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# ULTRAPRODUCTS OF CROSSED PRODUCT VON NEUMANN ALGEBRAS

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Abstract. We study a relationship between the ultraproduct of a crossed product von Neumann algebra and the crossed product of an ultraproduct von Neumann algebra. As an application, the continuous core of an ultraproduct von Neumann algebra is described.

## 1. Preliminary

**1.1. Ultraproduct.** Our references are [1], [7]. In this paper, we denote by $\omega$ a fixed free ultrafilter on $\mathbb{N} = \{1, 2, \dots\}$. By $M$, we always denote a von Neumann algebra with separable predual. The automorphism group of a von Neumann algebra $N$ is denoted by $\mathrm{Aut}(N)$, and the center of $N$ is by $Z(N)$.

Denote by $\ell^\infty(M)$ the unital C$^*$-algebra which consists of all norm bounded sequences $(x^\nu) = (x^1, x^2, \dots)$, $x^\nu \in M$. An element $(x^\nu) \in \ell^\infty(M)$ is said to be *$\omega$-trivial* when $x^\nu$ converges to 0 in the strong$*$ topology as $\nu \to \omega$. By $\mathscr{I}_\omega(M)$, we denote the set of all $\omega$-trivial sequences. It is known that $\mathscr{I}_\omega(M)$ is a C$^*$-subalgebra of $\ell^\infty(M)$, but it is not an ideal when $M$ is infinite. Hence, we consider its normalizer $\mathscr{M}^\omega(M)$ defined by

$$\mathscr{M}^\omega(M) := \left\{x \in \ell^\infty(M) \mid x\mathscr{I}_\omega(M) + \mathscr{I}_\omega(M)x \subset \mathscr{I}_\omega(M)\right\}.$$

Then the quotient C$^*$-algebra $M^\omega := \mathscr{M}^\omega(M)/\mathscr{I}_\omega(M)$ is in fact a W$^*$-algebra that is called an *ultraproduct von Neumann algebra*. We denote by $(x^\nu)^\omega$ the equivalence class $(x^\nu) + \mathscr{I}_\omega(M)$ for $(x^\nu) \in \mathscr{M}^\omega(M)$.

Note that $M$ is regarded as a von Neumann subalgebra of $M^\omega$ by mapping $x \in M$ to its constant sequence $(x, x, \dots)^\omega =: x^\omega$. Since the norm unit

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ball of $M$ is $\sigma$-weakly compact, each $(x^\nu) \in \ell^\infty(M)$ has the $\sigma$-weak ultralimit $\lim_{\nu\to\omega} x^\nu$. This gives us a well-defined map $E_M : M^\omega \to M$ defined by $E_M((x^\nu)) := \lim_{\nu\to\omega} x^\nu$. Then $E_M$ is actually a faithful normal conditional expectation. For a weight $\varphi$ on $M$, we denote by $\varphi^\omega$ the ultraproduct weight of $\varphi$ on $M^\omega$, that is, $\varphi^\omega := \varphi \circ E_M$.

An element $(x^\nu) \in \ell^\infty(M)$ is said to be *$\omega$-central* if $x^\nu\varphi - \varphi x^\nu \in M_*$ converges to 0 in norm as $\nu \to \omega$ for all $\varphi \in M_*$, where we use the usual notation $a\varphi(x) := \varphi(xa)$ and $\varphi a(x) := \varphi(ax)$ for $a, x \in M$ and $\varphi \in M_*$. Then $\mathscr{C}_\omega(M)$, the set of all $\omega$-central sequences, is a unital C$^*$-subalgebra of $\ell^\infty(M)$ and contains $\mathscr{I}_\omega(M)$. We denote by $M_\omega$ the quotient C$^*$-algebra $\mathscr{C}_\omega(M)/\mathscr{I}_\omega(M)$ that is a W$^*$-subalgebra of $M^\omega$. We will call $M_\omega$ the *asymptotic centralizer* of $M$.

**1.2. Action and crossed product.** Let $G$ be a locally compact Hausdorff group that is always assumed to be second countable. We use the usual notation $C_c(G)$ and $L^2(G)$ for the set of compactly supported continuous functions on $G$ and the Hilbert space associated with a fixed left invariant Haar measure on $G$. The $*$-algebra operations of $C_c(G)$ are defined as usual $f * g(s) := \int_G f(t)g(t^{-1}s)\,dt$ and $f^*(s) := \Delta(s)^{-1}\overline{f(s^{-1})}$ for $f, g \in C_c(G)$ and $s \in G$, where $\Delta$ denotes the modular function of $G$ and $dt$ the left invariant Haar measure.

An action of $G$ on $M$ means a group homomorphism $\alpha : G \ni s \mapsto \alpha_s \in \mathrm{Aut}(M)$ such that $\|\varphi \circ \alpha_s - \varphi\|_{M_*} \to 0$ for all $\varphi \in M_*$ if $s \to e$ in $G$, where $e$ denotes the neutral element of $G$. The fixed point algebra $M^\alpha$ means the collection of all $x \in M$ such that $\alpha_s(x) = x$ for all $s \in G$. We next introduce the crossed product von Neumann algebra $M \rtimes_\alpha G$ as follows. Suppose $M$ is acting on a Hilbert space $H$. We define the operators $\pi_\alpha(x)$ and $\lambda^\alpha(s)$ on the tensor product Hilbert space $H \otimes L^2(G) = L^2(G, H)$ as follows: for $x \in M$, $s, t \in G$ and $\xi \in L^2(G, H)$,

$$\bigl(\pi_\alpha(x)\xi\bigr)(t) := \alpha_{t^{-1}}(x)\xi(t), \qquad \bigl(\lambda^\alpha(s)\xi\bigr)(t) := \xi\bigl(s^{-1}t\bigr).$$

We can also write $\lambda^\alpha(s) = 1 \otimes \lambda(s)$ for $s \in G$, where $\lambda$ denotes the left regular representation on $L^2(G)$. Then $M \rtimes_\alpha G$ denotes the von Neumann algebra generated by $\pi_\alpha(M)$ and $\lambda^\alpha(G)$. For $f \in C_c(G)$, we denote $\lambda^\alpha(f) := \int_G f(s)\lambda^\alpha(s)\,ds$. Note that $\lambda^\alpha$ is a $*$-representation of $C_c(G)$ on $H \otimes L^2(G)$.

