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ON LOGARITHMIC DIFFERENTIAL OPERATORS AND EQUATIONS IN THE PLANE

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ABSTRACT. Let k be a field of characteristic zero. Let $f \in k[x_0, y_0]$ be an irreducible polynomial. In this article, we study the space of polynomial partial differential equations of order one in the plane, which admit f as a solution. We provide algebraic characterizations of the associated graded $k[x_0, y_0]$-module (by degree) of this space. In particular, we show that it defines the general component of the tangent space of the curve $\{f = 0\}$ and connect it to the V-filtration of the logarithmic differential operators of the plane along $\{f = 0\}$.

1. Introduction

Let k be a field of characteristic zero and $f \in \mathbf{A} := k[x_0, y_0]$ be an irreducible polynomial.

1.1. Let E be a *polynomial partial differential equation* of order one such that

$$\sum_{i,j\geq 0} a_{i,j}(x_0, y_0)\partial_{x_0}(f)^i\partial_{y_0}(f)^j = a(x_0, y_0)f.$$

We call such a datum a *logarithmic* polynomial PDE along $\{f = 0\}$. To such a differential equation, we attach the polynomial $\mathrm{Sb}(E) \in \mathbf{A}_1 := k[x_0, y_0, x_1, y_1]$ defined by $\mathrm{Sb}(E) = \sum_{i,j\geq 0} a_{i,j}x_1^i y_1^j$, and call it the *symbol of* E. In particular, we have $\mathrm{Sb}(E)(\partial_{x_0}(f), \partial_{y_0}(f)) \in \langle f\rangle\mathbf{A}$. The degree of E is that of its symbol $\mathrm{Sb}(E)$; it is said to be *homogeneous* if its symbol is homogeneous (as a polynomial in x_1, y_1 with coefficients in $\mathbf{A}$). We justify this terminology by the analogy with the one variable case where every object which is homogeneous of degree one corresponds to an ordinary differential equation of the form $f'(x)/f(x) = a$ with $a \in k(x)$. The set of the logarithmic polynomial PDE

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can be identified, *via* symbols, with an ideal of $\mathbf{A}_1$ (the inverse image of the ideal $\langle f\rangle\mathbf{A}$ by the morphism of k-algebras $\mathbf{A}_1 \to \mathbf{A}$ defined by $x_1 \mapsto \partial_{x_0}(f)$, $y_1 \mapsto \partial_{y_0}(f)$) that we denote by $\mathscr{E}_k(f)$. For every integer $i \in \mathbf{N}$, we denote by $\mathscr{E}_k^{(i)}(f)$ the $\mathbf{A}$-module generated by the elements of $\mathscr{E}_k(f)$ homogeneous of degree i. It is a finite $\mathbf{A}$-module. We consider the associated homogeneous ideal in the ring $\mathbf{A}_1$

$$\hat{\mathscr{E}}_k(f) = \bigoplus_{i \geq 0} \mathscr{E}_k^{(i)}(f).$$

In particular, every element of $\hat{\mathscr{E}}_k(f)$ is a combination with coefficients in $\mathbf{A}$ of the symbols of homogeneous elements of $\mathscr{E}_k(f)$, but in general this inclusion is strict. The aim of this article is to study the ideal $\hat{\mathscr{E}}_k(f)$ and to show that this object appears in various contexts of algebra or geometry.

1.2. We introduce the polynomial $\Delta(f) \in \mathbf{A}_1$ by the following formula

$$\Delta(f) := \partial_{x_0}(f)x_1 + \partial_{y_0}(f)y_1.$$

Recall that, for every polynomial $h \in \mathbf{A}$, the ideal $(\langle f, \Delta(f)\rangle : h^\infty)$ of the ring $\mathbf{A}_1$ is formed by the polynomials $P \in \mathbf{A}_1$ such that there exists an integer $N \in \mathbf{N}$ satisfying $h^N P \in \langle f, \Delta(f)\rangle$. Let $\tau\colon \mathbf{A}_1 \to \mathbf{A}_1$ be the $\mathbf{A}$-automorphism defined by

$$\begin{cases} \tau(x_1) = y_1, \\ \tau(y_1) = -x_1. \end{cases} \tag{1.1}$$

We show the following main technical statement, which is the first algebraic characterization of $\hat{\mathscr{E}}_k(f)$:

Proposition 1.1. *Let k be a field of characteristic zero. Let $\mathscr{C}$ be an integral affine curve of the affine plane $\mathbf{A}_k^2$ defined by the datum of the irreducible polynomial $f \in \mathbf{A}$. Let us denote by $\partial(f)$ a* nonzero *partial derivative of f. Then we have $\tau(\hat{\mathscr{E}}_k(f)) = (\langle f, \Delta(f)\rangle : \partial(f)^\infty)$ (see formula (1.1)).*

1.3. One usually attaches to the integral affine plane curve $\mathscr{C} = \operatorname{Spec}(\mathbf{A}/\langle f\rangle)$ its *tangent space* $\pi\colon T_{\mathscr{C}/k} := \operatorname{Spec}(\operatorname{Sym}(\Omega^1_{\mathcal{O}(\mathscr{C})/k})) \to \mathscr{C}$. Recall that, one has an irreducible decomposition of $T_{\mathscr{C}/k}$ given by

$$(T_{\mathscr{C}/k})_{\mathrm{red}} = \overline{\pi^{-1}\big(\operatorname{Reg}(\mathscr{C})\big)} \cup \Bigg(\bigcup_{x \in \operatorname{Sing}(\mathscr{C})} \pi^{-1}(x)\Bigg).$$

We call $\overline{\pi^{-1}(\operatorname{Reg}(\mathscr{C}))}$ the *general component* of $T_{\mathscr{C}/k}$ by analogy with the theory of ODE. We obtain the following consequence of Proposition 1.1.

Corollary 1.2. *Let k be a field of characteristic zero. Let $\mathscr{C}$ be an integral affine curve of the affine plane $\mathbf{A}_k^2$ defined by the datum of the irreducible polynomial $f \in \mathbf{A}$. The general component of $T_{\mathscr{C}/k}$ is isomorphic to the* (*reduced*) *closed subscheme $V(\hat{\mathscr{E}}_k(f))$ of $\mathbf{A}_k^4$.*

1.4. Recall that a *differential operator* D is the datum of a combination with coefficients in $\mathbf{A}$ of the form $D = \sum_{i,j\geq 0} a_{i,j}(x_0, y_0)\partial_{x_0}^i \partial_{y_0}^j$. The *order* of D then is the maximum of the integers $i+j$ for $a_{i,j} \neq 0$. We call *(total) symbol of* D the underlying polynomial $\mathrm{Sb}(D) := \sum_{i,j\geq 0} a_{i,j} x_1^i y_1^j$. Let $\mathscr{D}$ be the ring of the differential operators on $\mathbf{A}$. We recall that one can endow it with the *V-filtration $V_\star^{\mathscr{C}}$ along $\mathscr{C}$* defined as follows: for every integer $s \in \mathbf{N}$, one sets

$$V_s^{\mathscr{C}} = \left\{ D \in \mathscr{D} \mid \forall \ell \in \mathbf{Z}\ D\big(\langle f \rangle^{\ell}\big) \subset \langle f \rangle^{\ell - s} \right\}. \tag{1.2}$$

In this formula, one adopts the convention that $\langle f \rangle^t = \mathbf{A}$ for every negative integer $t \in \mathbf{Z}$. We obtain the following characterization of $\hat{\mathscr{E}}_k(f)$.

THEOREM 1.3. *Let k be a field of characteristic zero. Let $\mathscr{C}$ be an integral affine curve of the affine plane $\mathbf{A}_k^2$ defined by the datum of the irreducible polynomial $f \in \mathbf{A}$. Let $d \in \mathbf{N}$. Let $P = \sum_{j=0}^{d} a_j(x_0, y_0) x_1^j y_1^{d-j}$ be an homogeneous polynomial in $\mathbf{A}_1$ with (total) degree (in x_1, y_1) equal to d. Let $D_P = \sum_{j=0}^{d} a_j(x_0, y_0)\partial_{x_0}^j \partial_{y_0}^{d-j}$ be the associated differential operator. The following assertions are equivalent*:

(1) *The polynomial P is the symbol of an homogeneous element of $\mathscr{E}_k(f)$ of degree d, that is, $P(\partial_{x_0}(f), \partial_{y_0}(f))$ belongs to $\langle f \rangle \mathbf{A}$.*
(2) *The differential operator D_P belongs to $V_{d-1}^{\mathscr{C}}$.*
(3) *The differential operator D_P satisfies $D_P(f^d) \in \langle f \rangle$.*

