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A NOTE ON THE SIMULTANEOUS WARING RANK OF MONOMIALS

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Abstract. In this paper, we study the complex simultaneous Waring rank for collections of monomials. For general collections, we provide a lower bound, whereas for special collections we provide a formula for the simultaneous Waring rank. Our approach is algebraic and combinatorial. We give an application to ranks of binomials and maximal simultaneous ranks. Moreover, we include an appendix of scripts written in the algebra software `Macaulay2` to experiment with simultaneous ranks.

1. Introduction

The problem of determining a minimal simultaneous decomposition for several homogeneous polynomials dates back to the work of Terracini [18], appeared in 1915, where even the existence of defective cases was observed; for more defective cases, see [11]. More precisely, let $S = \mathbb{C}[x_0, \dots, x_n]$ and let $\mathcal{F} = \{F_1, \dots, F_s\}$ be a collection of homogeneous polynomials, or *forms*, $F_1, \dots, F_s \in S$. The *simultaneous Waring rank* over $\mathbb{C}$ of $\mathcal{F}$, denoted by $\mathrm{rk}_{\mathbb{C}}\mathcal{F}$, is the minimal integer r such that there exists linear forms $L_1, \dots, L_r$ with the property that every F_i may be written as

$$F_i = \sum_{j=1}^{r} \lambda_j L_j^{d_i}, \quad \text{where } d_i = \deg F_i, \lambda_j \in \mathbb{C}.$$

Equivalently, the linear forms L_j minimally decompose simultaneously all the forms F_i; note that the degrees d_i need not be necessarily the same. The simultaneous decompositions as above are a direct generalization of the *simultaneous diagonalization* of matrices to the context of homogeneous polynomials

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and, more generally, of tensors. They have recently appeared in statistics, in the context of decompositions of moments [1, Section 3].

Terracini's geometric approach led to the notion of *Grassmann defectivity* of projective varieties; see [12, Definition 1.1] for Veronese surfaces. In his article, Terracini observed that the Veronese variety of dimension n and degree d has (k,h)-*Grassmann secant variety* filling up all the space if and only if the generic collection of $k+1$ forms of degree d can be written as a linear combination of the powers of $h+1$ linear forms. More recently, Angelini, Galuppi, Mella, and Ottaviani [2] studied identifiability properties for simultaneous decompositions of general collections.

This note is inspired by the problem of explicitly determining the complex simultaneous Waring rank of special collections of *monomials*. Since monomials are the sparsest forms and are particularly amenable to combinatorial approaches, our point of view to investigate this question is mainly algebraic and combinatorial.

The structure of this note is the following. In Section 2, we recall a useful lower bound (Proposition 2.2) for a collection of monomials $\mathcal{M}$. We interpret it combinatorially in Proposition 2.3 as an *inclusion–exclusion formula* [15, Chapter 2] of complex ranks of greatest common divisors of monomials varying in all subsets of $\mathcal{M}$. In Section 3, we derive a formula for the complex simultaneous Waring rank for special pairs (Theorem 3.2). This extends to an inclusion–exclusion formula of complex ranks of greatest common divisors for an arbitrary number of special monomials in Proposition 3.4 and Proposition 3.5. Finally, for collections given by all first derivatives of a given monomial whose exponents are strictly larger than one, we determine the complex simultaneous Waring rank in Proposition 3.8. In particular, this shows that not for every collection the complex simultaneous Waring rank is an inclusion–exclusion as before. In Section 4, we give an application of the simultaneous Waring rank to ranks of binomials and maximal simultaneous ranks of forms. Finally, in the Appendix we include scripts in the algebra software `Macaulay2` [13], which are hopefully useful to implement our results.

2. A lower bound

In this section, we recall a useful lower bound for the simultaneous Waring rank of a collection of special monomials. Before we proceed, we keep the notation from above and we let $T=\mathbb{C}[X_0,\dots,X_n]$ act by differentiation on S. Given a form $F\in S$, the *apolar ideal* of F is the ideal $F^\perp=\{G\in T\mid G(F)=0\}\subset T$. The apolarity lemma [14, Lemma 1.15] states that, whenever an ideal of reduced points $I_{\mathbb{X}}$ is contained in $F^\perp$, F admits a Waring decomposition with linear forms dual to the points in $\mathbb{X}$. In such a case, $\mathbb{X}$ is an *apolar scheme* to F.

We are now ready to introduce some lower bounds.

PROPOSITION 2.1. *Let $\mathcal{F} = \{F_j\}_{j\in J}$ be a collection of forms. The simultaneous Waring rank* $\mathrm{rk}_{\mathbb{C}}\mathcal{F}$ *is at least the dimension of the finite $\mathbb{C}$-algebra $A = T/I$, where*

$$I = (L) + \bigcap_{j\in J} \big(F_j^{\perp} : (L)\big)$$

and L is any linear form.

