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Estimates on the effective resistance in a long-range percolation on $\mathbb{Z}^d$

By

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Abstract

We give several estimates on *volumes* and *effective resistances* in a long-range percolation on a vertex set of a d-dimensional square lattice. When $d = 1$, our results imply some kind of discontinuity in the long-range percolation model; more precisely, in the order of the effective resistance. Our another consequence is that, when $d \geq 2$ and $s \in (d, d+2)$, where s is the parameter determining the magnitude of the range, the order of the effective resistance corresponds to the α-stable process with $\alpha = s - d$.

1. Introduction and Results

Random walks on the long-range percolation clusters in $\mathbb{Z}^d$ are well-studied recently. The long-range percolation is, roughly speaking, the model in which any pair of two points is connected by a random bond independently. In [3], the recurrence and transience of the random walks on the random graphs generated in the long-range percolation are studied. In [7], strongly recurrent random walks on random media are discussed in general context, and as an application, Gaussian on-diagonal heat kernel estimates are given in a one-dimensional long-range percolation, under the condition that the effects of long bonds are relatively small. In Section 2 in [7], the key of the proof was to show the estimates on the *volume* and the *effective resistance*.

In this paper, we extend the volume and resistance estimates to the case of a long-range percolation on $\mathbb{Z}^d$ with $d \geq 1$, $s > d$; see Theorems 1.1, 1.2 below. When $d = 1$, we prove that the order of the effective resistance is discontinuous at $s = 2$, where s is a parameter which determines the magnitude of the range and will be given in Section 1.1. When $d \geq 2$ and $s \in (d, d+2)$, we can prove that the effective resistance has the order corresponding to the α-stable process with $\alpha = s - d$. Hence, it is expected that the scaling limit of the random walk may become a stable process in such a case. This is quite different from the case that the behavior of the random walk is asymptotically Gaussian, which

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is true for large s. This is because the effects of long bonds are large and can not be ignored when $s \in (d, d+2)$. In the transient case, the argument in [7] does not work, and so, we can not deduce heat kernel estimates directly from volume and resistance estimates obtained in this paper. But the estimates on the effective resistance have an importance giving the information for the Green function. In this respect, we expect that the estimates given in this paper will be useful in future studies of the random walks on the long-range percolation clusters.

1.1. The model

Here, we formulate the long-range percolation model on $\mathbb{Z}^d$, $d \geq 1$. Let $\mathbf{p} = \{p(n)\}_{n=1}^{\infty}$ be a sequence of real numbers in $[0,1]$. We assume that $\mathbf{p}$ satisfies $p(n) \sim \beta n^{-s}$ for some $s > 0$, $\beta > 0$, in the following sense:

$$\lim_{n\to\infty} \frac{p(n)}{\beta n^{-s}} = 1. \tag{1.1}$$

Each unoriented pair of distinct points $x, y \in \mathbb{Z}^d$ is connected by an unoriented bond with probability $p(x,y) = p(y,x) = p(|x-y|)$, independently of other pairs. Here, $|x-y| = \sum_{i=1}^{d} |x_i - y_i|$. Let μ_{xy} be a $\{0,1\}$-valued random variable, which takes 1 if x and y are connected by a bond and takes 0 otherwise. We note $\mu_{xy} = \mu_{yx}$, and $\mu_{xx} = 0$. The sum $\mu_x = \sum_{y\in\mathbb{Z}^d} \mu_{xy}$ stands for the number of bonds which have x as an endpoint. For $A \subset \mathbb{Z}^d$, we denote $\mu(A) = \sum_{x\in A} \mu_x$. Now, $G = \mathbb{Z}^d$ is the vertex set and $E = \{\langle x,y\rangle | \mu_{xy} = 1\}$ is the edge set of the corresponding random graph. We identify $\langle x,y\rangle = \langle y,x\rangle$.

It is known that the random graph is locally finite (i.e. $\mu_x < \infty$ for all $x \in \mathbb{Z}^d$) almost surely if and only if $s > d$, and in this case we can define the simple random walk on the random graph. Here, the simple random walk means the discrete-time or continuous-time Markov process in which a particle at the point jumps to one of the points connected by a bond with an equal probability. If the random graph has a connected component containing infinitely many points, such a component is called an ∞-*cluster*. Let $\mathbb{P}$ be the probability measure by which the long-range percolation model is defined, and we denote $P_\infty = \mathbb{P}[\text{there exists an } \infty\text{-cluster}]$.

