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# Mod $p$ decompositions of non-simply connected Lie groups

By

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**Abstract**

A generalization of Wilkerson's mod $p$ decompositions of compact, connected, semi-simple Lie groups [7] to the non-simply connected case is given. As an application, H-spaces with interesting fundamental groups are constructed.

## 1. Introduction

Let $p$ be a prime. Mod $p$ decompositions of compact Lie groups were given by many authors. In particular, Nishida [4] gave a mod $p$ decomposition of $U(n)$ by exploiting unstable Adams operations of Sullivan [6]. Later, Wilkerson [7] generalized this result to arbitrary compact, connected, semi-simple Lie groups without $p$-torsion in the integral homology and showed that each piece in the decompositions is indecomposable. Then one can see that the above decompositions are best. However it does not contain the case that the fundamental groups have $p$-torsion. On the other hand, Harper [3] gave mod $p$ decompositions of finite H-spaces with $p$-torsion in the fundamental groups under a restrictive condition.

The purpose of this paper is to generalize the mod $p$ decompositions of Wilkerson [7] to the case that the fundamental groups have $p$-torsion. One can see that these decompositions also generalize the result of Harper [3] in a sense.

Let $-_{(p)}$ denote the localization at $p$ and let $-_{(0)}$ mean the rationalization. We call a finite H-space $X$ is of type $(n_1, \ldots, n_l)$ if

$$X_{(0)} \simeq \prod_{i=1}^{l} S^{2n_i-1}_{(0)}.$$

We define a set $N(X, i)$ for $i = 1, \ldots, p-1$ by

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$$N(X,i) = \{k \mid k \text{ is in the entry of the type of } X \text{ and } k \equiv 0 \ (p-1)\}.$$

We shall prove :

**Theorem 1.1.** *Let $G$ be a compact, connected, semi-simple Lie group, not necessarily simply connected, such that $H_*(\widetilde{G};\mathbf{Z})$ has no $p$-torsion, where $\widetilde{G}$ is the universal covering group of $G$. Then there exist H-spaces $X_i$ of type $N(G,i)$ for $i = 1,\ldots,p-1$ such that*

$$G_{(p)} \simeq \prod_{i=1}^{p-1} X_i.$$

*Moreover, $X_i$ is simply connected if $i \neq 1$, and each $X_i$ is indecomposable if $G$ is simple.*

By Theorem 1.1, we can construct H-spaces with interesting fundamental groups as follows.

**Corollary 1.1.** *Let $p$ be an odd prime. Then there exists an H-space of the same type as $SU(n)$ such that the order of the fundamental group is divided by any $p^r$ with $p^r \leq n$.*

*Proof.* Let $r$ be the maximal integer such that $p^r \leq n$. By applying Theorem 1.1 to $PSU(p^r)$, we obtain an H-space $X$ of type

$$N = \{k \mid 2 \leq k \leq n,\ k \equiv 1 \ (p-1)\}$$

with $\pi_1(X) = \mathbf{Z}/p^r$. Then

$$X \times \prod_{k \in \{2,\ldots,n\} \setminus N} S^{2k-1}_{(p)}$$

is the desired H-space. □

Let $X$ be a connected, homotopy associative H-space. Borel [2] studied a restriction to $\pi_1(X)$ due to the type of $X$. Ohsita [5] improved the result of Borel in the case that $X$ has the same type as the exceptional Lie group. Then, regarding $\pi_1(X)$ when $X$ is of the same type as the exceptional Lie group, the only problem left is to show whether $X$ of the same type as $E_6$ with $\pi_1(X) = \mathbf{Z}/15$ exists or not. The following corollary gives the answer.

