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Effective calculation of the geometric height and the Bogomolov conjecture for hyperelliptic curves over function fields

By

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Introduction

Let us begin with a brief survey on the Bogomolov conjecture. Let A be an abelian variety over a field K. We assume that K is a number field for a while. Let L be a symmetric ample line bundle on A and $\hat{h}_L$ the Néron-Tate height on $A(\overline{K})$, where $\overline{K}$ is the algebraic closure of K. For a closed subvariety V of $A \otimes_K \overline{K}$, we set

$$V(\epsilon) := \{P \in V(\overline{K}) \mid \hat{h}_L(P) \leq \epsilon\}.$$

Theorem ([14]. Generalized Bogomolov conjecture). *Suppose that V is not the translation of an abelian subvariety by a torsion point. Then there exists $\epsilon > 0$ such that $V(\epsilon)$ is not Zariski dense in V.*

This theorem was proved by Zhang in [14]. The original version due to Bogomolov deals with a curve V embedded in its Jacobian variety, which was proved by Ullmo in [8].

Let us recall the proof of Zhang. In [13], he introduced the notion of admissible metric on a line bundle on V, and defined the admissible intersection numbers and the admissible height, which are compatible with the Néron-Tate height. Then, he found a key inequality called the fundamental inequality:

$$\sup_{W \subsetneq V} \left\{ \inf_{x \in (V \setminus W)(\overline{K})} \hat{h}_L(x) \right\} \geq \text{“the admissible height of } V\text{”},$$

where W ranges over all proper closed subvarieties of V. If the admissible height is proved to be positive, the fundamental inequality leads the Bogomolov conjecture immediately, but in general it is quite hard to calculate. To avoid this difficulty, Zhang proved the equidistribution theorem which says that a certain kind of sequence of small points should be equidistributed in the complex

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analytic space over an archimedean place. Then he showed that a sequence of small points in V arising from a counter-example of the Bogomolov conjecture should not be equidistributed although it should satisfy the conditions of the equidistribution theorem. That contradiction leads us to the proof of the conjecture.

The Bogomolov conjecture has been proved over other global fields. Moriwaki in [7] invented a general arithmetic height function over a finitely generated field K over $\mathbb{Q}$, which coincides with the classical one if K is a number field. If K is transcendental over $\mathbb{Q}$, this height is determined after a choice of, so called "polarization" of K. It should be remarked that Moriwaki's arithmetic height for a "big" polarization has a contribution of archimedean places. Following ideas of Zhang, he proved the generalized Bogomolov conjecture for this new height over finitely generated fields, in which proof, the equidistribution theorem at an archimedean place played a crucial role too.

It is believed that the Bogomolov conjecture over function fields with respect to the classical geometric height also holds true under a natural additional condition. However, the lack of archimedean places prevented us from using an analogue of Zhang's proof. Indeed, the geometric height is a special case of Moriwaki's arithmetic height, but the polarization giving it is far from a big polarization. In spite of such a situation, Gubler recently proved it in [2] under the assumption of the existence of a place v at which the abelian variety is totally degenerated. His proof follows Zhang's one replacing the equidistribution theorem on the complex analytic space over an archimedean place by that over the tropical analytic geometry over v. His proof shows us that Zhang's idea can be applied to a certain geometric case, but the tropical variety is not so rich at the general place that we cannot enjoy the equidistribution theorem.

In spite of that, an effective version of the Bogomolov conjecture has already been proved for some curves in its Jacobian. The proofs are due to the calculation of the admissible pairing in [12] on a curve. Let us recall it here. Let Y be a nonsingular projective curve over an algebraically closed field k, and let $f : X \to Y$ be a generically smooth semistable curve of genus $g \geq 2$ over Y, where we assume X to be nonsingular. Let K denote the function field of Y, $\overline{K}$ the algebraic closure of K, and let C denote the generic fiber of f. Let J_C be the Jacobian variety of C, $j : C(\overline{K}) \to J_C(\overline{K})$ a morphism defined by $j(x) = (2g-2)x - \omega_C$ where ω_C is the canonical divisor class of C, and let $\|\cdot\|_{NT}$ be the semi-norm arising from the Néron-Tate pairing on $J_C(\overline{K})$. We set

$$B_C(P; r) := \left\{ x \in C(\overline{K}) \;\middle|\; \|j(x) - P\|_{NT} \leq r \right\}$$

for $P \in J_C(\overline{K})$ and $r \geq 0$, and set

$$r_C(P) := \begin{cases} -\infty & \text{if } \#\big(B_C(P;0)\big) = \infty, \\ \sup \left\{ r \geq 0 \;\middle|\; \#\big(B_C(P;r)\big) < \infty \right\} & \text{otherwise.} \end{cases}$$

Note that, with this notation, the Bogomolov conjecture is nothing but the statement that $r_C(P) > 0$ for all P. The fundamental inequality here can be translated into the following important inequality, which says that, if $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a > 0$, then we have

$$\inf_{P \in J_C(\overline{K})} r_C(P) \geq \sqrt{(g-1)(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a},$$

where $\omega^a_{X/Y}$ is the admissible dualizing sheaf and $(\cdot)_a$ is the admissible pairing (cf. [12, Theorem 5.6], [5, Corollary 2.3] and [3, Theorem 2.1]). Then, we have the following conjecture, which is well-known as the effective version of the Bogomolov conjecture.

Conjecture (Effective version of the geometric Bogomolov conjecture)**.** If f is non-isotrivial, then there exists an effectively calculated positive number r_0 with

$$\inf_{P \in J_C(\overline{K})} r_C(P) \geq r_0.$$

The assumption of non-isotriviality is necessary, since a point in k-trace has height 0. Here "effectively calculated" means that a concrete algorithm or a formula to find r_0 is required. It is not expected in the equidistributional approach.

If we can calculate the admissible pairing effectively, then we obtain the effective Bogomolov conjecture immediately by the above inequality. As in [12], the admissible pairing is given by

$$(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a = (\omega_{X/Y} \cdot \omega_{X/Y}) - \sum_{y \in Y} \epsilon_y,$$

where ϵ_y is the admissible constant arising from the harmonic analysis on the reduction graph over y (cf. Subsection 1.3). Therefore, our problem is reduced to the comparison of $(\omega_{X/Y} \cdot \omega_{X/Y})$ and ϵ_y's, and in fact, there are some earlier results on the effective Bogomolov conjecture obtained in that way. In [3], Moriwaki gave an answer for curves of genus 2. He also gave answers in [5], [4] and finally in [6], for a curve f of which fibers are trees of irreducible components in the case of $\mathrm{char}(k) = 0$. The author gave an answer for non-hyperelliptic curves of genus 3 in [9].

In this paper, we will give an affirmative answer to the effective geometric Bogomolov conjecture for hyperelliptic semistable curve f following the way as above (cf. Theorem 4.1 and Corollary 4.2). It is fortunate, in hyperelliptic case, that we have a necessary explicit description of $(\omega_{X/Y} \cdot \omega_{X/Y})$ as in [10] or [11]. Thus our main task is reduced to the calculation of the admissible constants, and the most part of this paper are devoted to it.

This paper is organized as follows. In Section 1, we first fix the notion on graphs and give some basic properties. After that, we recall what the admissible constants are. In Section 2, we introduce an important class of graphs, called hyperelliptic graphs. They will play a central role in this article. Section 3 will be mainly occupied by the struggle to obtain the concrete description of the admissible constants of hyperelliptic graphs. In the last section, we will estimate the admissible constants, and obtain our result.

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1. Admissible constants of graphs

1.1. Graphs

A *graph* $G = (V, E, \partial)$ consists of two finite sets V and E, and a map $\partial : E \to S^2V$, where S^2V is the 2nd symmetric power of V. An element of V is called a *vertex* and an element of E is called an *edge*. The map ∂ is called the *incidence relation*, and for an edge e, the image ∂e is called the *boundary* of e. We can naturally regard S^2V as a subset of the power set of V, and hence a boundary is regarded as a subset of V. A vertex in ∂e is called an *extremity* of e. An edge e is called a *line segment* if it has two extremities and is called a *self-loop* if it has only one extremity. For $v, v' \in V$, a finite sequence $(e_1, \ldots, e_n)$ of edges such that $v \in \partial e_1, v' \in \partial e_n$ and $\partial(e_i) \cap \partial(e_{i+1}) \neq \emptyset$ for any $i = 1, \ldots, n-1$, is called a *path* from v to v'. If there exists path from v to v' for any distinct two $v, v' \in V$, we say G is *connected*. A connected graph without edges called a *one-point graph*. It is represented by its unique vertex.

For a graph G, we denote by $\mathrm{Vert}(G)$ the set of vertices and by $\mathrm{Ed}(G)$ the set of edges. A *subgraph* of G means a graph G' such that $\mathrm{Vert}(G') \subset \mathrm{Vert}(G)$, $\mathrm{Ed}(G') \subset \mathrm{Ed}(G)$, with the incidence relation which is the restriction of that of G. For a set S of edges, the *subgraph generated by* S is the subgraph such that its vertices are the extremities of the edges in S and its set of edges is S.

A subgraph G' is said to be *saturated* if any edge $e \in \mathrm{Ed}(G)$ with $\partial e \subset \mathrm{Vert}(G')$ is necessarily an edge of G'. For a subset V' of $\mathrm{Vert}(G)$, there exists a unique saturated subgraph H with $\mathrm{Vert}(H) = V'$. We call it the *saturation* of V'.

For subgraphs G_1 and G_2, we can define the intersection $G_1 \cap G_2$ and the union $G_1 \cup G_2$ as subgraphs in an obvious way. We say that G is a *one-point sum* of G_1 and G_2 if neither G_1 nor G_2 is one-point graphs, $G = G_1 \cup G_2$, and $G_1 \cap G_2$ is a one-point graph, say $\{v\}$. We write $G = G_1 \vee_v G_2$ or simply $G = G_1 \vee G_2$. If G is a succession of one-point sums of subgraphs, we say G is a *sum* of them. Note that if G is a connected graph without self-loop and if $G = G_1 \vee G_2$, then G_1 and G_2 are saturated.

Definition 1.1. Let G be a connected graph. We say G is *reducible at* v if there exist subgraphs G_1 and G_2 with $G = G_1 \vee_v G_2$. We say G is *irreducible at* v if it is not reducible at v. We call a vertex at which G is reducible a *jointing vertex*. We denote by $J(G)$ the set of jointing vertices. We say G is *reducible* if $J(G) \neq \emptyset$, and say G is *irreducible* if $J(G) = \emptyset$.

Proposition 1.2. *Let G be a connected graph.*

(1) *If $G = G_1 \vee_v G_2$, then any path connecting a vertex in* $\mathrm{Vert}(G_1) \setminus \{v\}$ *to a vertex in* $\mathrm{Vert}(G_2) \setminus \{v\}$ *passes through v.*

(2) *If G is irreducible at $v \in \mathrm{Vert}(G)$ then, for any $w_1, w_2 \in \mathrm{Vert}(G) \setminus \{v\}$, there exists a path not passing through v but connecting w_1 and w_2.*

Proof. To show (1) we may assume there exist vertices $w_1 \in \mathrm{Vert}(G_1) \setminus \{v\}$ and $w_2 \in \mathrm{Vert}(G_2) \setminus \{v\}$. Let $l = (e_1, \ldots, e_n)$ be an arbitrary path from w_1 to w_2. Let k be the maximal integer with $e_k \in \mathrm{Ed}(G_1)$. Since v is the only common vertex of G_1 and G_2, we have $k < n$, and $e_{k+1} \in \mathrm{Ed}(G_2)$. Let w be a common extremity of e_k and e_{k+1}. Then it is contained in both G_1 and G_2 and hence $w = v$. Thus we have (1).

Let us prove (2). We may assume there is no self-loop with v as the extremity. Suppose contrary that for some $w_1, w_2 \in \mathrm{Vert}(G) \setminus \{v\}$, any path connecting w_1 and w_2 necessarily passes through v. Let V_1 be the set of vertices of G to which we can connect w_1 by a path without meeting v halfway (allowing v to appear as the terminus of the path), and let G_1 be the saturation of V_1. Then G_1 is a connected subgraph with w_1 and v as vertices but without w_2. Let G' be the saturation of $(\mathrm{Vert}(G) \setminus V_1) \cup \{v\}$. Then G' has w_2 and v as vertices. In particular, neither G_1 nor G' is a one-point graph. Accordingly, it is enough to show $G = G_1 \vee_v G_2$. By the definition of G_1 and G', we have $G_1 \cap G' = \{v\}$, and we also have $\mathrm{Ed}(G_1) \cap \mathrm{Ed}(G') = \emptyset$ since there is no self-loop with v. Therefore we are reduced to show $\mathrm{Ed}(G_1) \cup \mathrm{Ed}(G') = \mathrm{Ed}(G)$. Let us take any $e \in \mathrm{Ed}(G) \setminus \mathrm{Ed}(G')$. Then at least one extremity of e lies in G_1. From the construction of the set V_1 of vertices of G_1, the other vertex must be in G_1. Since G_1 is saturated, we have $e \in \mathrm{Ed}(G_1)$. Thus we have $\mathrm{Ed}(G_1) \cup \mathrm{Ed}(G') = \mathrm{Ed}(G)$. □

Corollary 1.3. *Suppose that $G = G_1 \vee_v G_2$ and that G_1 is irreducible at v. Let H be a connected subgraph of G irreducible at v with $G_1 \subset H$. Then $H = G_1$.*

Proof. We may assume that there is no self-loop at v. It is enough to show $H \cap G_2 = \{v\}$. Suppose that H and G_2 has a common vertex w other than v. Let v_1 be a vertex of G_1 other than v. Since H is irreducible at v, we can connect w and v_1 by a path in H not passing through v by Proposition 1.2. This path is a one connecting a vertex in G_1 and that in G_2, keeping away from v. That contradicts, again by Proposition 1.2, to the assumption $G = G_1 \vee_v G_2$. □

If G is reducible, we can write $G = G_1 \vee_{v_1} G_2$. If G_1 is again reducible, we can write G_1 as a one-point sum of its subgraphs. Repeating this process until anyone becomes irreducible, we can write G as a sum of irreducible graphs:

$$G = (\cdots((G_1 \vee G_2) \vee G_3) \vee \cdots \vee G_n).$$

The right-hand side is usually written as $G_1 \vee \cdots \vee G_n$ simply. We call it the *irreducible decomposition* of G. A subgraph of G appearing in the irreducible decomposition called an *irreducible component* of G.

Remark 1.4. We have only mentioned the existence of an irreducible decomposition, but we can actually show the uniqueness. We can therefore say "*the* irreducible decomposition".

Let $G = (V, E, \partial)$ be a graph and let S be a subset of E. We would like to define a *contraction*. We set $E_S := E \setminus S$ and $V_S := V/\sim$, where $v \sim v'$ if and only if there is an edge $e \in S$ such that $\{v, v'\} \subset \partial e$. Then we have a natural injective map $E_S \to E$ and natural surjective map $V \to V_S$, and hence we have an associated incidence relation $\partial_S : E_S \to S^2 V_S$ with ∂. Thus we have a graph $G_S := (V_S, E_S, \partial_S)$. We call that operation or G_S itself the *contraction* of S. With complement, we write G^S for $G_{\mathrm{Ed}(G)\setminus S}$.

There is a natural correspondence, denoted by contr_S, from vertices and edges of G to those of G_S: For $v \in \mathrm{Vert}(G)$, let $\mathrm{contr}_S(v)$ be the corresponding vertex of G_S by the natural surjection. For $e \in \mathrm{Ed}(G) \setminus S$, then $e \in \mathrm{Ed}(G_S)$ and hence put $\mathrm{contr}_S(e) := e$. For $e \in S$, set $\mathrm{contr}_S(e) := \mathrm{contr}_S(v)$ where v is an extremity of e. We write contr^S for $\mathrm{contr}_{\mathrm{Ed}(G)\setminus S}$ with complement.

Remark 1.5. We can regard an irreducible component not only as a subgraph but also as a contraction. Indeed, if $G = G_1 \vee G_2$, then G_1 can be canonically identified with $G^{\mathrm{Ed}(G_1)}$. This point of view will later lead us to a reasonable definition of the irreducible decomposition of polarized graphs.

Let $\mathrm{Div}_{\mathbb{R}}(G)$ be the $\mathbb{R}$-vector space with basis $\mathrm{Vert}(G)$. Its element is called an $\mathbb{R}$-*divisor* or a *polarization* on G. For $D = \sum_{v\in\mathrm{Vert}(G)} d_v v$, define $\deg(D) := \sum_{v\in\mathrm{Vert}(G)} d_v$. For a polarization $D = \sum_{v\in\mathrm{Vert}(G)} d_v v$ on G, we have the polarization

$$D_S := \sum_{v\in\mathrm{Vert}(G)} d_v \,\mathrm{contr}_S(v)$$

associated with D. Note that $\deg(D_S) = \deg(D)$. We write D^S for $D_{\mathrm{Ed}(G)\setminus S}$ as well.