1.2. Volume and Resistance

We give the definition of the volume and the effective resistance associated with our long-range percolation model. For $x \in \mathbb{Z}^d$, $R > 0$,

$$B(x,R) = \{y \in \mathbb{Z}^d : |x-y| < R\} \tag{1.2}$$

is the Euclidean ball with center x and radius R. We call

$$V(x,R) = \mu(B(x,R)) \tag{1.3}$$

the *volume* of $B(x,R)$. We denote $B_R = B(0,R)$, $V_R = V(0,R)$. For $f,g : \mathbb{Z}^d \to \mathbb{R}$, we define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{x,y\in\mathbb{Z}^d} (f(x)-f(y))(g(x)-g(y))\mu_{xy}. \tag{1.4}$$

Set $H^2 = \{f \in \mathbb{R}^{\mathbb{Z}^d} : \mathcal{E}(f,f) < \infty\}$. Let A, B be disjoint subsets of $\mathbb{Z}^d$. We define the *effective resistance* between A and B by

$$R_{\mathrm{eff}}(A,B)^{-1} = \inf\{\mathcal{E}(f,f) : f \in H^2, f|_A = 1, f|_B = 0\}. \tag{1.5}$$

We simply denote $R_{\mathrm{eff}}(x,y) = R_{\mathrm{eff}}(\{x\},\{y\})$.

1.3. Results

In this paper, we use the notation c_i as positive constants which depend on $d, \mathbf{p}$. We note that the values of c_i's may change from line to line. First, we state the result for the estimate on the volume.

Theorem 1.1. *Let $d \geq 1$, $s > d$, and $P_\infty = 1$ under $\mathbf{p}$. Then,*

$$\tilde{\mathbb{P}}[\lambda^{-1}R^d \leq V_R \leq \lambda R^d] \geq 1 - \exp\{-c_1\lambda\}, \tag{1.6}$$

where $\tilde{\mathbb{P}} = \mathbb{P}[\cdot||C(0)| = \infty]$ is the conditional probability. Here, $|C(0)| = \infty$ means the origin is contained in an ∞-cluster.

This theorem implies that

$$V_R \approx R^d \tag{1.7}$$

as $R \to \infty$ independently of s. Next, we present the estimate on the effective resistance $R_{\mathrm{eff}}(B_R, B_{2R}^c)$, which is the main result of this paper.

Theorem 1.2. *Let $\tilde{\mathbb{P}}$ be the conditional probability measure introduced in Theorem 1.1, let $p_c(\mathbb{Z}^d)$ be the critical probability of the bond percolation, and $\gamma = (2d-s)/(3d-s)$.*
(1) *Let $d = 1$.* (i) *For $1 < s < 2$, if $P_\infty = 1$ under $\mathbf{p}$,*

$$\tilde{\mathbb{P}}[\lambda^{-1}R^{s-2} \leq R_{\mathrm{eff}}(B_R, {B_{2R}}^c) \leq \lambda R^{s-2}] \geq 1 - \exp\{-c_1\lambda^\gamma\}. \tag{1.8}$$

(ii) *For $s = 2$, if $p(n) \in [0,1)$ $(n \geq 1)$,*

$$\mathbb{P}[\lambda^{-1} \leq R_{\mathrm{eff}}(B_R, {B_{2R}}^c)] \geq 1 - \exp\{-c_2\lambda\}. \tag{1.9}$$

(iii) *For $s > 2$, if $p(1) = 1$,*

$$\mathbb{P}[\lambda^{-1}R \leq R_{\mathrm{eff}}(B_R, {B_{2R}}^c) \leq c_3R] \geq 1 - c_4\lambda^{-q}. \tag{1.10}$$

Here, $q = 1$ for $s > 3$, and q is any value taken from $(0, s-2)$ for $2 < s \leq 3$.

