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COMMON HYPERCYCLIC VECTORS FOR CERTAIN FAMILIES OF DIFFERENTIAL OPERATORS

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ABSTRACT. Let (k_n) be a strictly increasing sequence of positive integers. If $\sum_{n=1}^{+\infty}\frac{1}{k_n}=+\infty$, we establish the existence of an entire function f such that for every $\lambda\in(0,+\infty)$ the set $\{\lambda^{k_n}f^{(k_n)}(\lambda z):n=1,2,\dots\}$ is dense in the space of entire functions endowed with the topology of uniform convergence on compact subsets of the complex plane. This provides the best possible strengthened version of a corresponding result due to Costakis and Sambarino (*Adv. Math.* **182** (2004) 278–306). From this, and using a non-trivial result of Weyl which concerns the uniform distribution modulo 1 of certain sequences, we also derive an entire function g such that for every $\lambda\in J$ the set $\{\lambda^{k_n}g^{(k_n)}(\lambda z):n=1,2,\dots\}$ is dense in the space of entire functions, where J is "almost" equal to the set of non-zero complex numbers. On the other hand, if $\sum_{n=1}^{+\infty}\frac{1}{k_n}<+\infty$ we show that the conclusions in the above results fail to hold.

1. Introduction

Let X be a topological space. A sequence of functions $T_n:X\to X$, $n=1,2,\dots$ is called hypercyclic provided there exists a vector $x\in X$ such that the set $\{T_n(x):n=1,2,\dots\}$ is dense in X. Such a vector x is called hypercyclic for the sequence (T_n) and the set of hypercyclic vectors for (T_n) is denoted by $\mathcal{HC}((T_n)_{n\in\mathbb{N}})$ or simply $\mathcal{HC}(T_n)$. For a thorough study on this subject, we refer to the books [5], [13]. We stress here that we introduce the

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most abstract definition of hypercyclicity in the sense that we do not assume continuity of T_n's. This will be of use to us in the proof of Proposition 13. For this abstract setting see also [11]. As usual the symbols $\mathbb{C}$ and $\mathbb{N}$ stand for the sets of complex and natural numbers, respectively. Let us consider the set of entire functions

$$\mathcal{H}(\mathbb{C}) := \{f : \mathbb{C} \to \mathbb{C} \mid f \text{ is holomorphic}\}$$

endowed with the topology $\mathcal{T}_u$ of uniform convergence on compact subsets of $\mathbb{C}$. Let $D : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ be the classical differentiation operator, i.e. $D(f) := f'$ for $f \in \mathcal{H}(\mathbb{C})$, where f' is the usual derivative of f. The operators D^n, $n = 1, 2, \dots$ acting on the space of entire functions are continuous and linear, and they are defined inductively as follows:

$$D^1 := D \quad \text{and}$$
$$D^{n+1} = D^n \circ D \ \big(\text{the usual composition in } \mathcal{H}(\mathbb{C})\big), \quad n = 1, 2, \dots.$$

For $\lambda \in \mathbb{C}$, we consider the dilation function $\varphi_\lambda : \mathbb{C} \to \mathbb{C}$, $\varphi_\lambda(z) = \lambda z$, $z \in \mathbb{C}$, which is linear. Now, for $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ we consider the linear and continuous operator $T_{n,\lambda} : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ defined by the following formula:

$$T_{n,\lambda}(f) := D^n(f \circ \varphi_\lambda) \quad \text{for every } f \in \mathcal{H}(\mathbb{C}),$$

where $f \circ \varphi_\lambda$ denotes the usual composition of the functions f and φ_λ. It is well known that $(T_{n,\lambda})$ is hypercyclic when $\lambda \neq 0$, and that the set of hypercyclic vectors for the sequence $(T_{n,\lambda})$, that is

$$\mathcal{HC}(T_{n,\lambda}) := \big\{f \in \mathcal{H}(\mathbb{C}) \mid \overline{\{T_{n,\lambda}(f), n = 1, 2, \dots\}} = \mathcal{H}(\mathbb{C})\big\},$$

is a dense, G_δ subset of $(\mathcal{H}(\mathbb{C}), T_u)$ when $\lambda \neq 0$, see for instance [5], [10] and [13].