1.5. In the end, Theorem 1.3 and Proposition 1.1 improve the understanding of the scheme structure of the arc scheme $\mathscr{L}(\mathscr{C})$ associated with the (integral) affine plane curve $\mathscr{C}$. Recall that the k-scheme $\mathscr{L}(\mathscr{C})$ is classically defined by the following adjunction formula $\mathrm{Hom}_{\mathsf{Sch}_k}(T, \mathscr{L}(\mathscr{C})) \cong \mathrm{Hom}_{\mathsf{Sch}_k}(T \hat{\otimes}_k k[[t]], \mathscr{C})$ for every affine k-scheme T. (We refer, e.g., to [6], [8] for the details on arc scheme.) In [10], we proved in particular that $\mathscr{L}(\mathscr{C})$ is reduced if and only if the curve $\mathscr{C}$ is smooth (see also [3], [9]). The following statement provides a *new* characterization for a polynomial $P \in \mathbf{A}_1$ to induce a nilpotent function in $\mathcal{O}(\mathscr{L}(\mathscr{C}))$ in terms of differential operators.

COROLLARY 1.4. *Keep the notation of Proposition 1.1. Let $P \in \mathcal{O}(\mathscr{L}(\mathbf{A}_k^2))$. Let us assume that the polynomial P belongs to $\mathbf{A}_1$ with (total) degree d (in x_1, y_1). Then P induced a nilpotent element in $\mathcal{O}(\mathscr{L}(\mathscr{C}))$ if and only if the differential operator $D_{\tau(P)}$ associated with $\tau(P)$ is a combination with coefficients in $\mathbf{A}$ of homogeneous differential operators D_i of order i ($i \leq d$) such that $D_i(f^i) \in \langle f \rangle$.*

Let us stress that, in the smooth case, the ideal $(\langle f, \Delta(f) \rangle : \partial(f)^\infty)$ coincides with $\langle f, \Delta(f) \rangle$; hence, Corollaries 1.2 and 1.4 are obvious. (See also [5] for related topics.)

1.6. Conventions, notation. Let k be a field of characteristic zero. Let $f \in \mathbf{A} := k[x_0, y_0]$ be an irreducible polynomial. Let $\mathbf{A}_1 := k[x_0, y_0, x_1, y_1]$. A polynomial in $\mathbf{A}_1$ will (always) be considered as a polynomial with variables x_1, y_1 and coefficients in $\mathbf{A}$. The *degree* of a polynomial in $\mathbf{A}_1$ refers to the total degree in x_1, y_1. The notion of *homogeneity* in the polynomial ring $\mathbf{A}_1$ must be understood as the homogeneity with respect to the variables x_1, y_1 (and the corresponding degree).

2. The ideal $\mathscr{M}(f)$

Let k be a field of characteristic zero and $f \in \mathbf{A}$ be an irreducible polynomial. In particular, one of its partial derivatives is nonzero. We fix such a partial derivative and denote it by $\partial(f)$. Let $\mathscr{C} = \mathrm{Spec}(\mathbf{A}/\langle f \rangle)$. In this section, we state properties of the ideal

$$\mathscr{M}^{\partial}(f) := \big(\big\langle f, \Delta(f) \big\rangle : \partial(f)^{\infty}\big),$$

which will be useful for the proof of our main statements.

2.1. The degree function on $\mathbf{A}_1$ induces an increasing filtration $(\mathscr{M}^{\partial}_{\leq i}(f))_{i \in \mathbf{N}}$ with

$$\mathscr{M}^{\partial}_{\leq i}(f) = \big\{ P \in \mathscr{M}^{\partial}(f), \deg(P) \leq i \big\} \tag{2.1}$$

of the ideal $\mathscr{M}^{\partial}(f)$ which is exhaustive. We set

$$\mathscr{M}^{\partial}_{i}(f) := \mathscr{M}^{\partial}_{\leq i}(f) / \mathscr{M}^{\partial}_{\leq i-1}(f)$$