For Abelian $G$, $M \rtimes_\alpha G$ admits the dual action $\widehat{\alpha}$ of the dual group $\widehat{G}$ satisfying

$$\widehat{\alpha}_p\bigl(\pi_\alpha(x)\bigr) = \pi_\alpha(x), \qquad \widehat{\alpha}_p\bigl(\lambda^\alpha(s)\bigr) = \overline{\langle s, p\rangle}\lambda^\alpha(s), \quad x \in M, s \in G, p \in \widehat{G},$$

where $\langle\cdot,\cdot\rangle$ denotes the dual coupling of $G$ and $\widehat{G}$. By Takesaki duality, we have an isomorphism $\Gamma_\alpha$ from $(M \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ onto $M \otimes B(L^2(G))$ such that

$$\Gamma_\alpha\bigl(\pi_{\widehat{\alpha}}\bigl(\pi_\alpha(x)\bigr)\bigr) = \pi_\alpha(x), \qquad \Gamma_\alpha\bigl(\pi_{\widehat{\alpha}}\bigl(\lambda^\alpha(s)\bigr)\bigr) = 1 \otimes \lambda(s),$$
$$\Gamma_\alpha\bigl(\lambda^{\widehat{\alpha}}(p)\bigr) = 1 \otimes \overline{\langle p, \cdot\rangle}$$

for $x \in M$, $s \in G$ and $p \in \widehat{G}$. As for the bidual action $\widehat{\widehat{\alpha}}$, we have $\Gamma_\alpha \circ \widehat{\widehat{\alpha}}_s = \alpha_s \otimes \operatorname{Ad}\rho(s)$ for $s \in G$, where $\rho$ denotes the right regular representation on $L^2(G)$.

## 2. Main result

**2.1. Equicontinuous parts.** Readers are referred to [5, Chapter 3]. Note that basic results introduced there are concerned with $\mathbb{R}$, but they also hold for a general locally compact Hausdorff groups.

DEFINITION 2.1. Let $\alpha$ be an action of a locally compact Hausdorff group $G$ on a von Neumann algebra $M$. A norm bounded sequence $(x^\nu)$, $x^\nu \in M$, is said to be $(\alpha,\omega)$-*equicontinuous* when the following holds: for every $\sigma$-strong* neighbourhood $V$ of $0 \in M$, there exist a neighbourhood $U$ of the neutral element $e \in G$ and $A \in \omega$ such that $\alpha_s(x^\nu) - x^\nu \in V$ for all $s \in U$ and $\nu \in A$.

Denote by $\mathscr{E}_\alpha^\omega(M)$ the collection of all $(\alpha,\omega)$-equicontinuous sequences. Set

$$M_\alpha^\omega := \big(\mathscr{E}_\alpha^\omega(M) \cap \mathscr{M}^\omega(M)\big)/\mathscr{I}_\omega(M)$$

and

$$M_{\omega,\alpha} := \big(\mathscr{E}_\alpha^\omega(M) \cap \mathscr{C}^\omega(M)\big)/\mathscr{I}_\omega(M),$$

which we will call the *equicontinuous parts* of $M^\omega$ and $M_\omega$, respectively. Note that $M_{\omega,\alpha} \subset M_\alpha^\omega$, $M \subset M_\alpha^\omega$ and they are von Neumann subalgebras which admit the $G$-action $\alpha^\omega$ defined by $\alpha_s^\omega((x^\nu)^\omega) := (\alpha_s(x^\nu))^\omega$ for $s \in G$ and $(x^\nu)^\omega \in M_\alpha^\omega$.

Note the crucial fact that $\mathscr{M}^\omega(M)$ coincides with $\mathscr{E}_\alpha^\omega(M)$ for $\alpha := \sigma^\varphi$, the modular automorphism group of a given faithful normal state $\varphi$ on $M$. (See [1, Proposition 4.11] and [6, Theorem 1.5] for its proof.)

A useful tool to construct an equicontinuous sequence is to average a norm bounded sequence by $L^1$-function. To be precise, we let $f \in L^1(G)$ and $(x^\nu) \in \ell^\infty(M)$. Then $(\alpha_f(x^\nu))$ is $(\alpha,\omega)$-equicontinuous, where $\alpha_f(y) = \int_G f(s)\alpha_s(y)\,ds$ for $y \in M$. Note that the averaging and the ultraproduct of an equicontinuous sequence are commutative operations, that is, for $x := (x^\nu)^\omega \in M_\alpha^\omega$, we have $\alpha_f^\omega(x) = (\alpha_f(x^\nu))^\omega$. In particular, the set which consists of $(\alpha_f(x^\nu))^\omega$, $f \in L^1(G)$ and $(x^\nu) \in \mathscr{M}^\omega(M)$ is $\sigma$-weakly dense in $M_\alpha^\omega$.

EXAMPLE 2.2. Consider the action $\alpha$ of $G$ on $M := L^\infty(G)$ by left translation. Then $M_\alpha^\omega$ is actually nothing but $M$. This fact tells us that equicontinuous parts could be small. We need the following claim to show this: if a uniformly norm bounded net $\{f_n\}_{n\in I}$ in $M$ converges to 0 in the $\sigma$-weak topology, then the convolution $g * f_n$ converges to 0 compact uniformly for all

$g \in L^1(G)$. Then for $t \in G$

$$g * f_n(t) = \int_G g(ts) f_n(s^{-1})\, ds = \langle g_{t^{-1}}, \tilde{f}_n \rangle,$$

where $g_r(s) := g(r^{-1}s)$, $\tilde{f}_n(s) := f_n(s^{-1})$, and $\langle \cdot, \cdot \rangle$ denotes the pairing of $L^1(G)$ and $L^\infty(G)$. It is trivial that $\tilde{f}_n \to 0$ $\sigma$-weakly, and $g * f_n$ converges to 0 pointwise.

Let $K \subset G$ be a compact set. The map $K \ni t \mapsto g_{t^{-1}} \in L^1(G)$ is norm-continuous. Thus for $\varepsilon > 0$, there exist $t_1, \dots, t_k \in K$ such that for any $t \in K$, $\|g_{t^{-1}} - g_{t_i^{-1}}\| < \varepsilon$ for some $t_i$. Take $n_0 \in I$ so that $|\langle g_{t_i^{-1}}, \tilde{f}_n \rangle| < \varepsilon$ for all $i = 1, \dots, k$ and $n \geq n_0$. For $t \in K$, take $t_i$ so that $\|g_{t^{-1}} - g_{t_i^{-1}}\| < \varepsilon$. When $n \geq n_0$, we have

$$\begin{aligned} \big| g * f_n(t) \big| &\leq \big| \langle g_{t^{-1}} - g_{t_i^{-1}}, \tilde{f}_n \rangle \big| + \big| \langle g_{t_i^{-1}}, \tilde{f}_n \rangle \big| \\ &\leq \varepsilon \|f_n\|_\infty + \big| \langle g_{t_i^{-1}}, \tilde{f}_n \rangle \big| < \big( \|f_n\|_\infty + 1 \big) \varepsilon. \end{aligned}$$

So, we have proved the claim.

