeq\Omega B(11,15)\times Q'$. First consider the composite

$$\Omega^2 S^{19}\xrightarrow{3^2}\Omega^2 S^{19}\longrightarrow Q\longrightarrow \Omega B(11,15).$$

The decomposition of Q is defined so that the composite $\Omega^2 S^{19} \longrightarrow Q \longrightarrow \Omega B(11,15)$ is homotopic to the composite $\Omega^2 S^{19} \xrightarrow{\Omega\partial} \Omega F_4\langle 3\rangle \longrightarrow \Omega B(11,15)$. By the leftmost square in Lemma 6.1, $\Omega\partial$ has order 3^2, so g composes trivially to $\Omega B(11,15)$. Next consider the composite

$$\Omega^2 S^{19} \xrightarrow{3^2} \Omega^2 S^{19} \longrightarrow Q \longrightarrow Q'.$$

By [CMN], [N1], multiplication by 3^2 on $\Omega^2 S^{19}$ factors as a composite $\Omega^2 S^{19} \longrightarrow S^{17} \xrightarrow{3} S^{17} \xrightarrow{E^2} \Omega^2 S^{19}$. Thus the projection of g to Q' factors through a composite $S^{17} \xrightarrow{3} S^{17} \xrightarrow{t} Q'$ for some map t. In the homotopy fibration $A \longrightarrow Q' \longrightarrow \Omega S^{23}\{3\}$, the space $\Omega S^{23}\{3\}$ is 20-connected, so the 18-skeleton of Q' is the same as that of A, which is the Moore space $P^{18}(3)$. Thus $t \circ 3$ represents an element in $\pi_{17}(P^{18}(3)) = \mathbb{Z}/3\mathbb{Z}$, and so must be null homotopic. Hence g projected to Q' is null homotopic.

With g null homotopic, we obtain a lift

$$\begin{array}{ccccc} & & \Omega^2 S^{19} & & \\ & \swarrow & \downarrow{\scriptstyle 3^2} & & \\ R & \longrightarrow & \Omega^2 S^{19} & \longrightarrow & Q. \end{array}$$

This allows us to apply Lemma 5.1 to show $\exp(R) \leq 3^2 \cdot \max(\exp(S^{19}), \exp(Q))$. As $\exp(S^{19}) = 3^9$ and $\exp(Q) \leq 3^8$, we have $\exp(R) \leq 3^{11}$.

Step 4: Finally, we prove the exponent bound for Y. Consider the homotopy fibration $R \longrightarrow \Omega Y \longrightarrow \Omega^2 S^{23}$. Taking vertical connecting maps in the pullback defining R in Step 1 we obtain a homotopy commutative diagram

$$\begin{array}{ccc} \Omega^2 F_4\langle 3\rangle & = & \Omega^2 F_4\langle 3\rangle \\ \downarrow & & \downarrow{\scriptstyle 3\circ f} \\ \Omega Y & \longrightarrow & \Omega^2 S^{23}. \end{array}$$

By Lemma 6.2, there is a map $\Omega S^{23} \xrightarrow{t} \Omega F_4\langle 3\rangle$ such that $f \circ t \simeq 3$. Combining Ωt and the diagram above, we obtain a homotopy commutative diagram

$$\begin{array}{ccc} \Omega^2 S^{23} & & \\ \downarrow & \searrow^{3^2} & \\ \Omega Y & \longrightarrow & \Omega^2 S^{23}. \end{array}$$

By Lemma 5.1 we then have $\exp(Y) \leq 3^2 \cdot \max(\exp(S^{23}), \exp(R))$. As $\exp(S^{23}) = 3^{11}$ and $\exp(R) \leq 3^{11}$, we have $\exp(Y) \leq 3^{13}$, as desired. $\square$

Proof of Theorem 1.1. First consider the homotopy fibration $X \longrightarrow E_7 \longrightarrow S^{35}$. By Proposition 6.2, the composite $S^{35} \xrightarrow{c_3} E_7 \longrightarrow S^{35}$ has degree 3^5, so by Lemma 5.1 there is a homotopy fibration $\Omega X \times \Omega S^{35} \longrightarrow \Omega E_7 \longrightarrow S^{35}\{3^5\}$. Thus $\exp(E_7) \leq 3^5 \cdot \max(\exp(X), \exp(S^{35}))$. Since $\exp(S^{35}) = 3^{17}$, if $\exp(X) \leq 3^{17}$ then $\exp(E_7) \leq 3^{23}$ and we are done.

It remains to show that $\exp(X) \leq 3^{17}$. Consider the homotopy fibration $Y \longrightarrow X \longrightarrow S^{27}$. By Proposition 6.2, the composite $S^{27} \xrightarrow{c_2} X \longrightarrow S^{27}$ has degree 3^4. Since X is not necessarily an H-space, we loop and use Lemma 5.2 to obtain a homotopy fibration $\Omega^2 Y \times \Omega^2 S^{27} \longrightarrow \Omega^2 Y \longrightarrow \Omega S^{27}\{3^4\}$. Thus $\exp(X) \leq 3^4 \cdot \max(\exp(Y), \exp(S^{27}))$. Now $\exp(S^{27}) = 3^{13}$ and by Lemma 7.1, $\exp(Y) \leq 3^{13}$. So $\exp(X) \leq 3^{17}$ and we are done. □

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