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Unconditional uniqueness of the derivative nonlinear Schrödinger equation in energy space

By

Yin Yin Su WIN

Abstract

We consider the unconditional uniqueness for the Cauchy problem of the derivative nonlinear Schrödinger equation. The proof is based on the Gauge transformation and the Fourier restriction method.

1. Introduction

We consider the Cauchy problem of the derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_{xx} u = g(u) \quad \text{on } (0,T)\times\mathbb{R},\ T>0, \tag{1.1}$$

$$u(0,x) = u_0(x) \in H^1(\mathbb{R}), \tag{1.2}$$

where $g(u) = i\delta\partial_x(|u|^2u)$ and $u(t,x)$ is a complex valued function of $(0,T)\times\mathbb{R}, \partial_x = \frac{\partial}{\partial x}$, $\delta\in\mathbb{R}$.

Our purpose is to study the unconditional well-posedness of (1.1)–(1.2) in energy space. In [10], Kato introduced the concept of the unconditional well-posedness of the Cauchy problem with power nonlinearity. He explained that the solution need the auxiliary condition to ensure the well-posedness, but in some cases it can be removed. We consider particularly the model case of the pure power nonlinearity $g(u) = |u|^\alpha u$ with $\alpha \le \frac{4}{N-2}$ $(N\ge 3)$. Let $u_0 \in H^1(\mathbb{R}^N)$.

1. Assume $\alpha < \frac{4}{N-2}$. There exists a unique solution $u \in C([0,T), H^1(\mathbb{R}^N)) \cap L^q((0,T), L^r(\mathbb{R}^n))$, for some $T\ge 0$, where (q,r) is admissible pair.
2. Assume $\alpha = \frac{4}{N-2}$. There exists a unique solution $u \in C([0,T], H^1(\mathbb{R}^N)) \cap L^q((0,T), L^r(\mathbb{R}^n))$, for some $T\ge 0$, where (q,r) is admissible pair.

In (i), the auxiliary space $L^q((0,T),L^r)$ can be removed, it is bonus in the words of Kato, which may or may not appear in the theorem. Hence we say that (i) is unconditional well-posedness. But in (ii), the auxiliary space $L^q((0,T),L^r)$ is essential part of the well-posedness because we might not prove the uniqueness without the auxiliary conditions. Hence we say that, (ii) is conditional well-posedness in $H^1(\mathbb{R}^N)$ with the auxiliary space $L^q((0,T),L^r)$.

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We first recall the some known results of (1.1)–(1.2). In [14], Tsutsumi and Fukuda proved that the solution of (1.1)–(1.2) is global existence in $L^\infty(0,T;H^s)\cap C(0,T;H^{s-1})$, $s>3/2$. Their results was improved by Hayashi [7] and he proved the uniqueness in $C(\mathbb{R},H^1(\mathbb{R}))\cap L^{12}_{\text{loc}}(\mathbb{R},H^{1,3}(\mathbb{R}))$. He developed the gauge transformation technique to reduce the derivative nonlinearity to another nonlinearity, but the new equation still contains the derivative in the form of $u^2\bar{u}_x$. In [13], Takaoka used the Fourier restriction method to handle the gauge equivalent equation containing the derivative and he proved the Cauchy problem (1.1)–(1.2) is sharp local well-posedness in $H^s(s\geq 1/2)$. In spatial periodic case, [9] Herr proved the same result as Takaoka. In our study, we emphasize the gauge equivalent equation to prove the unconditional well-posedness.

We give some notations and function spaces which are used in this work.

We let $\|.\|_p$ be the norm of $L^p(\mathbb{R})$ where $1\leq p<\infty$. The space time mixed norm $L^p_tL^q_x$ is defined by

$$\|f\|_{L^p_tL^q_x} := \left(\int \|f\|^p_{L^q_x}dt\right)^{1/p}$$

with the usual modifications when $p=\infty$.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space. Given $s\in\mathbb{R}$, one defines

$$H^{s,p}(\mathbb{R}) := \{f(x)\in\mathcal{S}'(\mathbb{R}); (1+|\xi|^2)^{\frac{s}{2}}\widehat{u}\in L^p(\mathbb{R})\}$$

and

$$\|u\|_{H^{s,p}} := \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\widehat{u}\|_{L^p}.$$

When $p=2$, we write $H^{s,2}=H^s$.

We define the Fourier transform in time by

$$\mathcal{F}_\tau(f)(\tau) = \int_{\mathbb{R}} e^{-it\tau}f(t)dt$$

and the spatial Fourier transform by

$$\mathcal{F}_\xi(f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi}f(x)dx.$$

Let $\mathcal{F}f=\mathcal{F}_\tau\mathcal{F}_\xi f$. We also denote it by $\widehat{f}$.

For $s,b\in\mathbb{R}$. We define $X_{s,b}(\mathbb{R}\times\mathbb{R})$ which is very useful to estimate the derivative nonlinearity,

$$X_{s,b} := \{f\in\mathcal{S}' : \langle\xi\rangle^s\langle\tau+\xi^2\rangle^b\widehat{f}\in L^2_{\tau,\xi}\}.$$

where $\langle.\rangle=(1+|.|^2)^{\frac{1}{2}}$. In $X_{s,b}$, it has two weights, the one $\langle\xi\rangle$ is elliptic and the other is parabolid $\langle\tau-\xi^2\rangle$ which it is important to consider geometrically.

For any time interval $(0,T)$, $T>0$, we define the restricted space $X_{s,b}((0,T)\times\mathbb{R})$ by

$$\|u\|_{X^T_{s,b}} := \inf\left\{||\widetilde{u}||_{X_{s,b}} : \widetilde{u}|_{((0,T)\times\mathbb{R})}=u\right\}.$$

In complex conjugate case, $\|\bar{u}\|_{X^{s,b}} = \|u\|_{X^-_{s,b}}$.