Recall that $M_\alpha^\omega$ has the $\sigma$-weakly total set which consists of $(\alpha_g(f^\nu))$, $g \in L^1(G)$ and $(f^\nu) \in \ell^\infty(M)$. Putting $f := \lim_{\nu \to \omega} f^\nu$, we see $\alpha_g(f^\nu) = g * f^\nu \to g * f$ compact uniformly. In particular, $(\alpha_g(f^\nu))^\omega$ equals the constant sequence $\alpha_g(f)^\omega$.

LEMMA 2.3. *There exists a unique faithful normal conditional expectation $E_\alpha$ from $M^\omega$ onto $M_\alpha^\omega$ such that for an arbitrary faithful normal semifinite weight $\varphi$ on $M$, one has $\varphi^\omega = \varphi^\omega \circ E_\alpha$.*

*Proof.* Since $M_\alpha^\omega$ contains $M$, $\varphi^\omega$ is semifinite on $M_\alpha^\omega$. We will show that $M_\alpha^\omega$ is globally invariant by $\sigma^{\varphi^\omega}$. By [1, Theorem 4.1], [3, Proposition 2.1] or [8, Theorem 2.1], we obtain $\sigma_t^{\varphi^\omega}((x^\nu)^\omega) = (\sigma_t^\varphi(x^\nu))^\omega$ for $(x^\nu)^\omega \in M^\omega$. Then for $t \in \mathbb{R}$, $s \in G$, $(x^\nu)^\omega \in M_\alpha^\omega$ and $\nu \in \mathbb{N}$, we have

$$\begin{aligned} \alpha_s\big(\sigma_t^\varphi\big(x^\nu\big)\big) &= \sigma_t^{\varphi \circ \alpha_{s^{-1}}}\big(\alpha_s\big(x^\nu\big)\big) \\ &= [D\varphi \circ \alpha_{s^{-1}} : D\varphi]_t \sigma_t^\varphi\big(\alpha_s\big(x^\nu\big)\big) [D\varphi \circ \alpha_{s^{-1}} : D\varphi]_t^*. \end{aligned}$$

This implies that $(\sigma_t^\varphi(x^\nu))$ is $(\alpha, \omega)$-equicontinuous for each $t \in \mathbb{R}$ since $(x^\nu)$ is an element of $\mathscr{M}^\omega(M)$. (See [5, Lemma 3.6].) Hence, $M_\alpha^\omega$ is globally invariant by $\sigma^{\varphi^\omega}$. Thanks to Takesaki's criterion [10, p. 309], we can take a faithful normal conditional expectation $E_\alpha$ from $M^\omega$ onto $M_\alpha^\omega$ so that $\varphi^\omega = \varphi^\omega \circ E_\alpha$. This equality implies that $E_M = E_M \circ E_\alpha$, and $E_\alpha$ is unique. □

**2.2. Main results.** The canonical embedding $\pi_\alpha$ of $M$ into $M \rtimes_\alpha G$ induces $\pi_\alpha^\infty : \mathscr{E}_\alpha^\omega(M) \cap \mathscr{M}^\omega(M) \to \ell^\infty(M \rtimes_\alpha G)$ by putting $\pi_\alpha^\infty((x^\nu)) := (\pi_\alpha(x^\nu))$.

LEMMA 2.4. *If $(x^\nu) \in \mathscr{E}_\alpha^\omega(M) \cap \mathscr{M}^\omega(M)$, then $\pi_\alpha^\infty((x^\nu)) \in \mathscr{M}^\omega(M \rtimes_\alpha G)$.*

*Proof.* Let $(y^\nu)$ be an $\omega$-trivial sequence in $M \rtimes_\alpha G$ with $\|y^\nu\| \le 1$ for all $\nu$. It suffices to show that $\|y^\nu \pi_\alpha(x^\nu)(\xi \otimes f)\| \to 0$ as $\nu \to \omega$ for $\xi \in H$ and $f \in C_c(G)$ with compact support $K \subset G$.

Let $\varepsilon > 0$. Since $(x^\nu)$ is $(\alpha, \omega)$-equicontinuous, we can take $W \in \omega$ and a open neighborhood $V$ of $e \in G$ so that if $t^{-1}s \in V$, $s, t \in K$ and $\nu \in W$, then $\|\alpha_{s^{-1}}(x^\nu)\xi - \alpha_{t^{-1}}(x^\nu)\xi\| < \varepsilon$.

Take $s_1, \dots, s_N \in K$ so that $K \subset s_1V \cup \cdots \cup s_NV$. Let $\{h_1, \dots, h_N\}$ be a partition of unity on $K$ subordinate to the cover $\{s_1V, \dots, s_NV\}$. (See [9, Theorem 2.13].) Then for $\nu \in W$, we obtain the following:

$$\begin{aligned}
&\left\| \left( \pi_\alpha(x^\nu) - \sum_{j=1}^{N} \left(\alpha_{s_j^{-1}}(x^\nu) \otimes h_j\right) \right) (\xi \otimes f) \right\|^2 \\
&= \int_K \left\| \left( \alpha_{s^{-1}}(x^\nu) - \sum_{j=1}^{N} \alpha_{s_j^{-1}}(x^\nu) h_j(s) \right) \xi \right\|^2 |f(s)|^2 \, ds \\
&= \int_K \left\| \sum_{j=1}^{N} h_j(s) \left( \alpha_{s^{-1}}(x^\nu) - \alpha_{s_j^{-1}}(x^\nu) \right) \xi \right\|^2 |f(s)|^2 \, ds \\
&\le \int_K \left( \sum_{j=1}^{N} h_j(s) \left\| \alpha_{s^{-1}}(x^\nu) - \alpha_{s_j^{-1}}(x^\nu)\xi \right\| \right)^2 |f(s)|^2 \, ds \\
&\le \varepsilon^2 \|f\|_2^2.
\end{aligned}$$

