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Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave

By

Yuming Chu and Xiaoming Zhang

Abstract

In the article, established are necessary and sufficient conditions such that the extended mean values are Schur-convex and Schur-concave.

1. Introduction

The histories of mean values and inequalities are long [3]. The mean values are related to the Mean Values Theorems for derivative or for integral, which are the bridge between the local and global properties of functions (cf. [4]). The arithmetic-mean-geometric-mean inequality is probably the most important inequality, and certainly a keystone of the theory of inequalities [1]. Inequalities of mean values are one of the main parts of theory of inequalities, they have explicit geometric meanings [4]. The theory of mean values plays an important role in the whole mathematics, since many norms in mathematics are always means (cf. [4]).

In 1975, the extended mean values $E(r,s;x,y)$ were defined in [13] by K. B. Stolarsky as follows.

$$
\begin{aligned}
&E(r,s;x,y)=\left(\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-y^r}\right)^{\frac{1}{s-r}}, \quad rs(r-s)(x-y)\neq 0;\\
&E(r,0;x,y)=\left(\frac{1}{r}\cdot\frac{y^r-x^r}{\log y-\log x}\right)^{\frac{1}{r}}, \quad r(x-y)\neq 0;\\
&E(r,r;x,y)=\frac{1}{e^{\frac{1}{r}}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}}, \quad r(x-y)\neq 0;\\
&E(0,0;x,y)=\sqrt{xy}, \qquad x\neq y;\\
&E(r,s;x,y)=x, \qquad x=y.
\end{aligned}
$$

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Here $x, y > 0$ and $r, s \in R$.

It is easy to see that the extended mean values $E(r, s; x, y)$ are continuous on the domain $\{(r, s; x, y) \mid r, s \in R; x, y > 0\}$.

They are of symmetry between r and s and between x and y.

Many basic properties have been researched by E. B. Leach and M. C. Sholander in [6].

Study of $E(r, s; x, y)$ is not only interesting but also important, because most of the two-variables mean values are special cases of $E(r, s; x, y)$ and it is challenging to study a function whose formulation is so indeterminate [8].

Let $\Omega \subseteq R^n$ be a symmetric convex set with nonempty interior. A real-valued function f on Ω is called a Schur-convex function if $f(x) \le f(y)$ for each two n-tuples $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \Omega$ such that $x \prec y$, i.e.

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \qquad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $1 \le k \le n-1$ and $x_{[i]}$ denotes the ith largest component in x.

A real-valued function f is called Schur-concave if $-f$ is Schur-convex.

The theory of Schur-convex functions is one of the most important theory in the fields of inequalities. It can be used in combinatorial optimization [5], isoperimetric problem for ploytopes [14], linear regression [12], graphs and matrices [2] and other related fields.

The Schur-convexity of the extended mean values $E(r, s; x, y)$ with respect to (r, s) and (x, y) are investigated in [9], [10], and [11]. F. Qi first obtained the following result in [9].

Theorem A. *For fixed $(x, y) \in (0, \infty) \times (0, \infty)$ with $x \ne y$, the extended mean values $E(r, s; x, y)$ are Schur-concave on $[0, +\infty) \times [0, +\infty)$ and Schur-convex on $(-\infty, 0] \times (-\infty, 0]$ with respect to (r, s).*

In [10], F. Qi, J. Sándor, S. S. Dragomir and A. Sofo tried to obtain the Schur-convexity of the extended mean values $E(r, s; x, y)$ with respect to (x, y) for fixed (r, s) and declared an incorrect conclusion as follows: For given (r, s) with $r, s \notin (0, \frac{3}{2})$ (or $r, s \in (0, 1]$, resp.), the extended mean values $E(r, s; x, y)$ are Schur-concave (or schur-convex, resp.) with respect to (x, y) on $(0, \infty) \times (0, \infty)$. H.-N. Shi, Sh.-H. Wu and F. Qi observed that the above conclusion is wrong and obtained the following Theorem B in [11].