Since the uniqueness of solution is a main part of the unconditional well-posedness, we consider the uniqueness only. We recall the results of Takaoka [13] and Herr [9]. Takaoka [13] proved that the gauge equivalent solution is unique in $X_{s,\frac{1}{2}+} \cap L^4_T(W^{s,\infty}_x)$ for $s \geq \frac{1}{2}$ and Herr [9] proved in $X_{s,\frac{1}{2}} \cap Y_{s,0}$ for $s \geq \frac{1}{2}$ where the norm of auxiliary $Y_{s,b}$-space is related to the norm of $X_{s,b}$, with L^2-integral for the spatial frequency variable ξ and L^1-integral for the time frequency variable τ. While the Sobolev embedding $X^T_{s,b} \hookrightarrow L^\infty(H^s)$ holds for $b > \frac{1}{2}$, we need the extra conditions such as the space $Y_{s,b}$. To explain our result, we start the following observation. We first see that if $u \in L^\infty(0,T;H^1(\mathbb{R}))$ then we have $u \in L^2(0,T;H^1(\mathbb{R}))$, which means $u \in X_{1,0}$. On the other hand, by taking the Fourier transform on both sides of (1.1), we can see $\mathcal{F}u = \frac{\mathcal{F}(|u|^2u)_x}{(\tau+|\xi|^2)} \in L^2(\mathbb{R}^2)$, i.e., $u \in X_{0,1}$. By using the interpolation between above two cases, we get the solution $u \in X_{\frac{1}{2},\frac{1}{2}}$. Hence, if the solution is unique in $X_{\frac{1}{2},\frac{1}{2}}$, the problem (1.1)–(1.2) is unconditionally well-posed in H^1.

The unconditional uniqueness is a concept of uniqueness which does not depend on how to construct the solution. For example, we consider (1.1)–(1.2) with zero Drichlet boundary condition on the ball B_R centered at the origin with radius $R > 0$. Let u_0 be a compactly supported function in $H^1(\mathbb{R})$. If $\|u_0\|_{L^2(B_R)}$ is small, we have a global solution $u_R \in L^\infty(\mathbb{R};H^1_0(B_R))$ such that

$$\|u(t)\|_{L^2(B_R)} = \|u_0\|_{L^2(B_R)}$$

and

$$E(u(t)) \leq E(u_0)$$

where $E(u(t)) = \|\partial_x u\|^2_{L^2(B_R)} - \frac{1}{2}\text{Im}\int_{B_R} u\bar{u}u\partial_x\bar{u}dx$. Indeed, because we have by the Gagliardo-Nirenberg inequality,

$$\left|\text{Im}\int_{B_R} u\bar{u}u\partial_x\bar{u}dx\right| \leq C\|u\|^2_{L^2(B_R)}\|u\|^2_{H^1(B_R)},$$

there exists a solution $u \in L^\infty(\mathbb{R},H^1_0(B_R))$ when $\|u_0\|_{L^2(B_R)}$ is sufficiently small. Then the passage to the limit as $R \to \infty$ leads to a solution in $L^\infty(\mathbb{R};H^1(\mathbb{R}))$. In this case, the proof of existence does not imply that a such solution is in the auxiliary spaces. But our Theorem 1.1 below show that when $u_0 \in H^1_0$ this solution is identical to the solution given by Hayashi [7] and Takaoka [13]. Our main result is as follows.

Theorem 1.1. *Let $u_0 \in H^1(\mathbb{R})$ and $T > 0$, assume that u and v are two solutions of* (1.1)–(1.2) *in $L^\infty(0,T;H^1(\mathbb{R}))$ with the same initial data. Then $u(t) = v(t)$, $t \in [0,T]$.*

The plan of this paper is composed as follows. In section 2, we give some useful linear estimates. In section 3, we derived the estimates of nonlinearity on Bourgain space. Finally, in section 4 we present the gauge transformation and prove that the solution of gauge equivalent equation is unique in $X^T_{\frac{1}{2},\frac{1}{2}}$ and the solution of (1.1)–(1.2) is unique in $L^\infty(0,T;H^1)$.

2. Linear estimates

Let $s_1, s_2, b_1, b_2 \in \mathbb{R}$. There exists $c > 0$ such that

$$\|u\|_{X_{s_1,b_1}} \le c\|u\|_{X_{s_2,b_2}}, \tag{2.1}$$

where $s_2 \ge s_1,\ b_2 \ge b_1$. We have known the usual Strichartz estimates

$$\|u\|_{L^6_{t,x}} \le c\|u\|_{X_{0,b}},\ b > \frac{1}{2} \tag{2.2}$$

and

$$\|u\|_{L^2_{t,x}} = \|u\|_{X_{0,0}}. \tag{2.3}$$

By using the interpolation of (2.2) and (2.3),

$$\|u\|_{L^4_{t,x}} \le c\|u\|_{X_{0,b^*}},\ where\ b^* > \frac{3}{8}. \tag{2.4}$$

The solution of (1.1)–(1.2) with the initial data $u(0) = u_0$ admits the equivalent integral equation

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')\partial_x(|u|^2u)(t')dt' \tag{2.5}$$

where $U(t) = e^{it\triangle}$, is the free Schrödinger operator. In this proof we want to consider the local solution in time, for that purpose we introduce time cut off function in (2.5). Let $\psi_1 \in C^\infty(\mathbb{R})$ with $0 \le \psi_1 \le 1$. $\psi_1(t) = 1$ on $|t| \le 1$, $\psi_1(t) = 0$ on $|t| \ge 2$ and $\psi_T(t) = \psi_1(t/T)$ for any $0 < T \le 1$. Then

$$u(t) = \psi_1(t)U(t)u_0 - i\psi_T(t)\int_0^t U(t-t')f(t')dt', \tag{2.6}$$

where $f(t) = \partial_x(|u|^2u)(t)$. Since the relation of $X_{s,b}$ and $H^{s,b}$

$$\|u\|_{X_{s,b}} = \|U(-t)u\|_{H^{s,b}},$$

there exists $c > 0$ such that

$$\begin{aligned} \|\psi_1(t)U(t)u_0\|_{X_{s,b}} &= \|\psi_1(t)u_0\|_{H^{s,b}} \\ &\le c\|u_0\|_{H^s_x}. \end{aligned} \tag{2.7}$$

We next consider the convolution part of (2.6)

$$\|\psi_T(t)(U * f)\|_{X_{s,b}} \le c\|f\|_{X_{s',b'}}$$

where * is convolution in time and it is equivalent to

$$\|Lf\|_{H^{s,b}} \leq c\|f\|_{H^{s',b'}}$$

with some constant c ,where $X_{s',b'}$ and $H^{s',b'}$ are the dual of $X_{s,b}$ and $H^{s,b}$. The operator L is defined by

$$(Lf) = \psi_T(t)\int_0^t f(t')dt'. \tag{2.8}$$

Lemma 2.1. *Let $s \in \mathbb{R}$ and the following estimates are hold.*

1. $\|Lf\|_{H_t^{\frac{1}{2}}} \leq c\|f\|_{H_t^{-\frac{1}{2}}} + c\int_{|\tau|T\geq 1} |\tau|^{-1}|\widehat{f}(\tau)|d\tau$,

2. $\|\psi_T(U*f)\|_{X_{s,\frac{1}{2}}} \leq c\|f\|_{X_{s,-\frac{1}{2}}} + c\left\{\int \langle\xi\rangle^{2s}(\int \langle\tau-\xi^2\rangle^{-1}|\widehat{f}_+(\tau,\xi)|d\tau)^2 d\xi\right\}^{\frac{1}{2}}$,

where some constant c and $\widehat{f}_+(\tau,\xi) = \chi\{|\tau-\xi^2|T \geq 1\}\widehat{f}(\tau,\xi)$.