(2) *Let* $d = 2$. (i) *For* $2 < s < 4$, *if* $P_\infty = 1$ *under* $\mathbf{p}$,

$$\tilde{\mathbb{P}}[\lambda^{-1} R^{s-4} \le R_{\mathrm{eff}}(B_R, B_{2R}{}^c) \le \lambda R^{s-4}] \ge 1 - \exp\{-c_5 \lambda^\gamma\}. \tag{1.11}$$

(ii) *For* $s = 4$, *if* $p(1) > \frac{1}{2}$,

$$\tilde{\mathbb{P}}\left[\lambda^{-1} \frac{1}{\log R} \le R_{\mathrm{eff}}(B_R, B_{2R}{}^c) \le \lambda\right] \ge 1 - \exp\{-c_6 \lambda^{1/2}\}. \tag{1.12}$$

(iii) *For* $s > 4$, *if* $p(1) > \frac{1}{2}$,

$$\tilde{\mathbb{P}}[\lambda^{-1} \le R_{\mathrm{eff}}(B_R, B_{2R}{}^c) \le \lambda] \ge 1 - \exp\{-c_7 \lambda^{1/2}\}. \tag{1.13}$$

(3) *Let* $d \ge 3$. (i) *For* $d < s < d + 2$, *if* $P_\infty = 1$ *under* $\mathbf{p}$,

$$\tilde{\mathbb{P}}[\lambda^{-1} R^{s-2d} \le R_{\mathrm{eff}}(B_R, B_{2R}{}^c) \le \lambda R^{s-2d}] \ge 1 - \exp\{-c_8 \lambda^\gamma\}. \tag{1.14}$$

(ii) *For* $s = d + 2$, *if* $P_\infty = 1$ *under* $\mathbf{p}$,

$$\tilde{\mathbb{P}}\left[\lambda^{-1} \frac{R^{2-d}}{\log R} \le R_{\mathrm{eff}}(B_R, B_{2R}{}^c) \le \lambda R^{2-d}\right] \ge 1 - \exp\{-c_9 \lambda^\gamma\}. \tag{1.15}$$

(iii) *For* $s > d + 2$, *if* $p(1) > p_c(\mathbb{Z}^d)$,

$$\tilde{\mathbb{P}}[\lambda^{-1} R^{2-d} \le R_{\mathrm{eff}}(B_R, B_{2R}{}^c) \le \lambda R^{2-d}] \ge 1 - \exp\{-c_{10} \lambda^{1/2}\}. \tag{1.16}$$

Roughly speaking, this theorem implies that, when $d = 1$,

$$R_{\mathrm{eff}}(B_R, B_{2R}^c) \approx \begin{cases} R^{s-2} & 1 < s < 2, \\ R & s > 2, \end{cases}$$

and when $d \ge 2$,

$$R_{\mathrm{eff}}(B_R, B_{2R}^c) \approx \begin{cases} R^{s-2d} & d < s < d + 2, \\ R^{2-d} & s > d + 2, \end{cases}$$

as $R \to \infty$. We note that there exists a large gap at $s = 2$, in the case $d = 1$.

Remark 1. (1) Two probability measures $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are same, if $p(1) = 1$.
(2) In the above statements, the decaying orders of probabilities are not necessarily the best ones.
(3) Theorem 1.2 (1)(ii) is most delicate. If we assume $p(1) = 1$, we can prove that the order of the effective resistance is at most R. But we could not determine the exact order of the resistance in general.
(4) Theorem 1.2 (1)(iii) is essentially the fact shown in Section 2 in [7]. In this case, it is not difficult to see that $R_{\mathrm{eff}}(B_R, B_{2R}{}^c)$ and $R_{\mathrm{eff}}(0, B_R)$ are of the same order, though this is not true in general.
(5) The assertion in Theorem 1.2 (3) is a natural extension of Theorem 1.2 (2), except some technical differences.

1.4. Known results and Backgrounds

Here, we state some additional comments on the backgrounds of our problem. First, consider the case $d = 1$. When $s \leq 2$ and $p(n) \in [0, 1)$, $n \geq 1$, we can find $\mathbf{p}$ under which $P_\infty = 1$. On the other hand, when $s > 2$ and $p(n) \in [0, 1)$, $n \geq 1$, P_∞ always equals to 0. Thus, there is a phase transition in a certain sense. In [7], the case $s > 2$, $p(1) = 1$ is considered. Of course, $P_\infty = 1$ if $p(1) = 1$. The phenomenon at $s = 2$ is most non-trivial. In [1], the discontinuity of the percolation density at $s = 2$ is shown. By the result in [7], together with the transience result in [3], it is implied that the order of the heat kernel is discontinuous at $s = 2$, though we do not have a rigorous proof of the heat kernel estimate for $1 < s \leq 2$. In the recent study in [2], it is shown that the order of the mixing time changes discontinuously when $s = 2$. In Theorem 1.2, we see the discontinuity at $s = 2$ in the sense of the effective resistance.