By this fact, the fact that the space $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$ is a complete metric space and Baire's category theorem we conclude that for every sequence (λ_m), $m = 1, 2, \dots$ of non-zero complex numbers, the set $\bigcap_{m=1}^{+\infty} \mathcal{HC}(T_{n,\lambda_m})$ is a dense, G_δ subset of $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$. Now the following question arises naturally.

Let $I \subseteq \mathbb{C} \setminus \{0\}$ be uncountable. Is it true that:

$$\bigcap_{\lambda \in I} \mathcal{HC}(T_{n,\lambda}) \neq \emptyset?$$

Costakis and Sambarino [10] showed that the above question has a positive answer for $I := \mathbb{C} \setminus \{0\}$ (the greatest possible set I). Later on, Costakis [9] examined some refinements of the above problem in the setting of translation operators. Following this refinement, we can interpret the previous mentioned problem in the context of differential operators along *sparse* powers as follows:

Fix a subsequence (k_n) of natural numbers, a non-zero complex number z and consider the set $\mathcal{HC}(T_{k_n,z})$ of hypercyclic vectors for the sequence $(T_{k_n,z})$, that is

$$\mathcal{H}C(T_{k_n,\lambda}) := \left\{ f \in \mathcal{H}(\mathbb{C}) \mid \overline{\left\{ T_{k_n,\lambda}(f), n = 1, 2, \dots \right\}} = \mathcal{H}(\mathbb{C}) \right\}.$$

Let $I \subseteq \mathbb{C} \setminus \{0\}$, be uncountable. Is it true that:

$$\bigcap_{\lambda \in I} \mathcal{H}C(T_{k_n,\lambda}) \neq \emptyset?$$

In this direction, we prove in Section 2 the following proposition.

Proposition 1. *Let* (k_n), $n = 1, 2, \dots$ *be a strictly increasing sequence of positive integers such that*:

$$\sum_{n=1}^{+\infty} \frac{1}{k_n} = +\infty.$$

Then the set

$$\bigcap_{\lambda \in (0,+\infty)} \mathcal{H}C(T_{k_n,\lambda})$$

is residual in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$, *i.e. it contains a dense* G_δ *set.*

Further, we can ask the following:
Is it true that

$$\bigcap_{z \in \mathbb{C} \setminus \{0\}} \mathcal{H}C(T_{k_n,z}) \neq \emptyset?$$

We do not know the answer to this question. However, we are able to give a positive answer for a set $J \subseteq \mathbb{C} \setminus \{0\}$ that is nearly equal to $\mathbb{C} \setminus \{0\}$ in the sense of 2-dimensional Lebesque measure. For the sequel, we denote by m_1 the usual Lebesgue measure on the real line. In order to describe the set J we just mentioned, let us give the following definitions:

Definition 2. For $J \subseteq \mathbb{C} \setminus \{0\}$ and $\lambda > 0$ we define the set

$$J_\lambda := \left\{ \theta \in [0, 1) \mid \exists z \in J : z = \lambda \cdot e^{2\pi i \theta} \right\}$$

and we call J_λ *the set of arcs of* J *with respect to* λ.

It is evident that for every subset $J \subseteq \mathbb{C} \setminus \{0\}$ the sets J_λ are well defined. However, we may have $J_\lambda = \emptyset$ for certain $\lambda \in (0, +\infty)$.

Definition 3. We say that a set $J \subseteq \mathbb{C} \setminus \{0\}$ has *radially full measure* if for every $\lambda > 0$, $m_1(J_\lambda) = 1$.

Now we are ready to state the main result of this paper.