Proof. We first prove (i), we may define

$$\int_0^t f(t')dt' = c\int_{-\infty}^{\infty} \frac{e^{it\tau}-1}{i\tau}\widehat{f}(\tau)d\tau \tag{2.9}$$

and we split $f = f_+ + f_-$ where

$$\widehat{f}_+(\tau) = \widehat{f}(\tau)\chi(|\tau|T \geq 1),$$
$$\widehat{f}_-(\tau) = \widehat{f}(\tau)\chi(|\tau|T \leq 1).$$

Then (2.9) becomes

$$\psi_T\int_0^t f(t')dt' = I + II + III$$

where

$$I = \psi_T\sum_{k=1}^{\infty}\frac{t^k}{k!}\int \widehat{f}_-(\tau)(i\tau)^{(k-1)}d\tau \tag{2.10}$$

$$II = \psi_T\mathcal{F}^{-1}(\widehat{f}_+(\tau)(i\tau)^{-1}), \tag{2.11}$$

$$III = -\psi_T\int \frac{\widehat{f}_+(\tau)}{i\tau}d\tau. \tag{2.12}$$

The first contribution is bounded by

$$\|I\|_{H_t^{\frac{1}{2}}} \leq c\sum_{k=1}^{\infty}\left\|\langle\tau\rangle^{\frac{1}{2}}\mathcal{F}_t\left(\psi_T t^k\right)\right\|_{L_\tau^2}\left(\int |\widehat{f}_-(\tau)||\tau|^{k-1}d\tau\right). \tag{2.13}$$

Hence the first norm is estimated by T^k, since the support of ψ. The second norm is bounded by

$$\begin{aligned}\int |\widehat{f}_-(\tau)||\tau|^{k-1}d\tau &\leq cT^{1-k}\|f\|_{H_t^{-\frac{1}{2}}}\left(\int_{|\tau|T\leq 1}\langle\tau\rangle d\tau\right)^{\frac{1}{2}}\\ &\leq cT^{-k}\|f\|_{H_t^{-\frac{1}{2}}}.\end{aligned} \tag{2.14}$$

Combining them, the first contribution is bounded by

$$\|I\|_{H_t^{\frac{1}{2}}} \leq c\|f\|_{H_t^{-\frac{1}{2}}}. \tag{2.15}$$

When we apply $H^{\frac{1}{2}}$ norm on II, we see that

$$\begin{aligned}\|II\|_{H_t^{\frac{1}{2}}} &= \left\|\langle\tau\rangle^{\frac{1}{2}}\left(\mathcal{F}_t\psi_T * \mathcal{F}_t f_+(\tau)(i\tau)^{-1}\right)\right\|_{L^2_\tau} \\ &\leq c\left\||\tau|^{\frac{1}{2}}\widehat{\psi}_T * \mathcal{F}_t f_+(\tau)(i\tau)^{-1}\right\|_{L^2_\tau} + c\left\|\widehat{\psi}_T * \langle\tau\rangle^{\frac{1}{2}}\mathcal{F}_t f_+(\tau)(i\tau)^{-1}\right\|_{L^2_\tau}\end{aligned}$$

By using Young inequality,

$$\begin{aligned}\|II\|_{H_t^{\frac{1}{2}}} &\leq c\|\widehat{\psi}_T|\tau|^{\frac{1}{2}}\|_{L^1_\tau}\|\mathcal{F}_t f_+(\tau)|\tau|^{-1}\|_{L^2_\tau} + c\|\widehat{\psi}_T\|_{L^1_\tau}\|\langle\tau\rangle^{\frac{1}{2}}\mathcal{F}_t f_+(\tau)|\tau|^{-1}\|_{L^2_\tau} \\ &\leq c(\sup_{|\tau|\geq T^{-1}}(|\tau|^{-1+\frac{1}{2}})T^{-\frac{1}{2}} + \sup_{|\tau|\geq T^{-1}}|\tau|^{\frac{1}{2}-1+\frac{1}{2}})\|f\|_{H_t^{-\frac{1}{2}}} \\ &\leq c\|f\|_{H_t^{-\frac{1}{2}}}.\end{aligned} \tag{2.16}$$

We apply $H_t^{\frac{1}{2}}$ norm on III and by using Young inequality, we get

$$\begin{aligned}\|III\|_{H_t^{\frac{1}{2}}} &\leq c\|\psi_T\|_{H_t^{\frac{1}{2}}}\int|\tau|^{-1}|\widehat{f}_+(\tau)|d\tau \\ &\leq c\int_{|\tau|T\geq 1}|\tau|^{-1}|\widehat{f}(\tau)|d\tau.\end{aligned} \tag{2.17}$$

Combining I, II and III, we get

$$\|Lf\|_{H_t^{\frac{1}{2}}} \leq c\|f\|_{H_t^{-\frac{1}{2}}} + c\int_{|\tau|T\geq 1}|\tau|^{-1}|\widehat{f}(\tau)|d\tau. \tag{2.18}$$

Fix ξ and multiplying each by $\langle\xi\rangle^{2s}$ and taking the L^2 norm over ξ. We obtain

$$\|\psi_T U * f\|_{H^{s,\frac{1}{2}}} \leq c\|f\|_{H^{s,-\frac{1}{2}}} + c\left\{\int\langle\xi\rangle^{2s}\left(\int_{|\tau|T\geq 1}|\tau|^{-1}|\widehat{f}(\tau)|d\tau\right)^2 d\xi\right\}^{\frac{1}{2}}. \tag{2.19}$$

Substitute $U(-t)f$ for f, then we get (ii). □

3. Nonlinear estimates

In this section we derive the nonlinear estimates in the framework of L^2 based Bourgain spaces. Let f be a nonnegative function such that

$$\begin{aligned}f_i(\tau_i,\xi_i) &= \langle\tau_i+\xi_i^2\rangle^{\frac{1}{2}}\langle\xi_i\rangle^{\frac{1}{2}}|\widehat{u}(\tau_i,\xi_i)|, \\ f_3(\tau_3,\xi_3) &= \langle\tau_3-\xi_3^2\rangle^{\frac{1}{2}}\langle\xi_3\rangle^{\frac{1}{2}}|\widehat{u}(\tau_3,\xi_3)|\end{aligned}$$

where i=1,2 and $\tau, \xi \in \mathbb{R}$. Here the scales of time and space are different for the Schrödinger equation. Let $\tau = \Sigma_{i=1}^{3}\tau_i$, $\xi = \Sigma_{i=1}^{3}\xi_i$.