for every positive integer i. In particular, we obviously have $\mathscr{M}^{\partial}_{0}(f) = \langle f \rangle \mathbf{A}$. For every polynomial $P \in \mathbf{A}_1$, whose homogeneous decomposition is $P = \sum_{i \geq 0} P_i$ with $\deg(P_i) = i$, we observe that $P \in \mathscr{M}^{\partial}(f)$ if and only if $P_i \in \mathscr{M}^{\partial}(f)$ (hence, in $\mathscr{M}^{\partial}_{i}(f)$), for every integer $i \in \mathbf{N}$, by the very definition of $\mathscr{M}^{\partial}(f)$ and the fact that $\partial(f)$, f, $\Delta(f)$ are homogeneous respectively of degree 0, 0 and 1. Thus, the ideal $\mathscr{M}^{\partial}(f)$ is homogeneous.

2.2. By the relation (which directly follows from the expression of $\Delta(f)$)

$$\partial_{x_0}(f) x_1 \equiv -\partial_{y_0}(f) y_1 \mod\big(\big\langle \Delta(f) \big\rangle\big), \tag{2.2}$$

we easily obtain, for every homogeneous polynomial $P \in \mathbf{A}_1$, with $\deg(P) = d \geq 1$, the following formulas

$$\begin{aligned} \big(\partial_{x_0}(f)\big)^d P &\equiv y_1^d P\big(-\partial_{y_0}(f), \partial_{x_0}(f)\big) \mod\big(\big\langle \Delta(f) \big\rangle\big), \\ \big(\partial_{y_0}(f)\big)^d P &\equiv (-x_1)^d P\big(-\partial_{y_0}(f), \partial_{x_0}(f)\big) \mod\big(\big\langle \Delta(f) \big\rangle\big). \end{aligned} \tag{2.3}$$

We can deduce from relations (2.3) that a homogeneous polynomial $P \in \mathbf{A}_1$, with $\deg(P) = d \geq 1$, belongs to the ideal $\mathscr{M}^{\partial}(f)$ if and only if the polynomial f divides $P(-\partial_{y_0}(f), \partial_{x_0}(f))$ in the ring $\mathbf{A}$. In particular we observe that the ideal $\mathscr{M}^{\partial}(f)$ *does not depend* on the choice of the nonzero partial derivative $\partial(f)$. From now on, we simply denote it by

$$\mathscr{M}(f) := \mathscr{M}^{\partial}(f) := \big(\langle f, \Delta(f)\rangle : \partial(f)^{\infty}\big).$$

REMARK 2.1. By Section 1.3, we know that the ideal $\langle f, \Delta(f)\rangle$ is not prime in general, but the ideal $\mathscr{M}(f)$ is always prime. Indeed, let $P, Q \in \mathbf{A}_1$ such that $PQ \in \mathscr{M}(f)$. Then, by Section 2.2, we conclude that

$$P\big(-\partial_{y_0}(f), \partial_{x_0}(f)\big)Q\big(-\partial_{y_0}(f), \partial_{x_0}(f)\big) \in \langle f\rangle.$$

Since the polynomial f is irreducible, we conclude that the polynomial $P(-\partial_{y_0}(f), \partial_{x_0}(f))$ or the polynomial $Q(-\partial_{y_0}(f), \partial_{x_0}(f))$ belongs to the ideal $\langle f\rangle$, which concludes the proof of our claim. This property and the fact that the ideal $\mathscr{M}(f)$ does not depend on the choice of $\partial(f)$ can also be deduced from classical results of differential algebra (e.g., see [2, IV/17/Proposition 10] and [2, IV/9/Lemma 2], [7, §12, page 30]).

2.3. The next lemma explains theorem 1.3 in the special case where $d = 1$.

LEMMA 2.2. *Let k be a field of characteristic zero and $f \in \mathbf{A}$ be an irreducible polynomial. Let $P = ax_1 + by_1 \in \mathbf{A}_1$ be a homogeneous polynomial of degree* 1. *Then the following assertions are equivalent*:

(1) *There exists a polynomial $\alpha \in \mathbf{A} \setminus \langle f\rangle$ such that the Kähler differential form $\omega = adx_0 + bdy_0 \in \Omega^1_{\mathbf{A}/k}$ satisfies $\alpha\omega \in f\Omega^1_{\mathbf{A}/k} + \mathbf{A}df$.*
(2) *The k-derivation $D = b\partial_{x_0} - a\partial_{y_0} \in \mathrm{Der}_k(\mathbf{A})$ satisfies $D(f) \in \langle f\rangle$.*
(3) *The polynomial P belongs to $\mathscr{M}_1(f)$.*
(4) *The polynomial P is the symbol of a logarithmic polynomial partial differential equation E which belongs to $\hat{\mathscr{E}}^{(1)}_k(f)$.*