Thus for all $\nu \in W$, we have

$$\left\| y^\nu \pi_\alpha(x^\nu)(\xi \otimes f) \right\| \le \varepsilon \|f\|_2 + \left\| y^\nu \sum_{j=0}^{N-1} \left(\alpha_{s_j^{-1}}(x^\nu) \otimes h_j\right)(\xi \otimes f) \right\|.$$

In the last term, we know that $(\alpha_{s_j^{-1}}(x^\nu) \otimes 1)$ belongs to $\mathscr{M}^\omega(M \otimes B(L^2(G)))$ by the proof of [5, Lemma 2.8]. In particular, the last term converges to 0 in the strong topology as $\nu \to \omega$. Hence, the above inequality implies that

$$\lim_{\nu \to \omega} \left\| y^\nu \pi_\alpha(x^\nu)(\xi \otimes f) \right\| \le \varepsilon \|f\|_2.$$

Thus, we are done. □

The map $\pi_\alpha^\infty$ induces a well-defined map $\pi_\alpha^\omega$ from $M_\alpha^\omega$ into $(M \rtimes_\alpha G)^\omega$ such that $\pi_\alpha^\omega((x^\nu)^\omega) := (\pi_\alpha(x^\nu))^\omega$ for $(x^\nu)^\omega \in M_\alpha^\omega$. In the proof of Lemma 2.4, we have shown $\pi_\alpha^\omega$ is actually a map from $M_\alpha^\omega$ into $(M \otimes B(L^2(G)))^\omega$. Recall the isomorphism $\Psi$ from $(M \otimes B(L^2(G)))^\omega$ onto $M^\omega \otimes B(L^2(G))$ that is given in the proof of [5, Lemma 2.8]. Note that the map $\Psi$ is naturally defined so that for $f, g \in L^2(G)$ and $x = (x^\nu)^\omega \in (M \otimes B(L^2(G)))^\omega$, we have $(\mathrm{id} \otimes \phi_{f,g})(\Psi(x)) = ((\mathrm{id} \otimes \phi_{f,g})(x^\nu))^\omega$, where $\phi_{f,g}$ denotes the normal functional $\langle \cdot f, g\rangle$ on $B(L^2(G))$.

Lemma 2.5. *For any* $x \in M_\alpha^\omega$, *one has* $\Psi(\pi_\alpha^\omega(x)) = \pi_{\alpha^\omega}(x)$.

*Proof.* Let $f, g \in L^2(G)$ and $x = (x^\nu)^\omega \in M_\alpha^\omega$. On the one hand, we have

$$(\mathrm{id} \otimes \phi_{f,g})\big(\Psi\big(\pi_\alpha^\omega(x)\big)\big) = \big((\mathrm{id} \otimes \phi_{f,g})\big(\pi_\alpha\big(x^\nu\big)\big)\big)^\omega.$$

On the other hand, using the equicontinuity (cf. [5, Lemma 3.15]), we have

$$\begin{aligned}(\mathrm{id} \otimes \phi_{f,g})\big(\pi_{\alpha^\omega}(x)\big) &= \int_G f(s)\overline{g(s)}\alpha_{s^{-1}}^\omega(x)\,ds = \left(\int_G f(s)\overline{g(s)}\alpha_{s^{-1}}\big(x^\nu\big)\,ds\right)^\omega \\ &= \big((\mathrm{id} \otimes \phi_{f,g})\big(\pi_\alpha\big(x^\nu\big)\big)\big)^\omega.\end{aligned}$$

Thus, we are done. □

We now prove the main result of this paper which strengthens [6, Theorem 1.10]. Note that a generalization of Example 2.2 to the crossed product for $G$ being Abelian.

Theorem 2.6. *Let* $\alpha$ *be an action of a second countable locally compact Hausdorff group* $G$ *on a von Neumann algebra* $M$ *with separable predual. Then the following statements hold*:

(1) *There exists a canonical embedding* $\Phi_\alpha$ *of* $M_\alpha^\omega \rtimes_{\alpha^\omega} G$ *into* $(M \rtimes_\alpha G)^\omega$ *such that* $\Phi_\alpha(\pi_{\alpha^\omega}(x)) = \pi_\alpha^\omega(x)$ *and* $\Phi_\alpha(\lambda^{\alpha^\omega}(s)) = \lambda^\alpha(s)^\omega$, *respectively, for all* $x \in M_\alpha^\omega$ *and* $s \in G$.
(2) *If* $G$ *is Abelian, the map* $\Phi_\alpha$ *induces the isomorphism from* $M_\alpha^\omega \rtimes_{\alpha^\omega} G$ *onto* $(M \rtimes_\alpha G)_{\hat{\alpha}}^\omega$.

*Proof.* (1) Put $N := \pi_\alpha^\omega(M_\alpha^\omega) \vee \{\lambda^\alpha(t)^\omega \mid t \in G\}''$ that is a von Neumann subalgebra of $(M \rtimes_\alpha G)_{\hat{\alpha}}^\omega$. We will show that there exists a canonical isomorphism from $M_\alpha^\omega \rtimes_{\alpha^\omega} G$ onto $N$.

Let $\varphi$ be a faithful normal semifinite weight on $M$ and $\psi$ the dual weight of $\varphi$ on $M \rtimes_\alpha G$. It is obvious that $\psi^\omega$ is semifinite on $N$ since $N$ contains $M \rtimes_\alpha G$. Then for $(x^\nu)^\omega \in M_\alpha^\omega$, $s \in G$ and $t \in \mathbb{R}$, we have

$$\sigma_t^{\psi^\omega}\big(\lambda^\alpha(s)^\omega\big) = \big(\sigma_t^\psi\big(\lambda^\alpha(s)\big)\big)^\omega = \Delta_G(s)^{it}\lambda^\alpha(s)^\omega\pi_\alpha\big([D\varphi \circ \alpha_s : D\varphi]_t\big)^\omega$$

and

$$\begin{aligned}\sigma_t^{\psi^\omega}\big(\pi_\alpha^\omega\big(\big(x^\nu\big)^\omega\big)\big) &= \sigma_t^{\psi^\omega}\big(\big(\pi_\alpha\big(x^\nu\big)\big)^\omega\big) = \big(\sigma_t^\psi\big(\pi_\alpha\big(x^\nu\big)\big)\big)^\omega \\ &= \big(\pi_\alpha\big(\sigma_t^\varphi\big(x^\nu\big)\big)\big)^\omega = \pi_\alpha^\omega\big(\big(\sigma_t^\varphi\big(x^\nu\big)\big)^\omega\big),\end{aligned}$$

where we note that the last term is well-defined from the proof of Lemma 2.3. This observation implies $N$ is globally invariant under $\sigma^{\psi^\omega}$. Thanks to Takesaki's theorem, we can take a faithful normal conditional expectation from $(M \rtimes_\alpha G)^\omega$ onto $N$. In particular, the restriction of the modular conjugation $J_{\psi^\omega}$ on $L^2(N, \psi^\omega)$ gives the modular conjugation associated with $\psi^\omega\restriction_N$.