Proof. Let $I_{\mathbb{X}} \subset \bigcap_{j\in J} F_j^{\perp}$. In order to show the statement, we have to establish a lower bound on the cardinality of $\mathbb{X}$. We adapt the proof of [10, Theorem 3.3]. Note that the simultaneous rank is at least the dimension of the algebra

$$T/\big(I_{\mathbb{X}} : (L) + L\big)$$

for any linear form L. Let $J = \bigcap_{j\in J} F_j^{\perp}$. Since $I_{\mathbb{X}} \subset J$, we have that

$$I_{\mathbb{X}} : (L) + (L) \subset J : (L) + (L)$$

and the conclusion follows. □

Note that Proposition 2.1 provides a tool to experiment with the lower bound on any collection of forms; this is made possible by the first script of the Appendix. Moreover, for special collections of form we are able to derive more explicit lower bounds.

PROPOSITION 2.2. *Let $\mathcal{M} = \{M_j\}_{j\in J}$ be a collection of monomials, where each monomial is of the form $M_i = x_0^{a_{0,j}} x_1^{a_{1,j}} \cdots x_n^{a_{n,j}}$, and $1 \le a_{i,j}$ for every i, j. The simultaneous Waring rank* $\mathrm{rk}_{\mathbb{C}}\mathcal{M}$ *is at least the dimension of the finite $\mathbb{C}$-algebra $A = T/I$, where $I = (X_i) + \bigcap_{j\in J} M_j^{\perp}$ for any i.*

Proof. By Proposition 2.1, the sum of the Hilbert function in all degrees of the ideal

$$I = (X_i) + \left(\bigcap_{j\in J} M_j^{\perp}\right) : (X_i) = (X_i) + \bigcap_{j\in J} M_j^{\perp} : (X_i) = (X_i) + \bigcap_{j\in J} M_j^{\perp}$$

gives a lower bound to the degree of an ideal of reduced points in $\bigcap_{j\in J} M_j^{\perp}$. □

As we have established a lower bound for the simultaneous Waring rank of monomials, let us record the following observation concerning an upper bound.

REMARK 2.1. Let $\mathcal{M} = \{M_1, \ldots, M_s\}$ and let $M = \mathrm{lcm}(M_1, \ldots, M_s)$ be their least common multiple. We have $\mathrm{rk}_{\mathbb{C}}\mathcal{M} \le \mathrm{rk}_{\mathbb{C}} M$. Indeed, $M^{\perp} \subseteq M_i^{\perp}$, for each $1 \le i \le s$; hence any ideal of reduced points $I_{\mathbb{X}} \subset M^{\perp}$ would be contained in each $M_i^{\perp}$ as well, thus providing a simultaneous decomposition. This upper bound is far from being optimal in general; see Example 3.3.

By a further specialization of our collection we can give am explicit combinatorial description of the dimension of the algebra A of Proposition 2.2.

PROPOSITION 2.3. *Let $\mathcal{M} = \{M_j\}_{j\in J}$, where $M_j = x_0^{a_{0,j}} x_1^{a_{1,j}} \cdots x_n^{a_{n,j}}$, and $1 \le a_{0,j} \le a_{i,j}$ for every i, j. Let $M_{j_1,\dots,j_k}$ denote the greatest common divisor of $M_{j_1},\dots,M_{j_k}$. The dimension of the finite $\mathbb{C}$-algebra $A = T/I$, where $I = (X_0) + \bigcap_{j\in J} M_j^{\perp}$ is given by the alternating sum*

$$\dim_{\mathbb{C}} A = \sum_{\emptyset\neq[k]\subset J} (-1)^{k+1}\mathrm{rk}_{\mathbb{C}} M_{j_1,\dots,j_k}.$$

Proof. We show that the *standard monomials* of A, that is, the monomials *not* in I, are exactly the monomials dividing at least one of the monomials $\tilde{M}_j = X_1^{a_{1,j}} \cdots X_n^{a_{n,j}}$. Suppose N is a monomial dividing $\tilde{M}_j$ for some j. Then N cannot be a monomial in I, since it does not annihilate $\tilde{M}_j$. Conversely, suppose $N \notin I$. Hence there exists $\tilde{M}_j$ such that N does not annihilate it. This implies that all the exponents of N are at most those of $\tilde{M}_j$. In other words, N is a divisor of $\tilde{M}_j$.