Next, when $d \geq 2$, by comparing the random graph with the bond percolation clusters, we can see that, if we choose $p(1) \in (p_c(\mathbb{Z}^d), 1)$, $P_\infty = 1$ for any s. Here, $p_c(\mathbb{Z}^d) \in (0, 1)$ is the critical probability of the bond percolation. Especially, $p_c = \frac{1}{2}$ when $d = 2$ ([6]). In Theorem 1.2, we can see that, when $d \geq 2$, $s \in (d, d + 2)$, the order of the effective resistance corresponds to the α-stable process with $\alpha = s - d$. A stable-like behavior in a long-range percolation is also studied recently in [4].

We will give the proof of Theorems 1.1 and 1.2 in Section 2.

2. Proof of the results

First, we give two lemmas on some large deviation estimates.

Lemma 2.1. *Let $d \geq 1$, $s > d$. For all $R \geq 1$,*

$$\mathbb{P}[c_1 R^d \leq V_R \leq c_2 R^d] \geq 1 - \exp\{-c_3 R^d\}. \tag{2.1}$$

Proof. For $x, y \in \mathbb{Z}^d$, $x \neq y$, We denote $x \prec y$ if for some $i \in \{0, 1, \cdots, d-1\}$,

$$\begin{aligned} x_j &= y_j \quad (\forall j \leq i), \\ x_{i+1} &< y_{i+1}. \end{aligned}$$

Then, we can represent

$$V_R = \sum_{x \in B_R} \{\mu_x^+ + \mu_x^-\},$$

where $\mu_x^+ = \sum_{y \succ x} \mu_{xy}$, $\mu_x^- = \sum_{y \prec x} \mu_{xy}$. Note that μ_x^+ is independent for each x, and so is μ_x^-. From the large deviation principle for i.i.d. random variables, for sufficiently small $c_4 > 0$,

$$\begin{aligned} \mathbb{P}[V_R \leq c_4 |B_R|] &\leq \mathbb{P}[\sum_{x \in B_R} \mu_x^+ \leq c_4 |B_R|] \\ &\leq \exp\{-c_5 |B_R|\}. \end{aligned}$$

And for sufficiently large $c_6 > 0$,

$$\begin{aligned}\mathbb{P}[V_R \geq c_6|B_R|] &= \mathbb{P}[\sum_{x\in B_R} \mu_x^+ + \sum_{x\in B_R} \mu_x^- \geq c_6|B_R|] \\ &\leq \mathbb{P}\left[\sum_{x\in B_R} \mu_x^+ \geq \frac{1}{2}c_6|B_R|\right] + \mathbb{P}\left[\sum_{x\in B_R} \mu_x^- \geq \frac{1}{2}c_6|B_R|\right] \\ &\leq \exp\{-c_7|B_R|\}.\end{aligned}$$

So, the assertion holds. □

Lemma 2.2. *Let $d \geq 1$, $d < s < 2d$. For all $R \geq 1$,*

$$\mathbb{P}[c_1 R^{2d-s} \leq \sum_{x\in B_R} \sum_{y\in B_{2R}{}^c} \mu_{xy} \leq c_2 R^{2d-s}] \geq 1 - \exp\{-c_3 R^{2d-s}\}. \tag{2.2}$$

Proof. For $R \in \mathbb{N}$, $x \in B_R$, we denote

$$\mu_{x,R} \equiv \sum_{y\in B_{2R}^c} \mu_{xy}.$$

We have the following estimates, uniformly in x.