Theorem B. *For fixed $(r, s) \in R^2$,*

(1) if $2 < 2r < s$ or $2 \le 2s \le r$, then the extended mean values $E(r, s; x, y)$ are Schur-convex with respect to $(x, y) \in (0, \infty) \times (0, \infty)$;

(2) if $(r, s) \in \{r < s \le 2r, 0 < r \le 1\} \cup \{s < r \le 2s, 0 < s \le 1\} \cup \{0 < s < r \le 1\} \cup \{0 < r < s \le 1\} \cup \{s \le 2r < 0\} \cup \{r \le 2s < 0\}$, then the extended mean values $E(r, s; x, y)$ are Schur-concave with respect to $(x, y) \in (0, \infty) \times (0, \infty)$.

The main purpose of this article is to establish the necessary and sufficient conditions such that the extended mean values $E(r, s; x, y)$ are Schur-convex or Schur-concave with respect to (x, y) for fixed (r, s). Our main result is the following.

Theorem 1.1. *For fixed* $(r, s) \in R^2$,

(1) *the extended mean values* $E(r, s; x, y)$ *are Schur-convex with respect to* $(x, y) \in (0, \infty) \times (0, \infty)$ *if and only if* $(r, s) \in \{s \geq 1, r \geq 1, s + r \geq 3\}$;

(2) *the extended mean values* $E(r, s; x, y)$ *are Schur-concave with respect to* $(x, y) \in (0, \infty) \times (0, \infty)$ *if and only if* $(r, s) \in \{r \leq 1, s + r \leq 3\} \cup \{s \leq 1, s + r \leq 3\}$.

2. Lemmas

In this section we introduce and establish several lemmas, which are used in the proof of Theorem 1.1.

Lemma 2.1 ([7]). *Let* $A \subseteq R^n$ *be a symmetric convex set with non-empty interior* $\mathrm{int}A$, $\varphi : A \to R$ *is a continuous symmetric function on* A. *If* φ *is differentiable on* $\mathrm{int}A$, *then* φ *is Schur-convex on* A *if and only if*

$$(x_i - x_j)\left(\frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j}\right) \geq 0$$

for all $(x_1, x_2, \ldots, x_n) \in \mathrm{int}A$ *and* $i, j = 1, 2, \ldots, n$ *with* $i \neq j$.

Lemma 2.2. *Let* $s, r \in R, s \neq 0$ *and* $f(t) = \frac{r}{s}[(s-r)(t^{s+r-1} - 1) - s(t^{s-1} - t^r) + r(t^{r-1} - t^s)]$. *Then the following statements hold.*

(a) *If* $s > r \geq 1$ *and* $s + r - 3 \geq 0$, *then* $f(t) \geq 0$ *for* $t \in [1, \infty)$;

(b) *if* $s > r > 1$ *and* $s + r - 3 < 0$, *then there exist* $t_1, t_2 \in (1, \infty)$ *such that* $f(t_1) > 0$ *and* $f(t_2) < 0$;

(c) *if* $r < 1, r \neq 0$ *and* $s + r - 3 > 0$, *then there exist* $t_3, t_4 \in (1, \infty)$ *such that* $f(t_3) > 0$ *and* $f(t_4) < 0$;

(d) *if* $s > 0, s > r, r < 1$ *and* $s + r - 3 \leq 0$, *then* $f(t) \leq 0$ *for* $t \in (1, \infty)$;

(e) *if* $r < s < 0$, *then* $f(t) \leq 0$ *for* $t \in [1, \infty)$.

Proof. (a) Let $g(t) = t^{2-r} f'(t)$ and $h(t) = t^{2+r-s} g''(t)$, then simple computation yields

$$f(1) = 0, \tag{2.1}$$

$$\begin{aligned} f'(t) = {} & \frac{r}{s}(s-r)(s+r-1)t^{s+r-2} - r(s-1)t^{s-2} \\ & + r^2 t^{r-1} + \frac{r^2}{s}(r-1)t^{r-2} - r^2 t^{s-1}, \end{aligned} \tag{2.2}$$