Thus, the set of standard monomials coincides with the divisors of all of the monomials $\tilde{M}_j = X_1^{a_{1,j}} \cdots X_n^{a_{n,j}}$. The number of these divisors is given by the alternating sum $\sum_{\emptyset\neq[k]\subset J}(-1)^{k+1}\mathrm{rk}_{\mathbb{C}} M_{j_1,\dots,j_k}$. Indeed, the number of divisors of $\tilde{M}_j$ is $\mathrm{rk}_{\mathbb{C}} M_j$. Moreover, the number of common divisors of $M_{j_1},\dots,M_{j_k}$ is $\mathrm{rk}_{\mathbb{C}} M_{j_1,\dots,j_k}$. Applying the inclusion–exclusion principle [15, Chapter 2.1] we finish the proof. □

3. Special collections of monomials

We begin with special pairs of monomials and we determine their complex simultaneous Waring rank.

PROPOSITION 3.1. *Let $\mathcal{M} = \{M_1, M_2\}$, $M_1 = {x_0}^c x_1^{a_1} \cdots x_n^{a_n}$ and $M_2 = {x_0}^c x_1^{b_1} \cdots x_n^{b_n}$ with $1 \le c \le a_i$ and $1 \le c \le b_i$, for $i = 1,\dots,n$. Let $M_{12} = \gcd(M_1, M_2)$ be the greatest common divisor of M_1, M_2. Suppose $|a_i - b_i| = 0, 1$ or $|a_i - b_i| \ge c+1$. Then*

$$\mathrm{rk}_{\mathbb{C}}\mathcal{M} \le \mathrm{rk}_{\mathbb{C}} M_1 + \mathrm{rk}_{\mathbb{C}} M_2 - \mathrm{rk}_{\mathbb{C}} M_{12}.$$

Proof. For each $i > 0$, consider the ideals $A_i = (X_0^{c+1}, X_i^{a_i+1})$ and $B_i = (X_0^{c+1}, X_i^{b_i+1})$. Our argument is for each fixed i.

Let us fix an arbitrary $i = 1,\dots,n$ and suppose $a_i \le b_i$. Let $F_i \in (A_i)_{a_i+1}$ be a square-free element and note that all monomials of F_i belong to B_i except possibly a scalar multiple of $X_i^{a_i+1}$. Now consider the colon ideal

$$J_i = B_i : \left(X_i^{a_i+1}\right) = \left(X_0^{c+1}, X_i^{b_i-a_i}\right).$$

Since $|a_i - b_i| = 0, 1$ or $|a_i - b_i| \ge c+1$, there exists $H_i \in J_i$ square-free of minimal degree with no common factors with F_i. Indeed, if $a_i - b_i = 0$, we may take H_i to be a scalar. If $b_i - a_i = 1$, we may take $H_i = X_i$, as F_i is general and hence avoids such a zero. If $b_i - a_i \geq c+1$, we may take $H_i = X_0^{b_i-a_i} - X_i^{b_i-a_i}$, as F_i is general and hence avoids such zeros. Thus, $G_i = F_i H_i$ is a square-free element of $(B_i)_{b_i+1}$.

The case $a_i > b_i$ is analogous. Namely, we pick $F_i \in (B_i)_{b_i+1}$ and construct $G_i = F_i H_i$ square-free in $(A_i)_{a_i+1}$.

In conclusion, for each $i > 0$, we construct square-free binary forms $F_i \in A_i$ and $G_i \in B_i$ such that one of the two divides the other. Now, let us consider the ideals

$$I_1 = (F_1, \dots, F_n) \subset M_1^\perp \quad \text{and} \quad I_2 = (G_1, \dots, G_n) \subset M_2^\perp.$$

Note that $I_1 = I(\mathbb{X}_1)$ and $I_2 = I(\mathbb{X}_2)$, where $\mathbb{X}_i$ is a minimal apolar scheme of points for M_i. By construction, $\mathbb{X}_1 \cup \mathbb{X}_2$ is apolar to M_1 and M_2. Moreover, the ideal of $\mathbb{X}_1 \cap \mathbb{X}_2$ is precisely

$$I_1 + I_2 = \big(\gcd(F_i, G_i) : i = 1, \dots, n\big).$$

The scheme $\mathbb{X}_1 \cap \mathbb{X}_2$ is a minimal apolar scheme for M_{12}. Hence, $|\mathbb{X}_1 \cup \mathbb{X}_2| = \mathrm{rk}_{\mathbb{C}} M_1 + \mathrm{rk}_{\mathbb{C}} M_2 - \mathrm{rk}_{\mathbb{C}} M_{12}$. This completes the proof. ☐