Using this notion, we make the following definition (cf. Remark 1.5).

Definition 1.6. An *irreducible component of a polarized graph* (G, D) is a polarized graph of form $\left(G^{\mathrm{Ed}(G')}, D^{\mathrm{Ed}(G')}\right)$ for some irreducible component G' of G.

For a graph G, let $\mathcal{W}(G)$ denote the dual vector space of the $\mathbb{R}$-vector space with basis $\mathrm{Ed}(G)$. We put

$$\mathcal{W}_{>0}(G) := \{\lambda \in \mathcal{W}(G) \mid \lambda(e) > 0 \text{ for any } e \in \mathrm{Ed}(G)\}.$$

We call its element a *weight*. For a weight $\lambda \in \mathcal{W}_{>0}(G)$, we call $\lambda(e)$ the length of e. We usually denote a weight by

$$\lambda = (\lambda_e)_{e\in\mathrm{Ed}(G)} = (\lambda_e),$$

which indicates that the length of e is λ_e.

Suppose that G' is a subgraph or a contraction of G. Then we have $\mathrm{Ed}(G') \subset \mathrm{Ed}(G)$ canonically and hence we have canonical maps $\mathcal{W}(G) \to \mathcal{W}(G')$ and $\mathcal{W}_{>0}(G) \to \mathcal{W}_{>0}(G')$. Thus a weight λ on G induces a weight on G', denoted by $\lambda|_{G'}$. If G' is an irreducible component, then it has two induced weights as a subgraph and as a contraction, but one coincides with the other.

Definition 1.7. Let (G, D, λ) be a polarized weighted graph. A polarized weighted graph (G', D', λ') is called an *irreducible component* of (G, D, λ) if (G', D') is an irreducible component of (G, D) and $\lambda' = \lambda|_{G'}$.

Let $G = (V, E, \partial, \lambda)$ be a weighted graph. We mean, by a *realization* of G, a metrized graph M equipped with two data, an inclusion $V \hookrightarrow M$ and a family $\{e^\circ\}_{e\in E}$ of subsets of M indexed by E, satisfying the following conditions:

(1) e° is a 1-cell of M and $\{\{v\}\}_{v\in V}, \{e^\circ\}_{e\in E}$ gives a cell decomposition of M.

(2) For any $e \in E$,

$$cl(e^\circ) \setminus e^\circ = \partial e,$$

where $cl(e^\circ)$ is the closure of e° in M and ∂e is regarded as a subset of M.

(3) The length of e°, and hence that of $cl(e^\circ)$, equal $\lambda(e)$.

Any weighted graph has a realization of it, and for two realizations, there is an isometry between them compatible with the equipped data of cell decomposition. Note that a weighted subgraph of G can be realized as a metrized subgraph of a realization M of G and a contraction of weighted graphs can be realized as a quotient space of M. For an edge e, a continuous map

$$s_e : [0, \lambda(e)] \to M$$

which induces an isometry from $(0, \lambda(e))$ to e° is called an *arc-length parameter* of e.

1.2. Remarks on the admissible constants

Let us recall several facts on a Green function on a metrized graph. For details, see [12].

Let M be a connected metrized graph and let μ be an arbitrary measure on M with total volume 1. Then, there exists a unique function $g_\mu(x, y)$ on $M \times M$ satisfying the following conditions.

(a) g_μ is continuous, piecewise smooth in both x and y and symmetric in x and y.

(b) For a fixed x, regard $g_\mu(x, y)$ as a function on y, and we have

$$\Delta g_\mu = \delta_x - \mu,$$
$$\int_M g_\mu \mu = 0.$$

We call this function g_μ the *Green function* for μ.

Remark 1.8. Its uniqueness comes from some weaker conditions: Let M, μ and g_μ be as above. Fix an $x \in M$. Let h be a function on M such that $\Delta h = \delta_x - \mu$ and $\int_M h\mu = 0$. Put $f(y) := h(y) - g_\mu(x, y)$. Then f is a constant function by [12, Lemma a.4], and since $\int_M h\mu = 0$, it must be 0. Thus we have $h(y) = g_\mu(x, y)$ for all $y \in M$.

Let D be an $\mathbb{R}$-divisor on M. If $\deg(D) \neq -2$, then there exists a unique measure $\mu_{(M,D)}$ of total volume 1 on G such that

$$g_{\mu_{(M,D)}}(D, y) + g_{\mu_{(M,D)}}(y, y) \tag{1.1}$$

is a constant function on $y \in M$. We call this measure $\mu_{(M,D)}$ the *admissible metric* of (M, D) and call $g_{\mu_{(M,D)}}$ the *admissible Green function*. Since the admissible Green function is determined from (M, D), we write $g_{(M,D)}$ for $g_{\mu_{(M,D)}}$. We denote the constant (1.1) by $c(M, D)$ and set

$$\epsilon(M, D) = 2 \deg(D) c(M, D) - g_{(M,D)}(D, D).$$

We call this number the *admissible constant* of (M, D).

Let (G, D) be a polarized graph, λ a weight, and let $\bar{G}^\lambda$ be the realization of (G, λ). Let v and w be vertices of G. Since a realization is unique up to isometry compatible with the graph structure, the value $g_{(\bar{G}^\lambda,D)}(v, w)$ does not depend on the choice of realizations. Accordingly the admissible constants also independent of the choice of realizations. Here we define functions $g_{(G,D)}(v, w)$ and $\epsilon(G, D)$ on $\mathcal{W}_{>0}(G)$ by

$$g_{(G,D)}(v, w)(\lambda) := g_{(\bar{G}^\lambda,D)}(v, w), \quad \epsilon(G, D)(\lambda) := \epsilon(\bar{G}^\lambda, D).$$

We also define a function

$$r_G(v, w) : \mathcal{W}_{>0}(G) \to \mathbb{R}$$

by $r_G(v, w)(\lambda) := g_{\delta_v}(w, w)$, where δ_v is the dirac measure supported at v. It is the resistance between v and w if $\bar{G}^\lambda$ is regarded as an electric circuit in a natural way, and $r_G(v, w)$ is a rational function.

Remark 1.9. Let G_S be the contraction of $S \subset \mathrm{Ed}(G)$. Then it is immediate to see

$$\lim_{\lambda_e \to 0 \text{ for } e \in S} r_G(v, w)(\lambda) = r_{G_S}(\mathrm{contr}_S(v), \mathrm{contr}_S(w))\,(\lambda|_{G_S})$$

if they are considered as a resistance in an electric circuit. Namely, the resistance is compatible with the contractions. We will see later that the admissible constants are also compatible with contractions.

We recall an explicit formula of the admissible metric in [12]. Let (G, D) be a polarized graph with $\deg(D) \neq -2$ and let λ be a weight on G. For

$e \in \mathrm{Ed}(G)$, let $G \setminus e^\circ$ be the subgraph generated by $\mathrm{Ed}(G) \setminus \{e\}$ and let v and w be the extremities of e. Put

$$r_{G,e}(\lambda) := r_{G \setminus e^\circ}(v, w) \left(\lambda|_{G \setminus e^\circ}\right). \tag{1.2}$$

Then, [12, Lemma 3.7] says that the admissible metric on a realization $\bar{G}^\lambda$ is given by

$$\mu_{(G,D)} = \frac{1}{\deg(D) + 2} \left(\delta_D - \delta_K + \sum_{e \in \mathrm{Ed}(G)} \frac{2}{\lambda(e) + r_{G,e}(\lambda)} d\lambda|_e \right), \tag{1.3}$$

where $d\lambda|_e$ is the Lebesgue measure on e associated with the arc-length parameter. The coefficient of the Lebesgue measure on each edge is a rational function on λ, and it is compatible with contractions.

We end this subsection by showing a useful formula on admissible constants.

Proposition 1.10. *Let (G, D) be a polarized graph with $\deg(D) \neq -2$. Suppose G_1 and G_2 are subgraphs with $G_1 \vee G_2 = G$. Let us identify G_i with $G^{\mathrm{Ed}(G_i)}$ for $i = 1, 2$. Let D_i be the polarization on G_i defined as $D^{\mathrm{Ed}(G_i)}$. Then we have*

$$\epsilon(G, D)(\lambda) = \epsilon(G_1, D_1)(\lambda_1) + \epsilon(G_2, D_2)(\lambda_2)$$

for any $\lambda \in \mathcal{W}(G)_{>0}$, where $\lambda_i := \lambda|_{G_i}$.

Proof. Let M be a realization of (G, λ), and let M_1 and M_2 be metrized subgraphs of M realizing (G_1, λ_1) and (G_2, λ_2) respectively. By (1.3), we have

$$\mu_{(M,D)} = \mu_{(M_1,D_1)} + \mu_{(M_2,D_2)} - \delta_o, \tag{1.4}$$

where $\{o\} = M_1 \cap M_2$. Consider the following function on M:

$$g(x) := \begin{cases} g_{(M_1,D_1)}(o, x) + g_{(M_2,D_2)}(o, o) & \text{if } x \in M_1, \\ g_{(M_2,D_2)}(o, x) + g_{(M_1,D_1)}(o, o) & \text{if } x \in M_2. \end{cases}$$

Then, we can easily check that g is continuous on M, $\Delta(g) = \delta_o - \mu_{(M,D)}$, and $\int_M g\mu_{(M,D)} = 0$. Thus we have $g_{(M,D)}(o, x) = g(x)$ (cf. Remark 1.8). Therefore, by [4, Lemma 4.1], we obtain the formula. □

1.3. Admissible constants arising from a semistable fibration

Let Y be a smooth projective curve over k and let $f : X \to Y$ be a generically smooth semistable curve of genus $g \geq 2$. We assume that X is nonsingular. Let $X \to \overline{X}$ be the contraction of the (-2)-curves in the fibers of f. Then we have the stable model $\overline{f} : \overline{X} \to Y$.

Let us recall the metrized dual graph arising from the fiber over $y \in Y(k)$. Let G_y be the dual graph by configuration of the fiber $\overline{X}_y$, that is, the graph such that $\mathrm{Vert}(G_y)$ is the set of irreducible components of $\overline{X}_y$, $\mathrm{Ed}(G_y)$ is the set of node of $\overline{X}_y$, and the extremities of $e \in \mathrm{Ed}(G_y)$ are the irreducible components which contain the branches making the node e. Further G_y is endowed with a natural weight λ_y such that $\lambda_y(e)+1$ coincides with the number of (-2)-curves contracted to e under $X \to \overline{X}$, or in other words, the node e is given formally by the equation $xy = t^{\lambda_y(e)+1}$ in $\overline{X}$, where t is a regular local parameter at y on Y. Then the metrized dual graph $\bar{G}_y$ in [12] is the realization of (G_y, λ_y).

Let $\omega_{\overline{X}/Y}$ be the relative dualizing sheaf of $\overline{f}$. We define a divisor ω_y on G_y by

$$\omega_y := \sum_{v \in \mathrm{Vert}(G_y)} (\omega_{\overline{X}/Y} \cdot v)v,$$

where $(\omega_{\overline{X}/Y} \cdot v)$ means the intersection number of $\omega_{\overline{X}/Y}$ and a curve v in $\overline{X}$. Then, the admissible constant ϵ_y over y is defined to be the admissible constant of the polarized metrized graph $(\bar{G}_y, \omega_y)$:

$$\epsilon_y := \epsilon(\bar{G}_y, \omega_y) = \epsilon(G_y, \omega_y)(\lambda_y).$$

It is a very important quantity in this paper.

2. Hyperelliptic graphs

In the sequel, let us fix a finite group $\langle\iota\rangle$ of order 2 with the generator ι.

2.1. Definitions and first properties

An *action of* $\langle\iota\rangle$ *on a graph* G is a pair of action on $\mathrm{Vert}(G)$ and that on $\mathrm{Ed}(G)$ compatible with the incidence relation. If the action on $\mathrm{Ed}(G)$ is free, we can naturally construct the quotient graph $G/\langle\iota\rangle$ such that $\mathrm{Vert}(G/\langle\iota\rangle) = \mathrm{Vert}(G)/\langle\iota\rangle$ and $\mathrm{Ed}(G/\langle\iota\rangle) = \mathrm{Ed}(G)/\langle\iota\rangle$ with the incidence relation induced by that of G.

Definition 2.1. A connected graph G equipped with an action of $\langle\iota\rangle$ is called a *hyperelliptic graph* if it satisfies the following conditions:

(a) G is not a one-point graph.

(b) Any edge is a line segment.

(c) $\iota(e) \neq e$ for any $e \in G$.

(d) The quotient graph $G/\langle\iota\rangle$ is a tree, that is, a graph without circuits.

(e) If a vertex v is not ι-fixed, then there exist at least 3 branches away from v. In other words, the valence is at least 3 at any vertex.

Remark 2.2. There is no end in a hyperelliptic graph by (c) and (e), and any end of $G/\langle\iota\rangle$ lies under an ι-fixed vertex of G by (e).

For each $v \in \mathrm{Vert}(G)$ and $e \in \mathrm{Ed}(G)$, let $[v] \in \mathrm{Vert}(G/\langle\iota\rangle)$ and $[e] \in \mathrm{Ed}(G/\langle\iota\rangle)$ denote their image by the quotient map in the sequel. More generally, we sometimes, for an object $*$ concerning G, denote by $[*]$ the corresponding one of $G/\langle\iota\rangle$.

Definition 2.3. Let G be a hyperelliptic graph. We say $[v] \in \mathrm{Vert}(G/\langle\iota\rangle)$ is *fixed* (resp. *mobile*) if its representative $v \in \mathrm{Vert}(G)$ is ι-fixed (resp. not ι-fixed). We denote by $\mathrm{FV}(G/\langle\iota\rangle)$ the set of fixed vertices and $\mathrm{MV}(G/\langle\iota\rangle)$ by that of mobile ones.

The next proposition characterizes the jointing vertices of hyperelliptic graphs.

Proposition 2.4. *Let G be a hyperelliptic graph and let v be a vertex of G. Then the following statements are equivalent to each other.*
(a) *v is a jointing vertex.*
(b) *$[v]$ is not an end of $G/\langle\iota\rangle$ but a fixed vertex.*

Proof. First suppose (a). To show that $[v]$ is a fixed vertex, suppose contrary that $\iota(v) \neq v$. Since $G/\langle\iota\rangle$ is a tree and $[v]$ is not an end (cf. Remark 2.2), there exists ι-stable subgraphs G_1 and G_2 of G such that $G/\langle\iota\rangle = (G_1/\langle\iota\rangle) \vee_{[v]} (G_2/\langle\iota\rangle)$. Let $[w_i]$ be an end of $G/\langle\iota\rangle$ with $w_i \in G_i$ for $i = 1, 2$. Note that $\iota(w_i) = w_i$ by Remark 2.2 again. Then, lifting up the geodesic in $G/\langle\iota\rangle$ connecting $[\iota(v)](= [v])$ and $[w_i]$, we can connect $\iota(v)$ and w_i by a path not through v. Accordingly we can connect w_1 and w_2 by a path not through v. In a similar way, we find that any vertex of G_i other than v can be connected to w_i by a path not through v. Thus we see that any two vertices of G other than v can be connected by a path not through v. That contradicts to the assumption of v being a jointing vertex by Proposition 1.2. Thus we have $\iota(v) = v$.

Suppose that $[v]$ is an end. Take any $w_1, w_2 \in \mathrm{Vert}(G)$. Let v' be an ι-fixed vertex other than v. Then we can connect $[w_1]$ and $[v']$ by a path not through $[v]$, and do the same thing for $[w_2]$ and $[v']$. Using the lifts of these paths, we can connect w_1 and w_2 by a path not through v but through v'. That is a contradiction by Proposition 1.2, and hence $[v]$ is not an end. Thus we obtain (b).

To show the other direction, we will prove that $[v]$ is an end of $G/\langle\iota\rangle$ if v is not a jointing vertex and $\iota(v) = v$. Let us take two arbitrary $w_1, w_2 \in \mathrm{Vert}(G)$ other than v. Then by Proposition 1.2, we can connect w_1 and w_2 by a path not through v. Pushing it down to the quotient, we have a path which connect $[w_1]$ and $[w_2]$, but it does not pass through $[v]$ since v is the only vertex over $[v]$. That tells us that any two vertices of $G/\langle\iota\rangle$ other than $[v]$ can be connected by a path not through $[v]$, which implies $[v]$ must be an end. □

We have the following result as an immediate corollary.

Corollary 2.5. *Let G be an irreducible hyperelliptic graph and let v be a vertex of G. Then $[v]$ is a fixed vertex if and only if it is an end of $G/\langle\iota\rangle$.*

The next proposition tells us one-point sum is compatible with the action of $\langle\iota\rangle$.