$$g(1) = f'(1) = 0, \tag{2.3}$$

$$\begin{aligned} g'(t) = {} & r(s-r)(s+r-1)t^{s-1} \\ & - r(s-1)(s-r)t^{s-r-1} + r^2 - r^2(s-r+1)t^{s-r}, \end{aligned} \tag{2.4}$$

$$g'(1) = 0, \tag{2.5}$$

(2.6)
$$g''(t) = r(s-r)(s+r-1)(s-1)t^{s-2} - r(s-1)(s-r)(s-r-1)t^{s-r-2} - r^2(s-r+1)(s-r)t^{s-r-1}$$

and

$$h'(t) = r^2(s-r)(s+r-1)(s-1)t^{r-1} - r^2(s-r+1)(s-r). \tag{2.7}$$

If $s > r \geq 1, s+r-3 \geq 0$, then from (2.6) and (2.7) we see that

$$h(1) = g''(1) = r^2(s-r)(s+r-3) \geq 0 \tag{2.8}$$

and

$$h'(t) \geq h'(1) = r^2 s(s-r)(s+r-3) \geq 0 \tag{2.9}$$

for $t \geq 1$. Then Lemma 2.2(a) follows from (2.1)–(2.9).

(b) If $s > r > 1$ and $s+r-3 < 0$, then $h'(1) = r^2 s(s-r)(s+r-3) < 0$ by (2.7), this and the continuity of $h'(t)$ imply that there exists $\delta_1 > 0$ such that $h'(t) < 0$ for $t \in [1, 1+\delta_1)$. Hence $h(t) \leq h(1) = r^2(s-r)(s+r-3) < 0$ for $t \in [1, 1+\delta_1)$, from (2.1)-(2.5) we clearly see that $f(t) < 0$ for $t \in (1, 1+\delta_1)$.

On the other hand, it is easy to see that $\lim_{t\to+\infty} f(t) = +\infty$. Hence Lemma 2.2(b) is true.

(c) If $r < 1, r \neq 0$ and $s+r-3 > 0$, then $s > r, s > 0$ and $h'(1) = r^2 s(s-r)(s+r-3) > 0$ by (2.7). The continuity of $h'(t)$ implies that there exists $\delta_2 > 0$ such that $h'(t) > 0$ for $t \in [1, 1+\delta_2)$, this leads to $h(t) > h(1) = g''(1) = r^2(s-r)(s+r-3) > 0$ for $t \in (1, 1+\delta_2)$, from (2.1)-(2.5) we see that $f(t) > 0$ for $t \in (1, 1+\delta_2)$.

On the other hand, it is easy to see that $\lim_{t\to+\infty} f(t) = -\infty$. Hence Lemma 2.2(c) is true.

(d) If $s > 0, s > r, r < 1, s+r-3 \leq 0$ and $t \in [1, \infty)$. Then we claim that $h'(t) \leq 0$, and from this we can get Lemma 2.2(d) by a similar argument as in Lemma 2.2(a). In fact, if $(s+r-1)(s-1) \geq 0$, then clearly (2.7) gives that

$$h'(t) \leq h'(1) = r^2 s(s-r)(s+r-3) \leq 0;$$

if $(s+r-1)(s-1) < 0$, then again (2.7) yields that

$$h'(t) \leq -r^2(s-r+1)(s-r) \leq 0.$$

(e) If $r < s < 0, t \geq 1$. Let $f_1(t) = t^{-s-r+1} f(t), f_2(t) = t^{1+s} f_1'(t)$ and

$f_3(t) = t^{-s+r+2} f_2''(t)$, then simple computation yields

$$f_1(1) = f(1) = 0, \tag{2.10}$$

$$f_1'(t) = -\frac{r}{s}(s-r)(-s-r+1)t^{-s-r} + r^2 t^{-r-1} + r(-s+1)t^{-s} - r^2 t^{-s-1} - \frac{r^2}{s}(-r+1)t^{-r}, \tag{2.11}$$

$$f_2(1) = f_1'(1) = 0, \tag{2.12}$$

$$f_2'(t) = -\frac{r}{s}(s-r)(-s-r+1)(1-r)t^{-r} + r^2(s-r)t^{s-r-1} + r(-s+1) - \frac{r^2}{s}(-r+1)(1+s-r)t^{s-r}, \tag{2.13}$$