THEOREM 3.2. *Let $\mathcal{M} = \{M_1, M_2\}$, $M_1 = {x_0}^c x_1^{a_1} \cdots x_n^{a_n}$ and $M_2 = {x_0}^c x_1^{b_1} \cdots x_n^{b_n}$, with $1 \leq c \leq a_i$ and $1 \leq c \leq b_i$, for $i = 1, \dots, n$. Let $M_{12} = \gcd(M_1, M_2)$ be the greatest common divisor of M_1, M_2. Suppose $|a_i - b_i| = 0, 1$ or $|a_i - b_i| \geq c+1$. Then the simultaneous Waring rank satisfies*

$$\mathrm{rk}_{\mathbb{C}} \mathcal{M} = \mathrm{rk}_{\mathbb{C}} M_1 + \mathrm{rk}_{\mathbb{C}} M_2 - \mathrm{rk}_{\mathbb{C}} M_{12}.$$

Proof. Proposition 3.1 gives the upper bound. We need to show the lower bound. Proposition 2.2 and Proposition 2.3 give the desired lower bound. ☐

REMARK 3.1. In the statement of Theorem 3.2, the conditions $|a_i - b_i| = 0, 1$ or $|a_i - b_i| \geq c+1$ are always satisfied for $c = 1$.

The following example shows how the construction in Proposition 3.1 gives a minimal simultaneous decomposition for special pairs.

EXAMPLE 3.3. We keep the notation of Proposition 3.1 and Theorem 3.2. Let $\mathcal{M} = \{M_1, M_2\}$, where $M_1 = x_0 x_1^3 x_2^4 x_3^7$ and $M_2 = x_0 x_1^4 x_2^2 x_3^5$. In this case, we have $c = 1$ and $M_{12} = x_0 x_1^3 x_2^2 x_3^5$. We consider the ideals $A_1 = (X_0^2, X_1^4)$, $B_1 = (X_0^2, X_1^5)$, $A_2 = (X_0^2, X_2^5)$, $B_2 = (X_0^2, X_2^3)$, $A_3 = (X_0^2, X_3^8)$, $B_3 = (X_0^2, X_3^6)$. We can choose general square-free binary forms $F_1 \in A_1, F_2 \in B_2, F_3 \in B_3$, so that, for $H_1 = X_1$, $H_2 = X_2^2 - X_0^2$, $H_3 = X_3^2 - X_3^2$, the binary forms $G_1 = F_1 H_1 \in B_1, G_2 = F_2 H_2 \in A_2, G_3 = F_3 H_3 \in A_3$ are square-free. The zeros of G_1, G_2, G_3 provides the minimal apolar scheme to the collection $\mathcal{M}$. Let $M = \mathrm{lcm}(M_1, M_2) = x_0 x_1^4 x_2^4 x_3^7$. Thus $\mathrm{rk}_{\mathbb{C}} \mathcal{M} \leq \mathrm{rk}_{\mathbb{C}} M = 200$. On the other hand, we have $\mathrm{rk}_{\mathbb{C}} \mathcal{M} = \mathrm{rk}_{\mathbb{C}} M_1 + \mathrm{rk}_{\mathbb{C}} M_2 - \mathrm{rk}_{\mathbb{C}} M_{12} = 178$.

We introduce two definitions in order to extend Theorem 3.2 to more general collections of monomials; these definitions are crucial to apply our proof techniques.

DEFINITION 3.1 ((1,1)-*free collections*). Let $\mathcal{M}=\{M_j\}_{j\in J}$ be a set of monomials, where $M_j=x_0x_1^{a_{1,j}}\cdots x_n^{a_{n,j}}$. For each $1\leq i\leq n$, let us order linearly the exponents of the variable x_i appearing in the monomials $M_j\in\mathcal{M}$; let $\mathcal{E}_i=\{a_{i,k_1}\leq a_{i,k_2}\leq\cdots\leq a_{i,k_{|J|}}\}$ be the set of those exponents. Let $\mathcal{D}_i$ denote the multiset of differences $\{a_{i,k_{s+1}}-a_{i,k_s}\}$ of successive elements in $\mathcal{E}_i$. The collection $\mathcal{M}$ is said to be $(1,1)$-*free* if, for every $i\in\{1,\dots,n\}$, the multiset $\mathcal{D}_i$ contains 1 at most one time.

DEFINITION 3.2 (*Free collections*). Let $\mathcal{M}=\{M_j\}_{j\in J}$ be a set of monomials, where $M_j=x_0^cx_1^{a_{1,j}}\cdots x_n^{a_{n,j}}$. As in Definition 3.1, let $\mathcal{D}_i$ denote the multiset of differences $\{a_{i,k_{s+1}}-a_{i,k_s}\}$ of successive elements in $\mathcal{E}_i$. The collection $\mathcal{M}$ is said to be *free* if $\mathcal{M}$ is $(1,1)$-free and, for every i, the multiset $\mathcal{D}_i$ does not contain nonzero integers smaller than or equal to c.