Proposition 2.6. *Let G be a hyperelliptic graph. If $G = G_1 \vee_v G_2$, then $\iota(G_1) = G_1$ and $\iota(G_2) = G_2$.*

Proof. Let us show that G_1 is stable by ι. We may assume that G_1 is irreducible at the jointing vertex v. Let $[G_1]$ be the image of G_1 in $G/\langle\iota\rangle$. Let us take arbitrary vertices w_1 and w_1' of G_1 other than v. Since G_1 is irreducible at v, there exists a path not passing through v but connecting w_1 and w_1' by Proposition 1.2. Since v is the only vertex over $[v]$, the image of that path does not pass through $[v]$ but connects $[w_1]$ and $[w_1']$. That implies that any two vertex other than $[v]$ can be connected by a path not through $[v]$, and hence $[v]$ is an end of a tree $[G_1]$.

Let H be the largest subtree of $G/\langle\iota\rangle$ such that $[G_1] \subset H$ and that $[v]$ is an end of H. Then we see that the pull-back $\tilde{H}$ of H by the quotient map $G \to G/\langle\iota\rangle$ is a hyperelliptic subgraph having v as an ι-fixed vertex. Accordingly, by virtue of Proposition 2.4, $\tilde{H}$ is irreducible at v, and hence we have $\tilde{H} = G_1$ by Corollary 1.3. Thus G_1 is ι-stable. □

The following corollary says that the notion of one-point sum behaves well among the hyperelliptic graphs.

Corollary 2.7. *Let G be a hyperelliptic graph. If $G = G_1 \vee G_2$, then G_1 and G_2 are naturally hyperelliptic graphs.*

Proof. Immediate from Proposition 2.4 and Proposition 2.6. □

Definition 2.8. Let i be an integer. We say that $[e] \in \mathrm{Ed}(G/\langle\iota\rangle)$ is *i-jointed* if the number of fixed extremities of $[e]$ is equal to i. We denote by $\mathrm{Ed}_i(G/\langle\iota\rangle)$ the set of i-jointed edges. By abuse of words, we say $e \in \mathrm{Ed}(G)$ is i-jointed if so is $[e]$.

Here we give some examples.

Example 2.9. Let G be an irreducible hyperelliptic graph with a 2-jointed edge $[e]$. Then the extremities $[v]$ and $[w]$ of $[e]$ are fixed vertices. Since G is irreducible, we see, by Corollary 2.5, that $G/\langle\iota\rangle$ is the tree with a unique edge $[e]$. The configuration of G looks like Figure 1.

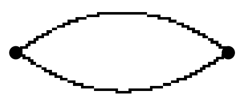

Figure 1.

Example 2.10. Let G be an irreducible hyperelliptic graph only with 1-jointed edges. Since G is irreducible, each edge of $G/\langle\iota\rangle$ has exactly one fixed vertex and any fixed vertex is an end of $G/\langle\iota\rangle$ by Corollary 2.5. Therefore, $G/\langle\iota\rangle$ is a tree such that the number of edges coincides with that of ends. Since such a graph is uniquely determined by this number, we can deduce G is also determined by the number of edges. The configuration of G looks like Figure 2, where ι acts vertically.

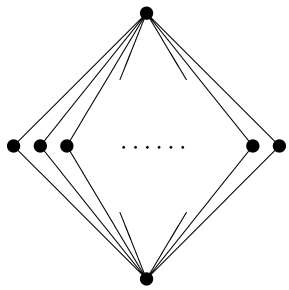

Figure 2.

Let S be an ι-stable subset of edges of a hyperelliptic graph G. It is not difficult to see that the contraction G' of S is canonically a hyperelliptic graph too.

Remark 2.11. Let G be a hyperelliptic graph. For $S \subset \mathrm{Ed}(G/\langle\iota\rangle)$, let $\tilde{S}$ be the pull-back of S by the quotient map. By abuse of notation, we write G_S and G^S for $G_{\tilde{S}}$ and $G^{\tilde{S}}$. We also call G_S the contraction of S by abuse of words. In the most part of the sequel, only this kind of contractions appears.

Example 2.12. Let G be an irreducible hyperelliptic graph without 2-jointed edges. Let e be a 0-jointied edge. Then we can see that the contraction $G_{\{[e]\}}$ is also an irreducible hyperelliptic graph, and the genus does not change. Contracting all the 0-jointied edges, we obtain an irreducible hyperelliptic graph with 1-jointied edges only, which appeared in the previous example.

For the class of irreducible hyperelliptic graphs, let us consider the partial order such that G' is smaller than G if and only if G' is the contraction of a ι-stable subset of edges of G. Then above three examples tell us that, in the class of irreducible hyperelliptic graphs, the ones in Example 2.9 and 2.10 are the minimal irreducible hyperelliptic graphs with respect to this order.

Definition 2.13. Let G be an irreducible hyperelliptic graph. We say G is *minimal* if it has no 0-jointed edges. We call $G_{\mathrm{Ed}_0(G)}$ the *minimal model* of G.

According to the above examples, we see that the minimal hyperelliptic graphs are classified by the genus. More precisely, if G is a minimal hyperelliptic graph of genus n, then we have the following.

(1) If $n = 1$, then G consists of two 2-jointed edges e and $\iota(e)$, as Example 2.9.

(2) If $n > 1$, then any edge is one-jointed, and $\#\,\mathrm{FV}(G/\langle\iota\rangle) = n + 1$ and $\#\,\mathrm{MV}(G/\langle\iota\rangle) = 1$, as Example 2.10.

By contrast with minimality, we introduce the notion of maximality.

Definition 2.14. Let G be an irreducible hyperelliptic graph. We call it a *maximal* hyperelliptic graph if $b_v = 3$ for any v with $[v] \in \mathrm{MV}(G/\langle\iota\rangle)$, where b_v denotes the valence at v.

Generally, let G be a graph and let v be a vertex of G with $b_v > 3$. Let k be an integer with $3 \leq k < b_v$. Then, there exists a graph G' and an edge e' of G' with the following properties (cf. [9, Figure 2]).

(a) e' is a line segment

(b) G is the contraction of $\{e'\}$.

(c) If w and w' are the extremities of e', then $b_w = k$ and hence $b_{w'} = b_v - k + 2$.

Let us return to the hyperelliptic case. Any vertex v of G with $\iota(v) \neq v$ has at least 3 branches. Suppose $b_v > 3$. Then, applying the above operation ι-equivariantly at v and $\iota(v)$, we find that there exist an irreducible hyperelliptic graph G_1 and a 0-jointied edge e with the following properties:

(a) $(G_1)_{\{[e]\}} = G$.

(b) If w and w' are the extremities of e, we have $3 \leq b_w, b_{w'} < b_v$.

Therefore, applying this process for all non-ι-fixed vertices successively, we can achieve a maximal hyperelliptic graph:

Proposition 2.15. *Let G be an irreducible hyperelliptic graph. Then there exist a maximal hyperelliptic graph G' and a set of 0-jointied edges S of $G'/\langle\iota\rangle$ such that $G'_S = G$.*

When we talk on hyperelliptic graphs, natural weights to be considered are ι-invariant weights. The canonical surjection $\mathrm{Ed}(G) \to \mathrm{Ed}(G/\langle\iota\rangle)$ lets us consider $\mathcal{W}(G/\langle\iota\rangle)$ as a subspace of $\mathcal{W}(G)$ and $\mathcal{W}_{>0}(G/\langle\iota\rangle)$ as a subspace of $\mathcal{W}_{>0}(G)$. Then $\mathcal{W}(G/\langle\iota\rangle)$ is the set of the ι-invariants. We call a member λ of $\mathcal{W}_{>0}(G/\langle\iota\rangle)$ a *hyperelliptic weight.* We usually write

$$\lambda = \left(\lambda_{[e]}\right)_{[e]\in\mathrm{Ed}(G/\langle\iota\rangle)} = \left(\lambda_{[e]}\right),$$

where $\lambda_{[e]}$ indicates the length of e or $[e]$ with respect to the weight λ.

When G is a hyperelliptic graph and D is an ι-invariant polarization, we call (G, D) a *polarized hyperelliptic graph.* As we mentioned before, we see that for a polarized hyperelliptic graph (G, D), such quantities defined after realization taken as the values of the admissible Green function at vertices, the resistance between vertices, and hence the admissible constant, are functions on $\lambda \in \mathcal{W}_{>0}(G/\langle\iota\rangle)$. We denote these functions by

$$g_{(G,D)}(v, w), \quad r_G(v, w), \quad \epsilon(G, D) \tag{2.1}$$

respectively.

2.2. Seriesization of an extreme circuit

In this subsection, we assume, for simplicity, that G is a maximal irreducible hyperelliptic graph of genus more than 2. (General cases can be treated as the contraction of edges of a maximal model.)

Definition 2.16. A set $\{[e_0], [e_1]\}$ of 1-jointed edges of $G/\langle\iota\rangle$ is called an *extreme circuit* if $[e_0] \neq [e_1]$, and if $[e_0]$ and $[e_1]$ has a common extremity.

Let $[v]$ be the common extremity of $[e_0]$ and $[e_1]$. Then by the maximality, $[v]$ is an end of the subtree $[H]$ generated by $\mathrm{Ed}_0(G/\langle\iota\rangle)$. In particular, there exists $[e_2] \in \mathrm{Ed}_0(G/\langle\iota\rangle)$ uniquely with an extremity $[v]$. On the other hand, in G, the edges e_0, e_1, $\iota(e_0)$ and $\iota(e_1)$ generate a circuit. Thus $\{[e_0], [e_1]\}$ is considered as the data expressing a circuit at an extreme position.

Now let us introduce an operation to "seriesize" an extreme circuit. Let $\{[e_0], [e_1]\}$ be an extreme circuit, $[v]$ the common extremity, and let $[e_2]$ be the 0-jointed edge with an extremity $[v]$. After a suitable choice of their representatives, we may assume v is the common extremity of e_0, e_1 and e_2 in G. Let G_1 be the subgraph generated by $\{e_0, e_1, e_2, \iota(e_0), \iota(e_1), \iota(e_2)\}$ and let G_1' be the graph with an ι-action characterized by the following conditions: G_1' is a tree consisting of 2 edges, and ι-action on the set of edges is free. Then we can write $\{e_0', \iota(e_0')\}$ for the set of edges and $\{v_0', v_1', \iota(v_1')\}$ for that of vertices, where v_0' is the ι-fixed vertex on e_0' and v_1' is the other one on e_0'. Now we construct another hyperelliptic graph G' which we obtain from G by replacing the subgraph G_1 with G_1': namely, remove G_1 from G and joint G_1' so as v_1' and $\iota(v_1')$ to make up with the missing vertices v_1 and $\iota(v_1)$ (cf. Figure 3).

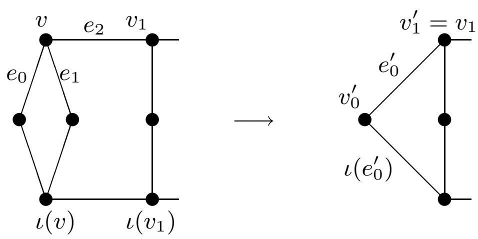

Figure 3. Seriesization

Then, G' is an irreducible hyperelliptic graph.

Definition 2.17. We call this G' the *seriesization* of an extreme circuit $\{[e_0], [e_1]\}$.

Remark 2.18. Let n and n' be the genus of G and G' respectively. Then we have $n' = n - 1$. There is a natural identification

$$\mathrm{Ed}(G'/\langle\iota\rangle) \setminus \{[e_0']\} = \mathrm{Ed}(G/\langle\iota\rangle) \setminus \{[e_0], [e_1], [e_2]\}.$$

We usually denote this set by $\mathcal{E}$ in the sequel.

It is more essential to treat seriesizations with weights. Let us denote by $\mathcal{E}$ the common subset of edges in Remark 2.18. For a hyperelliptic weight $\lambda = (\lambda_{[e]})$ of G, we define a hyperelliptic weight λ' of G' as follows: For $e' \in \mathrm{Ed}(G')$, if $[e'] = [e'_0]$ in $\mathrm{Ed}(G'/\langle\iota\rangle)$, then put

$$\lambda'_{[e']} := \frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} + \lambda_{[e_2]}, \tag{2.2}$$

and otherwise, we have $[e'] \in \mathcal{E}$ and put

$$\lambda'_{[e']} = \lambda_{[e']}.$$

We call G' with such a weight λ' the *seriesization* of (G, λ). From the definition of λ', it has the following properties (cf. (1.2) for notation).

(1) For any $e \in \mathcal{E}$, we have

$$r_{(G',e)}(\lambda') = r_{(G,e)}(\lambda). \tag{2.3}$$

(2) We have

$$\lambda'_{[e'_0]} + r_{(G',e'_0)}(\lambda') = \lambda_{[e_2]} + r_{(G,e_2)}(\lambda). \tag{2.4}$$

3. The admissible constants of hyperelliptic graphs

3.1. The admissible metric

For a hyperelliptic graph G with a hyperelliptic weight λ, the involution ι can acts on its realization as an isometry, and the quotient is the realization of the weighted tree $(G/\langle\iota\rangle, \lambda)$. We call such metrized graph with an isometry of order 2, a *hyperelliptic metrized graph.* In this subsection, we would like to describe the admissible metric of hyperelliptic metrized graphs.

Let n denote the genus of G. For a subset S of $\mathrm{Ed}(G/\langle\iota\rangle)$ with $\#(S) = n$, we define a real number, denoted by δ^S_G or simply δ^S, in the following way: If G^S is a sum of minimal hyperelliptic graphs of genus 1, then we put $\delta^S_G = \delta^S := 1$. Otherwise, we put $\delta^S_G = \delta^S := 0$. Note that $\delta^S = 1$ if and only if all edges of G^S are 2-jointed. Using it, we define a polynomial function L_G on $\mathcal{W}(G/\langle\iota\rangle)$ by

$$L_G(\lambda) := \sum_S \delta^S \prod_{[e]\in S} \lambda_{[e]},$$

where S ranges over all subsets of $\mathrm{Ed}(G/\langle\iota\rangle)$ with $\#S = n$. Further, for each $e \in \mathrm{Ed}(G)$, we define $P^{[e]}_G$ to be the polynomial function such that

$$L_G(\lambda) - P^{[e]}_G(\lambda)\lambda_{[e]}$$

does not contain the valuable $\lambda_{[e]}$.

Example 3.1. Let G be the minimal hyperelliptic graph of genus n.

(1) If $n = 1$, then $L_G(\lambda) = \lambda_{[e]}$.

(2) Suppose $n > 1$. Then $\mathrm{Ed}(G/\langle\iota\rangle)$ consists of $(n+1)$ 1-jointied edges, namely, $[e_0], [e_1], \dots, [e_n]$. Put $\lambda = (\lambda_0, \dots, \lambda_n)$, where $\lambda(e_i) = \lambda_i$. Taking account that a contraction of $\{e_i, \iota(e_i)\}$ is sum of n copies of minimal hyperelliptic graphs of genus 1, we see

$$L_G(\lambda) = \sum_{i=0}^{n} \prod_{k \neq i} \lambda_k,$$

which is the n-the elementary symmetric polynomial on $\lambda_0, \dots, \lambda_n$.

The polynomial L_G has the following properties.

Lemma 3.2. *Let G be a hyperelliptic graph.*

(1) *If $G = G_1 \vee G_2$, then $L_G = L_{G_1} L_{G_2}$. If in addition $e \in \mathrm{Ed}(G_1)$, then $P_G^{[e]} = P_{G_1}^{[e]} L_{G_2}$.*

(2) *Let G' be the contraction of $\{e, \iota(e)\}$ and let λ' be the associated weight on G' with λ. Then $L_{G'}(\lambda') = \lim_{\lambda_{[e]} \to 0} L_G(\lambda)$.*

Proof. The last equality in (1) immediately follows from the first one. For $S \subset \mathrm{Ed}(G/\langle\iota\rangle)$, put $S_i := S \cap \mathrm{Ed}(G_i/\langle\iota\rangle)$ for $i = 1, 2$. Then, G^S is a sum of minimal hyperelliptic graphs of genus 1 if and only if so are both ${G_1}^{S_1}$ and ${G_2}^{S_2}$. Now it is obvious that $L_G = L_{G_1} L_{G_2}$.