$$\mathbb{E}[\exp\{\mu_{x,R}\}] \leq \exp\{c_4 R^{d-s}\}, \tag{2.3}$$

$$\mathbb{E}[\exp\{-\mu_{x,R}\}] \leq \exp\{-c_5 R^{d-s}\}. \tag{2.4}$$

In fact, for (2.3),

$$\begin{aligned}\mathbb{E}\left[\exp\left\{\sum_{y\in B_{2R}^c} \mu_{xy}\right\}\right] &= \prod_{y\in B_{2R}^c} \mathbb{E}[\exp\{\mu_{xy}\}] \\ &= \prod_{y\in B_{2R}^c} \{ep(x,y) + (1-p(x,y))\} \\ &\leq \prod_{y\in B_{2R}^c} \{1 + (e-1)c_6|y-x|^{-s}\} \equiv I.\end{aligned}$$

We have used the fact $p(n) \sim \beta n^{-s}$ in the last inequality. Furthermore,

$$\begin{aligned}\log I &= \sum_{y\in B_{2R}^c} \log\{1 + (e-1)c_6|y-x|^{-s}\} \\ &\leq \sum_{y\in B_{2R}^c} (e-1)c_6|y-x|^{-s} \leq c_7 R^{d-s}.\end{aligned}$$

We can check (2.4) in the same manner.

Next, by (2.3),

$$\begin{aligned}\mathbb{E}\left[\exp\left\{\sum_{x\in B_R} \mu_{x,R}\right\}\right] &= \prod_{x\in B_R} \mathbb{E}[\exp\{\mu_{x,R}\}] \\ &\leq \{\exp(c_8 R^{d-s})\}^{|B_R|} \leq \exp(c_9 R^{2d-s}).\end{aligned}$$

On the other hand, by Chebyshev's inequality, for $c_{10} > 0$,

$$\mathbb{E}\left[\exp\left\{\sum_{x\in B_R}\mu_{x,R}\right\}\right] \geq \exp\{c_{10}R^{2d-s}\}\mathbb{P}\left[\sum_{x\in B_R}\mu_{x,R} \geq c_{10}R^{2d-s}\right].$$

Combining these estimates, the assertion for the upper bound holds when we choose sufficiently large c_{10}. Similarly, by (2.4),

$$\begin{aligned}\exp\{-c_{11}R^{2d-s}\} &\geq \mathbb{E}[\exp\{-\sum_{x\in B_R}\mu_{x,R}\}] \\ &\geq \exp\{-c_{12}R^{2d-s}\}\mathbb{P}\left[-\sum_{x\in B_R}\mu_{x,R} \geq -c_{12}R^{2d-s}\right]\end{aligned}$$

and we obtain the assertion for the lower bound by choosing sufficiently small $c_{12} > 0$. □

Proof of Theorem 1.1. We show the volume estimate by using Lemma 2.1. First, we show the upper bound.

$$\begin{aligned}V_R &= \sum_{x\in B_R}\mu_x \\ &= \sum_{x\in B_R}\sum_{y\succ x}\mu_{xy} + \sum_{x\in B_R}\sum_{y\prec x}\mu_{xy} \equiv V_+ + V_-,\end{aligned}$$

where $\succ$ is the same as in Lemma 2.1. By Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}[V_+ > \lambda R^d] &= \mathbb{P}[R^{-d}V_+ > \lambda] \\ &\leq \exp\{-c_1\lambda\}\mathbb{E}[\exp\{c_1R^{-d}V_+\}] \\ &= \exp\{-c_1\lambda\}\prod_{x\in B_R}\prod_{y\succ x}\mathbb{E}[\exp\{c_1R^{-d}\mu_{xy}\}] \\ &\leq \exp\{-c_1\lambda\}\prod_{x\in B_R}\prod_{y\succ x}\{1+(\exp\{c_1R^{-d}\}-1)c_2|x-y|^{-s}\}.\end{aligned}$$

Now,

$$\begin{aligned}&\log\prod_{y\succ x}\{1+(\exp\{c_1R^{-d}\}-1)c_2|x-y|^{-s}\} \\ &\leq \sum_{y\succ x}(\exp\{c_1R^{-d}\}-1)c_2|x-y|^{-s} \\ &= (\exp\{c_1R^{-d}\}-1)c_2\sum_{n=1}^{\infty}n^{-s}\times c_3n^{d-1} \leq c_4R^{-d}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{P}[V_+ > \lambda R^d] &\leq \exp\{-c_1\lambda\}\{\exp\{c_4R^{-d}\}\}^{|B_R|} \\ &\leq c_5\exp\{-c_1\lambda\}.\end{aligned}$$