Theorem 4. *Let* (k_n) *be a subsequence of natural numbers such that* $\sum_{n=1}^{+\infty} \frac{1}{k_n} = +\infty$. *Then there exists a set* $J \subseteq \mathbb{C} \setminus \{0\}$ *with* radially full measure *such that*:

$$\bigcap_{z \in J} \mathcal{H}C(T_{k_n,z}) \neq \emptyset.$$

We mention that the assumption on the divergence of the series in the above theorem cannot be removed. In particular, given a sequence (k_n) of positive integers such that the series $\sum_{n=1}^{+\infty} \frac{1}{k_n}$ converges, then for any interval I of the positive (or negative) real line we have $\bigcap_{\lambda \in I} \mathcal{H}C(T_{k_n,\lambda}) = \emptyset$. We do not give a proof of this result. Instead, we refer to the proof of item (i) of Theorem 1.1 in [21], which can be easily adapted to our case.

The proof of Theorem 4 relies on: Proposition 1, a classical result of Weyl on uniform distribution of sequences and a variation of Proposition 5.2 from [3]. The proof of Proposition 1 refines the argument in the proof of the common hypercyclicity criterion from [10] and a variant, towards a more abstract setting, of the very interesting Proposition 5.2 from [3] is needed in order to close our argument.

A similar approach has been recently developed by the author in [21] for certain families of backward shift operators. Specifically, the crucial divergent series condition with which we have the positive result for non degenerate intervals on the real line is the same for both types of operators (differentiation type or backward shift type). The negative result is treated similarly for both problems.

We stress that new common hypercyclicity criteria are available, due to the work of Bayart and Matheron [4] and Shkarin [18]. Unfortunately, it is not clear to us if these criteria are applicable in our case. Frederic Bayart presented a thorough study on the subject of common hypercyclic vectors in his recent significant paper [2]. In fact, Bayart works in a quite general framework where the parameters live in $\mathbb{R}^n$, whereas in our setting the parameters live in $\mathbb{C}$. Similar problems for translation type operators are treated in a series of recent papers, [2], [9], [10], [11], [12], [22], [20], [19], which require a different approach at a technical level. Further results on common hypercyclic vectors can be found in [1], [4], [3], [6], [7], [8], [15], [16], [17], [18].

2. The solution of our problem in the basic case

In order to prove Proposition 1 we need some preparation. Let $\Psi := \{p_1, p_2, \ldots\}$ be an enumeration of all non-zero complex polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. For every $\rho = 2, 3, \ldots$ we set $\Delta_\rho := [\frac{1}{\rho}, \rho]$. Obviously, $\bigcup_{\rho=2}^{+\infty} \Delta_\rho = (0, +\infty)$. From now on we fix some strictly increasing sequence (k_n) of natural numbers and for every $n, j, m, \rho, s \in \mathbb{N}$, $\rho > 2$, $s > 1$ we define the set

$$E(n,\rho,j,s,m) := \left\{ f \in \mathcal{H}(\mathbb{C}) \mid \forall \lambda \in \Delta_\rho \ \exists v \in \mathbb{N}, v \leq m : \left\| T_{k_v,\lambda}(f) - p_j \right\|_{C_n} < \frac{1}{s} \right\}.$$

Lemma 5. *For every $n, j, m, \rho, s \in \mathbb{N}$ and ρ, $s > 2$ the set $E(n,\rho,j,s,m)$ is open in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$.*

LEMMA 6. *We set*

$$G := \bigcap_{n=1}^{+\infty} \bigcap_{\rho=1}^{+\infty} \bigcap_{j=1}^{+\infty} \bigcap_{s=1}^{+\infty} \bigcup_{m=1}^{+\infty} E(n,\rho,j,s,m).$$

Then

$$G \subseteq \bigcap_{\lambda \in (0,+\infty)} \mathcal{HC}(T_{k_n,\lambda}).$$

LEMMA 7. *For every $n,j,\rho,s \in \mathbb{N}$ and $\rho,s>2$, the set $\bigcup_{m=1}^{+\infty} E(n,\rho,j,s,m)$ is dense in $(\mathcal{H}(\mathbb{C}),\mathcal{T}_u)$.*

The proofs of Lemmas 5 and 6 are very easy; therefore are left to the interested reader. We will only prove Lemma 7; this is the non-trivial part of the proof of Proposition 1. A series of lemmas is needed for the proof of Lemma 7 and this is what we do below.