The equality in (2) is obvious from the definition. □

Let G be a maximal hyperelliptic graph of genus more than 2, with a weight λ. Let $\{[e_0], [e_1]\}$ be an extreme circuit and let G' be the seriesization, with the induced weight λ'. We adopt the notation in Subsection 2.2, and recall

$$\mathcal{E} = \mathrm{Ed}(G'/\langle\iota\rangle) \setminus \{[e_0']\} = \mathrm{Ed}(G/\langle\iota\rangle) \setminus \{[e_0], [e_1], [e_2]\}.$$

Lemma 3.3. *With the notation above, we have*

$$\left(\lambda_{[e_0]} + \lambda_{[e_1]}\right) L_{G'}(\lambda') = L_G(\lambda),$$
$$\left(\lambda_{[e_0]} + \lambda_{[e_1]}\right) P_{G'}^{[e_0']}(\lambda') = P_G^{[e_2]}(\lambda),$$

and if $[e] \in \mathcal{E}$, then

$$\left(\lambda_{[e_0]} + \lambda_{[e_1]}\right) P_{G'}^{[e]}(\lambda') = P_G^{[e]}(\lambda).$$

Proof. Let S be a set of n edges of $G/\langle\iota\rangle$. First note that if $\delta^S = 1$, then $S \cap \{[e_0], [e_1], [e_2]\}$ coincides with one of

$$\{[e_0]\}, \quad \{[e_1]\}, \quad \{[e_0], [e_1]\}, \quad \{[e_0], [e_2]\}, \quad \{[e_1], [e_2]\}. \tag{3.1}$$

For such an S, we define a subset of $\mathrm{Ed}(G'/\langle\iota\rangle)$ by

$$S' := \begin{cases} \mathcal{E} \cap S & \text{if } \#(S \cap \{[e_0], [e_1], [e_2]\}) = 1 \\ (\mathcal{E} \cap S) \cup \{[e_0']\} & \text{otherwise.} \end{cases}$$

Now, if $\delta^S = 1$ then $\#(S') = n-1$ and $\delta^{S'} = 1$. Conversely, under the condition that $S \cap \{[e_0], [e_1], [e_2]\}$ is one of (3.1), suppose $\#(S') = n-1$ and $\delta^{S'} = 1$. Then it is easy to see $\delta^S = 1$. This observation leads us to our formulas. $\square$

Now we can show an explicit formula for the admissible metric.

Proposition 3.4. *Let (G, D) be a polarized hyperelliptic graph, λ a weight, and let K be the canonical divisor of G. Suppose* $\deg(D) \neq -2$. *Then, the admissible metric on the realization $\bar{G}^\lambda$ is given by*

$$\mu_{(\bar{G}^\lambda, D)} = \frac{1}{\deg(D)+2}\left(\delta_D - \delta_K + \sum_{e \in \mathrm{Ed}(G)} \frac{P_G^{[e]}(\lambda)}{L_G(\lambda)}(d\lambda|_e)\right).$$

Proof. By (1.4) and Lemma 3.2 (1), it is enough to show it for irreducible hyperelliptic graphs. Then our assertion is an immediate consequence of the following lemma and (1.3). $\square$

Lemma 3.5. *Let G be an irreducible hyperelliptic graph and let λ be a weight. Then for any $e \in \mathrm{Ed}(G)$, we have*

$$\frac{2}{\lambda_{[e]} + r_{(G,e)}(\lambda)} = \frac{P_G^{[e]}(\lambda)}{L_G(\lambda)}.$$

Proof. We will prove our assertion by induction on the genus n of G. If $n \leq 2$, then G is minimal and we can obtain it by direct calculations.

Now, suppose that $n > 2$. Since the both sides of the equality is compatible with the contraction (cf. Remark 1.9, (1.2) and Lemma 3.2)), we may assume that G is maximal. Then, under the assumption of $n > 2$, there are at least 2 extreme circuit of $G/\langle\iota\rangle$. Let $([e_0], [e_1])$ be an extreme circuit such that $[e_0] \neq [e]$ and $[e_1] \neq [e]$. We adopt the notation in Subsection 2.2.

By (2.3), (2.4) and the induction hypothesis, we have

$$\frac{2}{\lambda_{[e]} + r_{(G,e)}(\lambda)} = \begin{cases} \dfrac{P_{G'}^{[e_0']}(\lambda')}{L_{G'}(\lambda')} & \text{if } [e] = [e_2], \\[2ex] \dfrac{P_{G'}^{[e]}(\lambda')}{L_{G'}(\lambda')} & \text{otherwise.} \end{cases} \tag{3.2}$$

Thus by Lemma 3.3, we obtain our equality. $\square$

3.2. The Green function as a piecewise quadratic function

Let (G, D) be a polarized hyperelliptic graph and let $\lambda = (\lambda_{[e]})$ be a hyperelliptic weight. Let $\bar{G} = \bar{G}^\lambda$ be the realization with the isometric natural action of ι and let $\pi : \bar{G} \to \bar{G}/\langle\iota\rangle$ be the quotient.

Let us fix $[o] \in \mathrm{FV}(G/\langle\iota\rangle)$. The admissible metric on $\bar{G}$ is of form

$$\mu_{(\bar{G},D)} = \sum_{v \in \mathrm{Vert}(G)} a_{[v]}\delta_v + \sum_{e \in \mathrm{Ed}(G)} b_{[e]} d\lambda|_e.$$

We define a measure ν on $\bar{G}/\langle\iota\rangle$ by

$$2\nu = \sum_{v\in\mathrm{Vert}(G)} a_{[v]}\delta_{[v]} + \sum_{e\in\mathrm{Ed}(G)} b_{[e]}d\lambda|_{[e]},$$

where λ is regarded as a weight on $G/\langle\iota\rangle$ and hence $d\lambda|_{[e]}$ is the induced Lebesgue measure on the edge $[e]$. If h is a piecewise smooth function on $\bar{G}/\langle\iota\rangle$ satisfying

$$\begin{aligned} \Delta h &= \frac{1}{2}\delta_{[o]} - \nu \\ \int h\pi_*\mu_{(\bar{G},D)} &= 0, \end{aligned} \tag{3.3}$$

it is easy to check that

$$\begin{aligned} \Delta\pi^* h &= \delta_o - \mu_{(\bar{G},D)} \\ \int \pi^* h\mu_{(\bar{G},D)} &= 0, \end{aligned}$$

and hence by Remark 1.8 we have

$$\pi^* h(x) = g_{(\bar{G},D)}(o,x).$$

Thus we are interested in the solution of (3.3).

With the above notation, assume in addition that G is an irreducible hyperelliptic graph of genus $n \geq 2$. We define a partial order $\prec$ in $\mathrm{Vert}(G/\langle\iota\rangle)$ and $\mathrm{Ed}(G/\langle\iota\rangle)$ with respect to $[o]$ as follows. First note that there exists a unique geodesic connecting any two vertices since $G/\langle\iota\rangle$ is a tree. For $[v],[w] \in \mathrm{Vert}(G/\langle\iota\rangle)$, we write $[v] \prec [w]$ if $[v]$ is in a halfway of the geodesic connecting $[o]$ and $[w]$. Similarly, for $[e],[e'] \in \mathrm{Ed}(G/\langle\iota\rangle)$, we write $[e] \prec [e']$ if $[e] \neq [e']$ and $[e]$ is a part of the geodesic connecting $[o]$ and a vertex on $[e']$.

Let $\left\{s_{[e]}\right\}_{[e]\in\mathrm{Ed}(G/\langle\iota\rangle)}$ be a collection of arc-length parameters $s_{[e]} : \left[0,\lambda_{[e]}\right] \to [e]$ such that $s_{[e]}(0) \prec s_{[e]}\left(\lambda_{[e]}\right)$, where $[e] \in \mathrm{Ed}(G/\langle\iota\rangle)$ is regarded as a closed line segment in the realization. For a function h on $\bar{G}/\langle\iota\rangle$, let $h_{[e]}$ denote $h\circ s_{[e]}$ for simplicity. Then we have a collection $\left\{h_{[e]}\right\}_{[e]\in\mathrm{Ed}(G/\langle\iota\rangle)}$ of functions, where each $h_{[e]}$ is defined on a closed interval $\left[0,\lambda_{[e]}\right]$.

Let D be a polarization of form

$$D = \sum_v (b_v - 2)v + \sum_{[e]\in\mathrm{Ed}_1(G/\langle\iota\rangle)} 2d_{[e]}w_{[e]}, \tag{3.4}$$

where v ranges over the vertices of G with $\iota(v) \neq v$, b_v is the valence at v, and $w_{[e]}$ is the ι-fixed extremity of e. For this kind of polarization, we would like to describe the function h satisfying (3.3) in terms of the collection $\left\{h_{[e]}\right\}_{[e]\in\mathrm{Ed}(G/\langle\iota\rangle)}$ of functions on intervals. To do that, we are going to define, for all $[e] \in \mathrm{Ed}(G/\langle\iota\rangle)$, the rational functions $\alpha_{(G,D),[e]}$, $\beta_{(G,D),[e]}$ and $\gamma_{(G,D),[e]}$ on λ.

In $G/\langle\iota\rangle$, there is exactly one edge starting from $[o]$, which is denoted by $[e_0]$. First we put

$$\alpha_{(G,D),[e]}(\lambda) = \alpha_{[e]}(\lambda) := \frac{P_G^{[e]}(\lambda)}{2(\deg(D)+2)L_G(\lambda)} \tag{3.5}$$

for each $[e] \in \mathrm{Ed}(G/\langle\iota\rangle)$. Note that we have

$$2\sum_{[e]\in\mathrm{Ed}(G/\langle\iota\rangle)} 2\alpha_{G,[e]}(\lambda)\lambda_{[e]} + \sum_{[e]\in\mathrm{Ed}_1(G/\langle\iota\rangle)} \frac{2d_{[e]}}{\deg(D)+2} = 1 \tag{3.6}$$

since the admissible metric has total volume 1 (cf. Proposition 3.4). Next for any $[e] \in \mathrm{Ed}_1(G/\langle\iota\rangle) \setminus \{[e_0]\}$, we put

$$\beta_{(G,D),[e]}(\lambda) = \beta_{[e]}(\lambda) := -\frac{d_{[e]}}{\deg(D)+2} - 2\alpha_{(G,D),[e]}(\lambda)\lambda_{[e]}. \tag{3.7}$$

For other $[e] \in \mathrm{Ed}(G/\langle\iota\rangle)$, we define $\beta_{(G,D),[e]}(\lambda)$ by the following descending induction. Since it is defined for the other 1-jointed edges than $[e_0]$, we already have $\beta_{(G,D),[e]}(\lambda)$ for the maximal edges with respect to $\prec$. Suppose that $\beta_{(G,D),[e']}(\lambda)$ is defined for any $[e']$ that is next to $[e]$ with respect to $\prec$. Then, we put

$$\beta_{(G,D),[e]}(\lambda) = \beta_{[e]}(\lambda) := \sum_{[e']} \beta_{(G,D,),[e']}(\lambda) - 2\alpha_{(G,D),[e]}(\lambda)\lambda_{[e]}, \tag{3.8}$$

where $[e']$ ranges over the edges of $G/\langle\iota\rangle$ with $[e] \prec [e']$. Finally, let $\gamma_{(G,D),[e]} = \gamma_{[e]}$'s be the solutions of the following equations:

$$\gamma_{(G,D),[e]}(\lambda) = \alpha_{(G,D),[e'']}(\lambda){\lambda_{[e'']}}^2 + \beta_{(G,D),[e'']}(\lambda)\lambda_{[e'']} + \gamma_{(G,D),[e'']}(\lambda) \tag{3.9}$$

for all $[e'']$ and $[e]$ with $[e''] \prec [e]$, and

$$\begin{aligned}
&2\sum_{[e]\in\mathrm{Ed}(G/\langle\iota\rangle)} \left(\frac{1}{3}\alpha_{(G,D),[e]}(\lambda){\lambda_{[e]}}^2 + \frac{1}{2}\beta_{(G,D),[e]}(\lambda)\lambda_{[e]}\right. \\
&\qquad\qquad \left. + \gamma_{(G,D),[e]}(\lambda)\right)(2\alpha_{(G,D),[e]}(\lambda)\lambda_{[e]}) \\
&\quad + \sum_{[e]\in\mathrm{Ed}_1(G/\langle\iota\rangle)\setminus\{[e_0]\}} \frac{2d_{[e]}}{\deg(D)+2}\big(\alpha_{(G,D),[e]}(\lambda){\lambda_{[e]}}^2 + \beta_{(G,D),[e]}(\lambda)\lambda_{[e]} \\
&\qquad\qquad + \gamma_{(G,D),[e]}(\lambda)\big) \\
&\quad + \frac{2d_{[e_0]}}{\deg(D)+2}\gamma_{(G,D),[e]}(\lambda) \\
&= 0.
\end{aligned} \tag{3.10}$$

It is not difficult to see that this system of equations on $\gamma_{[e]}(\lambda)$'s have a unique solution (cf. (3.14)).

By the equation (3.9), there exists a function h on $\bar{G}/\langle\iota\rangle$ such that

$$h_{[e]}(t) = \alpha_{[e]}(\lambda)t^2 + \beta_{[e]}(\lambda)t + \gamma_{[e]}(\lambda).$$

Such a function is unique and piecewise smooth, and moreover, taking account of Proposition 3.4, we can check by a messy but straightforward computation that h satisfies (3.3). Thus we can describe the admissible Green function as a piecewise quadratic function.

The values of the Green function at vertices can be given by $\gamma_{[e]}(\lambda)$'s: Let e be an edges of G, and let v be the nearer extremity of e to $[o]$. Then form the construction above, we have

$$g_{(G,D)}(o,v)(\lambda) = \gamma_{[e]}(\lambda). \tag{3.11}$$

Thus, the $\gamma_{[e]}$'s are important for the calculation of the admissible constants.

Remark 3.6. Here let us consider $\lambda = \left(\lambda_{[e]}\right)$ as a system of valuables, and let $\mathbb{R}[\lambda]$ be the polynomial ring of them. Let $\mathbb{R}[\lambda]_{(k)}$ denote the subset of homogeneous ones of degree k. From the above description, we can see

$$\alpha_{[e]}(\lambda) \in \frac{1}{L_G(\lambda)}\mathbb{R}[\lambda]_{(n-1)}$$

and

$$\beta_{[e]}(\lambda) \in \frac{1}{L_G(\lambda)}\mathbb{R}[\lambda]_{(n)}.$$

Accordingly we have, by (3.14) which follows from (3.9) and (3.10),

$$\gamma_{[e]}(\lambda) - \gamma_{[e_0]}(\lambda) \in \frac{1}{L_G(\lambda)}\mathbb{R}[\lambda]_{(n+1)}. \tag{3.12}$$

It is also easy to see that $\gamma_{[e]}(\lambda)$ is a rational function on λ. In fact, a little more calculation (cf. (3.14) again) tells us

$$\gamma_{[e]}(\lambda) \in \frac{1}{L_G(\lambda)^2}\mathbb{R}[\lambda]_{(n+1)}.$$

Let us carry out the calculation for minimal graphs here.

Example 3.7. Let G be a minimal hyperelliptic graph of genus $n > 1$. We use the same notation as that in Example 3.1. Let w_i denote the ι-fixed vertex of e_i and let $D = \sum_{i=0}^{n} 2d_i w_i$ be a polarization with $\deg(D) \neq -2$. We fix $o := w_0$. Let $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n)$ be a weight, where $\lambda_i = \lambda(e_i)$. Under those notations, let us describe $\alpha_i := \alpha_{[e_i]}(\lambda)$, $\beta_i := \beta_{[e_i]}(\lambda)$ and $\gamma_i := \gamma_{[e_i]}(\lambda)$. Let σ_k denote the k-th elementary symmetric polynomial on $\lambda_0, \lambda_1, \dots, \lambda_n$ and let $\sigma_k^{(i)}$

denote the k-th elementary symmetric polynomial on $\lambda_0, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n$. Then, by Example 3.1 and (3.5), we have

$$\alpha_i = \frac{\sigma_{n-1}^{(i)}}{2(\deg(D)+2)\sigma_n},$$

and by (3.7) and (3.8), we see

$$\beta_i = \begin{cases} \dfrac{d_0}{\deg(D)+2} - \dfrac{1}{2} & \text{if } i = 0, \\ -\dfrac{d_i}{\deg(D)+2} - \dfrac{\sigma_{n-1}^{(i)}\lambda_i}{(\deg(D)+2)\sigma_n} & \text{otherwise.} \end{cases}$$

From (3.9) and (3.10) we can obtain

$$\begin{aligned} \gamma_0 = &-\sum_{i=1}^{n}\left(\frac{2}{3}\alpha_i{\lambda_i}^2 + \beta_i\lambda_i\right)(2\alpha_i\lambda_i) - \sum_{i=1}^{n}\frac{2d_i}{\deg(D)+2}(\alpha_i{\lambda_i}^2 + \beta_i\lambda_i) \\ &+ \left(\frac{4}{3}\alpha_0{\lambda_0}^2 + \beta_0\lambda_0\right)(2\alpha_0\lambda_0) + \left(\frac{2d_0}{\deg(D)+2} - 1\right)(\alpha_0{\lambda_0}^2 + \beta_0\lambda_0). \end{aligned}$$

Using equalities

$$\begin{gathered} \sigma_{n-1}^{(i)}{\lambda_i}^2 = \sigma_n\lambda_i - \sigma_{n+1} \\ \sum_{i=0}^{n}\bigl(\sigma_{n-1}^{(i)}\bigr)^2{\lambda_i}^3 = \sigma_n\bigl(\sigma_n\sigma_1 - (2n+1)\sigma_{n+1}\bigr), \end{gathered}$$

we have

$$\gamma_0 = \frac{2}{3(\deg(D)+2)^2}\left(\sum_{i=0}^{n}(1+3d_i+3{d_i}^2)\lambda_i + (n-1)\frac{\sigma_{n+1}}{\sigma_n}\right) - \left(\frac{2d_0+1}{\deg(D)+2} - \frac{1}{2}\right)\lambda_0.$$

Using the description as quadratic functions, let us show that the values of Green function and hence the admissible constants are compatible with contraction. Let G be an irreducible hyperelliptic graph with such a polarization D as (3.4), and let λ be a hyperelliptic weight. For an edge $e \in \mathrm{Ed}(G)$, let $\mathrm{contr} : G \to G'$ be the contraction of $\{e, \iota(e)\}$, and let D' and λ' be the associated polarization and weight on G'.