We consider [13],

$$\sigma - \sigma_1 - \sigma_2 - \sigma_3 = 2(\xi - \xi_1)(\xi - \xi_2), \tag{3.1}$$

where $\sigma_i = \tau_i + \xi_i^2, i = 1,2$ and $\sigma_3 = \tau_3 - \xi_3^2$. So either of the following two cases happens

$$|\xi - \xi_1| \leq 1 \textit{ or } |\xi - \xi_2| \leq 1, \tag{3.2}$$

$$|\xi - \xi_1| \ > 1 \textit{ and } |\xi - \xi_2| \ > 1. \tag{3.3}$$

Lemma 3.1. *Let $u(t)$ be supported on $\{t; |t| \leq T\}$ with $0 < T \leq 1$. Then, there exist $c, \acute{\epsilon} \geq 0$ such that the following estimate holds.*

$$\|u_1 \ u_2 \ \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} \leq cT^{\acute{\epsilon}}\|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}}\|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}}\|u_3\|_{X_{\frac{1}{2},\frac{1}{2}}}. \tag{3.4}$$

Proof. Consider

$$\begin{aligned} &\|u_1u_2\partial_x\bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} \\ &\leq \left\| \int \cdots \int_{\mathbb{R}^4} \left(\frac{\langle \xi \rangle^{\frac{1}{2}} \langle \xi_3 \rangle}{\Pi_{i=1}^{3} \langle \xi_i \rangle^{\frac{1}{2}}} \right) \frac{f_1(\tau_1,\xi_1) f_2(\tau_2,\xi_2) f_3(\tau_3,\xi_3)}{\langle \sigma \rangle^{\frac{1}{2}} \Pi_{i=1}^{3} \langle \sigma_i \rangle^{\frac{1}{2}}} d\mu_1 d\mu_2 \right\|_{L^2_{\tau,\xi}} \end{aligned} \tag{3.5}$$

where $d\mu = d\tau d\xi$.

Case 1: In the region $|\xi - \xi_1| \leq 1$ or $|\xi - \xi_2| \leq 1$, we have,

$$\frac{\langle \xi \rangle^{\frac{1}{2}} \langle \xi_3 \rangle}{\langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}} \langle \xi_3 \rangle^{\frac{1}{2}}} \leq 1. \tag{3.6}$$

Then

$$\begin{aligned} \|u_1u_2\partial_x\bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq \left\| \int \cdots \int_{\mathbb{R}^4} \frac{f_1(\tau_1,\xi_1) f_2(\tau_2,\xi_2) f_3(\tau_3,\xi_3)}{\langle \sigma \rangle^{\frac{1}{2}} \Pi_{i=1}^{3} \langle \sigma_i \rangle^{\frac{1}{2}}} d\mu_1 d\mu_2 \right\|_{L^2_{\tau,\xi}} \\ &= c \left\| \langle \sigma \rangle^{-\frac{1}{2}} \left(\langle \xi_1 \rangle^{\frac{1}{2}} |\hat{u}_1| * \langle \xi_2 \rangle^{\frac{1}{2}} |\hat{u}_2| * \langle \xi_3 \rangle^{\frac{1}{2}} |\hat{\bar{u}}_3| \right) \right\|_{L^2_{\tau,\xi}}. \end{aligned} \tag{3.7}$$

By Plancherel theorem,

$$\begin{aligned} \|u_1u_2\partial_x\bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq c \left\| \langle P \rangle^{-\frac{1}{2}} \Pi_{i=1}^{3} D_x^{\frac{1}{2}} u_i \right\|_{L^2_{t,x}} \\ &= \left\| \Pi_{i=1}^{3} D_x^{\frac{1}{2}} u_i \right\|_{X_{0,-\frac{1}{2}}} \end{aligned} \tag{3.8}$$

where $P = i\partial_t + \partial_{xx}$. Then we apply the $L^4_{t,x}$ dual Strichartz estimate and Hölder inequality,

$$\begin{aligned} \left\|\Pi_{i=1}^3 D_x^{\frac{1}{2}} u_i\right\|_{X_{0,-\frac{1}{2}}} &\leq c\left\|\Pi_{i=1}^3 D_x^{\frac{1}{2}} u_i\right\|_{L^{\frac{4}{3}}_{t,x}} \\ &\leq c\Pi_{i=1}^3 \left\|D_x^{\frac{1}{2}} u_i\right\|_{L^4_{t,x}} \\ &\leq c\|D_x^{\frac{1}{2}} u_i\|_{X_{0,\frac{3}{8}+\epsilon}}. \end{aligned} \tag{3.9}$$

We conclude that

$$\|u_1\ u_2\ \partial_x\bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} \leq cT^{\epsilon}\|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}}\|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}}\|u_3\|_{X_{\frac{1}{2},\frac{1}{2}}}. \tag{3.10}$$

Case 2: In the region $|\xi-\xi_1|\ >1$ and $|\xi-\xi_2|\ >1$.