Noting that $\mathbb{P}[V_R > \lambda R^d] \le \mathbb{P}[V_+ \ge \frac{\lambda}{2} R^d] + \mathbb{P}[V_- \ge \frac{\lambda}{2} R^d]$, the assertion under $\mathbb{P}$ follows. Since $\tilde{\mathbb{P}}[V_R > \lambda R^d] \le \mathbb{P}[|C(0)| = \infty]^{-1}\mathbb{P}[V_R > \lambda R^d]$, we can replace $\mathbb{P}$ by $\tilde{\mathbb{P}}$.

Secondly, we show the lower bound.

$$\begin{aligned}\tilde{\mathbb{P}}[V_R < \lambda^{-1} R^d] &= \tilde{\mathbb{P}}[R^d {V_R}^{-1} > \lambda] \le \exp\{-c_6\lambda\}\tilde{\mathbb{E}}[\exp\{c_6 R^d {V_R}^{-1}\}] \\ &= \exp\{-c_6\lambda\} \\ &\quad \times \left(\tilde{\mathbb{E}}[\exp\{c_6 R^d {V_R}^{-1}\}; V_R > c_7 R^d] + \tilde{\mathbb{E}}[\exp\{c_6 R^d {V_R}^{-1}\}; V_R \le c_7 R^d]\right) \\ &\equiv \exp\{-c_6\lambda\}(E_1 + E_2).\end{aligned}$$

$E_1 \le \exp\{c_6 {c_7}^{-1}\}$ is obvious. For E_2, by noting that $V_R \ge 1$ if $|C(0)| = \infty$, and by Lemma 2.1,

$$\begin{aligned}E_2 &\le \exp\{c_6 R^d\}\tilde{\mathbb{P}}[V_R \le c_7 R^d] \\ &\le \exp\{c_6 R^d\} \times \exp\{-c_8 R^d\}.\end{aligned}$$

In the above, c_7, c_8 are the constants corresponding to Lemma 2.1, and by taking c_6 such that $c_6 < c_8$, we obtain the result. □

Proof of Theorem 1.2. We write $R_{\text{eff}} = R_{\text{eff}}(B_R, {B_{2R}}^c)$, for convenience. The part (1)(iii) follows by the same way as the estimate of $R_{\text{eff}}(0, {B_R}^c)$ in [7]. For the part of the upper bound in (1)(i), (2)(i), (3)(i)(ii),

$$\begin{aligned}\tilde{\mathbb{P}}[R_{\text{eff}} \ge \lambda R^{s-2d}] &\le e^{-c_1\lambda^\gamma}\tilde{\mathbb{E}}[\exp\{c_1(R_{\text{eff}}/R^{s-2d})^\gamma\}] \\ &\equiv e^{-c_1\lambda^\gamma}(I_1 + I_2),\end{aligned}$$

where

$$\begin{aligned}I_1 &= \tilde{\mathbb{E}}[\exp\{c_1(R_{\text{eff}}/R^{s-2d})^\gamma\} : R_{\text{eff}} < c_2 R^{s-2d}], \\ I_2 &= \tilde{\mathbb{E}}[\exp\{c_1(R_{\text{eff}}/R^{s-2d})^\gamma\} : R_{\text{eff}} \ge c_2 R^{s-2d}].\end{aligned}$$

We have $I_1 \le c_3$, and

$$\begin{aligned}I_2 &\le \exp\{c_4 R^{(3d-s)\gamma}\}\tilde{\mathbb{P}}[R_{\text{eff}} \ge c_2 R^{s-2d}] \\ &\le \exp\{c_4 R^{(3d-s)\gamma}\}\tilde{\mathbb{P}}\left[\sum_{x\in B_R}\sum_{y\in B_{2R}^c} \mu_{xy} \le c_5 R^{2d-s}\right] \\ &\le \exp\{c_4 R^{(3d-s)\gamma}\}\exp\{-c_6 R^{2d-s}\} \le c_7,\end{aligned}$$

for sufficiently small c_1. We have used the fact that $R_{\text{eff}} \le c_8 R^d$ if $|C(0)| = \infty$, and used Lemma 2.2. In the above, c_5,c_6 are the constants corresponding to Lemma 2.2.