Let $\mathscr{C} = \mathrm{Spec}(\mathbf{A}/\langle f\rangle)$. These equivalences provide an isomorphism of $\mathbf{A}$-modules

$$\mathrm{Tors}\big(\Omega^1_{\mathcal{O}(\mathscr{C})/k}\big) \cong \mathscr{M}_1(f)/\big\langle fx_1, fy_1, \Delta(f)\big\rangle,$$

where we denote by $\mathrm{Tors}(\Omega^1_{\mathcal{O}(\mathscr{C})/k})$ the torsion submodule of the module $\Omega^1_{\mathcal{O}(\mathscr{C})/k}$ of the Kähler differential forms of the ring $\mathcal{O}(\mathscr{C})$.

See [1], [10] for related topics.

Proof. Equivalence (1) $\Leftrightarrow$ (2) can be proved by a direct argument of linear algebra. Equivalence (3) $\Leftrightarrow$ (4) is a direct consequence of the criterion established in Section 2.2. The equivalence (2) $\Leftrightarrow$ (4) is obvious by the very definition of $\hat{\mathscr{E}}^{(1)}_k(f)$.

Let us construct the isomorphism. We consider the $\mathbf{A}$-linear map $\mathscr{M}_1(f) \to \Omega^1_{\mathbf{A}/k}$ which sends $\omega_1 x_1 + \omega_2 y_1$ to $\omega_1 dx_0 + \omega_2 dy_0$, and compose it by the surjective $\mathbf{A}$-linear map $\Omega^1_{\mathbf{A}/k} \to \Omega^1_{\mathcal{O}(\mathscr{C})/k}$. The obtained $\mathbf{A}$-linear map θ takes its values in $\mathrm{Tors}(\Omega^1_{\mathcal{O}(\mathscr{C})/k})$ by (3) $\Rightarrow$ (1). Its kernel coincides with

$\langle fx_1, fy_1, \Delta(f)\rangle$ by the very definition of θ. The surjectivity directly follows from (1) $\Rightarrow$ (3). $\square$

2.4. Let $\sigma\colon \mathbf{A} \to \mathbf{A}$ be a ring automorphism. We construct a ring morphism

$$T(\sigma) := \sigma_1\colon \mathbf{A}_1 \to \mathbf{A}_1, \tag{2.4}$$

which extends σ, by setting

$$\begin{cases} \sigma_1(x_1) = x_1 \partial_{x_0}\sigma(x_0) + y_1 \partial_{y_0}\sigma(x_0), \\ \sigma_1(y_1) = x_1 \partial_{x_0}\sigma(y_0) + y_1 \partial_{y_0}\sigma(y_0). \end{cases}$$

(Let us stress indeed that $\sigma(x_0)$, $\sigma(y_0)$ are polynomials in $\mathbf{A}$; hence, their expressions *a priori* depend on both variables x_0 and y_0.) It is easy to observe that $\sigma_1(\Delta(f)) = \Delta(\sigma(f))$; hence, the morphism σ_1 induces an isomorphism of $\mathbf{A}$-algebras:

$$\sigma_1\colon \mathbf{A}_1/\big\langle f, \Delta(f)\big\rangle \to \mathbf{A}_1/\big\langle \sigma(f), \Delta\big(\sigma(f)\big)\big\rangle$$

whose inverse is, in a similar way, associated with the morphism σ^{-1}, that is, $(\sigma_1)^{-1} = (\sigma^{-1})_1$.

LEMMA 2.3. *Keep the notation of Section 2.4. We have* $\sigma_1(\mathscr{M}(f)) = \mathscr{M}(\sigma(f))$.