Let $\chi$ be the dual weight of $\varphi^\omega\restriction_{M_\alpha^\omega}$ on $P := M_\alpha^\omega \rtimes_{\alpha^\omega} G$. We will compare the GNS Hilbert spaces $L^2(P, \chi)$ and $L^2(N, \psi^\omega)$. The definition left ideals are as usual denoted by $n_\chi$ and $n_{\psi^\omega}$, respectively. (See [11, Lemma VII.1.2].) Denote by $\Lambda_\chi : n_\chi \to L^2(P,\chi)$ and $\Lambda_{\psi^\omega} : n_{\psi^\omega} \to L^2(N,\psi^\omega)$ the canonical embeddings.

Let us introduce a map $V$ which maps $\Lambda_\chi(\lambda^{\alpha^\omega}(f)\pi_{\alpha^\omega}(x))$ to $\Lambda_{\psi^\omega}(\lambda^\alpha(f)^\omega\pi_\alpha^\omega(x))$ for $f \in C_c(G)$ and $x \in M_\alpha^\omega$. We claim that $V$ extends to an isometry from $L^2(P,\chi)$ into $L^2(N,\psi^\omega)$. Take $f, g \in C_c(G)$ and $x, y \in M_\alpha^\omega$. Then we have

$$\begin{aligned}\left\langle \Lambda_{\psi^\omega}\left(\lambda^\alpha(f)^\omega\pi_\alpha^\omega(x)\right), \Lambda_{\psi^\omega}\left(\lambda^\alpha(g)^\omega\pi_\alpha^\omega(y)\right)\right\rangle &= \psi^\omega\left(\pi_\alpha^\omega\left(y^*\right)\lambda^\alpha\left(g^* * f\right)^\omega\pi_\alpha^\omega(x)\right)\\ &= \psi\Big(\lim_{\nu\to\omega}\pi_\alpha\left(\left(y^\nu\right)^*\right)\lambda^\alpha(h)\pi_\alpha\left(x^\nu\right)\Big),\end{aligned}$$

where $h := g^* * f$. Let $F$ be the support of $h$. Then for each $\nu \in \mathbb{N}$, we have

$$\pi_\alpha\left(\left(y^\nu\right)^*\right)\lambda^\alpha(h)\pi_\alpha\left(x^\nu\right) = \int_F h(s)\pi_\alpha\left(\left(y^\nu\right)^*\alpha_s\left(x^\nu\right)\right)\lambda^\alpha(s)\,ds.$$

Using [5, Lemma 3.3], we know that

$$\lim_{\nu\to\omega}\pi_\alpha\left(\left(y^\nu\right)^*\right)\lambda^\alpha(h)\pi_\alpha\left(x^\nu\right) = \int_F h(s)\pi_\alpha\Big(\lim_{\nu\to\omega} y^\nu\alpha_s\left(x^\nu\right)\Big)\lambda^\alpha(s)\,ds.$$

Hence it follows from the definition of the dual weight $\psi$ the following:

$$\begin{aligned}\left\langle \Lambda_{\psi^\omega}\left(\lambda^\alpha(f)^\omega\pi_\alpha^\omega(x)\right), \Lambda_{\psi^\omega}\left(\lambda^\alpha(g)^\omega\pi_\alpha^\omega(y)\right)\right\rangle &= h(e)\varphi\Big(\lim_{\nu\to\omega}\left(y^\nu\right)^*x^\nu\Big)\\ &= h(e)\varphi^\omega\left(y^*x\right),\end{aligned}$$

which equals to $\langle\Lambda_\chi(\lambda^{\alpha^\omega}(f)\pi_{\alpha^\omega}(x)), \Lambda_\chi(\lambda^{\alpha^\omega}(g)\pi_{\alpha^\omega}(y))\rangle$ again by the definition of the dual weight $\chi$. Thus, we have proved the existence of the isometry $V$.

We next claim that $K$, the image of $V$, is $N'$-invariant. For a $\sigma^{\varphi^\omega}$-analytic $y \in M_\alpha^\omega$ and $t \in G$, we obtain the followings for all $f \in C_c(G)$ and $x \in M_\alpha^\omega$:

$$J_{\psi^\omega}\sigma_{i/2}^{\psi^\omega}\left(\pi_\alpha^\omega(y)\right)^* J_{\psi^\omega}\Lambda_{\psi^\omega}\left(\lambda^\alpha(f)^\omega\pi_\alpha^\omega(x)\right) = \Lambda_{\psi^\omega}\left(\lambda^\alpha(f)^\omega\pi_\alpha^\omega(xy)\right) \in K$$

and

$$J_{\psi^\omega}\sigma_{i/2}^{\psi^\omega}\left(\lambda^\alpha(t)^\omega\right)^* J_{\psi^\omega}\Lambda_{\psi^\omega}\left(\lambda^\alpha(f)^\omega\pi_\alpha^\omega(x)\right) = \Lambda_{\psi^\omega}\left(\lambda^\alpha(g)^\omega\pi_\alpha^\omega\left(\alpha_{t^{-1}}^\omega(x)\right)\right) \in K,$$

where $g(s) := \Delta_G(t)^{-1}f(st^{-1})$. Hence, $K$ is $N'$-invariant.