$$f_2'(1) = 0, \tag{2.14}$$

(2.15)

$$f_2''(t) = \frac{r^2}{s}(s-r)(-s-r+1)(1-r)t^{-r-1} + r^2(s-r)(s-r-1)t^{s-r-2} - \frac{r^2}{s}(-r+1)(1+s-r)(s-r)t^{s-r-1},$$

$$f_3(1) = f_2''(1) = r^2(s-r)(s+r-3) < 0, \tag{2.16}$$

and

(2.17)

$$\begin{aligned} f_3'(t) &= \frac{r^2}{s}(s-r)(-s-r+1)(1-r)(1-s)t^{-s} - \frac{r^2}{s}(-r+1)(1+s-r)(s-r) \\ &\le \frac{r^2}{s}(s-r)(-s-r+1)(1-r)(1-s) - \frac{r^2}{s}(-r+1)(1+s-r)(s-r) \\ &= r^2(s-r)(1-r)(s+r-3) < 0. \end{aligned}$$

Now, Lemma 2.2(e) follows from (2.10)–(2.17). □

Lemma 2.3. *For $r \in R$ and $t \ge 1$, let $h(t) = -r(t^{r-1} + t^r)\log t + (t^{r-1}+1)(t^r - 1)$. If $1 < r < \frac{3}{2}$, then there exists $t_1, t_2 \in (1, \infty)$ such that $h(t_1) < 0$ and $h(t_2) > 0$.*

Proof. For $t \ge 1$, let $h_1(t) = t^{2-r}h'(t)$. If $1 < r < \frac{3}{2}$, then simple computation yields

$$\begin{aligned} &h'(t) = -r[(r-1)t^{r-2} + rt^{r-1}]\log t + (2r-1)t^{2r-2} - (2r-1)t^{r-2}, \\ &h_1(1) = h'(1) = 0, \end{aligned} \tag{2.18}$$

$$h_1'(t) = -r^2 \log t - r\left(\frac{r-1}{t} + r\right) + r(2r-1)t^{r-1},$$

$$h_1'(1) = 0, \tag{2.19}$$

$$h_1''(t) = -\frac{r^2}{t} + \frac{r(r-1)}{t^2} + r(2r-1)(r-1)t^{r-2}$$

and

$$h_1''(1) = r^2(2r-3) < 0. \tag{2.20}$$

By (2.20) and the continuity of $h_1''(t)$ we know that there exists $\delta_3 > 0$ such that $h_1''(t) < 0$ for $t \in [1, 1+\delta_3)$, this together with (2.19) imply that $h_1'(t) < h_1'(1) = 0$ for $t \in (1, 1+\delta_3)$. Then (2.18) and $h(1) = 0$ lead to $h(t) < 0$ for $t \in (1, 1+\delta_3)$.

On the other hand, it is easy to see that $\lim_{t\to+\infty} f(t) = +\infty$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *For $t \geq 1$, let $f(t) = r(1+t^{r-1})\log t - t^r - t^{r-1} + 1 + \frac{1}{t}$. If $r > 3$, then there exist $t_1, t_2 \in (1, \infty)$ such that $f(t_1) > 0$ and $f(t_2) < 0$.*

Proof. Let $g(t) = tf(t)$ and $h(t) = tg''(t)$. If $r > 3$, then simple computation yields

$$g(1) = f(1) = 0, \tag{2.21}$$

$$g'(t) = r(1+rt^{r-1})\log t - (r+1)(t^r - 1), \tag{2.22}$$

$$g'(1) = 0, \tag{2.23}$$

$$g''(t) = r^2(r-1)t^{r-2}\log t + r\left(\frac{1}{t} + rt^{r-2}\right) - r(r+1)t^{r-1}, \tag{2.24}$$

$$h(1) = g''(1) = 0, \tag{2.25}$$

$$h'(t) = r^2(r-1)^2t^{r-2}\log t + 2r^2(r-1)t^{r-2} - r^2(r+1)t^{r-1},$$

and

$$h'(1) = r^2(r-3) > 0. \tag{2.26}$$

From (2.26) and the continuity of $h'(t)$ we see that there exists $\delta > 0$ such that $h'(t) > 0$ for $t \in [1, 1+\delta)$, this together with (2.21)–(2.25) imply that $f(t) > 0$ for $t \in (1, 1+\delta)$.