Proof. We only have to prove that $\sigma_1(\mathscr{M}(f)) \subset \mathscr{M}(\sigma(f))$. Indeed, the other inclusion can be deduced from the former inclusion applied to σ_1^{-1} and $\sigma(f)$. Let $P \in \mathscr{M}(f)$. We have to prove that $\sigma_1(P) \in \mathscr{M}(\sigma(f))$. By assumption, there exists an integer N such that the polynomials $\partial_{x_0}(f)^N P$, $\partial_{y_0}(f)^N P$ belong to the ideal $\langle f, \Delta(f)\rangle$. Then, we check that

$$\partial_{x_0}\big(\sigma(f)\big)^{2N}\sigma_1(P) = \big(\partial_{x_0}\sigma(x_0)\sigma(\partial_{x_0}f) + \partial_{x_0}\sigma(y_0)\sigma(\partial_{y_0}f)\big)^{2N}\sigma_1(P),$$

which equals the polynomial:

$$\sigma_1(P)\sum_{j=0}^{2N} \mathbf{C}_{2N}^j \partial_{x_0}\big(\sigma(x_0)\big)^j \sigma_1\big(\partial_{x_0}(f)^j\big)\partial_{x_0}\big(\sigma(y_0)\big)^{2N-j}\sigma_1\big(\partial_{y_0}(f)^{2N-j}\big).$$

It belongs to $\langle \sigma(f), \Delta(\sigma(f))\rangle$ since, for every integer $j \in \{0, \dots, 2N\}$, the integer j or the integer $2N - j$ is bigger than N. $\square$

3. The proof of our main statements

3.1. Let us prove that $\tau(\hat{\mathscr{E}}_k(f)) = \mathscr{M}(f)$ which is the statement of proposition 1.1. For simplicity we assume that $\partial(f) = \partial_{x_0}(f)$. Let $P \in \hat{\mathscr{E}}_k(f)$. Then, by Section 2.1, we may assume that P is homogeneous of degree d. By formula (2.3), we deduce that $\partial_{x_0}(f)^d \tau(P) \equiv y_1^d P(\partial_{x_0}(f), \partial_{y_0}(f)) \pmod{\langle \Delta(f)\rangle}$. Now, by assumption, we know that $P(\partial_{x_0}(f), \partial_{y_0}(f)) \equiv 0 \pmod{\langle f\rangle}$. So, we conclude that

$$\partial_{x_0}(f)^d \tau(P) \in \big\langle f, \Delta(f)\big\rangle$$

which proves that $\tau(P) \in \mathscr{M}(f)$ (in fact, that $\tau(\mathscr{E}_k^{(d)}(f)) \subset \mathscr{M}_d(f)$). Conversely, let $P \in \mathscr{M}_d(f)$. Then, there exists an integer $N \geq d$ such that $\partial_{x_0}(f)^N P(-\partial_{y_0}(f), \partial_{x_0}(f)) \in \langle f \rangle$, since $\Delta(f)(-\partial_{y_0}(f), \partial_{x_0}(f)) = 0$. Since the polynomial f is assumed to be irreducible, we have $P(-\partial_{y_0}(f), \partial_{x_0}(f)) \in \langle f \rangle$. Let us set $Q := P(-y_1, x_1)$. Then, we have $P = \tau(Q)$ and $Q \in \mathscr{E}_k^{(d)}(f)$; hence, $\tau(\mathscr{E}_k^{(d)}(f)) \supset \mathscr{M}_d(f)$. We have proved the assertion.

3.2. Let us prove Corollary 1.2. Let $\pi\colon T_{\mathscr{C}/k} \to \mathscr{C}$ be the canonical morphism. By classical arguments, one knows that $\overline{\pi^{-1}(\mathrm{Reg}(\mathscr{C}))}$ is an irreducible component of $T_{\mathscr{C}/k}$. We also know by Section 2.2 that the closed subscheme of $T_{\mathscr{C}/k}$ corresponding to $V(\mathscr{M}(f))$ is irreducible. For every field extension K of k, a K-point in $T_{\mathscr{C}/k}(K)$ is called a 1*-jet* of $\mathscr{C}$ and corresponds to a morphism of k-schemes in $\mathscr{C}(K[t]/\langle t^2 \rangle)$, that is, to a pair $(\gamma_1(t), \gamma_2(t)) \in (K[t]/\langle t^2 \rangle)^2$ which satisfies the equation $f(\gamma_1(t), \gamma_2(t)) \equiv 0 \pmod{\langle t^2 \rangle}$. Furthermore, the datum of every k-scheme morphism in $\mathscr{C}(K[t]/\langle t^2 \rangle)$ is by construction that of a morphism of k-algebras $\mathcal{O}(\mathscr{C}) \to K[t]/\langle t^2 \rangle$, which can also be seen, in an equivalent way, to be that of a morphism of k-algebras $\mathbf{A}_1/\langle f, \Delta(f) \rangle \to K$. With the latter description and the very definition of the ideal $\mathscr{M}(f)$, we easily observe that $\overline{\pi^{-1}(\mathrm{Reg}(\mathscr{C}))} \subset V(\mathscr{M}(f))$, which implies that $\overline{\pi^{-1}(\mathrm{Reg}(\mathscr{C}))} = V(\mathscr{M}(f))$.