LEMMA 8. *For fixed $m_0 \in \mathbb{N}$, $\lambda_0 > 0$ and a fixed non-zero polynomial $p(z) = \sum_{j=0}^{\ell_0} \beta_j z^j$, where $\beta_j \in \mathbb{C}$, $j = 0,1,\ldots,\ell_0$, $\ell_0 \in \mathbb{N}\cup\{0\}$, $z \in \mathbb{C}$, the polynomial*

$$f(z) := \sum_{j=0}^{\ell_0} \frac{j!}{(j+m_0)!} \frac{\beta_j}{\lambda_0^{j+m_0}} z^{j+m_0}, \quad z \in \mathbb{C},$$

is a solution of the differential equation

$$T_{m_0,\lambda_0}(y) = p.$$

Proof. The proof is just a straightforward computation, using the obvious equality $T_{n,\lambda}(f) = \lambda^n (f^{(n)} \circ \varphi_\lambda)$, $n \in \mathbb{N}$, $\lambda \in (0,+\infty)$, $f \in \mathcal{H}(\mathbb{C})$. $\square$

At this point, we introduce some terminology. Let $m_0 \in \mathbb{N}$, $\lambda_0 > 0$ and consider a polynomial $p(z) = \sum_{j=0}^{\ell_0} \beta_j z^j$, $\ell_0 = \deg p(z)$ $p \neq 0$, $z \in \mathbb{C}$. Recall that the polynomial

$$f(z) = \sum_{j=0}^{\ell_0} \frac{j!}{(j+m_0)!} \frac{\beta_j}{\lambda_0^{j+m_0}} z^{j+m_0}, \quad z \in \mathbb{C}$$

is a solution of the differential equation $T_{m_0,\lambda_0}(y) = p$. In other words, f is the solution with data (m_0,λ_0,p) of the above differential equation and we simply say the solution (m_0,λ_0,p) of the equation $T_{m_0,\lambda_0}(y) = p$, or the solution (m_0,λ_0,p).

LEMMA 9. *Fix $m_0 \in \mathbb{N}$, $\lambda_0 > 0$ and a non-zero polynomial $p(z) = \sum_{j=0}^{\ell_0} \beta_j z^j$, $\beta_{\ell_0} \neq 0$, $\ell_0 \in \{0,1,2,\ldots\}$, $z \in \mathbb{C}$. Let f be the solution (m_0,λ_0,p) of the differential equation*

$$T_{m_0,\lambda_0}(y) = p.$$

We set $N_0 := \deg f(z) = m_0 + \ell_0$. We fix some positive number $\varepsilon_0 \in (0,1)$ and some positive number $R_0 > 1$ and let us define $M_0 := \max\{|\beta_j|, j =$

$0,1,\dots,\ell_0\}$, $M_1:=M_0\cdot\sum_{j=0}^{\ell_0}R_0^j$. *It holds that* $M_0>0$, $M_1>0$. *Then, for every positive number* $\lambda\in[\lambda_0,\lambda_0\cdot\sqrt[N_0]{1+\varepsilon_0/M_1})$ *the following inequality holds*:

$$\big\|T_{m_0,\lambda_0}(f)-T_{m_0,\lambda}(f)\big\|_{\overline{D}_{R_0}}\le\varepsilon_0,$$

where $\overline{D}_{R_0}=\overline{D}(0,R_0):=\{z\in\mathbb{C}\mid|z|\le R_0\}$.