The upper bound parts of (2)(ii)(iii), (3)(iii) are proved using the following result about bond percolation; if we see only nearest-neighbor bonds, then, (see Theorem 7.68 in [5])

$$\mathbb{P}[F_R] \ge 1 - \exp\{-c_1 R^{d-1}\}, \tag{2.5}$$

where F_R is the event that, in $[-R,R]^d$ there are at least c_2R^{d-1} left-right crossing paths which are edge-disjoint each other. Thus, by considering a flow on such good paths, we have

$$\mathbb{P}[R_{\text{eff}} \le c_3R^{2-d}] \ge 1 - \exp\{-c_4R^{d-1}\}. \tag{2.6}$$

So, by using (2.6) instead of Lemma 2.2, we obtain the desired estimate in the same way as the part we have already proved.

Finally, we see the lower bound in (1)(i)(ii), (2), (3). Let

$$g(x) = 1_{\{|x|<R\}} + 1_{\{R\le|x|<2R\}}\left(2 - \frac{|x|}{R}\right), \tag{2.7}$$

and

$$\phi(R) = \begin{cases} R^{s-2d} & d < s < d+2, \\ R^{2-d}/\log R & s = d+2, \\ R^{2-d} & s > d+2. \end{cases}$$

We have

$$\begin{aligned}
&\mathbb{P}[R_{\text{eff}} \le \lambda^{-1}\phi(R)] \\
&\quad = \mathbb{P}[\inf\{\mathcal{E}(f,f) : f \in H^2, f|_{B_R} = 1, f|_{B^c_{2R}} = 0\} \ge \lambda\phi(R)^{-1}] \\
&\quad \le \mathbb{P}[\mathcal{E}(g,g)\phi(R) \ge \lambda] \\
&\quad \le e^{-\lambda}\mathbb{E}[\exp\{\mathcal{E}(g,g)\phi(R)\}] \\
&\quad = e^{-\lambda}\prod_{x,y\in\mathbb{Z}^d}\mathbb{E}[\exp\{\phi(R)|g(x)-g(y)|^2\mu_{xy}\}] \\
&\quad \le e^{-\lambda}\prod_{x,y\in\mathbb{Z}^d}\{1 + \{\exp(\phi(R)|g(x)-g(y)|^2) - 1\}c_1|x-y|^{-s}\} \\
&\quad \equiv e^{-\lambda}\prod_{x,y\in\mathbb{Z}^d} W.
\end{aligned}$$

We have a desired estimate if we can check $\prod_{x,y\in\mathbb{Z}^d} W \le c_2$. First,

$$\begin{aligned}
\prod_{x,y\in B_{2R}\setminus B_R} W &= \prod_{x,y\in B_{2R}\setminus B_R}\{1 + \{\exp(\phi(R)R^{-2}|x-y|^2) - 1\}c_1|x-y|^{-s}\} \\
&= \prod_{n=1}^{R}\prod_{x,y\in B_{2R}\setminus B_R, |x-y|=n}\left\{1 + \{\exp\{\phi(R)R^{-2}n^2\} - 1\}c_1n^{-s}\right\} \\
&\le \prod_{n=1}^{R}\{1 + \{\exp\{\phi(R)R^{-2}n^2\} - 1\}c_1n^{-s}\}^{c_3R^dn^{d-1}} \equiv a_R,
\end{aligned}$$

and

$$\log a_R \le \sum_{n=1}^{R} c_4R^dn^{d-1}\{\exp\{\phi(R)R^{-2}n^2\} - 1\}n^{-s} \equiv U.$$

When $\phi(R) = R^{s-2d}$, $s < d+2$, $U \le c_5 R^{s-d-2}\sum_{n=1}^{R} n^{d-s+1} \le c_6$. When $\phi(R) = R^{2-d}/\log R$, $d \ge 2$, $s = d+2$, $U \le \sum_{n=1}^{R} c_7 R^d n^{d-1-s} R^{-d} n^2/\log R \le c_8$. When $\phi(R) = R^{2-d}$, $d \ge 2$, $s > d+2$, $\log a_R \le \sum_{n=1}^{R} c_9 n^{d-s+1} \le c_{10}$.