Remark 3.1. If $n\colon \bar{\mathscr{C}} \to \mathscr{C}$ is the normalization of $\mathscr{C}$, the description above also implies that $\mathscr{M}(f)/\langle \Delta(f), f \rangle = \mathrm{Ker}(\mathrm{Sym}(n^\sharp))$.

3.3. The proof of Theorem 1.3 is based on the following lemma. We set $\partial_1 := \partial_{x_0}$ and $\partial_2 := \partial_{y_0}$.

Lemma 3.2. *Let $\alpha \in \mathbf{N}^2$. Let $\ell \in \mathbf{N}$ be an integer such that $\ell \geq |\alpha| := \alpha_1 + \alpha_2$. We denote by ∂_{ij}^α the differential operator on $\mathbf{A}$ of order $|\alpha|$ defined to be $\partial_i^{\alpha_1} \circ \partial_j^{\alpha_2}$ for every pair $(i,j) \in \{1,2\}$. Then, for every polynomials $g, P \in \mathbf{A}$, there exists a polynomial $q \in \mathbf{A}$ such that the following formula holds*

$$\partial_{ij}^\alpha(Pg^\ell) = \frac{\ell!}{(\ell - |\alpha|)!} Pg^{\ell - |\alpha|} \partial_i(g)^{\alpha_1} \partial_j(g)^{\alpha_2} + qg^{\ell - |\alpha| + 1}.$$

Let us stress that, by iterating derivations, the monomial $(\partial_i(g))^{\alpha_1}(\partial_j(g))^{\alpha_2}$ appears, in the formula, only one time.

Proof. We prove this assertion by induction on $|\alpha|$. If $|\alpha| \in \{1,2\}$, it is easy to check the formula, with $q = 0$ if $|\alpha| = 0$ and $q = \partial_i(P)$ if $\alpha = (1,0)$ and $q = \partial_j(P)$ if $\alpha = (0,1)$. Let $d \geq 1$. Let us assume that the formula holds true for every $\beta \in \mathbf{N}^2$ with $|\beta| < d$. Let α' with $|\alpha'| = d + 1$. Let $\ell \geq d + 1$.

∘ Let us assume that $\alpha' = (\alpha'_1, \alpha'_2 + 1)$. Let us note that $\alpha'_1 + \alpha'_2 = d$. Thus we have

$$\begin{aligned}\partial_{ij}^{\alpha'}\big(Pg^\ell\big) &= \partial_i^{\alpha'_1}\big(\partial_j^{\alpha'_2}\big(g^\ell\partial_j(P) + \ell P g^{\ell-1}\partial_j(g)\big)\big)\\ &= \partial_i^{\alpha'_1}\circ\partial_j^{\alpha'_2}\big(\partial_j(P)g^\ell\big) + \ell\partial_i^{\alpha'_1}\circ\partial_j^{\alpha'_2}\big(Pg^{\ell-1}\partial_j(g)\big).\end{aligned}$$

Then, we conclude the proof of this case by applying the induction hypothesis to the differential operator $\partial_i^{\alpha'_1}\circ\partial_j^{\alpha'_2}$ at each term of the former sum.