On the other hand, it is easy to see that $\lim_{t\to+\infty} f(t) = -\infty$. This completes the proof of Lemma 2.4. □

3. Proof of Theorem 1.1

Proof. For fixed $r, s \in R$, it is easy to see that $E(r, s; x, y)$ is differentiable with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ by the elementary theory of differential and integral calculus. We use Lemma 2.1 to discuss the nonpositivity and nonnegativity of $(y-x)(\frac{\partial E}{\partial y} - \frac{\partial E}{\partial x})$ for all $(x, y) \in (0, \infty) \times (0, \infty)$ and for fixed $(r, s) \in R^2$. Since $(y-x)(\frac{\partial E}{\partial y} - \frac{\partial E}{\partial x}) = 0$ for $x = y$ and $(y-x)(\frac{\partial E}{\partial y} - \frac{\partial E}{\partial x})$ is symmetric with respect to x and y, without loss of generality we assume $y > x$ in the following discussion.

Let

$$E_1 = \{(r,s) : r \geq 1, s \geq 1, r+s \geq 3\},$$
$$E_2 = \{(r,s) : r > 1, s > 1, r+s < 3\} \cup \{(r,s) : r < 1, s+r > 3\}$$
$$\cup \{(r,s) : s < 1, s+r > 3\}$$

and

$$E_3 = \{(r,s) : r \leq 1, s+r \leq 3\} \cup \{(r,s) : s \leq 1, s+r \leq 3\}.$$

Then $E_1 \cup E_2 \cup E_3 = R^2, E_1 \cap E_2 = \emptyset, E_3 \cap E_2 = \emptyset$ and $intE_1 \cap intE_3 = \emptyset$, where $intE_1$ and $intE_3$ are the interior of E_1 and E_3, respectively.

It is obvious that Theorem 1.1 is true if once we prove that $E(r,s;x,y)$ is Schur-convex, Schur-concave, and neither Schur-convex nor Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(r,s) \in E_1, E_3$ and E_2, respectively. We divide our proof into three cases.

Case 1. $(r,s) \in E_1$. Let $E_{11} = \{(r,s) : s+r \geq 3, s > r \geq 1\}$, $E_{12} = \{(r,s) : s+r \geq 3, r > s \geq 1\}$ and $F(r,s;x,y) = \frac{r}{s}\frac{y^s - x^s}{y^r - x^r}$, then

$$E_1 = \overline{E_{11}} \cup \overline{E_{12}}$$

and

(3.1)

$$(y-x)\left(\frac{\partial E}{\partial y} - \frac{\partial E}{\partial x}\right)$$
$$= \frac{1}{s-r}\frac{y-x}{(y^r-x^r)^2}x^{s+r-1}F^{\frac{1}{s-r}-1}$$
$$\times \frac{r}{s}\left[(s-r)\left(\left(\frac{y}{x}\right)^{s+r-1} - 1\right) - s\left(\left(\frac{y}{x}\right)^{s-1} - \left(\frac{y}{x}\right)^{r}\right) + r\left(\left(\frac{y}{x}\right)^{r-1} - \left(\frac{y}{x}\right)^{s}\right)\right]$$

for $(r,s) \in E_{11}$. From Lemma 2.1, Lemma 2.2 (a), (3.1) and the assumption $y > x$ we see that $E(r,s;x,y)$ is Schur-convex with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(r,s) \in E_{11}$. Then the continuity and symmetry of $E(r,s;x,y)$ with respect to (r,s) imply that $E(r,s;x,y)$ is Schur-convex with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(r,s) \in E_1$.