Lemma 3.8. *Under the situation above, let o be an ι-fixed vertex and let v be any vertex. Then we have*

$$\lim_{\lambda_{[e]}\to 0} g_{(G,D)}(o,v)(\lambda) = g_{(G',D')}\left(\mathrm{contr}(o), \mathrm{contr}(v)\right)\left(\lambda'\right)$$

and

$$\lim_{\lambda_{[e]}\to 0} \epsilon(G,D)(\lambda) = \epsilon\left(G',D'\right)\left(\lambda'\right).$$

Proof. To show the first equality, it is enough to see that

$$\begin{aligned}
\lim_{\lambda_{[e]}\to 0} \alpha_{(G,D),[e']}(\lambda) &= \alpha_{(G',D'),[e']}(\lambda') \\
\lim_{\lambda_{[e]}\to 0} \beta_{(G,D),[e']}(\lambda) &= \beta_{(G',D'),[e']}(\lambda') \\
\lim_{\lambda_{[e]}\to 0} \gamma_{(G,D),[e']}(\lambda) &= \gamma_{(G',D'),[e']}(\lambda'),
\end{aligned}$$

for any $[e'] \in \mathrm{Ed}(G/\langle\iota\rangle) \setminus \{[e]\}$, which can be straightforwardly verified by their definitions.

The second equality follows from the first equality and the fact that the resistances are compatible with the contractions. □

3.3. Denominator of the admissible constant

The purpose of this subsection is to show that $\epsilon(G, D)$ is a rational function with a denominator L_G, namely, $L_G\epsilon(G, D)$ is a polynomial function on $\mathcal{W}_{>0}(G/\langle\iota\rangle)$. To do that, we will show that for one $o \in \mathrm{Vert}(G)$ with $\iota(o) = o$ and any $v \in \mathrm{Vert}(G)$, the functions $g_{(G,D)}(o, w)$ and $r_G(o, w)$ have a denominator L_G.

Let us look at $g_{(G,D)}(o, v)$ first. As in Remark 3.6, we know that $\alpha_{G,[e]}$, $\beta_{G,[e]}$ and $\gamma_{G,[e]}$ are rational functions on the weight λ. Moreover, we see that $\alpha_{G,[e]}$ and $\beta_{G,[e]}$ have a denominator L_G. Then how about $\gamma_{G,[e]}$? Indeed, it will be technically the most important question for our goal. The next lemma gives us an answer.

Lemma 3.9. *Let G be an irreducible hyperelliptic graph of genus $n \geq 2$ and let D be a polarization as (3.4) with $\deg(D) \neq -2$. Let w be an ι-fixed vertex and w' be any vertex. Then $L_G(\lambda)g_{(G,D)}(w, w')(\lambda)$ is a homogeneous polynomial function on λ of degree $n + 1$.*

Proof. We show our assertion by induction on the genus n. If $n = 2$, then it is minimal and hence it follows from Example 3.7. Suppose we have our assertion for those of genus up to $n - 1$.

Let G be an irreducible hyperelliptic graph of genus $n(> 2)$. By virtue of Lemma 3.2 (2) and Lemma 3.8, we may assume that it is maximal. Let o_1 and o_2 be ι-fixed vertices of G. Taking account of (3.11) and (3.12), we see that if L_G is a denominator of $g_{(G,D)}(o_i, o_j)$ for some $i, j = 1, 2$, then so is it for any $i, j = 1, 2$. Therefore, it is enough to show that $L_G g_{(G,D)}(o, o)$ is a homogeneous polynomial of degree $n + 1$, for o such that there exists an extreme circuit $\{[e_0], [e_1]\}$ with $[o] \in [e_0]$.

Now we adopt the notation in Subsection 2.2: let $[e_0]$ and $[e_1]$ be the 1-jointed edges jointed at $[v]$, and let $[e_2]$ be the 0-jointed edge going away from $[v]$. Let G' be the seriesization of $\{[e_0], [e_1]\}$ and let λ' be the associated weight with λ. Recall that the genus of G' is $n - 1$.Let v'_0 be the ι-fixed vertex in G'_1

as before. We write $o' := v'_0$ to indicate that it is the "origin" of the new graph G'. We define a polarization D' on G' associated with D as (3.4) by

$$D = \sum_v (b_v - 2)v + \sum_{[e]\in \mathrm{Ed}_1(G'/\langle\iota\rangle)\setminus\{[e'_0]\}} 2d_{[e]}w_{[e]} + (2d_{[e_0]} + 2d_{[e_1]} + 2)w_{[e'_0]},$$

where we note $w_{[e'_0]} = [o'] = [v'_0]$. Note also that $\deg(D') = \deg(D)$.

To simplify the notation, we write $\alpha_{[e]}$, $\beta_{[e]}$ and $\gamma_{[e]}$ for $\alpha_{G,[e]}(\lambda)$, $\beta_{G,[e]}(\lambda)$ and $\gamma_{G,[e]}(\lambda)$ respectively, and similarly we write $\alpha'_{[e]}$, for $\alpha_{G',[e]}(\lambda')$, etc. Now our purpose is to show $L_G(\lambda)\gamma_{[e_0]}$ is a polynomial function on λ under the induction hypothesis that $L_{(G',\iota)}(\lambda')\gamma'_{[e'_0]}$ is a polynomial function on λ'.

Before proceeding the proof, recall that, as in Remark 2.18, there are identifications between sets of edges of $G/\langle\iota\rangle$ and sets of edges of $G'/\langle\iota\rangle$:

$$\mathcal{E} = \mathrm{Ed}(G'/\langle\iota\rangle) \setminus \{[e'_0]\} = \mathrm{Ed}(G/\langle\iota\rangle) \setminus \{[e_0], [e_1], [e_2]\}.$$

Further, we have naturally

$$\mathrm{Ed}_1(G'/\langle\iota\rangle) \setminus \{[e'_0]\} = \mathrm{Ed}_1(G/\langle\iota\rangle) \setminus \{[e_0], [e_1]\},$$

which is denoted by $\mathcal{E}_1$, and

$$\mathrm{Ed}_0(G'/\langle\iota\rangle) = \mathrm{Ed}_0(G'/\langle\iota\rangle) \setminus \{[e_2]\},$$

which is denoted by $\mathcal{E}_0$. Moreover, for $[e] \in \mathcal{E}$, we put

$$\mathcal{E}_0^{\prec[e]} := \{[e'] \in \mathcal{E}_0 \mid [e'] \prec [e]\}.$$

By the equations (3.9), we have

(3.13)
$$\gamma_{[e]} = \sum_{[e']\in\left(\mathcal{E}_0^{\prec[e]}\right)\cup\{[e_2]\}} \left(\alpha_{[e']}{\lambda_{[e']}}^2 + \beta_{[e']}\lambda_{[e']}\right) + \alpha_{[e_0]}{\lambda_{[e_0]}}^2 + \beta_{[e_0]}\lambda_{[e_0]} + \gamma_{[e_0]}.$$

Substituting (3.13) to (3.10) and taking account of (3.6),we find

$$
\begin{aligned}
&\alpha_{[e_0]}{\lambda_{[e_0]}}^2+\beta_{[e_0]}\lambda_{[e_0]}+\gamma_{[e_0]}\\
&\quad+2\sum_{[e]\in\mathcal{E}}\left(\frac{1}{3}\alpha_{[e]}{\lambda_{[e]}}^2+\frac{1}{2}\beta_{[e]}\lambda_{[e]}+S_{[e]}\right)(2\alpha_{[e]}\lambda_{[e]})\\
&\quad+2\left(\frac{1}{3}\alpha_{[e_2]}{\lambda_{[e_2]}}^2+\frac{1}{2}\beta_{[e_2]}\lambda_{[e_2]}\right)\\
&\quad+2\left(\frac{1}{3}\alpha_{[e_1]}{\lambda_{[e_1]}}^2+\frac{1}{2}\beta_{[e_1]}\lambda_{[e_1]}\right)\\
&\quad+2\left(-\frac{2}{3}\alpha_{[e_0]}{\lambda_{[e_0]}}^2-\frac{1}{2}\beta_{[e_0]}\lambda_{[e_0]}\right)\\
&\quad+\sum_{[e]\in\mathcal{E}_1}\frac{2d_{[e]}}{\deg(D)+2}\left(\alpha_{[e]}(\lambda){\lambda_{[e]}}^2+\beta_{[e]}(\lambda)\lambda_{[e]}+S_{[e]}\right)\\
&\quad+\frac{2d_{[e_1]}}{\deg(D)+2}\left(\alpha_{[e_1]}(\lambda){\lambda_{[e_1]}}^2+\beta_{[e_1]}(\lambda)\lambda_{[e_1]}\right)\\
&\quad-\frac{2d_{[e_0]}}{\deg(D)+2}(\alpha_{[e_0]}{\lambda_{[e_0]}}^2+\beta_{[e_0]}\lambda_{[e_0]})\\
&=0,
\end{aligned}
\tag{3.14}
$$

where we put

$$
S_{[e]}:=\sum_{[e']\in\left(\mathcal{E}_0^{\prec[e]}\right)\cup\{[e_2]\}}(\alpha_{[e']}{\lambda_{[e']}}^2+\beta_{[e']}\lambda_{[e']}).
$$

From (3.14), we see that ${L_G}^2$ is a denominator of $\gamma_{[e_0]}$. In the similar way, we find

$$
\begin{aligned}
&\alpha'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]}+\lambda_{[e_1]}}\right)^2+\beta'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]}+\lambda_{[e_1]}}\right)+\gamma'_{[e'_0]}\\
&\quad+2\sum_{[e]\in\mathcal{E}}\left(\frac{1}{3}\alpha'_{[e]}{\lambda_{[e]}}^2+\frac{1}{2}\beta'_{[e]}\lambda_{[e]}+S'_{[e]}\right)(2\alpha'_{[e]}\lambda_{[e]})\\
&\quad+2\left(\frac{1}{3}\alpha'_{[e'_0]}{\lambda'_{e'_0}}^2+\frac{1}{2}\beta'_{[e'_0]}\lambda'_{[e'_0]}\right.\\
&\qquad\left.-\left(\alpha'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]}+\lambda_{[e_1]}}\right)^2+\beta'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]}+\lambda_{[e_1]}}\right)\right)\right)\left(2\alpha'_{[e'_0]}\lambda'_{e'_0}\right)\\
&\quad+\sum_{[e]\in\mathcal{E}_1}\frac{2d_{[e]}}{\deg(D)+2}\left(\alpha_{[e]}(\lambda){\lambda_{[e]}}^2+\beta_{[e]}(\lambda)\lambda_{[e]}+S'_{[e]}\right)\\
&\quad-\frac{2d_{[e_0]}+2d_{[e_1]}+2}{\deg(D)+2}\left(\alpha'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]}+\lambda_{[e_1]}}\right)^2+\beta'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]}+\lambda_{[e_1]}}\right)\right)\\
&=0,
\end{aligned}
$$

where we put

$$S'_{[e]} := \sum_{[e'] \in \left(\mathcal{E}_0^{\prec [e]}\right)} (\alpha'_{[e']} {\lambda_{[e']}}^2 + \beta'_{[e']} \lambda_{[e']}) + \alpha'_{[e'_0]} {\lambda'_{[e'_0]}}^2 + \beta'_{[e'_0]} \lambda'_{[e'_0]}$$
$$- \left(\alpha'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right)^2 + \beta'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right) \right).$$

We would like to compare $\gamma_{[e_0]}$ with $\gamma'_{[e'_0]}$. First, by (3.2) and the definition of $\alpha_{[e]}$, we have $\alpha_{[e]} = \alpha'_{[e]}$ for $[e] \in \mathcal{E}$ and $\alpha_{[e_2]} = \alpha'_{[e'_0]}$. Then by (3.7) and (3.8), we have $\beta_{[e]} = \beta'_{[e]}$ for $[e] \in \mathcal{E}$. Therefore, by (3.8), we also have

$$\beta'_{[e'_0]} = \beta_{[e_2]} - 2\alpha'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right).$$

From these equalities, we can check easily

$$\alpha'_{[e'_0]} {\lambda'_{e'_0}}^2 + \beta'_{[e'_0]} \lambda'_{[e'_0]} - \left(\alpha'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right)^2 + \beta'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right) \right)$$
$$= \alpha_{[e_2]} {\lambda_{[e_2]}}^2 + \beta_{[e_2]} \lambda_{[e_2]},$$

which tells us $S_{[e]} = S_{[e']}$. Further, we can see

$$\begin{aligned}
&\left(\frac{1}{3} \alpha'_{[e'_0]} {\lambda'_{e'_0}}^2 + \frac{1}{2} \beta'_{[e'_0]} \lambda'_{[e'_0]} \right. \\
&\quad \left. - \left(\alpha'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right)^2 + \beta'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right) \right) \right) \left(2\alpha'_{[e'_0]} \lambda'_{e'_0} \right) \\
&= - \left(\frac{2}{3} \alpha'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right)^2 + \frac{1}{2} \beta'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right) \right) \left(2\alpha'_{[e'_0]} \left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}} \right) \right) \\
&\quad + \left(\frac{1}{3} \alpha_{[e_2]} {\lambda_{[e_2]}}^2 + \frac{1}{2} \beta_{[e_2]} \lambda_{[e_2]} \right) \left(2\alpha_{[e_2]} \lambda_{[e_2]} \right).
\end{aligned}$$

Accordingly, we find

(3.15)

$$\gamma_{[e_0]} = \gamma'_{[e'_0]} + \left(\frac{2d_{[e_0]}}{\deg(D)+2} - 1\right)\left(\alpha_{[e_0]}{\lambda_{[e_0]}}^2 + \beta_{[e_0]}\lambda_{[e_0]}\right)$$
$$- \left(\frac{2d_{[e_0]} + 2d_{[e_1]} + 2}{\deg(D')+2} - 1\right)\left(\alpha'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}}\right)^2 + \beta'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}}\right)\right)$$
$$+ 2\left(\frac{2}{3}\alpha_{[e_0]}{\lambda_{[e_0]}}^2 + \frac{1}{2}\beta_{[e_0]}\lambda_{[e_0]}\right)(2\alpha_{[e_0]}\lambda_{[e_0]})$$
$$- 2\left(\frac{1}{3}\alpha_{[e_1]}{\lambda_{[e_1]}}^2 + \frac{1}{2}\beta_{[e_1]}\lambda_{[e_1]}\right)(2\alpha_{[e_1]}\lambda_{[e_1]})$$
$$- 2\left(\frac{2}{3}\alpha'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}}\right)^2 + \frac{1}{2}\beta'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}}\right)\right)\left(2\alpha'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}}\right)\right).$$

Recall that our goal is to show

$$\gamma_{[e_0]} \in \frac{1}{L_G(\lambda)}\mathbb{R}[\lambda]_{(n+1)}.$$

As in Remark 3.6, we have

$$\gamma_{[e_0]} \in \frac{1}{L_G(\lambda)^2}\mathbb{R}[\lambda]_{(2n+1)}.$$

Since $\lambda_{[e_0]} + \lambda_{[e_1]}$ and $L_G(\lambda)$ are co-prime to each other, it is enough to show

$$(\lambda_{[e_0]} + \lambda_{[e_1]})^a L_G(\lambda)\gamma_{[e_0]} \in \mathbb{R}[\lambda]$$

for some $a \in \mathbb{N}$.

By the induction hypothesis, we have

$$L_{G'}(\lambda')\gamma'_{[e_0]} \in \mathbb{R}[\lambda'].$$

Therefore, taking account of Lemma 3.3, (3.5) and (3.7), we can see that the first two lines in the right-hand side of (3.15) lie in

$$\frac{1}{(\lambda_{[e_0]} + \lambda_{[e_1]})^a L_G(\lambda)}\mathbb{R}[\lambda]$$

for some $a \in \mathbb{N}$. Further, eliminating $\beta_{[e_1]}$ by

$$\beta_{[e_1]} = -\frac{d_{[e_1]}}{\deg(D)+2} - 2\alpha_{[e_1]}\lambda_{[e_1]},$$

we are reduced to show

$${\alpha_{[e_0]}}^2{\lambda_{[e_0]}}^3 + {\alpha_{[e_1]}}^2{\lambda_{[e_1]}}^3 - {\alpha'_{[e'_0]}}^2\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]} + \lambda_{[e_1]}}\right)^3 \in \frac{1}{(\lambda_{[e_0]} + \lambda_{[e_1]})^a L_G(\lambda)}\mathbb{R}[\lambda]$$

for some $a \in \mathbb{N}$.