PROPOSITION 3.4. *Let $\mathcal{M}=\{M_j\}_{j\in J}$, where $M_j=x_0^cx_1^{a_{1,j}}\cdots x_n^{a_{n,j}}$, with $a_{i,j}\geq c$, for every i, j. Assume $\mathcal{M}$ is a free collection. Let $M_{j_1,\dots,j_k}=\gcd(M_{j_1},\dots,M_{j_k})$ be the greatest common divisor of $M_{j_1},\dots,M_{j_k}$. The simultaneous Waring rank satisfies*

$$\mathrm{rk}_{\mathbb{C}}\mathcal{M}=\sum_{\emptyset\neq[k]\subset J}(-1)^{k+1}\mathrm{rk}_{\mathbb{C}}M_{j_1,\dots,j_k}.$$

Proof. Proposition 2.2 and Proposition 2.3 give the lower bound. We show the upper bound constructing an minimal apolar scheme of points $\mathbb{X}$. For each $1\leq i\leq n$, let us order linearly the exponents of the variable x_i appearing in the monomials $M_j\in\mathcal{M}$; let $\mathcal{E}_i=\{a_{i,k_1}\leq a_{i,k_2}\leq\cdots\leq a_{i,k_{|J|}}\}$ be the set of those exponents. Since the collection $\mathcal{M}$ is *free*, for every $1\leq i\leq n$, the multiset $\mathcal{D}_i$ of differences of successive elements in $\mathcal{E}_i$ is $(1,1)$-free and does not contain nonzero integers smaller than or equal to c. For each $1\leq j\leq|J|$, the set $\mathbb{X}_i^j$ denotes the set of x_i-coordinates of points apolar to M_j. The procedure presented in the proof of Proposition 3.1 produces the sets $\mathbb{X}_i^j$, such that $\mathbb{X}_i^{k_1}\subseteq\mathbb{X}_i^{k_2}\subseteq\cdots\subseteq\mathbb{X}_i^{k_j}\subseteq\cdots\subseteq\mathbb{X}_i^{k_{|J|}}$, for every i. Since the collection $\mathcal{M}$ is *free*, the construction gives a set of points $\mathbb{X}$, all of which differ in at least one coordinate.

Thus, for each monomial M_j we have a collection of points $\mathbb{X}_j$ that constitute a minimal apolar scheme for M_j. Hence $|\mathbb{X}_j|=\mathrm{rk}_{\mathbb{C}}M_j$. The intersection $\mathbb{X}_{j_1}\cap\cdots\cap\mathbb{X}_{j_k}$ is a minimal collection of points apolar to $M_{j_1,\dots,j_k}$. Since the points in the union $\mathbb{X}=\bigcup_{j\in J}\mathbb{X}_j$ decompose all monomials in $\mathcal{M}$, $\mathrm{rk}_{\mathbb{C}}\mathcal{M}\leq|\bigcup_{j\in J}\mathbb{X}_j|$. By the inclusion–exclusion principle [15, Chapter 2.1], the cardinality of the union of the sets $\mathbb{X}_j$ is given by the alternating sum

$$\sum_{\emptyset\neq[k]\subset J}(-1)^{k+1}|\mathbb{X}_{j_1}\cap\cdots\cap\mathbb{X}_{j_k}|=\sum_{\emptyset\neq[k]\subset J}(-1)^{k+1}\mathrm{rk}_{\mathbb{C}}M_{j_1,\ldots,j_k}.$$

This concludes the proof. □

REMARK 3.2. The assumption on the collection $\mathcal{M}$ being free is necessary for our construction. For instance, if the j th multiset of differences $\mathcal{D}_j$ contains 1 two times for some j, the construction in the proof of Proposition 3.4 does not produce distinct points. Hence, when a collection is not free, we do not know whether this upper bound is valid or not.

COROLLARY 3.5. *Let $\mathcal{M}=\{M_j\}_{j\in J}$, where $M_j=x_0x_1^{a_{1,j}}\cdots x_n^{a_{n,j}}$, with $a_{i,j}\geq 1$ for every i, j. Assume $\mathcal{M}$ is a $(1,1)$-free collection. Let $M_{j_1,\ldots,j_k}=\gcd(M_{j_1},\ldots,M_{j_k})$ be the greatest common divisor of $M_{j_1},\ldots,M_{j_k}$. The simultaneous Waring rank satisfies*

$$\mathrm{rk}_{\mathbb{C}}\mathcal{M}=\sum_{\emptyset\neq[k]\subset J}(-1)^{k+1}\mathrm{rk}_{\mathbb{C}}M_{j_1,\ldots,j_k}.$$

We now extend Theorem 3.2 to monomials having, possibly, different supports.