Now let us take a $\sigma^{\psi^\omega}$-analytic $y \in n_{\psi^\omega}$. Then

$$J_{\psi^\omega}\sigma_{i/2}^{\psi^\omega}(y)^* J_{\psi^\omega}\Lambda_{\psi^\omega}\left(\lambda^\alpha(f)^\omega\pi_\alpha^\omega(x)\right) = \lambda^\alpha(f)^\omega\pi_\alpha^\omega(x)\Lambda_{\psi^\omega}(y).$$

This implies that $\Lambda_{\psi^\omega}(y)$ is contained in the closure of $N'K$. Since $N'K \subset K$, $\Lambda_{\psi^\omega}(y)$ belongs to $K$. Thus, $K = L^2(N,\psi^\omega) \subset L^2((M \rtimes_\alpha G)^\omega, \psi^\omega)$. Then the map $N \ni x \mapsto V^*xV$ provides us with the isomorphism from $N$ onto $M_\alpha^\omega \rtimes_{\alpha^\omega} G$. More precisely, we can check $V^*\pi_\alpha^\omega(x)V = \pi_{\alpha^\omega}(x)$ and $V^*\lambda^\alpha(s)^\omega V = \lambda^{\alpha^\omega}(s)$ for $x \in M_\alpha^\omega$ and $s \in G$. Denote by $\Phi_\alpha$ its inverse map.

(2) Suppose that $G$ is Abelian. Then the image of $\Phi_\alpha$ is clearly contained in $(M \rtimes_\alpha G)^\omega_{\widehat{\alpha}}$. Let us apply the statement of (1) to the dual action $\beta := \widehat{\alpha}$ on $R := M \rtimes_\alpha G$. Then we have the embedding $\Phi_\beta$ of $R^\omega_\beta \rtimes_{\beta^\omega} \widehat{G}$ into $(R \rtimes_\beta \widehat{G})^\omega_{\widehat{\beta}}$.

Recall the isomorphism $\Psi$ from $(M \otimes B(L^2(G)))^\omega$ onto $M^\omega \otimes B(L^2(G))$ that is introduced in the remark before Lemma 2.5. Then $\Psi$ induces an isomorphism from $(M \otimes B(L^2(G)))^\omega_{\alpha\otimes \mathrm{Ad}\,\rho}$ onto $M^\omega_\alpha \otimes B(L^2(G))$. This fact can be directly proved or deduced from [5, Lemma 3.12] for general groups. In summary, we have the following diagram:

$$\begin{array}{ccccc}
R^\omega_\beta \rtimes_{\beta^\omega} \widehat{G} & \xhookrightarrow{\Phi_\beta} & (R \rtimes_\beta \widehat{G})^\omega_{\widehat{\beta}} & \xrightarrow{(\Gamma_\alpha)^\omega} & (M \otimes B(L^2(G)))^\omega_{\alpha\otimes\mathrm{Ad}\,\rho} \\
\uparrow{\scriptstyle \Phi_\alpha\otimes\mathrm{id}} & & & & \downarrow{\scriptstyle \Psi} \\
P \rtimes_{\widehat{\alpha^\omega}} \widehat{G} & \xleftarrow{(\Gamma_{\alpha^\omega})^{-1}} & M^\omega_\alpha \otimes B(L^2(G)) & \overset{f}{\dashrightarrow} & M^\omega_\alpha \otimes B(L^2(G))
\end{array}$$

where $(\Gamma_\alpha)^\omega$ is defined by $(\Gamma_\alpha)^\omega((x^\nu)^\omega) := (\Gamma_\alpha(x^\nu))^\omega$ for $(x^\nu)^\omega \in (R \rtimes_\beta \widehat{G})^\omega$ and $f$ denotes the composition of all of them.

We will show $f$ actually equals the identity map. This implies the surjectivity of $\Phi_\alpha \otimes \mathrm{id}$ in the diagram above, and we obtain $\Phi_\alpha(P) = R^\omega_\beta$ by taking the fixed point algebra of the dual action of $\beta^\omega$. Recall that $M^\omega_\alpha \otimes B(L^2(G))$ is generated by $\pi_{\alpha^\omega}(M^\omega_\alpha)$, $1 \otimes \lambda(G)$ and $1 \otimes L^\infty(G)$. We can directly check that $f$ identically maps $1 \otimes \lambda(G)$ and $1 \otimes L^\infty(G)$. For $x \in M^\omega_\alpha$, it is not difficult to show $\pi_{\alpha^\omega}(x)$ is mapped to $\pi^\omega_\alpha(x)$ in $(M \otimes B(L^2(G)))^\omega_{\alpha\otimes\mathrm{Ad}\,\rho}$, and it turns out from Lemma 2.5 that $f(\pi_{\alpha^\omega}(x)) = \pi_{\alpha^\omega}(x)$. □

It would be interesting to generalize the previous theorem to a locally compact Hausdorff group or quantum group by introducing the equicontinuity of their actions.

## 3. Applications

**3.1. Continuous or discrete crossed product decomposition of $M^\omega$.** Let $M = N \rtimes_\theta \mathbb{R}$ be the continuous crossed product decomposition of a properly infinite von Neumann algebra $M$, that is, $N$ is a semifinite von Neumann algebra that is endowed with the $\mathbb{R}$-action $\theta$ and a faithful normal tracial weight $\tau$ satisfying $\tau \circ \theta_s = e^{-s}\tau$ for $s \in \mathbb{R}$. Let $\varphi$ be the dual weight of $\tau$. Since the dual action $\widehat{\theta}$ is nothing but the modular automorphism $\sigma^\varphi$, the following result follows from Theorem 2.6, [1, Proposition 4.11] and [6, Theorem 1.5].

THEOREM 3.1. *Let $M = N \rtimes_\theta \mathbb{R}$ be the continuous crossed product decomposition of a properly infinite von Neumann algebra $M$. Then the continuous crossed product decomposition of $M^\omega$ is given by $M^\omega = N^\omega_\theta \rtimes_{\theta^\omega} \mathbb{R}$. In particular, the flow of weights of $M^\omega$ is given by the restriction of $\theta^\omega$ on $Z(N^\omega_\theta)$.*

The following result on a discrete crossed product decomposition is proved first by Ando–Haagerup in [1]. We will present another proof using Theorem 2.6.