We first assume $|\xi_1| \leq |\xi|/2$ and $|\xi_2| \leq |\xi|/2$. Since $|\xi| \leq |\xi-\xi_1|+|\xi_1|$, $|\xi| \leq |\xi-\xi_2|+|\xi_2|$, $|\xi| \leq c|\xi-\xi_1|^{\frac{1}{2}}|\xi-\xi_2|^{\frac{1}{2}}$. In addition, if $|\xi| \leq |\xi_3|/2$, since $|\xi_3| \leq |\xi-\xi_1|+|\xi-\xi_2|+|\xi|$, we get $|\xi_3| \leq c|\xi-\xi_1|^{\frac{1}{2}}|\xi-\xi_2|^{\frac{1}{2}}$. We conclude that

$$\frac{\langle\xi\rangle^{\frac{1}{2}}\langle\xi_3\rangle}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\xi_3\rangle^{\frac{1}{2}}\langle\xi-\xi_1\rangle^{\frac{1}{2}}\langle\xi-\xi_2\rangle^{\frac{1}{2}}} \leq \frac{c}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}}. \tag{3.11}$$

On the other hand $|\xi| << |\xi_1|$ and $|\xi| << |\xi_2|$, we have the identity

$$\frac{1}{\xi-\xi_1} = \frac{\xi}{(\xi-\xi_1)\xi_1} - \frac{1}{\xi_1}.$$

We can show that

$$\begin{aligned} \frac{1}{|\xi-\xi_1|} &\leq \frac{|\xi|}{|\xi-\xi_1||\xi_1|} + \frac{1}{|\xi_1|} \\ &\leq \frac{|\xi|^{1/2}}{|\xi-\xi_1|^{1/2}|\xi_1|^{1/2}}\frac{1}{|\xi_1|^{1/2}} \\ \frac{1}{|\xi-\xi_1|^{1/2}} &\leq \frac{c}{|\xi_1|^{1/2}}. \end{aligned}$$

Similarly, we can show that

$$\frac{1}{|\xi-\xi_2|^{1/2}} \leq \frac{c}{|\xi_2|^{1/2}}.$$

Since

$$|\xi_3| \leq |\xi|+|\xi_1|+|\xi_2| \leq c(|\xi_1|+|\xi_2|)$$

and if either $|\xi_1| \leq |\xi_2|$ or $|\xi_2| \leq |\xi_1|$ we then conclude

$$\frac{\langle\xi\rangle^{\frac{1}{2}}\langle\xi_3\rangle}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\xi_3\rangle^{\frac{1}{2}}\langle\xi-\xi_1\rangle^{\frac{1}{2}}\langle\xi-\xi_2\rangle^{\frac{1}{2}}} \leq \frac{c}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}}. \tag{3.12}$$

Next, we consider the four subcases according to the one of $\sigma' s$ is the largest.

Subcase 1. $(\langle\sigma\rangle \geq \max\{\langle\sigma_1\rangle, \langle\sigma_2\rangle, \langle\sigma_3\rangle\})$

By Plancherel theorem and using Hölder inequality,

(3.13)
$$\begin{aligned}\|u_1 u_2 \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq c \left\| \int \cdots \int_{\mathbb{R}^4} \frac{f_1(\tau_1,\xi_1) f_2(\tau_2,\xi_2) f_3(\tau_3,\xi_3)}{\langle\xi_1\rangle^{\frac{1}{2}} \langle\xi_2\rangle^{\frac{1}{2}} \Pi_{i=1}^3 \langle\sigma_i\rangle^{\frac{1}{2}}} d\mu_1 d\mu_2 \right\|_{L^2_{\tau,\xi}} \\ &= c \left\| u_1 u_2 D_x^{\frac{1}{2}} \bar{u}_3 \right\|_{L^2_{t,x}} \\ &\leq c \|u_1\|_{L^8_{t,x}} \|u_2\|_{L^8_{t,x}} \|D_x^{\frac{1}{2}} \bar{u}_3\|_{L^4_{t,x}}\end{aligned}$$

This estimate follows after applying the Sobolev embedding at both space and time for u_1, u_2 as well as L^4 Strichartz estimate for u_3. We use again Sobolev embedding, we get

$$\|u_1 u_2 \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} \leq c \Pi_{i=1}^3 T^{\epsilon} \|u_i\|_{X_{\frac{1}{2},\frac{1}{2}}}. \tag{3.14}$$

Subcase 2. $(\langle\sigma_1\rangle \geq \max\{\langle\sigma\rangle, \langle\sigma_2\rangle, \langle\sigma_3\rangle\})$

(3.15)
$$\begin{aligned}\|u_1 u_2 \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq c \left\| \int \cdots \int_{\mathbb{R}^4} \frac{f_1(\tau_1,\xi_1) f_2(\tau_2,\xi_2) f_3(\tau_3,\xi_3)}{\langle\xi_1\rangle^{\frac{1}{2}} \langle\xi_2\rangle^{\frac{1}{2}} \langle\sigma\rangle^{\frac{1}{2}} \langle\sigma_2\rangle^{\frac{1}{2}} \langle\sigma_3\rangle^{\frac{1}{2}}} d\mu_1 \, d\mu_2 \right\|_{L^2_{\tau,\xi}} \\ &= c \left\| \langle\sigma\rangle^{-\frac{1}{2}} \left(\langle\sigma_1\rangle^{\frac{1}{2}} |\hat{u}_1| * |\hat{u}_2| * \langle\xi_3\rangle^{\frac{1}{2}} |\hat{\bar{u}}_3| \right) \right\|_{L^2_{\tau,\xi}}.\end{aligned}$$

By Plancherel theorem and Sobolev embedding in time,

$$\begin{aligned}\|\langle P\rangle^{-\frac{1}{2}} (\langle P\rangle^{\frac{1}{2}} u_1 u_2 D_x^{\frac{1}{2}} \bar{u}_3)\|_{L^2_{t,x}} &\leq c \|\langle P\rangle^{\frac{1}{2}} u_1 u_2 D_x^{\frac{1}{2}} \bar{u}_3\|_{X_{0,-\frac{3}{8}}} \\ &\leq c \|\langle P\rangle^{\frac{1}{2}} u_1 u_2 D_x^{\frac{1}{2}} \bar{u}_3\|_{L^2_x, L_t^{\frac{8}{7}}} \\ &\leq c \|\langle P\rangle^{\frac{1}{2}} u_1\|_{L^2_t, L^8_x} \|u_2\|_{L^8_{t,x}} \|D_x^{\frac{1}{2}} \bar{u}_3\|_{L^4_{t,x}}.\end{aligned} \tag{3.16}$$

Applying Sobolev embedding again,

$$\begin{aligned}\|\langle P\rangle^{\frac{1}{2}} u_1\|_{L^2_T L^8_x} &\leq c \|\langle P\rangle^{\frac{1}{2}} u_1\|_{X_{\frac{3}{8},0}} \\ &\leq c \|u_1\|_{X_{\frac{3}{8},\frac{1}{2}}},\end{aligned} \tag{3.17}$$

We obtain the required estimate.

Subcase 3. $(\langle\sigma_2\rangle \geq \max\{\langle\sigma\rangle, \langle\sigma_1\rangle, \langle\sigma_3\rangle\})$. The proof is same as subcase 2.