**Corollary 1.2.** *There exists a connected, homotopy associative H-space $X$ of the same type as $E_6$ with $\pi_1(X) = \mathbf{Z}/15$.*

*Proof.* By applying Theorem 1.1 to $SU(10)$ for $p = 5$, we get a connected, homotopy associative H-space $V$ of type $(5,9)$ with $\pi_1(V) = \mathbf{Z}/5$. Then one has a connected, homotopy associative H-space $W = V \times F_4$ which is of the same type as $E_6$ with $\pi_1(W) = \mathbf{Z}/5$. Hence, by mixing the homotopy type of $W$ with $\mathrm{Ad}(E_6)$, one gets the desired H-space. □

## 2. Proof of Theorem 1.1

The case that $\pi_1(G_{(p)}) = 0$ is due to Wilkerson [7] and then we will deal with the case that $\pi_1(G_{(p)})$ has $p$-torsion.

We first observe the action of unstable Adams operations on $\pi_1(G_{(p)})$. Let $q$ be a prime with $(p, q) = 1$. By Theorem I of [7], one has the unstable Adams operation

$$\phi^q : BG_{(p)} \to BG_{(p)}$$

and the proof of Corollary 1.4 of [7] shows that we can choose a maximal torus $T$ of $G$ satisfying the following homotopy commutative diagram.

(1)
$$\begin{CD} BG^\wedge_{(p)} @>{(\phi^q)^\wedge}>> BG^\wedge_{(p)} \\ @A{Bi^\wedge_{(p)}}AA @AA{Bi^\wedge_{(p)}}A \\ BT^\wedge_{(p)} @>{\times q}>> BT^\wedge_{(p)}, \end{CD}$$

where $-^\wedge$ is the $p$-completion and $i : T \to G$ denotes the inclusion. By the classical result of Bott [2], one has

$$\pi_1(G/T) = 0.$$

Then, from the homotopy exact sequence of a fibration $G/T \to BT \to BG$, it follows that the map $(Bi^\wedge_{(p)})_* : \pi_2(BT^\wedge_{(p)}) \to \pi_2(BG^\wedge_{(p)})$ is epic. Hence, by (1), one can see that

$$(\phi^q)^\wedge_* = \times q : \pi_2(BG^\wedge_{(p)}) \to \pi_2(BG^\wedge_{(p)}).$$

Since $\pi_2(BG_{(p)})$ is $p$-torsion, the completion $BG_{(p)} \to BG^\wedge_{(p)}$ induces an isomorphism in $\pi_2$. Then we obtain :

**Proposition 2.1.** *The map $(\Omega\phi^q)_* : \pi_1(G_{(p)}) \to \pi_1(G_{(p)})$ is the $q$-power map.*

To prove Theorem 1.1, we shall follow the proof of Corollary III of [7] with a modification for considering fundamental groups by use of Proposition 2.1. Let $\widetilde{\phi^q} : B\widetilde{G}_{(p)} \to B\widetilde{G}_{(p)}$ denote a lift of $\phi^q$. Note that $\widetilde{\phi^q}$ is an unstable Adams operation. Let $u$ be an integer whose modulo $p$ reduction is a primitive $(p-1)$-st root of unity in $\mathbf{Z}/p$. Suppose that $u$ has a factorization $u = p_1^{k_1} \cdots p_n^{k_n}$ for primes $p_1, \ldots, p_n$. We denote the compositions $(\phi^{p_1})^{k_1} \circ \cdots \circ (\phi^{p_n})^{k_n}$ and $(\widetilde{\phi^{p_1}})^{k_1} \circ \cdots \circ (\widetilde{\phi^{p_n}})^{k_n}$ by $\phi^u$ and $\widetilde{\phi}^u$ respectively. By use of the group structure of $G$, we define a map $F_i : G_{(p)} \to G_{(p)}$ by the composition

$$\prod_{s \notin \bigcup_{k \neq i} N(G,k)} (\Omega\phi^u - u^s).$$

Define a map $\widetilde{F}_i : \widetilde{G}_{(p)} \to \widetilde{G}_{(p)}$ as well. Then we have a homotopy commutative diagram :

$$\begin{array}{ccc} \widetilde{G}_{(p)} & \xrightarrow{\widetilde{F}_i} & \widetilde{G}_{(p)} \\ {\scriptstyle \pi_{(p)}}\downarrow & & \downarrow{\scriptstyle \pi_{(p)}} \\ G_{(p)} & \xrightarrow{F_i} & G_{(p)}, \end{array} \tag{2}$$

where $\pi : \widetilde{G} \to G$ is the projection.