Case 2. $(r,s) \in E_2$. We divide the discussion of this case into seven subcases. Let

$$E_{21} = \{(r,s) : s > r > 1, s+r < 3\},$$
$$E_{22} = \{(r,s) : r > s > 1, s+r < 3\},$$
$$E_{23} = \left\{(r,s) : 1 < s = r < \frac{3}{2}\right\},$$
$$E_{24} = \{(r,s) : 1 > r \neq 0, s+r > 3\},$$
$$E_{25} = \{(r,s) : 1 > s \neq 0, s+r > 3\},$$
$$E_{26} = \{(r,s) : s = 0, r > 3\}$$

and

$$E_{27} = \{(r,s) : r = 0, , s > 3\}.$$

Then

$$E_2 = E_{21} \cup E_{22} \cup E_{23} \cup E_{24} \cup E_{25} \cup E_{26} \cup E_{27}. \tag{3.2}$$

Subcase 2.1. If $(r,s) \in E_{21}$. Then Lemma 2.1, Lemma 2.2 (b), (3.1) and the assumption $y > x$ imply that $E(r,s;x,y)$ is neither Schur-convex nor Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Subcase 2.2. If $(r,s) \in E_{22}$. Then the symmetry of $E(r,s;x,y)$ with respect to (r,s) and subcase 2.1 show that $E(r,s;x,y)$ is neither Schur-convex nor Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Subcase 2.3. If $(r,s) \in E_{23}$. Then

$$\begin{aligned}
&(y-x)\left(\frac{\partial E(r,r;x,y)}{\partial y} - \frac{\partial E(r,r;x,y)}{\partial x}\right) \\
&= \frac{y-x}{(x^r-y^r)^2} E(r,r;x,y) x^{2r-1} \\
&\times \left\{-r\left[\left(\frac{y}{x}\right)^{r-1} + \left(\frac{y}{x}\right)^r\right]\log\frac{y}{x} + \left[\left(\frac{y}{x}\right)^{r-1} + 1\right]\left[\left(\frac{y}{x}\right)^r - 1\right]\right\}.
\end{aligned} \tag{3.3}$$

Now, Lemma 2.1, Lemma 2.3, (3.3) together with the assumption $y > x$ imply that $E(r,s;x,y)$ is neither Schur-convex nor Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Subcase 2.4. If $(r,s) \in E_{24}$. Then Lemma 2.1, Lemma 2.2 (c), (3.1) and the assumption $y > x$ imply that $E(r,s;x,y)$ is neither Schur-convex nor Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Subcase 2.5. If $(r,s) \in E_{25}$. Then the symmetry of $E(r,s;x,y)$ with respect to (r,s) and subcase 2.4 imply that $E(r,s;x,y)$ is neither Schur-convex nor Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Subcase 2.6. If $(r,s) \in E_{26}$. Then

$$\begin{aligned}
&(y-x)\left(\frac{\partial E(r,0;x,y)}{\partial y} - \frac{\partial E(r,0;x,y)}{\partial x}\right) \\
&= \frac{\left(\frac{1}{r}\frac{y^r-x^r}{\log y - \log x}\right)^{\frac{1}{r}-1}}{r^2(\log y - \log x)^2}(y-x)x^{r-1} \\
&\times \left\{r\left[1 + \left(\frac{y}{x}\right)^{r-1}\right]\log\frac{y}{x} - \left(\frac{y}{x}\right)^r - \left(\frac{y}{x}\right)^{r-1} + 1 + \frac{1}{\frac{y}{x}}\right\}.
\end{aligned} \tag{3.4}$$

So, Lemma 2.1, Lemma 2.4, (3.4) together with the assumption $y > x$ show that $E(r,s;x,y)$ is neither Schur-convex nor Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Subcase 2.7. If $(r,s) \in E_{27}$. Then the symmetry of $E(r,s;x,y)$ with respect to (r,s) and subcase 2.6 imply that $E(r,s;x,y)$ is neither Schur-convex nor Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Now, (3.2) and subcases 2.1–2.7 show that $E(r,s;x,y)$ is neither Schur-convex nor Schur-concave with respect to $(x,y) \in (0,\infty)\times(0,\infty)$ for $(r,s) \in E_2$.