For simplicity, let us put $\lambda_0 := \lambda_{[e_0]}$, $\lambda_1 := \lambda_{[e_1]}$, $P_0 := P_G^{[e_0]}$, $P_1 := P_G^{[e_1]}$, $P_2 := P_G^{[e_2]}$ and $L := L_G$. Comparing (3.6) for G and that for G', we have

$$(3.16) \qquad 2\alpha'_{[e'_0]}\left(\frac{\lambda_{[e_0]}\lambda_{[e_1]}}{\lambda_{[e_0]}+\lambda_{[e_1]}}\right) = -\frac{1}{\deg(D)+2} + 2\alpha_{[e_0]}\lambda_{[e_0]} + 2\alpha_{[e_1]}\lambda_{[e_0]}.$$

Recalling

$$\alpha_{[e_0]} = \frac{P_0}{2(\deg(D)+2)L}\alpha_{[e_1]} = \frac{P_1}{2(\deg(D)+2)L}\alpha_{[e'_0]} = \frac{P_2}{2(\deg(D)+2)L},$$

we have by (3.16)

$$P_2\lambda_0\lambda_1 \equiv (\lambda_0+\lambda_1)(P_0\lambda_0+P_1\lambda_1) \qquad \bmod L$$

in $\mathbb{R}[\lambda]$. Thus, we are reduced to show

$$(\lambda_0+\lambda_1){P_0}^2{\lambda_0}^3 + (\lambda_0+\lambda_1){P_1}^2{\lambda_1}^3 - (P_0\lambda_0+P_1\lambda_1)^2\lambda_0\lambda_1 \equiv 0 \qquad \bmod L$$

in $\mathbb{R}[\lambda]$.

Let B be the polynomial such that $\lambda_0\lambda_1 B$ is the sum of all the monomials of L which are divisible by $\lambda_0\lambda_1$. Then, since L is symmetric on λ_0 and λ_1 and any monomial in L has the factor λ_0 or λ_1, we can see that there is another polynomial C such that $L = \lambda_0\lambda_1 B + (\lambda_0+\lambda_1)C$. Note that B and C are the polynomials which does not have the indeterminate λ_0 nor λ_1. By the definitions, we have $P_0 = \lambda_1 B + C$, $P_1 = \lambda_0 B + C$ and hence

$$P_0\lambda_0 + P_1\lambda_1 \equiv \lambda_0\lambda_1 B \qquad \bmod L.$$

Then, we see

$$\begin{aligned}
&(\lambda_0+\lambda_1){P_0}^2{\lambda_0}^3 + (\lambda_0+\lambda_1){P_1}^2{\lambda_1}^3 - (P_0\lambda_0+P_1\lambda_1)^2\lambda_0\lambda_1 \\
&\quad = (\lambda_0+\lambda_1)P_0{\lambda_0}^2(P_0\lambda_0+P_1\lambda_1) \\
&\qquad + (\lambda_0+\lambda_1)P_1\lambda_1(P_1{\lambda_1}^2 - P_0{\lambda_0}^2) - (P_0\lambda_0+P_1\lambda_1)^2\lambda_0\lambda_1 \\
&\quad \equiv (\lambda_0+\lambda_1)P_0{\lambda_0}^2(\lambda_0\lambda_1 B) \\
&\qquad + (\lambda_0+\lambda_1)P_1\lambda_1({\lambda_1}^2\lambda_0 B + {\lambda_1}^2 C - {\lambda_0}^2\lambda_1 B - {\lambda_0}^2 C) - (\lambda_0\lambda_1 B)^2\lambda_0\lambda_1 \\
&\hspace{30em} \bmod L \\
&\quad = (\lambda_0+\lambda_1)P_0{\lambda_0}^2(\lambda_0\lambda_1 B) + ({\lambda_1}^2 - {\lambda_0}^2)P_1\lambda_1(\lambda_0\lambda_1 B) \\
&\qquad + (\lambda_0+\lambda_1)({\lambda_1}^2 - {\lambda_0}^2)P_1\lambda_1 C - (\lambda_0\lambda_1 B)^2\lambda_0\lambda_1.
\end{aligned}$$

Here we have

$$\begin{aligned}
&(\lambda_0+\lambda_1)P_0{\lambda_0}^2 + ({\lambda_1}^2 - {\lambda_0}^2)P_1\lambda_1 - (\lambda_0\lambda_1 B)\lambda_0\lambda_1 \\
&\quad = (\lambda_0+\lambda_1)\big(\lambda_0(\lambda_0\lambda_1 B) + {\lambda_0}^2 C\big) - (\lambda_0\lambda_1 B)\lambda_0\lambda_1 + ({\lambda_1}^2 - {\lambda_0}^2)P_1\lambda_1 \\
&\quad = {\lambda_0}^2\big(\lambda_0\lambda_1 B + (\lambda_0+\lambda_1)C\big) + ({\lambda_1}^2 - {\lambda_0}^2)P_1\lambda_1 \\
&\quad \equiv ({\lambda_1}^2 - {\lambda_0}^2)P_1\lambda_1 \qquad \bmod L.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}&(\lambda_0+\lambda_1)P_0{}^2\lambda_0{}^3+(\lambda_0+\lambda_1)P_1{}^2\lambda_1{}^3-(P_0\lambda_0+P_1\lambda_1)^2\lambda_0\lambda_1\\&\quad\equiv(\lambda_1{}^2-\lambda_0{}^2)P_1\lambda_1(\lambda_0\lambda_1B)+(\lambda_0+\lambda_1)(\lambda_1{}^2-\lambda_0{}^2)P_1\lambda_1C\qquad\bmod L\\&\quad=(\lambda_1{}^2-\lambda_0{}^2)P_1\lambda_1\big(\lambda_0\lambda_1B+(\lambda_0+\lambda_1)C\big)\\&\quad\equiv 0\qquad\bmod L,\end{aligned}$$

thus, we complete the proof of our assertion. ☐

Next let us consider the resistances.

Lemma 3.10. *Let G be an irreducible hyperelliptic graph of genus n and let o be an ι-fixed vertex of G. Then, for any vertex v, the function*

$$L_G(\lambda)r_G(o,v)(\lambda)$$

on $\lambda\in\mathcal{W}_{>0}(G/\langle\iota\rangle)$ is a homogeneous polynomial function of degree $n+1$.

Proof. We proceed in four steps.

Step 1. *The case of $\iota(v)=v$.* In this case, it is elementary to see

$$r_G(o,v)(\lambda)=\frac{1}{2}\sum_{[e]}\lambda_{[e]},$$

where $[e]$ ranges over all edges of $G/\langle\iota\rangle$ between $[o]$ and $[v]$.

Step 2. *The case where o and v are the extremities of an edge.* We know

$$\frac{2}{\lambda_{[e]}+r_{G,[e]}(\lambda)}=\frac{P_G^{[e]}(\lambda)}{L_G(\lambda)}.$$

Since

$$\frac{1}{r_G(o,v)}=\frac{1}{\lambda_{[e]}}+\frac{1}{r_{G,[e]}(\lambda)}$$

by Ohm's Law, we obtain our assertion.

Step 3. *The case where v is the extremity of a* 1*-jonted edge e with $\iota(v)\neq v$.* Let o' be the other extremity of e. We know

$$\begin{aligned}r_G(o,v)&=g_{(G,D)}(o,o)-2g_{(G,D)}(o,v)+g_{(G,D)}(v,v)\\r_G(o',v)&=g_{(G,D)}(o',o')-2g_{(G,D)}(o',v)+g_{(G,D)}(v,v).\end{aligned}$$

Therefore by Lemma 3.9, $r_G(o,v)-r_G(o',v)$ has a denominator L_G. Since $r_G(o',v)$ has a denominator L_G by Step 2, and so does $r_G(o,v)$.

Step 4. *General case.* We will prove our assertion by induction on n. If $n \leq 2$, we are in a situation above.

Now let us assume $n > 2$. We may assume that $\iota(v) \neq v$ and that no 1-jointed edge has v as an extremity by virtue of Step 3. Then, the subtree H of $G/\langle\iota\rangle$ generated by $\mathrm{Ed}_0(G/\langle\iota\rangle)$ is not a chain, that is to say, it has at least 3 ends. Therefore, there exist two distinct extreme circuits $\{[e_{00}], [e_{01}]\}$ and $\{[e_{10}], [e_{11}]\}$ away from $[o]$. Then considering the seriesization of each of them, we can see by the induction hypothesis that both $(\lambda_{[e_{00}]} + \lambda_{[e_{01}]})^a L_G(\lambda)$ and $(\lambda_{[e_{10}]} + \lambda_{[e_{11}]})^a L_G(\lambda)$ are denominators of $r_G(o, v)(\lambda)$ for some $a \in \mathbb{N}$. Since $\lambda_{[e_{00}]} + \lambda_{[e_{01}]}$ and $\lambda_{[e_{10}]} + \lambda_{[e_{11}]}$ are co-prime to each other as polynomials, we find that $L_G(\lambda)$ is a denominator. Thus, we obtain our assertion. $\square$

Now we can obtain the following proposition.

Proposition 3.11. *Let G be a hyperelliptic graph of genus n and let D be a polarization as in* (3.4) *with* $\deg(D) \neq -2$. *Then,*

$$L_G(\lambda)\epsilon(G, D)(\lambda)$$

is a homogeneous polynomial function of degree $n + 1$ on the $\lambda \in \mathcal{W}(G)$.

Proof. By virtue of Lemma 3.2 and Proposition 1.10, we may assume G to be irreducible. Then our result follows from Lemma 3.9, Lemma 3.10 and [4, Lemma 4.1], except for $n = 1$. If $n = 1$, we can prove Theorem 3.14 directly and hence we omit it. $\square$

3.4. An explicit formula for the admissible constants

In this subsection, we will prove an explicit formula for the admissible constant of a hyperelliptic graph.

Let G be a hyperelliptic graph. We define a polynomial function M_G on $\lambda \in W(G/\langle\iota\rangle)$ as follows. For a subset T of $\mathrm{Ed}(G/\langle\iota\rangle)$ with $\#T = n + 1$ we define $c_G^T = c^T$ in the following way: If $\#\,\mathrm{MV}\left(G^{\tilde{T}}/\langle\iota\rangle\right) = 1$, then we put

$$c_G^T = c^T := \#\left(\mathrm{Ed}_1\left(G^T/\langle\iota\rangle\right)\right) - 2,$$

and otherwise we put $c_G^T = c^T := 0$. Note that $c^T = k - 1$ if and only if G^T is a sum of $n - k$ copies of minimal hyperelliptic graph of genus 1 and a minimal hyperelliptic graph of genus k. Then we define M_G by

$$M_G(\lambda) := \sum_T c^T \prod_{[e] \in T} \lambda_{[e]},$$

where T ranges over all subsets of $\mathrm{Ed}(G/\langle\iota\rangle)$ with $\#T = n + 1$. Note that $M_G = 0$ as a polynomial if G have only 2-jointed edges.

Example 3.12. Let G be the minimal hyperelliptic graph of genus n.
(1) If $n = 1$, then $M_G(\lambda) = 0$.

(2) Suppose $n > 1$. Then $\mathrm{Ed}(G/\langle\iota\rangle)$ consists of $(n+1)$ 1-jointied edges, namely, $[e_0], [e_1], \ldots, [e_n]$. We write $\lambda = (\lambda_0, \ldots, \lambda_n)$, where $\lambda(e_i) = \lambda_i$. Then we have $M_G(\lambda) = (n-1)\prod_{i=0}^{n} \lambda_i$.

The polynomial M_G is compatible with the operations of the one-point sum and the contraction:

Lemma 3.13. *Let G be a hyperelliptic graph.*
(1) *If $G = G_1 \vee G_2$, then $M_G = L_{G_1} M_{G_2} + L_{G_2} M_{G_1}$.*
(2) *Let G' be the contraction of $\{e, \iota(e)\}$ and let λ' be the associated weight with λ. Then $M_{G'}(\lambda') = \lim_{\lambda_{[e]} \to 0} M_G(\lambda)$.*

Proof. Let n denote the genus of G. Let T be a subset of $\mathrm{Ed}(G/\langle\iota\rangle)$ with $\#(T) = n+1$, and put $T_1 = T \cap \mathrm{Ed}(G_1/\langle\iota\rangle)$ and $T_2 = T \cap \mathrm{Ed}(G_2/\langle\iota\rangle)$. Then, by the definitions, we can see that $c_G^T = k-1$ if and only if "$\delta_{G_1}^{T_1} = 1$ and $c_{G_2}^{T_2} = k-1$" or "$\delta_{G_2}^{T_2} = 1$ and $c_{G_1}^{T_1} = k-1$". Accordingly we can find $M_{G_1} L_{G_2} + L_{G_1} M_{G_2} = M_G$. The equality in (2) is obvious from the definition. $\square$

Let D be a polarization on G as (3.4). For any $[e] \in \mathrm{Ed}(G/\langle\iota\rangle)$, the contraction $G^{\{[e]\}}$ is the minimal hyperelliptic graph of genus 1, and associated polarization on $G^{\{[e]\}}$ with D is of form $av + bw$, where v and w are the vertices and $a, b \in \mathbb{R}$. We put $\mathrm{tp}([e]) := \min\{a, b\}$. Now, we can explicitly express $\epsilon(G, D)$ by means of L_G and M_G:

Theorem 3.14. *Let G be a hyperelliptic graph, and let D be a polarization as* (3.4) *with $\deg(D) \neq -2$. Then, for any hyperelliptic weight $\lambda \in \mathcal{W}_{>0}(G/\langle\iota\rangle)$, we have*

$$\begin{aligned}
&\epsilon(G, D)(\lambda) \\
&= \sum_{[e] \in \mathrm{Ed}(G/\langle\iota\rangle)} \frac{2\deg(D) + 3\,\mathrm{tp}([e])(\deg(D) - \mathrm{tp}([e]))}{3(\deg(D)+2)} \lambda_{[e]} + \frac{2\deg(D)}{3(\deg(D)+2)} \frac{M_G(\lambda)}{L_G(\lambda)}.
\end{aligned}$$

Before going on to the proof, we check the theorem for minimal hyperelliptic graphs:

Lemma 3.15. *If G is minimal of genus n, then Theorem* 3.14 *holds.*

Proof. If $n = 1$, it is easy to see from [4, Proposition 4.2, Corollary 4.3].

Let G be the minimal hyperelliptic graph of genus $n > 1$. We adopt the notation in Example 3.7. In addition, let v be a vertex with $\iota(v) \neq v$. We can see that

$$\begin{aligned}
&r_G(w_0, v)(\lambda) = r_G(w_0, \iota(v))(\lambda) = \lambda_0 - \frac{\sigma_{n-1}^{(0)} {\lambda_0}^2}{2\sigma_n} = \frac{1}{2}\left(\lambda_0 + \frac{\sigma_{n+1}}{\sigma_n}\right), \\
&r_G(w_0, w_i)(\lambda) = \frac{1}{2}(w_0 + w_i)
\end{aligned}$$

by Ohm's law. Therefore taking account of Example 3.7, we obtain, for $i \neq 0$,

$$\begin{aligned}
&(\deg(D)+2)g_{(G,\iota,D)}(w_0,v)(\lambda)+r_G(w_0,v)(\lambda)\\
&\quad = \left(d_0-\frac{\deg(D)}{2}\right)\lambda_0+(\deg(D)+2)\gamma_0,\\
&(\deg(D)+2)g_{(G,D)}(w_0,w_i)(\lambda)+r_G(w_0,w_i)(\lambda)\\
&\quad = -d_i\lambda_i+\left(d_0-\frac{\deg(D)}{2}\right)\lambda_0+(\deg(D)+2)\gamma_0.
\end{aligned}$$

Thus by [4, Lemma 4.1],

$$\begin{aligned}
\epsilon(G,D)(\lambda) = {} & -\sum_{i=1}^{n}2{d_i}^2\lambda_i+(\deg(D)-2d_0)\left(d_0-\frac{\deg(D)}{2}\right)\lambda_0\\
& +(\deg(D))(\deg(D)+2)\gamma_0.
\end{aligned}$$

Now, our formula can be straightforwardly verified from Example 3.7. □

Let us start the proof of Theorem 3.14. Put

$$\begin{aligned}
&F_{(G,D)}(\lambda)\\
&:= \sum_{[e]\in \mathrm{Ed}(G/\langle\iota\rangle)}\frac{2\deg(D)+3\,\mathrm{tp}(e)(\deg(D)-\mathrm{tp}(e))}{3(\deg(D)+2)}\lambda_{[e]}+\frac{2\deg(D)}{3(\deg(D)+2)}\frac{M_G(\lambda)}{L_G(\lambda)}
\end{aligned}$$

and put

$$P_{(G,D)} := L_G\epsilon(G,D)-L_GF_{(G,D)}.$$

Let us show $P_{(G,D)}=0$ by induction on $l := \#\big(\mathrm{Ed}(G/\langle\iota\rangle)\big)$. If $l \leq 2$, then G is minimal, and it is nothing more than Lemma 3.15. Next, we assume that we have our assertion up to l ($l \geq 2$) and will prove it for $l+1$. First, we consider the case where G is reducible. Then, we can write $G = G_1 \vee G_2$ and G_i is a hyperelliptic graph for $i = 1, 2$. Put $D_i = D^{\mathrm{Ed}(G_i)}$, which is a polarization on G_i. Then by virtue of Proposition 1.10 and Lemma 3.2, we have

$$P_{(G,D)} = L_{G_2}P_{(G_1,D_1)}+L_{G_1}P_{(G_2,D_2)}.$$

Since $\#\big(\mathrm{Ed}(G_i/\langle\iota\rangle)\big) \leq l$ for $i = 1, 2$, we have $P_{(G,D)} = 0$ by the induction hypothesis.