Let $X_i$ and $\widetilde{X}_i$ denote the infinite telescopes $G_{(p)} \xrightarrow{F_i} G_{(p)} \xrightarrow{F_i} G_{(p)} \xrightarrow{F_i} \cdots$ and $\widetilde{G}_{(p)} \xrightarrow{\widetilde{F}_i} \widetilde{G}_{(p)} \xrightarrow{\widetilde{F}_i} \widetilde{G}_{(p)} \xrightarrow{\widetilde{F}_i} \cdots$ respectively. Then it is obvious that

$$(X_i)_{(0)} \simeq \prod_{k \in N(G,i)} S^{2k-1}_{(0)}.$$

By (2), one has a map $\pi' : \widetilde{X}_i \to X_i$ induced from the projection $\pi : \widetilde{G} \to G$. It is straightforward to check that $\pi'_* : \pi_*(\widetilde{X}_i) \to \pi_*(X_i)$ is an isomorphism if $* \neq 1$.

On the other hand, the result of Wilkerson [7] yields that the inclusion $\tilde{\iota} : \widetilde{G}_{(p)} \to \prod_{i=1}^{p-1} \widetilde{X}_i$ is a homotopy equivalence. Note that, for (2), we have a homotopy commutative diagram :

$$\begin{array}{ccc} \widetilde{G}_{(p)} & \xrightarrow{\tilde{\iota}} & \prod_{i=1}^{p-1} \widetilde{X}_i \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ G_{(p)} & \xrightarrow{\iota} & \prod_{i=1}^{p-1} X_i \end{array}$$

Then we obtain :

**Proposition 2.2.** $\iota_* : \pi_*(G_{(p)}) \to \pi_*(\prod_{i=1}^{p-1} X_i)$ *is an isomorphism for* $* \neq 1$.

Now let us consider the fundamental group of $X_i$. Suppose that $N(G,1) = \emptyset$. Then $H_*(\widetilde{X}_1; \mathbf{Z}/p) = 0$ and hence

$$H_*(X_1; \mathbf{Z}/p) \cong H_*(K(\pi_1(G_{(p)}), 1); \mathbf{Z}/p).$$

Since $\pi_1(G_{(p)})$ has $p$-torsion, $H_*(K(\pi_1(G_{(p)}), 1); \mathbf{Z}/p)$ is an infinite Hopf algebra. Then the above congruence contradicts to the finiteness of $H_*(X_1; \mathbf{Z}/p)$. Hence we obtain

$$N(G,1) \neq \emptyset.$$

Thus, by Proposition 2.1, one can see that the map $(F_i)_* : \pi_1(G_{(p)}) \to \pi_1(G_{(p)})$ is the zero map if $i \neq 1$, and an isomorphism if $i = 1$. This yields that

$$\pi_1(X_i) = 0$$

if $i \neq 1$, and the inclusion $G_{(p)} \hookrightarrow X_1$ induces an isomorphism in $\pi_1$. Then one has

$$\iota_* : \pi_1(G_{(p)}) \to \pi_1\left(\prod_{i=1}^{p-1} X_i\right)$$

is an isomorphism. Thus, together with Proposition 2.2,

$$\iota_* : \pi_*(G_{(p)}) \to \pi_*\left(\prod_{i=1}^{p-1} X_i\right)$$

is an isomorphism for all $*$ and, by the J.H.C. Whitehead theorem, we obtain a homotopy equivalence

$$\iota : G_{(p)} \xrightarrow{\simeq} \prod_{i=1}^{p-1} X_i.$$

When $G$ is simple, Wilkerson [7] showed that each $\widetilde{X}_i$ is indecomposable. Then so is each $X_i$. Hence the proof of Theorem 1.1 is completed.

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