∘ Let us assume that $\alpha' = (\alpha'_1 + 1, \alpha'_2)$. Let us note that $\alpha'_1 + \alpha'_2 = d$. Thus, by the induction hypothesis, we have

$$\begin{aligned}\partial_{ij}^{\alpha'}\big(Pg^\ell\big) &= \partial_i\big(\partial_i^{\alpha'_1}\circ\partial_j^{\alpha'_2}\big(g^\ell P\big)\big)\\ &= \partial_i\left(\frac{\ell!}{(\ell-d)!}Pg^{\ell-d}\big(\partial_i(g)\big)^{\alpha'_1}\big(\partial_j(g)\big)^{\alpha'_2} + qg^{\ell-d+1}\right).\end{aligned}$$

By differentiating the last expression, we also obtain a formula of the required type. □

Theorem 1.3 is a direct consequence of the following statement:

Corollary 3.3. *Let k be a field of characteristic zero. Let $f \in \mathbf{A}$ be an irreducible polynomial with $\mathscr{C} := \mathrm{Spec}(\mathbf{A}/\langle f\rangle)$. Let $D = \sum_{j=0}^{d} a_j(x_0, y_0)\partial_{x_0}^{j}\partial_{y_0}^{d-j} \in \mathscr{D}$ be a differential operator with order $d := \mathrm{ord}(D) \in \mathbf{N}$. Then the following assertions are equivalent*:

(1) *For every integer $\ell \geq d$, the differential operator D satisfies $D(\langle f^\ell\rangle) \subset \langle f^{\ell-d+1}\rangle$.*
(2) *The polynomial f divides the polynomial $D(f^d)$ in the the ring $\mathbf{A}$.*
(3) *The polynomial $\mathrm{Sb}(D)(\partial_{x_0}(f), \partial_{y_0}(f))$ belongs to the ideal $\langle f\rangle$ of the ring $\mathbf{A}$.*

Proof. (1) ⇒ (2) We prove this implication by applying (1) to $\ell = d$. (2) ⇔ (3) From Lemma 3.2 applied to $\ell = |\alpha| = d$, $g = f$ and $P = 1$, it follows that the polynomial $\mathrm{Sb}(D)(\partial_{x_0}(f), \partial_{y_0}(f))$ belongs to the ideal $\langle f\rangle$ if and only f divides the polynomial $D(f^d)$. (3) ⇒ (1) This implication follows from Lemma 3.2 applied to $|\alpha| = d$ and $g = f$. □

4. Example

4.1. Let us compute a system of generators of the ideal $\mathscr{M}(x_0^{2m+1} - y_0^2)$. For this aim, we introduce the following polynomials of $\mathbf{A}_1$:

$$\begin{cases}\delta_1 := x_0 y_1 - \frac{2m+1}{2}y_0 x_1 \in \mathbf{A}_1,\\ \delta_2 := y_1^2 - (\frac{2m+1}{2})^2 x_0^{2m-1}x_1^2 \in \mathbf{A}_1.\end{cases} \tag{4.1}$$

By the Buchberger algorithm, a direct computational argument implies the following fact: for every positive integer $m \in \mathbf{N}^*$, the family $\{f, \Delta(f)/2, \delta_1, \delta_2\}$ is the reduced Groebner basis of the ideal $\mathscr{M}(x_0^{2m+1} - y_0^2)$.

4.2. A polynomial $f \in k[x_0, y_0]$ of multiplicity $n \geq 2$ is said *cuspidal* if there exist a ring automorphism $\sigma\colon \mathbf{A} \to \mathbf{A}$ and a positive integer m, prime to n with $m > n$, such that $\sigma(f) = x_0^m - y_0^n$.

Example 4.1. Let us assume that the field k is algebraically closed of characteristic zero. By [4], one knows that every irreducible quasi-homogeneous polynomial $f \in \mathbf{A}$ of multiplicity $n \geq 2$ is cuspidal.

We assume from now on that $n = 2$. From Section 4.1, we deduce the following statement:

Proposition 4.2. *Let k be a field of characteristic zero. Let $\mathscr{C}$ be an integral affine plane curve defined by a cuspidal (irreducible) polynomial f of multiplicity two. Then, there exists a system of coordinates (x, y) in $\mathbf{A}$ such that every homogeneous differential operator D on $\mathbf{A}$ which satisfies $D(f^{\mathrm{ord}(D)}) \subset \langle f \rangle$ is a combination with coefficients in $\mathscr{D}$ of the following differential operators*:

$$\begin{cases} f(x,y), \\ 3x^2\partial_x - 2y\partial_y, \\ 2x\partial_x + 3y\partial_y, \\ 4\partial_x^2 - 9x\partial_y^2. \end{cases}$$

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