THEOREM 3.2 (Ando–Haagerup). *Let $M$ be a type III$_\lambda$ factor with $0 \leq \lambda < 1$. Let $M = N \rtimes_\theta \mathbb{Z}$ be the discrete crossed product decomposition. Then the discrete crossed product decomposition of $M^\omega$ is given by $M^\omega = N^\omega \rtimes_{\theta^\omega} \mathbb{Z}$. In particular, if $0 < \lambda < 1$, then $M^\omega$ is a type III$_\lambda$ factor.*

*Proof.* The dual action $\hat{\theta}$ of $\widehat{\mathbb{Z}}$ on $M$ satisfies $\hat{\theta}_t(x\lambda^\theta(m)) = e^{-imt}x\lambda^\theta(m)$ for $t \in \widehat{\mathbb{Z}}$, $x \in N$ and $m \in \mathbb{Z}$, where the usual coordinate $\widehat{\mathbb{Z}} = [0, 2\pi)$ is used. It turns out from Theorem 2.6 that $M^\omega_{\hat{\theta}} = N^\omega \rtimes_\theta \mathbb{Z}$. Hence, it suffices to show that $M^\omega_{\hat{\theta}} = M^\omega$. For $\lambda \neq 0$, $\hat{\theta}$ is nothing but the modular automorphism $\sigma^{\hat{\tau}}$, where $\tau$ denotes a faithful normal tracial weight on $N$ with $\tau \circ \theta = \lambda\tau$. Hence, we are done.

Suppose next that $\lambda = 0$. Take a faithful normal tracial weight $\tau$ on $N$ such that $\tau \circ \theta \leq \mu\tau$ with $0 < \mu < 1$. Let $H_n$ be the selfadjoint operator affiliated with $Z(N)$ such that $\tau \circ \theta^n = \tau_{\exp(H_n)}$ for $n \in \mathbb{Z}$. Then the spectrum of $H_n$ is contained in $(-\infty, n \log \mu]$ and $[n \log \mu^{-1}, \infty)$ when $n \geq 1$ and $n \leq -1$, respectively.

Let $\varphi := \hat{\tau}$ and $g_\beta(t) := \sqrt{\beta/\pi}\exp(-\beta t^2)$ for $\beta > 0$ and $t \in \mathbb{R}$ and $U_\beta := \widehat{g_\beta}(-\log \Delta_\varphi) = \int_{\mathbb{R}} g_\beta(t)\Delta_\varphi^{it}\,dt$, where $\widehat{g_\beta}(p) := \int_{\mathbb{R}} g_\beta(t)e^{-ipt}\,dt = \exp(-p^2/(4\beta))$, $p \in \mathbb{R}$. Then $U_\beta \to 1$ in the strong topology as $\beta \to \infty$.

Now we will show $M^\omega = M^\omega_{\hat{\theta}}$. Take $x = (x^\nu)^\omega \in M^\omega$. It suffices to show that $\sigma^{\varphi^\omega}_{g_\beta}(x)$ is contained in $M_{\hat{\theta}}$ since $\sigma^{\varphi^\omega}_{g_\beta}(x)$ converges to $x$ as $\beta \to \infty$ in the strong$*$ topology. Note that $\sigma^{\varphi^\omega}_{g_\beta}(x) = (\sigma^\varphi_{g_\beta}(x^\nu))^\omega$.

Let $y = \sum_{m\in\mathbb{Z}} y(m)\lambda^\theta(m)$ with $y(m) \in N$ be the formal decomposition of $y \in M$. Namely, we set $y(m) := \frac{1}{2\pi}\int_0^{2\pi} e^{imt}\hat{\theta}_t(y)\lambda^\theta(m)^*\,dt$ which we will call the Fourier coefficients of $y$. Note that $y = 0$ if and only if $y(m) = 0$ for all $m \in \mathbb{Z}$. By direct computation, we have the formal decomposition of $\sigma^\varphi_{g_\beta}(y)$ as follows:

$$\sigma^\varphi_{g_\beta}(y) = \sum_{m\in\mathbb{Z}} y(m)\lambda^\theta(m)\widehat{g_\beta}(-H_m).$$

Note the series in the right-hand side actually converges in the norm topology since $\|\widehat{g_\beta}(-H_m)\|_\infty \leq \exp(-m^2|\log \mu|^2/(4\beta))$ for all $m \in \mathbb{Z}$. Hence, the series in the right-hand side defines an element $z \in M$. By definition of $z$, all Fourier coefficients of $z$ equal those of $\sigma^\varphi_{g_\beta}(y)$, and the formal decomposition above is actually a genuine equality.

Now we fix $k \in \mathbb{N}$ and take a faithful state $\chi \in N_*$. Let $\hat{\chi}$ be the dual state of $\chi$ on $M$. Then for all $y \in M$, we have

$$\left\|\sigma^\varphi_{g_\beta}(y) - \sum_{|m|\leq k} y(m)\lambda^\theta(m)\widehat{g_\beta}(-H_m)\right\|_{\hat{\chi}} \leq \|y\| \sum_{|m|>k} \exp\bigl(-m^2|\log \mu|^2/(4\beta)\bigr).$$

Hence for $x=(x^\nu)^\omega\in M^\omega$ and $\beta>0$, we have

$$\begin{aligned}\Big\|\sigma_{g_\beta}^{\varphi^\omega}(x)-\sum_{|m|\le k}\big(x^\nu(m)\lambda^\theta(m)\widehat{g_\beta}(-H_m)\big)^\omega\Big\|_{\hat\chi}\\ \le\|x\|\sum_{|m|>k}\exp\big(-m^2|\log\mu|^2/(4\beta)\big).\end{aligned}$$

It is clear that $(x^\nu(m)\lambda^\theta(m)\widehat{g_\beta}(-H_m))^\omega$ is contained in $M_{\hat\theta}^\omega$, and so is $\sigma_{g_\beta}^{\varphi^\omega}(x)$. □

Thanks to [1, Theorem 6.11], we know $M^\omega$ is actually a type $\mathrm{III}_1$ factor when $M$ is. Hence, $N_\theta^\omega$ is a type $\mathrm{II}_\infty$ factor in this situation, but we could not directly prove this without appealing Ando–Haagerup's result.

**3.2. Description of $M_\omega$ and fullness of $M$.** Let $M$ be an infinite type III factor with separable predual and $M=N\rtimes_\theta\mathbb{R}$ be the continuous crossed product decomposition of $M$ as before. Then the following result holds.