Case 3. $(r,s) \in E_3$. We divide the discussion of this case into four subcases. Let

$$\begin{aligned} E_{31} &= \{(r,s) : s > 0, s > r, r < 1, r \neq 0, s + r < 3\}, \\ E_{32} &= \{(r,s) : r > 0, r > s, s < 1, s \neq 0, s + r < 3\}, \\ E_{33} &= \{(r,s) : 0 > s > r\} \end{aligned}$$

and

$$E_{34} = \{(r,s) : 0 > r > s\}.$$

Then

$$\overline{E_{31}} \cup \overline{E_{32}} \cup \overline{E_{33}} \cup \overline{E_{34}} = E_3. \tag{3.5}$$

Subcase 3.1. If $(r,s) \in E_{31}$. Then Lemma 2.1, Lemma 2.2 (d), (3.1) together with the assumption $y > x$ imply that $E(r,s;x,y)$ is Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Subcase 3.2. If $(r,s) \in E_{32}$. Then the symmetry of $E(r,s;x,y)$ with respect to (r,s) and subcase 3.1 lead to that $E(r,s;x,y)$ is Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Subcase 3.3. If $(r,s) \in E_{33}$. Then Lemma 2.1, Lemma 2.2 (e), (3.1) and the assumption $y > x$ imply that $E(r,s;x,y)$ is Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Subcase 3.4. If $(r,s) \in E_{34}$. Then the symmetry of $E(r,s;x,y)$ with respect to (r,s) and subcase 3.3 lead to that $E(r,s;x,y)$ is Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$.

Now, the continuity of $E(r,s;x,y)$, (3.5) together with subcases 3.1-3.4 imply that $E(r,s;x,y)$ is Schur-concave with respect to $(x,y) \in (0,\infty) \times (0,\infty)$ for $(r,s) \in E_3$

□

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Department of Mathematics
Huzhou Teachers College
313000 Huzhou, China
e-mail: chuyuming@hutc.zj.cn

Haining Radio and TV University
314400 Haining, China
e-mail: zjzxm79@tom.com

References

[1] E. F. Beckenbach and R. Bellman, Inequalities, Springer-Verlag, 1983.

[2] G. M. Constantine, *Schur-convex functions on the spectra of graphs*, Discrete Math. **45** no.2-3 (1983), 181–188.

[3] A. M. Fink, *An essay on the history of inequalities*, J. Math. Anal. Appl. **249** (2000), 118–134.

[4] B.-N. Guo and F. Qi, *Proofs of an integral inequality*, Mathematics and Informatics Quarterly **7**-4 (1997), 182–184.

[5] F. K. Hwang and U. G. Rothblum, *Partition-optimization with Schur convex sum objective functions*, SIAM J. Discrete Math. **18** (2004/2005), 512–524.

[6] E. B. Leach and M. C. Sholander, *Extended mean values*, Amer. Math. Monthly **85** (1978), 84–90.

[7] A. W. Marshall and I. Olkin, Inequalities: theory of majorization and its applications, Academic Press, 1979.

[8] Z. Páles, *Inequalities for differences of powes*, J. Math. Anal. Appl. **131** (1988), 271–281.

[9] F. Qi, *A note on Schur-convexity of extended mean values*, Rocky Mountain J. Math. **35** (2005), 1787–1793.

[10] F. Qi, J. Sándor, S. S. Dragomir and A. Sofo, *Notes on the Schur-convexity of the extended mean values*, Taiwanese J. Math. **9** (2005), 411–420.

[11] H.-N. Shi, Sh.-H. Wu and F. Qi, *An alternative note on the Schur-convexity of the extended mean values*, Math. Inequal. Appl. **9** (2006), 219–224.

[12] C. Stepniak, *Stochastic ordering and Schur-convex functions in comparison of linear experiments*, Metrika **36**-5 (1989), 291–298.

[13] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Mag. Math. **48** (1975), 87–92.

[14] X. M. Zhang, *Schur-convex functions and isoperimeric inequalities*, Proc. Amer. Math. Soc. **126** (1998), 461–470.