Finally, Subcase 4. $(\langle\sigma_3\rangle \geq \max\{\langle\sigma\rangle, \langle\sigma_1\rangle, \langle\sigma_2\rangle\})$

We have,

(3.18)
$$\begin{aligned}\|u_1 u_2 \partial_x \bar{u}_3\|_{X_{\frac{1}{2},-\frac{1}{2}}} &\leq c \left\| \int \cdots \int_{\mathbb{R}^4} \frac{f_1(\tau_1,\xi_1) f_2(\tau_2,\xi_2) f_3(\tau_3,\xi_3)}{\langle\xi_1\rangle^{\frac{1}{2}} \langle\xi_2\rangle^{\frac{1}{2}} \langle\sigma\rangle^{\frac{1}{2}} \langle\sigma_1\rangle^{\frac{1}{2}} \langle\sigma_2\rangle^{\frac{1}{2}}} d\mu_1 \, d\mu_2 \right\|_{L^2_{\tau,\xi}} \\ &= \|u_1 u_2 \mathcal{F}^{-1} f_3\|_{X_{0,-\frac{1}{2}}}.\end{aligned}$$

Applying the dual of Strichartz estimate and Hölder inequality, we get

$$
\begin{aligned}
\|u_1u_2\mathcal{F}^{-1}f_3\|_{L^{\frac{4}{3}}_{t,x}} &\le c\|u_1\|_{L^8_{t,x}}\|u_2\|_{L^8_{t,x}}\|\mathcal{F}^{-1}f_3\|_{L^2_{t,x}} \\
&\le cT^{\epsilon}\Pi_{i=1}^3\|u_i\|_{X_{\frac{1}{2},\frac{1}{2}}}.
\end{aligned} \tag{3.19}
$$

□

Lemma 3.2. *There exists $c, \epsilon > 0$ such that for $T \in (0,1)$, then the following estimate holds.*

$$
\left\{\int\langle\xi\rangle\left(\int\langle\sigma\rangle^{-1}|\widehat{F}_+(\tau,\xi)|d\tau\right)^2 d\xi\right\}^{\frac{1}{2}} \le c\,T^{\epsilon}\,\Pi_{i=1}^3\|u_i\|_{X_{\frac{1}{2},\frac{1}{2}}}, \tag{3.20}
$$

where each $u_j(t), j = 1,2,3$ has support in $(0,T)$.

Proof. We consider in the following two cases.

Case (1). In the region $|\xi-\xi_1| \le 1$ or $|\xi-\xi_2|\le 1$.

$$
\begin{aligned}
&\left\{\int\langle\xi\rangle\left(\int\langle\sigma\rangle^{-1}|\widehat{F}_+(\tau,\xi)|d\tau\right)^2 d\xi\right\}^{\frac{1}{2}} \\
&= c\left\|\int\cdots\int_{\mathbb{R}^4}\frac{\langle\xi\rangle^{\frac{1}{2}}|\xi_3|\Pi_{i=1}^3 f_i(\tau,\xi)}{\langle\sigma\rangle\Pi_{i=1}^3\langle\xi_i\rangle\Pi_{i=1}^3\langle\sigma_i\rangle^{\frac{1}{2}}}\right\|_{L^2_\xi,L^1_\tau}
\end{aligned} \tag{3.21}
$$

Applying Schwartz estimate in time, (3.21) is bounded by

$$
c\|\langle\sigma\rangle^{-1+\rho}\|_{L^2_\tau}\|\langle\sigma\rangle^{-\rho}\left(\langle\xi_1\rangle^{1/2}|\hat{u}_1| * \langle\xi_2\rangle^{1/2}|\hat{u}_2| * \langle\xi_3\rangle^{1/2}|\hat{\bar{u}}_3|\right)\|_{L^2_{\xi,\tau}}. \tag{3.22}
$$

where $\frac{1}{3}\le\rho\le\frac{1}{2}$, by case 1 of lemma 3.1, we obtain the required estimate.

Case (2). In the region of $|\xi-\xi_1| > 1$ or $|\xi-\xi_2| > 1$. We note that [9],

$$
\langle\tau-\xi^2\rangle^{\frac{1}{2}} \ge c\langle\tau_1-\xi_1^2\rangle^{\delta}\langle\tau_2-\xi_2^2\rangle^{\delta}\langle\tau_3-\xi_3^2\rangle^{\delta}\langle\xi-\xi_1\rangle^{\frac{1}{2}-3\delta}\langle\xi-\xi_2\rangle^{\frac{1}{2}-3\delta} \tag{3.23}
$$

We can estimate multipliers as the previous lemma,

$$
\frac{\langle\xi\rangle^{\frac{1}{2}}|\xi_3|}{\Pi_{i=1}^3\langle\xi_i\rangle^{\frac{1}{2}}\langle\xi-\xi_1\rangle^{1-3\delta}\langle\xi-\xi_2\rangle^{1-3\delta}} \le \frac{c}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}-3\delta}\langle\xi_3\rangle^{\frac{1}{2}-3\delta}}
$$

Next, we consider the four subcases according to the one of $\sigma's$ is the largest.

Subcase 1. $(\langle\sigma\rangle \ge \max\{\langle\sigma_1\rangle,\langle\sigma_2\rangle,\langle\sigma_3\rangle\})$

Hence (3.21) can be estimated by

$$
\left\|\int\cdots\int_{\mathbb{R}^4}\frac{f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)}{\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}-3\delta}\langle\xi_3\rangle^{\frac{1}{2}-3\delta}\langle\sigma_1\rangle^{\frac{1}{2}+\delta}\langle\sigma_2\rangle^{\frac{1}{2}+\delta}\langle\sigma_3\rangle^{\frac{1}{2}+\delta}}d\mu_1\,d\mu_2\right\|_{L^2_\xi,L^1_\tau} \tag{3.24}
$$

where $\delta \le \frac{1}{6}$. Fix ξ and applying Young inequality for variable τ, (3.24) is bounded by

$$\left\| \int_{\mathbb{R}^2} \langle\xi\rangle^{-\frac{1}{2}+3\delta} \Pi_{i=1}^{2} \left\| \frac{f_i}{\langle\xi_i\rangle^{\frac{1}{2}} \langle\sigma_i\rangle^{\frac{1}{2}+\delta}} \right\|_{L^1_\tau} \left\| \frac{f_3}{\langle\xi_3\rangle^{\frac{1}{2}-3\delta} \langle\sigma_3\rangle^{\frac{1}{2}+\delta}} \right\|_{L^1_\tau} d\xi_1 d\xi_2 \right\|_{L^2_\xi}. \tag{3.25}$$