Lemma 3.3. *The asymptotic centralizer $M_\omega$ is isomorphic to $(N_{\omega,\theta})^{\theta^\omega}$.*

*Proof.* Let $\tau$ be a faithful normal tracial weight on $N$ satisfying $\tau\circ\theta_s=e^{-s}\tau$ for $s\in\mathbb{R}$ and $\varphi$ the dual weight of $\tau$. Then by Theorem 3.1, we have $M^\omega=N_\theta^\omega\rtimes_{\theta^\omega}\mathbb{R}$. We will compute $(M'\cap M^\omega)^{\sigma^{\varphi^\omega}}$ which equals $M_\omega$. (Use [1, Proposition 4.35] and the Connes cocycle derivative.) Using $\lambda^\theta(t)\in M$, $t\in\mathbb{R}$, we have

$$M'\cap M^\omega\subset\pi_{\theta^\omega}\big(\big(N_\theta^\omega\big)^{\theta^\omega}\big)\vee\big\{\lambda^{\theta^\omega}(t)\mid t\in\mathbb{R}\big\}''. \tag{3.1}$$

This implies that $(M'\cap M^\omega)^{\sigma^{\varphi^\omega}}\subset\pi_{\theta^\omega}(N'\cap(N_\theta^\omega)^{\theta^\omega})$. Since the converse inclusion trivially holds and $N'\cap N^\omega=N_\omega$, we obtain $M_\omega=\pi_{\theta^\omega}((N_{\omega,\theta})^{\theta^\omega})$. □

A separable factor $M$ is said to be *full* when $M_\omega=\mathbb{C}$. The fullness of $M$ has been studied by several researchers in terms of the continuous core. See references cited in [4], [12]. Also see [2] for recent progress in study of fullness. Among them, Marrakchi in [4] shows that $N$ is full if and only if $M$ is a full type $\mathrm{III}_1$ factor with $\tau$-invariant being the usual topology of $\mathbb{R}$. The following theorem would suggest that the $\tau$-invariant could measure how continuously $\theta^\omega$ is acting on $N_\omega$. Our proof is similar to that of [12, Lemma 3].

Theorem 3.4. *Let $M$ be a type $III_1$ factor with the continuous crossed product decomposition $M=N\rtimes_\theta\mathbb{R}$ as before. Then $M$ is full if and only if $N_{\omega,\theta}=\mathbb{C}$.*

*Proof.* The "if" part is trivial from the previous lemma or [6, Corollary 1.9]. Suppose that $M$ is full. Let $p\in\mathbb{R}$ be an element of the Arveson spectrum of $\theta^\omega$ on $N_{\omega,\theta}$. By [5, Theorem 7.7], we can take a unitary $u\in N_{\omega,\theta}$ such that $\theta_t^\omega(u) = e^{ipt}u$ for all $t \in \mathbb{R}$. Let $(u^\nu) \in \ell^\infty(N)$ be a unitary representing sequence of $u$. Then $\mathrm{Ad}\,\pi_\theta(u^\nu)$ converges to $\hat{\theta}_p$ in $\mathrm{Aut}(M)$. The fullness of $M$ implies the innerness of $\hat{\theta}_p$, and it turns out that $p = 0$. Namely, $\theta^\omega$ is trivial on $N_{\omega,\theta}$, and the previous lemma implies $N_{\omega,\theta} = \mathbb{C}$. □

Remark 3.5. Ueda's problem asks if $M' \cap M^\omega = \mathbb{C}$ holds for any full factor $M$. This is affirmatively solved by Ando–Haagerup in [1, Theorem 5.2]. We would like to deduce this result by strengthening (3.1), but this approach has not been successful yet. Instead, let us present a short proof of the problem. Put $R := M' \cap M^\omega$. Suppose $R$ were non-trivial. Let $\varphi$ be a faithful normal state on $M$. Since $R_{\varphi^\omega} = M_\omega = \mathbb{C}$, $R$ would be a type $\mathrm{III}_1$ factor. We claim that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in R$ with $\|x\| \leq 1$ satisfies $\|x\varphi^\omega - \varphi^\omega x\|_{(M^\omega)_*} < \delta$, then $\|x - \varphi^\omega(x)\|_{\varphi^\omega} < \varepsilon$. By usual diagonal argument, we can show this claim by contradiction. Readers are referred to [7, Chapter 5] for this. Also note that $\|x\varphi^\omega - \varphi^\omega x\|_{(M^\omega)_*} = \lim_{\nu\to\omega} \|x^\nu\varphi - \varphi x^\nu\|$ for all $x = (x^\nu)^\omega \in M^\omega$. For proof of this fact, see [1, Lemma 4.36] or [5, Lemma 9.3]. However, since $R$ is a type $\mathrm{III}_1$ factor, there exist many non-trivial norm bounded sequences $(y^k) \in \ell^\infty(R)$ such that $\|y^k\varphi^\omega - \varphi^\omega y^k\| \to 0$ as $k \to \infty$, and we have a contradiction. The last claim is implied by the fact that $(R^\omega)_{\psi^\omega}$ is a type $\mathrm{II}_1$ factor, where $\psi := \varphi^\omega$ [1, Proposition 4.24].

## References


[1] H. Ando and U. Haagerup, *Ultraproducts of von Neumann algebras*, J. Funct. Anal. **266** (2014), 6842–6913. MR 3198856

[2] C. Houdayer, A. Marrakchi and P. Verraedt, *Fullness and Connes' $\tau$ invariant of type III tensor product factors*, available at arXiv:1611.07914.

[3] E. Kirchberg, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, J. Reine Angew. Math. **452** (1994), 39–77. MR 1282196

[4] A. Marrakchi, *Spectral gap characterization of full type III factors*, to appear in J. Reine Angew. Math.

[5] T. Masuda and R. Tomatsu, *Rohlin flows on von Neumann algebras*, Mem. Amer. Math. Soc. **244** (2016), no. 1153, ix + 111 pp. MR 3545107

[6] T. Masuda and R. Tomatsu, *Classification of actions of discrete Kac algebras on injective factors*, Mem. Amer. Math. Soc. **245** (2017), no. 1160, ix + 118 pp. MR 3592229

[7] A. Ocneanu, *Actions of discrete amenable groups on von Neumann algebras*, Lecture Notes in Mathematics, vol. 1138, Springer-Verlag, Berlin, 1985. MR 0807949

[8] Y. Raynaud, *On ultrapowers of non commutative $L_p$ spaces*, J. Operator Theory **48** (2002), no. 1, 41–68. MR 1926043

[9] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill, New York, 1987. MR 0924157

[10] M. Takesaki, *Conditional expectations in von Neumann algebras*, J. Funct. Anal. **9** (1972), 306–321. MR 0303307

[11] M. Takesaki, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, vol. 125, Operator Algebras and Non-commutative Geometry, vol. 6, Springer-Verlag, Berlin, 2003. MR 1943006

[12] R. Tomatsu and Y. Ueda, *A characterization of fullness of continuous cores of type $III_1$ free product factors*, Kyoto J. Math. **56** (2016), 599–610. MR 3542777

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