THEOREM 3.6. *Let $M_1=x_0y_1^{a_1}\cdots y_s^{a_s}z_1^{b_1}\cdots z_r^{b_r}$ and $M_2=x_0y_1^{c_1}\cdots y_s^{c_s}t_1^{d_1}\cdots t_\ell^{d_\ell}$. Here $b_i,d_i\geq 2$, and $a_i,c_i\geq 0$. Let $M_{12}=\gcd(M_1,M_2)$ be the greatest common divisor of M_1 and M_2. The simultaneous Waring rank of $\mathcal{M}=\{M_1,M_2\}$ satisfies*

$$\mathrm{rk}_{\mathbb{C}}\mathcal{M}=\mathrm{rk}_{\mathbb{C}}M_1+\mathrm{rk}_{\mathbb{C}}M_2-\mathrm{rk}_{\mathbb{C}}M_{12}.$$

Proof. As in Theorem 3.2, the lower bound is deduced from a variation of Proposition 2.2: the algebra $A=\mathbb{C}[X_0,Y_s,Z_r,T_\ell]/I$, where $I=(X_0)+M_1^\perp\cap M_2^\perp$, has dimension $\mathrm{rk}_{\mathbb{C}}M_1+\mathrm{rk}_{\mathbb{C}}M_2-\mathrm{rk}_{\mathbb{C}}M_{12}$. Indeed, the monomials not in I are formed by the union of those that are not in $(X_0)+M_1^\perp$ and those that are not in $(X_0)+M_2^\perp$. These two sets have in common the monomials not divisible by X_0 and dividing M_{12}. Hence, the dimension of the finite $\mathbb{C}$-algebra A is $\mathrm{rk}_{\mathbb{C}}M_1+\mathrm{rk}_{\mathbb{C}}M_2-\mathrm{rk}_{\mathbb{C}}M_{12}$. To see the upper bound, for both M_1 and M_2 we construct minimal apolar schemes containing the points $[1:\alpha_{1,i}:\cdots:\alpha_{s,i}:0:\cdots:0]$ (here the number of the last zeros is $r+\ell$), where $\sum_{i=1}^{\min\{a_i,c_i\}+1}\alpha_{k,i}=0$ for each $1\leq k\leq s$. Indeed, these points form a minimal apolar scheme for M_{12}. Moreover, they can be extended to minimal apolar schemes for both M_1 and M_2 by the assumption $b_i,d_i\geq 2$. Taking the union of those minimal apolar schemes for M_1 and M_2 yields the result. □

REMARK 3.3. In Theorem 3.6, assuming instead $b_i=1$ (or $d_i=1$) for some i, we do not know whether the upper bound is valid or not.

COROLLARY 3.7. *Let* $\mathcal{M} = \{M_1, M_2\}$, $M_1 = x_0 x_1^{a_1} \cdots x_n^{a_n}$ *and* $M_2 = x_0 y_1^{b_1} \cdots y_s^{b_s}$, *with* $a_i, b_i \geq 2$. *The simultaneous Waring rank of* $\mathcal{M}$ *satisfies*

$$\mathrm{rk}_{\mathbb{C}}\mathcal{M} = \mathrm{rk}_{\mathbb{C}} M_1 + \mathrm{rk}_{\mathbb{C}} M_2 - 1.$$

A natural case is when the collection of monomials $\mathcal{M}$ is given by all the first partial derivatives of a fixed monomial M.

PROPOSITION 3.8. *Let* $M = x_0^{a_0} \cdots x_n^{a_n}$ *be a monomial with* $a_i > 1$ *for all* i. *Let* $\mathcal{M} = \{\partial_{x_0} M, \ldots, \partial_{x_n} M\}$ *be the collection of its first derivatives. Then*

$$\mathrm{rk}_{\mathbb{C}} M = \mathrm{rk}_{\mathbb{C}} \mathcal{M}.$$

Proof. Let $H = \bigcap_{i=0}^{n} (\partial_{x_i} M)^{\perp}$ and $d = \deg M$. Since $M^{\perp} + T_d = H$, by the apolarity lemma [14, Lemma 1.15], we have $\mathrm{rk}_{\mathbb{C}} \mathcal{M} \leq \mathrm{rk}_{\mathbb{C}} M$. Again, by the apolarity lemma, proving the equality is now equivalent to showing that every reduced ideal of points in H is not supported on strictly less than $\mathrm{rk}_{\mathbb{C}} M$ points. Let us set $F = x_0^{a_0 - 1} x_1^{a_1} \cdots x_n^{a_n}$. Note that $\mathrm{rk}_{\mathbb{C}} F = \mathrm{rk}_{\mathbb{C}} M$ and, by definition of H, $H \subset F^{\perp}$. This concludes the proof. $\square$

4. Final remarks

Sub-additivity of the rank and rank of binomials. From the definition, it follows that $\mathrm{rk}_{\mathbb{C}}(F_1 + \cdots + F_r) \leq \mathrm{rk}_{\mathbb{C}} F_1 + \cdots + \mathrm{rk}_{\mathbb{C}} F_r$. The well-known Strassen's conjecture states that, whenever the forms are in disjoint sets of variables, then equality holds; see [16], [9], [17]. The simultaneous Waring rank enters the picture because

$$\mathrm{rk}_{\mathbb{C}}(F_1 + \cdots + F_r) \leq \mathrm{rk}_{\mathbb{C}}\{F_1, \ldots, F_r\} \leq \mathrm{rk}_{\mathbb{C}} F_1 + \cdots + \mathrm{rk}_{\mathbb{C}} F_r.$$

In particular, whenever the simultaneous rank is strictly less than the sum of the ranks, the rank is strictly sub-additive.