Since Cauchy Schwarz inequality

$$\left\| \frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\frac{1}{2}+\delta}} \right\|_{L^1_\tau} \le \left\| \langle\sigma_i\rangle^{-\frac{1}{2}-\epsilon} \right\|_{L^2} \left\| \frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\delta-\epsilon}} \right\|_{L^2}$$

(3.25) is bounded by

$$\left\| \langle\xi\rangle^{-\frac{1}{2}+3\delta} \right\|_{L^4_\xi} \left\| \Pi_{i=1}^{2} \langle\xi_i\rangle^{-\frac{1}{2}} \left\| \frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\delta-\epsilon}} \right\|_{L^2_\tau} \langle\xi_3\rangle^{-\frac{1}{2}+3\delta} \left\| \frac{f_3(\xi_3)}{\langle\sigma_3\rangle^{\delta-\epsilon}} \right\|_{L^2_\tau} \right\|_{L^4_\xi}. \tag{3.26}$$

We use again Young inequality for ξ, (3.26) is bounded by

$$\Pi_{i=1}^{2} \left\| \langle\xi_i\rangle^{-\frac{1}{2}} \left\| \frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\delta-\epsilon}} \right\|_{L^2_\tau} \right\|_{L^{\frac{4}{3}}_\xi} \left\| \langle\xi_3\rangle^{-\frac{1}{2}+3\delta} \left\| \frac{f_3(\xi_3)}{\langle\sigma_3\rangle^{\delta-\epsilon}} \right\|_{L^2_\tau} \right\|_{L^{\frac{4}{3}}_\xi} \tag{3.27}$$

Applying Hölder inequality and δ is chosen sufficiently small, (3.27) is bounded by

$$\begin{aligned} \Pi_{i=1}^{3} \|\langle\xi_i\rangle^{-\frac{1}{2}-\epsilon}\|_{L^4_\xi} \left\| \frac{f_i(\xi_i)}{\langle\sigma_i\rangle^{\delta-\epsilon}} \right\|_{L^2_{\tau,\xi}} &\le c\Pi_{i=1}^{3} \|\langle\xi_i\rangle^{\frac{1}{2}} \langle\sigma_i\rangle^{\frac{1}{2}-(\delta-\epsilon)} \hat{u}_i\|_{L^2_{\tau,\xi}} \\ &\le c\Pi_{i=1}^{3} \|u_i\|_{X_{\frac{1}{2},\frac{1}{2}-\epsilon}}. \end{aligned}$$

Similarly we can prove for the $\sigma_i's$ which is the largest of σ_j, $i \neq j$, where $1 \le i, j \le 3$ by using Cauchy-Schwarz inequality for τ. The required estimates are follow as (3.14), (3.16), (3.17) and (3.19). ◻

Lemma 3.3. *Let $s \in \mathbb{R}$, there exists $c, \acute{\epsilon} > 0$ such that for $T \in (0,1)$ then the following estimate holds.*

$$\left\| \Pi_{i=1}^{5} u_i \right\|_{X_{s,-\frac{1}{2}}} \le c\, T^{\acute{\epsilon}} \Pi_{i=1}^{5} \|u_i\|_{X_{s,\frac{1}{2}}}, \tag{3.28}$$

where each $u_i(t), i = 1, \dots, 5$ has support in $(0,T)$.

Proof. Let $\xi = \sum_{i=1}^{5} \xi_i$ then $\langle\xi\rangle \le \sum_{i=1}^{5} \langle\xi_i\rangle$. We can see simply

$$\left\| \Pi_{i=1}^{5} u_i \right\|_{X_{s,-\frac{1}{2}}} = \left\| \langle D_x\rangle^s\, \Pi_{i=1}^{5} u_i \right\|_{X_{0,-\frac{1}{2}}}. \tag{3.29}$$

Then

$$\begin{aligned}
\left\|\langle D_x\rangle^s\, \Pi_{i=1}^5 u_i\right\|_{X_{0,-\frac{1}{2}}} &\le C\sum_{i=1}^5 \left\|\langle D_x\rangle^s\, u_i\Pi_{k=1,i\neq k}^5 u_k\right\|_{X_{0,-\frac{1}{2}}}\\
&\le c\sum_{i=1}^5 \left\|D_x^s u_i\Pi_{k=1,i\neq k}^5\, u_k\right\|_{L_{t,x}^{\frac{4}{3}}}\\
&\le c\sum_{i=1}^5 \|D_x^s\, u_i\|_{L_{t,x}^4}\Pi_{k=1,i\neq k}^5\|u_k\|_{L_{t,x}^8}\\
&\le c\,T^{\epsilon}\,\Pi_{i=1}^5\|u_i\|_{X_{s,\frac{1}{2}}}
\end{aligned} \tag{3.30}$$

where the estimates are used by dual Strichartz estimate and Hölder inequality.
□

4. Gauge transformation

In this section we define the gauge transformation which is introduced by Hayashi [7] to derive from the derivative nonlinearity to some new nonlinearity.

Definition 4.1. Let G be the nonlinear map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ by

$$Gf(x) = e^{-i\int_{-\infty}^x |f(y)|^2 dy} f(x). \tag{4.1}$$

The inverse transform of f is given by

$$G^{-1}f(x) = e^{i\int_{-\infty}^x |f(y)|^2 dy} f(x). \tag{4.2}$$

Lemma 4.1. *The gauge transformation is a bicontinuous from H^1 to H^1.*

Proof. There exists $c > 0$ such that for any $f, g \in H^1$. Using the mean value theorem and there exists θ such that $0 < \theta < 1$, then

$$\begin{aligned}
Gf(x) - Gg(x) = e^{-i\int_{-\infty}^x |f(y)|^2 dy}&(f(x) - g(x))\\
&+ i\left(\int_{-\infty}^x |g(y)|^2 dy - \int_{-\infty}^x |f(y)|^2 dy\right) g(x)\\
&\int_0^1 \exp\left(-\theta i\int_0^x |f(y)|^2 dy - (1-\theta)i\int_0^x |g(y)|^2 dy\right) d\theta.
\end{aligned}$$

Taking L^2 norm on both sides and applying Hölder inequality,

$$\begin{aligned}
\|Gf - Gg\|_{L^2} &\le c\|f-g\|_{L^2} + \||f|^2 - |g|^2\|_{L^1}\|g\|_{L^2}\\
&\le c\|f-g\|_{L^2}(1 + \|f\|_{L^2}^2 + \|g\|_{L^2}^2).
\end{aligned} \tag{4.3}$$