Next we consider the case where G is irreducible. We may assume that G is not minimal by Lemma 3.15. Then, the genus n of G is less than l. Here we note the following claim.

Claim 1. Let P be a homogeneous polynomial on $Y_1, \ldots, Y_m$ of degree d with $d < m$. Suppose that $P(a_1, \ldots, a_m) = 0$ if $a_i = 0$ for some i. Then, we have $P = 0$ as a polynomial.

Proof. For a $(k_1, \ldots, k_d)$ with $1 \leq k_1 \leq k_2 \leq \cdots \leq k_d \leq m$, let $m(k_1, \ldots, k_d)$ be the coefficient of the monomial $Y_{k_1} \cdots Y_{k_d}$ in P, i.e., $P = \sum_{k_1 \leq k_2 \leq \cdots \leq k_d} m(k_1, \ldots, k_d) Y_{k_1} \cdots Y_{k_d}$. For any $(k_1, \ldots, k_d)$, let us take an integer $k \in \{1, \ldots, m\} \setminus \{k_1, \ldots, k_d\}$, and put

$$P_k(Y_1, \ldots, Y_m) := \sum_{k_1 \leq \cdots \leq k_d, k_i \neq k} m(k_1, \ldots, k_d) Y_{k_1} \cdots Y_{k_d}.$$

Then, we have

$$P_k(Y_1, \ldots, Y_m) = P(Y_1, \ldots, Y_{k-1}, 0, Y_{k+1}, \ldots, Y_m) = 0,$$

and hence $m(k_1, \ldots, k_d) = 0$. Thus we have our assertion. □

Take any $\lambda \in \mathcal{W}_{>0}(G/\langle \iota \rangle)$. By virtue of Proposition 3.2 and Lemma 3.8, we have

$$\lim_{\lambda([e]) \to 0} P_{(G,D)}(\lambda) = P_{\left(G_{\{[e]\}}, D_{\{\iota(e)\}}\right)} \left(\lambda_{G_{\{[e]\}}}\right)$$

for any $[e] \in (\mathrm{Ed}/\langle \iota \rangle)$. Since $\#\big(\mathrm{Ed}\big(G_{\{e,\iota(e)\}}/\langle \iota \rangle\big)\big) < l$, the right-hand side is equal to 0 by the induction hypothesis. Since $\deg\big(P_{(G,D)}\big) = n + 1$ is less than $(l+1)$, where n is the genus of G, we can see $P_{(G,D)} = 0$ by the above claim. Thus, we complete the proof of Theorem 3.14.

4. Effective version of the geometric Bogomolov conjecture for hyperelliptic curves

4.1. Main results

Let $f : X \to Y$ be a semistable curve of genus $g \geq 2$ as in the introduction. We say that f is *hyperelliptic* if there is an action of a finite group $\langle \iota \rangle$ of order 2 that induces the hyperelliptic involution on the generic fiber.

In order to describe our results, let us recall the notion of types of nodes. Let P be a node of a semistable curve Z of genus g over an algebraically closed field. We can assign a number i to the node P, called the *type* of P, in the following way. Let $\nu : Z_P \to Z$ be the partial normalization at P. If Z_P is connected, then $i = 0$. Otherwise, i is the minimum of arithmetic genera of the two connected components of Z_P. We denote by $\delta_i(Z)$ the number of nodes of type i, and by $\delta_i(X/Y)$ the number of nodes of type i in all the fibers of $f : X \to Y$, i.e., $\delta_i(X/Y) = \sum_{y \in Y} \delta_i(X_y)$.

In case of hyperelliptic curves, we can further define the notion of subtype for nodes of type 0. Let X_y be a fiber of f. It should be called a semistable hyperelliptic curve over k with the involution $\iota_y = \iota|_{X_y}$. Let P be a node of X_y of type 0. We can also assign a number j to the pair of nodes $(P, \iota(P))$ of type 0, which we call the *subtype* of $(P, \iota(P))$, in the following way (see [1, §4 (b)] and [11] for details). If $P = \iota(P)$, we set $j = 0$. If $P \neq \iota(P)$, then the partial normalization $(X_y)_{P,\iota(P)}$ at P and $\iota(P)$ has two connected components since $X_y/\langle \iota_y \rangle$ is a tree of smooth rational curves. We set j to be the minimum of the arithmetic genera of the two connected components of $(X_y)_{P,\iota(P)}$. We denote by $\xi_0(X_y)$ the number of *nodes* of type 0 and of subtype 0, and by $\xi_j(X_y)$ the number of *such pairs of nodes* of type 0 and of subtype j for $j \geq 1$. Note that $0 \leq j \leq \left[\frac{g-1}{2}\right]$ and

$$\delta_0(X_y) = \xi_0(X_y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2\xi_j(X_y).$$

Further, we put

$$\xi_j(X/Y) = \sum_{y \in Y} \xi_j(X_y).$$

The following theorem is our main result.

Theorem 4.1. *Let $f : X \to Y$ be a generically smooth semistable hyperelliptic curve of genus $g \geq 3$ in which Y is a nonsingular projective curve over an algebraically closed field k and X is also nonsingular. Assume that f is not smooth. Let r_1 be the number given below:*

(a) *If $g = 3, 4$, then*

$$r_1 = \frac{g-1}{g(2g+1)} \cdot \left(\frac{2g-5}{12}\xi_0(X/Y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} \big(2j(g-1-j)-1\big)\xi_j(X/Y) + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i(X/Y) \right).$$

(b) *If $g \geq 5$, then*

$$r_1 = \frac{g-1}{g(2g+1)} \cdot \left(\frac{2g-5}{12}\xi_0(X/Y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} \frac{6j(g-1-j)-2g-4}{3}\xi_j(X/Y) + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i(X/Y) \right).$$

Then we have

$$(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a \geq r_1 > 0,$$

where $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a$ is the admissible self-pairing of the admissible dualizing sheaf.

Recall that, as we mentioned in the introduction, we have

$$\inf_{P \in J_C(\overline{K})} r_C(P) \geq \sqrt{(g-1)(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a}$$

if $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a$ is positive. Thus we obtain an answer to the effective version of the geometric Bogomolov conjecture for hyperelliptic curves as a corollary:

Corollary 4.2. *In the same situation as above, let r_1 be the positive number in Theorem* 4.1. *Then we have*

$$\inf_{P \in J_C(\overline{K})} r_C(P) \geq \sqrt{(g-1)r_1}.$$

4.2. Estimate of the admissible constant from the above

The goal of this subsection is Proposition 4.5. To prove it, we have to prepare some more notations.

Let G be an irreducible hyperelliptic graph of genus > 1. Let $\mathcal{E}_0$ denote the power set of $\mathrm{Ed}_0(G/\langle\iota\rangle)$. (There is nothing to do with "$\mathcal{E}_0$" in the previous section.) Note that for any $S \in \mathcal{E}_0$, we have naturally $\mathrm{Ed}_1(G/\langle\iota\rangle) = \mathrm{Ed}_1\left(G_{((\mathrm{Ed}_0(G/\langle\iota\rangle))\setminus S)}/\langle\iota\rangle\right)$ and $S = \mathrm{Ed}_0\left(G_{((\mathrm{Ed}_0(G/\langle\iota\rangle))\setminus S)}/\langle\iota\rangle\right)$. For an $S \in \mathcal{E}_0$, put

$$\mathrm{MV}\{S\} := \mathrm{MV}\left(G_{((\mathrm{Ed}_0(G/\langle\iota\rangle))\setminus S)}/\langle\iota\rangle\right),$$

and for any $[v] \in \mathrm{MV}\{S\}$, let $E_1^S([v])$ be the set of those 1-jointed edges of $G/\langle\iota\rangle$ which, as edges of $G_{((\mathrm{Ed}_0(G/\langle\iota\rangle))\setminus S)}/\langle\iota\rangle$, have $[v]$ as an extremity. We set

$$\nu_1^S([v]) := \#E_1^S([v]).$$

Let $\sigma^S_{[v]}$ and $\tau^S_{[v]}$ be the $\left(\nu_1^S([v])-1\right)$-th and the $\nu_1^S([v])$-th elementary symmetric polynomials on the valuables $\{\lambda_{[e]}\}_{[e]\in E_1^S([v])}$ respectively. Further we put

$$\nu_0^S([v]) := \#\left\{ [e] \in \mathrm{Ed}_0\left(G_{((\mathrm{Ed}_0(G/\langle\iota\rangle))\setminus S)}/\langle\iota\rangle\right) \;\middle|\; [v] \in \partial[e]\right\}$$
$$\nu^S([v]) := \nu_0^S([v]) + \nu_1^S([v]).$$

By definition, $\nu^S([v])$ is nothing but the valence at v in $G_{((\mathrm{Ed}_0(G/\langle\iota\rangle))\setminus S)}$.

First, we note the following lemma.

Lemma 4.3. *In the above situation, we have the following equalities*:

$$L_G(\lambda) = \sum_{S\in\mathcal{E}_0}\left(\prod_{[v]\in\mathrm{MV}\{S\}} \sigma^S_{[v]}\right)\prod_{[e]\in S}\lambda_{[e]},$$
$$M_G(\lambda) = \sum_{S\in\mathcal{E}_0}\left(\sum_{[v]\in\mathrm{MV}\{S\}}\left((\nu^S([v])-2)\tau^S_{[v]}\prod_{[v']\in\mathrm{MV}\{S\}\setminus\{[v]\}}\sigma^S_{[v']}\right)\right)\prod_{[e]\in S}\lambda_{[e]}.$$

Proof. For a fixed $S \in \mathcal{E}_0$, let S' be a subset of $\mathrm{Ed}(G/\langle\iota\rangle)$ such that $\#S' = n$ and $S' \cap \mathrm{Ed}_0(G/\langle\iota\rangle) = S$. Note that

$$G^{S'} = \left(G_{\mathrm{Ed}_0(G/\langle\iota\rangle)\setminus S}\right)_{\left(\mathrm{Ed}_1\left(G_{\mathrm{Ed}_1(G/\langle\iota\rangle)\setminus S}\right)\setminus S'\right)}.$$

Then, it is not difficult to see that $\delta^{S'} = 1$ if and only if we have, for each $[v] \in \mathrm{MV}\{S\}$,

$$\#\left(E_1^S([v]) \setminus S'\right) = 1,$$

or equivalently,

$$\#\left(E_1^S([v]) \cap S'\right) = \nu_1^S([v]) - 1.$$

Our first equality follows from this observation.

Let us fix $S \in \mathcal{E}_0$. Let T be a subset of $\mathrm{Ed}(G/\langle\iota\rangle)$ such that $\#T = n+1$ and $T \cap \mathrm{Ed}_0(G/\langle\iota\rangle) = S$. Then we have also

$$G^T = \left(G_{\mathrm{Ed}_0(G/\langle\iota\rangle)\setminus S}\right)_{\left(\mathrm{Ed}_1\left(G_{\mathrm{Ed}_1(G/\langle\iota\rangle)\setminus S}\right)\setminus T\right)}.$$

Let $[v] \in \mathrm{MV}\{S\}$ and let l be a positive integer. Then it is not difficult to see that $c^T = l$ and $[v] \in \mathrm{MV}\left(G^T/\langle\iota\rangle\right)$, if and only if $E_1^S([v]) \subset T$, $l+2 = \nu^S([v])$ and

$$\#\left(E_1^S([v'])\right) \cap T = \nu_1^S([v']) - 1$$

for any $[v'] \in \mathrm{MV}\{S\} \setminus \{[v]\}$. From this observation, we can obtain our second equality. □

Using the expression in the above lemma, we can prove the following inequalities.

Lemma 4.4. *Let G be an irreducible hyperelliptic graph of genus n. For any $\lambda \in \mathcal{W}_{>0}(G/\langle\iota\rangle)$, we have*

$$\frac{M_G(\lambda)}{L_G(\lambda)} \leq \sum_{[e]\in\mathrm{Ed}_0(G/\langle\iota\rangle)} \lambda_{[e]} + \frac{1}{4}\sum_{[e]\in\mathrm{Ed}_1(G/\langle\iota\rangle)} \lambda_{[e]}.$$

Moreover, if $n \leq 4$, then we have

$$\frac{M_G(\lambda)}{L_G(\lambda)} \leq \frac{1}{2}\sum_{[e]\in\mathrm{Ed}_0(G/\langle\iota\rangle)} \lambda_{[e]} + \frac{1}{4}\sum_{[e]\in\mathrm{Ed}_1(G/\langle\iota\rangle)} \lambda_{[e]}.$$

Proof. By the above lemma, we have

$$\begin{aligned} &L_G(\lambda)\left(\sum_{[e]\in\mathrm{Ed}_0(G/\langle\iota\rangle)} \lambda_{[e]} + \frac{1}{4}\sum_{[e]\in\mathrm{Ed}_1(G/\langle\iota\rangle)} \lambda_{[e]}\right) \\ &\quad= \left(\sum_{S\in\mathcal{E}_0}\prod_{[v]\in\mathrm{MV}\{S\}}\sigma_{[v]}^S\prod_{[e]\in S}\lambda_{[e]}\right)\left(\sum_{[e]\in\mathrm{Ed}_0(G/\langle\iota\rangle)}\lambda_{[e]}\right) \\ &\qquad+ \left(\sum_{S\in\mathcal{E}_0}\prod_{[v]\in\mathrm{MV}\{S\}}\sigma_{[v]}^S\prod_{[e]\in S}\lambda_{[e]}\right)\left(\frac{1}{4}\sum_{[e]\in\mathrm{Ed}_1(G/\langle\iota\rangle)}\lambda_{[e]}\right). \end{aligned} \tag{4.1}$$

Taking account that $\mathrm{Ed}_1(G/\langle\iota\rangle) = \coprod_{[v]\in\mathrm{MV}\{S\}} E_1^S([v])$ and using an elementary inequality

$$(a_1 + \cdots + a_k)(a_1a_2\cdots a_{k-1} + a_2a_3\cdots a_k + \cdots + a_ka_1\cdots a_{k-2}) \geq k^2 a_1a_2\cdots a_k,$$

which holds for non-negative numbers, we have

$$\begin{aligned}
&\left(\prod_{[v]\in\mathrm{MV}\{S\}} \sigma_{[v]}^S\right)\left(\frac{1}{4}\sum_{[e]\in\mathrm{Ed}_1(G/\langle\iota\rangle)} \lambda_{[e]}\right)\\
&\quad = \frac{1}{4}\sum_{[v]\in\mathrm{MV}\{S\}}\left(\sigma_{[v]}^S\left(\sum_{[e]\in E_1^S([v])}\lambda_{[e]}\right)\prod_{[v']\in\mathrm{MV}\{S\}\setminus\{[v]\}}\sigma_{[v']}^S\right)\\
&\quad \geq \sum_{[v]\in\mathrm{MV}\{S\}}\left(\frac{\left(\nu_1^S([v])\right)^2}{4}\tau_{[v]}^S\prod_{[v']\in\mathrm{MV}\{S\}\setminus\{[v]\}}\sigma_{[v']}^S\right).
\end{aligned}$$

Let us go on to the estimate of (4.1). Suppose $S \neq \emptyset$. For any $[e] \in S$, let $\partial_+[e]$ and $\partial_-[e]$ denote the extremities of $[e]$. We put $c([e]) := \mathrm{contr}_{\mathrm{Ed}_0(G/\langle\iota\rangle)\setminus(S\setminus\{[e]\})}([e])$ for simplicity. Note that it is a vertex of $G_{\mathrm{Ed}_0(G/\langle\iota\rangle)\setminus(S\setminus\{[e]\})}/\langle\iota\rangle$. Then, we have

$$\sigma_{c([e])}^{S\setminus\{[e]\}} = \tau_{\partial_+[e]}^S\sigma_{\partial_-[e]}^S + \sigma_{\partial_+[e]}^S\tau_{\partial_-[e]}^S.$$