Secondly,

$$\begin{aligned}\prod_{x\in B_R, y\in {B_{2R}}^c} W &= \prod_{x\in B_R, y\in {B_{2R}}^c} \{1+(\exp(\phi(R))-1)c_1|x-y|^{-s}\}\\ &= \prod_{n=R}^{\infty} \prod_{x\in B_R, y\in {B_{2R}}^c, |x-y|=n} \{1+(\exp(\phi(R))-1)c_1 n^{-s}\}\\ &\le \prod_{n=R}^{\infty} \{1+(e^{\phi(R)}-1)c_1 n^{-s}\}^{c_3 R^d n^{d-1}} \equiv X,\end{aligned}$$

and when $\phi(R) = R^{s-2d}$, $d < s < d+2$, $\log X \le \sum_{n=R}^{\infty} c_4 R^d n^{d-1-s} R^{s-2d} = c_4 R^{s-d}\sum_{n=R}^{\infty} n^{d-s-1} \le c_5$. When $\phi(R) = R^{2-d}$, $s \ge d+2$, $d \ge 2$, $\log X \le \sum_{n=R}^{\infty} c_6 R^d n^{d-1-s} R^{2-d} \le c_7 R^{d-s+2} \le c_8$.

Thirdly,

$$\begin{aligned}&\prod_{x\in B_{2R}\setminus B_R, y\in {B_{2R}}^c} W\\ &= \prod_{x\in B_{2R}\setminus B_R, y\in {B_{2R}}^c} \{1+\{\exp\{\phi(R)R^{-2}(2R-|x|)^2\}-1\}c_1|x-y|^{-s}\}\\ &\equiv \prod_{x\in B_{2R}\setminus B_R}\left(\prod_{y\in {B_{2R}}^c} Y\right) = \prod_{x\in B_{2R}\setminus B_R} a_x,\end{aligned}$$

and

$$\begin{aligned}\log a_x &= \sum_{y\in {B_{2R}}^c} \log Y\\ &\le \{\exp\{\phi(R)R^{-2}(2R-|x|)^2\}-1\}\sum_{y\in B_{2R}^c} c_1|x-y|^{-s}\\ &\le \{\exp\{\phi(R)R^{-2}(2R-|x|)^2\}-1\}\sum_{n=2R-|x|}^{\infty} c_2 n^{-s} n^{d-1}\\ &\le c_3\phi(R)R^{-2}(2R-|x|)^2(2R-|x|)^{d-s}\end{aligned}$$

unless $d = 1$, $s > 2$. Then, $a_x \le \exp\{c_3\phi(R)R^{-2}(2R-|x|)^{d-s+2}\}$. So, the left-hand side is no larger than

$$\begin{aligned}
&\prod_{x\in B_{2R}\setminus B_R} \exp\{c_3\phi(R)R^{-2}(2R-|x|)^{d-s+2}\} \\
&\quad = \exp\left\{c_3\phi(R)R^{-2}\sum_{x\in B_{2R}\setminus B_R}(2R-|x|)^{d-s+2}\right\} \\
&\quad \le \exp\left\{c_4\phi(R)R^{-2}\sum_{n=R}^{2R-1}(2R-n)^{d-s+2}n^{d-1}\right\}.
\end{aligned}$$

When $d < s < d+2$, $\phi(R) = R^{s-2d}$, the right-hand side is no larger than

$$\exp\left\{c_5 R^{s-2d-2}\sum_{n=R}^{2R-1}R^{d-s+2}n^{d-1}\right\} = \exp\left\{c_5 R^{-d}\sum_{n=R}^{2R-1}n^{d-1}\right\} \le c_6. \tag{2.8}$$

When $s \ge d+2$, $\phi(R) = R^{2-d}$, the right-hand side is no larger than

$$\exp\left\{c_7 R^{-d}\sum_{n=R}^{2R-1}n^{d-1}\right\} \le c_8. \tag{2.9}$$

We can check $\prod_{x\in B_R, y\in B_{2R}\setminus B_R} W \le c_9$ in the same way as the estimate of $\prod_{x\in B_{2R}\setminus B_R, y\in B_{2R}^c} W$. We thus complete the proof of the lower bound. □

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