Similarly

$$\begin{aligned}
\|\partial_x(Gf - Gg)\|_{L^2} \le c\|\partial_x(f-g)\|_{L^2} &+ \|f-g\|_{L^2}\\
&\times\{\|\partial_x g\|_{L^2} + \|f\|_{L^2} + \|g\|_{L^2} + (\|f\|_{L^2} + \|g\|_{L^2})^4\}.
\end{aligned} \tag{4.4}$$

Then we get

$$\|Gf - Gg\|_{H^1} \le c\|f-g\|_{H^1}. \tag{4.5}$$

□

We can simply verify that let

$$v(x,t) = e^{-i\int_{-\infty}^{x}|u(y)|^2dy}u(x)$$

be solution in $X^T_{\frac{1}{2},\frac{1}{2}}$ of

$$\begin{aligned} &i\partial_t v + \partial_{xx} v = i\delta|v|^2\partial_x v + \delta^2|v|^4 v \quad \text{on } (0,T)\times\mathbb{R},\ T>0, \delta\in\mathbb{R} \\ &\qquad v(0,x) = v_0(x) \in H^1, \end{aligned} \tag{4.6}$$

which is equivalent to (1.1)–(2).

Theorem 4.1. *We assume that $v_0 \in H^1(\mathbb{R})$, there exists T such that $0 < T < 1$. Set $v = G(u)$. Let $v, \tilde{v} \in X^T_{\frac{1}{2},\frac{1}{2}}$ be two solutions of* (4.6) *with the same initial data in H^1. Then there exists a unique solution in $X^T_{\frac{1}{2},\frac{1}{2}}$ for all $t \in (0,T)$.*

Proof. For any $v_0 \in H^1$ and let $M > 0$ with $\|v_0\|_{H^1} \le M$ and there exists T such that $0 < T < 1$, we want to prove that the transformation

$$v(t) = \psi_1(t)U(t)v_0(t) + i\psi_T(t)\int_0^t U(t-s)(i\delta|v|^2\partial_x v + \delta^2|v|^4 v)ds \tag{4.7}$$

is a contraction. Let $v, \tilde{v} \in X^T_{\frac{1}{2},\frac{1}{2}}$ be two solutions of (4.6) with the same initial data. Let

$$\begin{aligned} \psi_T\tilde{v}(t) &= \psi_1(t)U(t)\tilde{v}_0(t) \\ &\qquad + i\psi_T(t)\int_0^t U(t-s)(i\delta\psi_T^3(s)|\tilde{v}|^2\partial_x\tilde{v} + \delta^2\psi_T^5(s)|\tilde{v}|^4\tilde{v})ds. \end{aligned} \tag{4.8}$$

Assume $\|v\|_{X^T_{\frac{1}{2},\frac{1}{2}}} = \|\psi_T\tilde{v}\|_{X^T_{\frac{1}{2},\frac{1}{2}}} \le M$. Then for all $t \in (0,T)$ with $0 < T < 1$, applying lemma (2.1), (3.1), (3.2) and (3.3), we get

$$\|v(t) - \psi_T\tilde{v}(t)\|_{X^T_{\frac{1}{2},\frac{1}{2}}} \le CT^{\epsilon}(M^2+M^4)\|v(t) - \psi_T\tilde{v}(t)\|_{X^T_{\frac{1}{2},\frac{1}{2}}}, \tag{4.9}$$

choose $T < \frac{1}{2C(M^2+M^4)^{\frac{1}{\epsilon}}}$ is sufficiently small, then

$$\|v(t) - \psi_T\tilde{v}(t)\|_{X^T_{\frac{1}{2},\frac{1}{2}}} \le \frac{1}{2}\|v(t) - \psi_T\tilde{v}(t)\|_{X^T_{\frac{1}{2},\frac{1}{2}}}. \tag{4.10}$$

Hence $v(t) = \psi_T\tilde{v}(t)$, we conclude that $v(t) = \tilde{v}(t)$ for all $t \in [0,T)$. □

Proof of Theorem 1.1. Let $u_0 \in H^1$. We define $v_0 = G(u_0) \in H^1$, since the gauge transformation is continuous. Let $v_0^{(n)} \in C^\infty$ with $v_0^{(n)} \longrightarrow v_0$ in H^1.

We define

$$u_n(t,x) = G^{-1}(v_n)(t,x),$$
$$u_0^{(n)}(t,x) = G^{-1}(v_0^{(n)})(t,x).$$

We have

$$\|u_n\|_{L_t^\infty(L_x^2)} = \|v_n\|_{L_t^\infty(L_x^2)}. \tag{4.11}$$

We can see that

$$\partial_x u_n = \left(\partial_x v_n(x) + |v_n|^2 v_n(x)\right) e^{-i\int_{-\infty}^{x} |v_n|^2 dy}.$$

Then

$$\|\partial_x u_n\|_{L_t^\infty(L_x^2)} \le c\|\partial_x v_n\|_{L_t^\infty(L_x^2)} + c\|v_n\|^3_{L_t^\infty(L_x^2)}.$$

By Gagliardo-Nirenberg's inequality,

$$\|\partial_x u_n\|_{L_t^\infty(L_x^2)} \le c\|\partial_x v_n\|_{L_t^\infty(L_x^2)} + c\|\partial_x v_n\|_{L_t^\infty(L_x^2)}\|v_n\|^2_{L_t^\infty(L_x^2)},$$

then

$$\|\partial_x u_n\|_{L_t^\infty(L_x^2)} \le c(1 + \|v_n\|^2_{L_t^\infty(L_x^2)})\|v_n\|_{L_t^\infty(H_x^1)}, \tag{4.12}$$

for all $v_n \in H^1$. Similarly,

$$\begin{aligned}\|u_n - u_m\|_{L_t^\infty(H_x^1)} &= \|G^{-1}(v_n) - G^{-1}(v_m)\|_{L_t^\infty(H_x^1)} \\ &\le C(M)\|v_n - v_m\|_{L_t^\infty(H_x^1)}.\end{aligned}$$

We conclude that there exists a unique solution $u \in L^\infty((0,T), H^1(\mathbb{R}))$ such that $u_n \to u$ in $L^\infty((0,T), H^1(\mathbb{R}))$, then we obtain the unique solution of (1.1)–(2). □

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Yin Yin Su Win
Department of Mathematics
Kyoto University
Kyoto 606-8502, Japan
e-mail: yyswinn@math.kyoto-u.ac.jp

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