By this formula, we see that

$$\begin{aligned}
&\sum_{[e]\in S}\left(\prod_{[v]\in\mathrm{MV}\{S\setminus\{[e]\}\}}\sigma_{[v]}^{S\setminus\{[e]\}}\right)\\
&\quad = \sum_{[e]\in S}\left(\tau_{\partial_+[e]}^S\sigma_{\partial_-[e]}^S\prod_{[v]\in\mathrm{MV}\{S\setminus\{[e]\}\}\setminus\{c([v])\}}\sigma_{[v]}^{S\setminus\{[e]\}}\right.\\
&\qquad\qquad \left. +\sigma_{\partial_+[e]}^S\tau_{\partial_-[e]}^S\prod_{[v]\in\mathrm{MV}\{S\setminus\{[e]\}\}\setminus\{c([v])\}}\sigma_{[v]}^{S\setminus\{[e]\}}\right)\\
&\quad = \sum_{[e]\in S}\left(\tau_{\partial_+[e]}^S\sigma_{\partial_-[e]}^S\prod_{[v]\in\mathrm{MV}\{S\}\setminus\{\partial_+[e],\partial_-[e]\}}\sigma_{[v]}^S\right. \qquad (4.2)\\
&\qquad\qquad \left. +\sigma_{\partial_+[e]}^S\tau_{\partial_-[e]}^S\prod_{[v]\in\mathrm{MV}\{S\}\setminus\{\partial_+[e],\partial_-[e]\}}\sigma_{[v]}^S\right)\\
&\quad = \sum_{[e]\in S}\left(\tau_{\partial_+[e]}^S\prod_{[v]\in\mathrm{MV}\{S\}\setminus\{\partial_+[e]\}}\sigma_{[v]}^S + \tau_{\partial_-[e]}^S\prod_{[v]\in\mathrm{MV}\{S\}\setminus\{\partial_-[e]\}}\sigma_{[v]}^S\right)\\
&\quad = \sum_{[v]\in\mathrm{MV}\{S\}}\left(\nu_0^S([v])\tau_{[v]}^S\prod_{[v']\in\mathrm{MV}\{S\}\setminus\{[v]\}}\sigma_{[v']}^S\right).
\end{aligned}$$

Here we set

$$\mathcal{E}_0^{(k)} := \{S \in \mathcal{E}_0 \mid \#(S) = k\},$$

for each k. Then we have

$$\begin{aligned}
&\left(\sum_{S\in\mathcal{E}_0^{(k)}} \prod_{[v]\in \mathrm{MV}\{S\}} \sigma_{[v]}^S \prod_{[e]\in S} \lambda_{[e]}\right)\left(\sum_{[e']\in \mathrm{Ed}_0(G/\langle\iota\rangle)} \lambda_{[e']}\right)\\
&\quad \geq \sum_{S\in\mathcal{E}_0^{(k)}} \left(\sum_{[e']\in \mathrm{Ed}_0(G/\langle\iota\rangle)\setminus S} \left(\prod_{[v]\in \mathrm{MV}\{S\}} \sigma_{[v]}^S \prod_{[e]\in S} \lambda_{[e]}\lambda_{[e']}\right)\right)\\
&\quad = \sum_{T\in\mathcal{E}_0^{(k+1)}} \left(\sum_{[e]\in T} \left(\prod_{[v]\in \mathrm{MV}\{T\setminus\{[e]\}\}} \sigma_{[v]}^{T\setminus\{[e]\}}\right) \prod_{[e]\in T} \lambda_{[e]}\right)\\
&\quad = \sum_{T\in\mathcal{E}_0^{(k+1)}} \left(\sum_{[v]\in \mathrm{MV}\{T\}} \left(\nu_0^S([v])\tau_{[v]}^T \prod_{[v']\in \mathrm{MV}\{T\}\setminus\{[v]\}} \sigma_{[v']}^T\right) \prod_{[e]\in T} \lambda_{[e]}\right),
\end{aligned}$$

where we used (4.2) at the last equality. Thus, we have

$$\begin{aligned}
&L_G(\lambda)\left(\sum_{[e]\in \mathrm{Ed}_0(G/\langle\iota\rangle)} \lambda_{[e]} + \frac{1}{4}\sum_{[e]\in \mathrm{Ed}_1(G/\langle\iota\rangle)} \lambda_{[e]}\right)\\
&\geq \sum_{S\in\mathcal{E}_0} \left(\sum_{[v]\in \mathrm{MV}\{S\}} \left(\left(\frac{\left(\nu_1^S([v])\right)^2}{4} + \nu_0^S([v])\right)\tau_{[v]}^S \prod_{[v']\in \mathrm{MV}\{S\}\setminus\{[v]\}} \sigma_{[v']}^S\right) \prod_{[e]\in S} \lambda_{[e]}\right).
\end{aligned}$$

Since

$$\frac{\left(\nu_1^S([v])\right)^2}{4} + \nu_0^S([v]) > \left(\nu_1^S([v]) - 1\right) + \left(\nu_0^S([v]) - 1\right) = \nu^S([v]) - 2$$

for any $[v]$, we obtain the first inequality.

We can also obtain

$$\begin{aligned}
&L_G(\lambda)\left(\frac{1}{2}\sum_{[e]\in \mathrm{Ed}_0(G/\langle\iota\rangle)} \lambda_{[e]} + \frac{1}{4}\sum_{[e]\in \mathrm{Ed}_1(G/\langle\iota\rangle)} \lambda_{[e]}\right)\\
&\geq \sum_{S\in\mathcal{E}_0} \left(\sum_{[v]\in \mathrm{MV}\{S\}} \left(\left(\frac{\left(\nu_1^S([v])\right)^2}{4} + \frac{\nu_0^S([v])}{2}\right)\tau_{[v]}^S \prod_{[v']\in \mathrm{MV}\{S\}\setminus\{[v]\}} \sigma_{[v']}^S\right) \prod_{[e]\in S} \lambda_{[e]}\right)
\end{aligned}$$

in the same way. If $n \leq 4$, then we can see $\#\left(\mathrm{Ed}_0(G/\langle\iota\rangle)\right) \leq 2$, and hence $\nu_0^S([v]) \leq 2$ for any $S \in \mathcal{E}_0$ and $[v] \in \mathrm{MV}\{S\}$. Therefore, we have

$$\frac{\left(\nu_1^S([v])\right)^2}{4} + \frac{\nu_0^S([v])}{2} \geq \left(\nu_1^S([v]) - 1\right) + \left(\nu_0^S([v]) - 1\right) = \nu^S([v]) - 2,$$

and we obtain the second inequality. □

Accordingly we have the following estimate.

Proposition 4.5. *With the same notation as that of Theorem* 3.14*, we have the following inequalities.*

(1) *For any hyperelliptic graph G and for any $\lambda \in \mathcal{W}_{>0}(G/\langle\iota\rangle)$, we have*

$$\epsilon(G,D)(\lambda) \leq \sum_{\mathrm{tp}([e])\neq 0} \frac{4\deg(D) + 3\,\mathrm{tp}([e])(\deg(D) - \mathrm{tp}([e]))}{3(\deg(D)+2)} \lambda_{[e]} + \sum_{\mathrm{tp}([e])=0} \frac{5\deg(D)}{6(\deg(D)+2)} \lambda_{[e]}.$$

(2) *If every irreducible component of G is of genus less than* 5*, then*

$$\epsilon(G,D)(\lambda) \leq \sum_{\mathrm{tp}([e])\neq 0} \frac{\deg(D) + \mathrm{tp}([e])(\deg(D) - \mathrm{tp}([e]))}{\deg(D)+2} \lambda_{[e]} + \sum_{\mathrm{tp}([e])=0} \frac{5\deg(D)}{6(\deg(D)+2)} \lambda_{[e]}.$$

Proof. By the compatibility with one-point sum, we may assume G to be irreducible. Since we know that e is 1-jointed if $\mathrm{tp}(e) = 0$, our inequalities are immediate from Theorem 3.14 and the above lemma. □

4.3. Proof of Theorem 4.1

Now we are ready to complete the proof of Theorem 4.1. We recall $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a = (\omega_{X/Y} \cdot \omega_{X/Y}) - \sum_{y\in Y} \epsilon_y$. Let us compare $(\omega_{X/Y} \cdot \omega_{X/Y})$ with the admissible constants. Since f is hyperelliptic, we have

$$(\omega_{X/Y} \cdot \omega_{X/Y}) = \frac{g-1}{2g+1}\xi_0(X/Y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} \frac{6j(g-1-j) + 2(g-1)}{2g+1}\xi_j(X/Y) + \sum_{i=1}^{\left[\frac{g}{2}\right]} \left(\frac{12i(g-i)}{2g+1} - 1\right)\delta_i(X/Y)$$

by virtue of [1, Proposition 4.7], or [11] for $\mathrm{char}(k) = 2$, with Noether's formula. There is no problem in this part.

Let us consider the admissible constants. Let $\overline{f} : \overline{X} \to Y$ be the stable model of $f : X \to Y$, G_y the dual graph of $\overline{X}_y = \left(\overline{f}\right)^{-1}(y)$ and let $\lambda = \lambda_y$ be the associated weight (cf. Subsection 1.3). The hyperelliptic involution ι naturally acts on G_y and λ is invariant under the action.

Lemma 4.6. *For G_y, we have the following properties.*

(1) *If $e \in E_y$ is a self-loop, then $\iota(e) = e$.*

(2) *If $b_v \leq 2$, then $\iota(v) = v$ and the irreducible component corresponding to v has positive genus.*

Proof. Since $\overline{X}_y/\langle\iota\rangle$ is a tree of smooth rational curves, an irreducible component of $\overline{X}_y$ with self-intersection is stable by the hyperelliptic involution. Thus we see that a self-loop of G_y is ι-fixed. The assertion (2) follows from the assumption of stability of $\overline{X}_y$ and that the hyperelliptic quotient has genus 0. $\square$

Let E_{old} be the subset of $\mathrm{Ed}(G_y)$ defined as follows: $e \in E_{old}$ if and only if "$\iota(e) \neq e$" or "e is not a self-loop, $\iota(e) = e$ and the action of ι on the extremities of e is the identity". Let $G_{y,old}$ be the subgraph generated by E_{old}. Then, since the nodes in $\mathrm{Ed}(G_y) \setminus E_{old}$ are the nodes mapped to the regular points by the quotient map $\overline{X}_y \to \overline{X}_y/\langle\iota\rangle$, we find that the quotient graph $G_{y,old}/\langle\iota\rangle$ is the dual graph of $\overline{X}_y/\langle\iota\rangle$. Thus $G_{y,old}/\langle\iota\rangle$ is a tree.

Now we define a graph G_y^+ to be the graph obtained from G_y by dividing all the edges in $\mathrm{Ed}(G_y) \setminus E_{old}$ into two line segments. Then we can write

$$\mathrm{Vert}(G_y^+) = V_{new} \amalg V_{old},$$

where $V_{old} = \mathrm{Vert}\,(G_y)$ and V_{new} denotes the set of vertices which newly appear as a division point of edges in $\mathrm{Ed}(G_y) \setminus E_{old}$, and

$$\mathrm{Ed}(G_y^+) = E_{new} \amalg E_{old},$$

where E_{new} is the set of edges which newly appear as fragments in the division of edges of G_y. Moreover, the hyperelliptic involution acts naturally on G_y^+ as follows: On V_{old} and on E_{old}, the action is the same as that on G_y. For $v \in V_{new}$, we have $\iota(v) = v$ and for $e \in E_{new}$, the edge $\iota(e)$ is the other fragment. We can induce a natural weight λ^+ on G_y^+ from λ so that if e_1 and e_2 are the fragments of e,

$$\lambda^+_{e_1} = \lambda^+_{e_2} = \frac{\lambda_e}{2}.$$

It is ι-invariant.

By the definition of the action and the fact that $G_{y,old}/\langle\iota\rangle$ is a tree, we find $G_y^+/\langle\iota\rangle$ is a tree. By the construction, the realization of $\left(G_y^+, \lambda^+\right)$ is the metrized graph $\bar{G}_y$ associated with the (semi)stable model as in [12].

Since the intersection of $\omega_{X/Y}$ with a (-2)-curve is 0, the canonical polarization ω_y can be regarded a divisor on G_y, i.e., it is supported in $\mathrm{Vert}(G_y)$. Therefore it is also regarded as a polarization of G_y^+.

Finally, let $G_{y,1}^+$ be the graph obtained by contracting all the ι-fixed edges of G_y^+. Then, taking account of Lemma 4.6 (2), we can see that $G_{y,1}^+$ with the ι-action is a hyperelliptic graph by its construction unless it is a one-point graph. Note that it is endowed with a natural weight $\lambda_1^+ := \lambda^+|_{G_{y,1}}$, which is a hyperelliptic weight. Let $\omega_{y,1}$ be the polarization of $G_{y,1}^+$ associated with ω_y on G_y^+. Since any vertex v of $G_{y,1}^+$ with $\iota(v) \neq v$ corresponds to smooth rational component, the coefficient of such v in $\omega_{y,1}$ is equal to $b_v - 2$.

To the contrary, let $G_{y,2}^+$, λ_2^+ and $\omega_{y,2}$ be the contractions of the non-ι-fixed edges. Taking account that a non-ι-fixed edge itself generate an irreducible component of G_y^+, we have

$$\epsilon_y = \epsilon(G_y^+, \omega_y)(\lambda^+) = \epsilon(G_{y,1}^+, \omega_{y,1})(\lambda_1^+) + \epsilon(G_{y,2}^+, \omega_{y,2})(\lambda_2^+)$$

by virtue of Proposition 1.10.

By the definition of the weight λ_1^+, we see

$$\sum_{[e]\in \mathrm{Ed}\left(G_{y,1}^+/\langle\iota\rangle\right), \mathrm{tp}(e)=0} \lambda_1^+([e]) = \frac{1}{2}\xi_0(X_y)$$

and

$$\sum_{[e]\in \mathrm{Ed}\left(G_{y,1}^+/\langle\iota\rangle\right), \mathrm{tp}(e)=j} \lambda_1^+([e]) = \xi_j(X_y)$$

Suppose $g \geq 5$. Then by Proposition 4.5 (1), we have

$$\epsilon(\bar{G}_{y,1}, \omega_{y,1}) \leq \frac{5(g-1)}{12g}\xi_0(X_y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} \frac{4(g-1)+6j(g-1-j)}{3g}\xi_j(X_y),$$

and by [6, Corollary 5.8], we have

$$\epsilon(\bar{G}_{y,2}, \omega_{y,2}) \leq \sum_{i=1}^{\left[\frac{g}{2}\right]} \left(\frac{4i(g-1)}{g} - 1\right)\delta_i(X_y),$$

where we used $\deg(\omega_y) = 2(g-1)$. Consequently we have

$$\epsilon_y \leq \frac{5(g-1)}{12g}\xi_0(X_y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} \frac{4(g-1)+6j(g-1-j)}{3g}\xi_j(X_y) + \sum_{i=1}^{\left[\frac{g}{2}\right]} \left(\frac{4i(g-1)}{g} - 1\right)\delta_i(X_y).$$

Therefore,

$$(\omega_{X/Y}^a \cdot \omega_{X/Y}^a)_a \geq \frac{(g-1)(2g-5)}{12g(2g+1)}\xi_0(X/Y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} \frac{2(g-1)(3j(g-1-j)-g-2)}{3g(2g+1)}\xi_j(X/Y) + \sum_{i=1}^{\left[\frac{g}{2}\right]} \frac{4(g-1)i(g-i)}{g(2g+1)}\delta_i(X/Y).$$

Now since $g \geq 5$, we have

$$3j(g-1-j) - g - 2 \geq 3(g-2) - g - 2 = 2(g-4) > 0,$$

which shows $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a > 0$. Thus, we obtain our main result for $g \geq 5$.

Secondly, let us consider the case of $g \leq 4$. Then the genus of G_y is at most 4. Therefore, we can apply the inequality of Proposition 4.5 (2), and by the same way as above, we obtain an inequality

$$\epsilon_y \leq \frac{5(g-1)}{12g}\xi_0(X_y) + \sum_{j=1}^{[\frac{g-1}{2}]} \frac{g-1+2j(g-1-j)}{g}\xi_j(X_y) + \sum_{i=1}^{[\frac{g}{2}]} \left(\frac{4i(g-1)}{g} - 1 \right) \delta_i(X_y),$$

and hence we find

$$(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a \geq \frac{(g-1)(2g-5)}{12g(2g+1)}\xi_0(X/Y) + \sum_{j=1}^{[\frac{g-1}{2}]} \frac{(g-1)(2j(g-1-j)-1)}{g(2g+1)}\xi_j(X/Y) + \sum_{i=1}^{[\frac{g}{2}]} \frac{4(g-1)i(g-i)}{g(2g+1)}\delta_i(X/Y).$$

All the coefficients of $\xi_j(X/Y)$ and $\delta_i(X/Y)$ are positive if $g \geq 3$, thus we have reached our main result for $g = 3, 4$.

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