As a direct consequence of Theorem 3.2, we obtain an upper bound for the rank of special binomials.

COROLLARY 4.1. *Let* M_1, M_2, M_{12} *be as in Theorem* 3.2. *Then*

$$\mathrm{rk}_{\mathbb{C}}(M_1 + M_2) \leq \mathrm{rk}_{\mathbb{C}} M_1 + \mathrm{rk}_{\mathbb{C}} M_2 - \mathrm{rk}_{\mathbb{C}} M_{12}.$$

EXAMPLE 4.2. Let $M_1 = x_0 x_1 x_2 x_3^2$ and $M_2 = x_0 x_1 x_2^2 x_3$, we have $\mathrm{rk}_{\mathbb{C}} M_1 = \mathrm{rk}_{\mathbb{C}} M_2 = 12$. Thus, $\mathrm{rk}_{\mathbb{C}}(M_1 + M_2) \leq 24$. Corollary 4.1 implies the improved upper bound

$$\mathrm{rk}_{\mathbb{C}}(M_1 + M_2) \leq 12 + 12 - 8 = 16.$$

However, in a private communication, Bruce Reznick informed us that $\mathrm{rk}_{\mathbb{C}}(M_1 + M_2) \leq 15$. Hence, this bound is not sharp.

High simultaneous rank. The study of *maximal* Waring ranks compared with *generic* Waring ranks has attracted a lot of attention; see, for instance,

[5], [7], [6]. A similar analysis could be performed in the case of the simultaneous Waring ranks. Here several difficulties arise due, for example, to the fact that generic simultaneous ranks are largely unknown.

REMARK 4.1. The simultaneous rank for a generic collection of $k+1$ forms of degree d, in $n+1$ variable, is $h+1$ if and only if the Segre–Veronese variety $\mathbb{P}^k \times \mathbb{P}^n$ embedded in bidegree $(1,d)$ is such that its h-secant variety fills up the ambient space. Indeed, the simultaneous rank for such a generic collection can be thought of the rank of the generic point of the projective k-dimensional linear space spanned by those $k+1$ forms of degree d. Thus, the knowledge of the generic simultaneous rank relates to the open problem of classifying defective Segre–Veronese varieties; see, for example, [3].

In [8], it is shown that monomials in three variables provide examples of forms having complex rank higher than the generic rank. An analogue result is the following:

PROPOSITION 4.3. *Let* $S = \mathbb{C}[x_0, x_1, x_2]$. *Then*

$$\operatorname{rk}_{\mathbb{C}}\{x_0x_1^t x_2^{t+1}, x_0x_1^{t+1}x_2^t\} = t^2 + 4t + 3,$$
$$\operatorname{rk}_{\mathbb{C}}\{x_0x_1^t x_2^{t+2}, x_0x_1^{t+2}x_2^t\} = t^2 + 6t + 5.$$

For $t \geq 1$, *these ranks are strictly higher than* $\operatorname{rk}_{\mathbb{C}}\{F_1, F_2\}$, *for generic* $F_1, F_2 \in S_d$.

Proof. The rank for the generic pair $F_1, F_2 \in S_d$ is

$$\left\lceil \frac{1}{2}\binom{d+2}{2} \right\rceil$$

unless $d = 3$, in which case the 5 th secant variety has codimension one, rather than filling up the ambient space; see [4, Theorem 1.3]. For $d = 2t+2$, the generic rank is

$$\left\lceil t^2 + \frac{7}{2}t + 3 \right\rceil.$$

For $d = 2t+3$, the generic rank is

$$\left\lceil t^2 + \frac{9}{2}t + 5 \right\rceil.$$

The result follows by a direct computation. □

Appendix

```
------ We use Apolarity script available at https://github.com/
       zteitler/ApolarIdeal.m2.
------ Script 1. Input: List of homogeneous polynomials of
       arbitrary degrees.
```

```
------ Output: Lower bound for the simultaneous Waring rank,
       by Proposition 2.1.

Lowerbound = method()
Lowerbound (List):=(L) -> (
    R:= ring(first L);
    J = sub(ideal(1),R);
    for i from 0 to #L-1 do
    J = intersect(J,Apolar(L_i));
    l = random(1,R);
    I = ideal(l)+(J:ideal(l));
    D = {};
    for f in L do D=append(D,degree(f));
    s = (max D)_0;
    lb=sum for i from 0 to s list hilbertFunction(i,I);
    return lb
    );

------ Script 2. Input: A monomial.
------ Output: Its complex Waring rank.

CWMon = method()
CWMon (RingElement):=(m)-> (
    Exponents=exponents(m);
    if(#Exponents > 1 ) then (
    error "Expected monomial as input";
  );
   Exponents=(exponents(m))_0;
   Exponents1={};
   for i in Exponents do if (i!=0) then Exponents1=
   append(Exponents1,i+1);
   return product(Exponents1)/(min(Exponents1))
   );

------ Script 3. Input: A monomial.
------ Output: Position(s) of the minimal exponent(s).

Minspot = method()
Minspot(RingElement):=(m)-> (
     if (#exponents(m)>1) then (
    error "Expected a monomial as input";
  );
    Minspt = {};
```

```
    for i from 0 to #(exponents(m))_0-1 do (
        if ( (exponents(m))_0_i == min((exponents(m))_0)) then
    Minspt = append(Minspt,i+1)
    );
    return set Minspt
    );

------ Script 4. Input: A list of monomials.
------ Output: Alternating sum of the complex Waring ranks of
       their gcd.

AlternatingSum = method()
AlternatingSum(List):=(L) -> (
t=#L;
  Alt = {};
 for k from 1 to t do (
        Subsets = subsets(t,k);
    for S in Subsets do (
            SList = toList S;
            G = L_SList_0;
            for j from 1 to #SList-1 do G = gcd(G,L_SList_j);
            rk = (-1)^(k+1)*CWMon(G);
         Alt = append(Alt, rk);
            );
        );
    return sum(Alt)
);

------ Script 5. Input: A list of monomials.
------ Output: Dimension of the algebra appearing in
       Proposition 2.3.

DimAlg = method()
DimAlg(List):=(L) -> (
    R:= ring(first L);
    for m in L do
    if (#exponents(m)>1) then (
    error "Expected list of monomials as input";
  );
for m in L do
    if (min((exponents(m))_0)==0) then (
    error "Expected list of monomials with full support";
  );
```

```
Min = {};
for m in L do Min=append(Min, Minspot(m));
Int = Min_0;
for i from 1 to #L-1 do Int = Int*Min_i;
if (#Int == 0) then (error "Monomials do not satisfy hypothesis
of Proposition 2.3";
);
IntList = toList Int;
r = IntList_0;
t = ideal (vars R)_{r-1};
J = sub(ideal(1),R);
for i from 0 to #L-1 do J = intersect(J,Apolar(L_i));
I = t+(J:t);
D = {};
for m in L do D=append(D,degree(m));
s = (max D)_0;
da = sum for i from 0 to s list hilbertFunction(i,I);
return da;
);

------ If L satisfies Proposition 2.3, then

DimAlg(L)==AlternatingSum(L)

------ is true

------ Script 6. Input: A list of monomials.
------ Output: Check if the list is a free collection, according
       to Definition 3.2.

Checkfree = method()
Checkfree(List):=(L) -> (
R:= ring(first L);
Min = {};
for m in L do Min=append(Min, Minspot(m));
Int = Min_0;
for i from 1 to #L-1 do Int = Int*Min_i;
if (#Int == 0) then (error "Monomials do not satisfy hypothesis
of Proposition 3.4";
);
IntList = toList Int;
r = IntList_0;
T = {};
```

```
for m in L do T = append(T,(exponents(m))_0_(r-1));
Tset = set T;
if (#Tset != 1) then (error "Monomials do not satisfy hypothesis
of Proposition 3.4";
    );
c = T_0;
v =  numgens R-1;
for i from 0 to v do (
    Exp = {};
    for j from 0 to #L-1 do Exp = append(Exp,
    (exponents(L_j))_0_i);
    Exp = sort Exp;
    Diff =  {};
    for k from 0 to #Exp-2 do Diff = append(Diff, Exp_(k+1)-
    Exp_k);
    Num1 =  {};
    Numc =  {};
    for l from 0 to #Diff-1 do (
    if (Diff_l == 1) then Num1=append(Num1,l)
    else if (Diff_l>c) then Numc=append(Numc,l)
    );
if (#Num1 > 1) then (error "Monomials do not satisfy hypothesis
of Proposition 3.4";
    );
if (c!=1 and #Numc > 0) then (error "Monomials do not satisfy
hypothesis of Proposition 3.4";
    );
);
return print "The collection of monomials is free"
);
```

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