Proof. It holds that $T_{n,\lambda}(f)=\lambda^n(f^{(n)}\circ\varphi_\lambda)$, for every $n\in\mathbb{N}$, $\lambda\in(0,+\infty)$and for every $f\in\mathcal{H}(\mathbb{C})$. From this equality and the definition of solution (m_0,λ_0,p), we get easily:

$$(1)\quad T_{m_0,\lambda}(f)(z)=\sum_{n=0}^{N_0-m_0}\frac{f^{(n+m_0)}(0)}{n!}\lambda^{n+m_0}z^n\quad\text{for every } z\in\mathbb{C},\lambda\ge\lambda_0.$$

Relation (1) implies

$$\begin{aligned}&\big|\big(T_{m_0,\lambda_0}(f)\big)(z)-\big(T_{m_0,\lambda}(f)\big)(z)\big|\\&\quad\le\sum_{n=0}^{\ell_0}\left|\left(\frac{\lambda}{\lambda_0}\right)^{n+m_0}-1\right|\cdot\left|\frac{\lambda_0^{n+m_0}f^{(n+m_0)}(0)}{n!}\right||z|^n\end{aligned}$$

for every $\lambda\ge\lambda_0$, $z\in\mathbb{C}$. Observe that

$$(2)\qquad\beta_n=\frac{\lambda_0^{n+m_0}f^{(n+m_0)}(0)}{n!}\quad\text{for every } n=0,1,\dots,\ell_0,$$

since f is the solution (m_0,λ_0,p). For every $\lambda\in[\lambda_0,\lambda_0\cdot\sqrt[N_0]{1+\varepsilon_0/M_1})$, we have

$$(3)\qquad M_1\cdot\left(\left(\frac{\lambda}{\lambda_0}\right)^{N_0}-1\right)<\varepsilon_0.$$

By (1), (2) and (3) and the definitions of M_0, M_1 the result follows. □

For the next lemma, we consider some data. More specifically, we fix two polynomials p and Q where $p(z)=\sum_{j=0}^{\ell_0}\beta_jz^j$, $p\ne0$, $\beta_i\in\mathbb{C}$, $\beta_{\ell_0}\ne0$, $i=0,1,\dots,\ell_0$, $\ell_0\in\{0,1,2,\dots\}$, $z\in\mathbb{C}$. Let $R_0\in(1,+\infty)$ and

$$M_0:=\max\{|\beta_j|\mid j=0,1,\dots,\ell_0\}.$$

We consider the sequence

$$\gamma_v:=\frac{(2R_0)^v}{v!},\quad v=1,2,\dots.$$

It is obvious that $\gamma_v\to0$. Then, there exists some natural number $N_0\in\mathbb{N}$ such that:

$$M_0\ell_0!\gamma_v<1$$

for every $v\in\mathbb{N}$, $v\ge N_0$. Let us define $N_1:=\max\{N_0,\deg Q,\ell_0\}+1$. We now consider two positive fixed numbers a_0,b_0, where $0<a_0<1<b_0<+\infty$, some fixed natural number $v_0>2$ and some fixed partition $\varDelta=\{a_0=\delta_1<\delta_2<\dots<\delta_{v_0}=b_0\}$ of the closed interval $[a_0,b_0]$. We also consider some fixed finite sequence of natural numbers $m_1, m_2, \ldots, m_{v_0}$, such that $m_1 < m_2 < \cdots < m_{v_0}$, $m_1 > N_1$ and $m_{i+1} - m_i > N_1$, $i = 1, 2, \ldots, v_0 - 1$. For every $i = 1, 2, \ldots, v_0$ the solution (m_i, δ_i, p) will be denoted by f_i, $i = 1, 2, \ldots, v_0$ for simplicity and we set

$$\Pi := \sum_{i=1}^{v_0} f_i + Q.$$

According to the above notations and terminology, we have the following lemma.

LEMMA 10. *Let $\lambda \in [a_0, b_0]$. We consider the unique $i \in \{1, 2, \ldots, v_0 - 1\}$ such that $\lambda \in [\delta_i, \delta_{i+1})$ (if it exists; otherwise define $\lambda = b_0$). Then the following holds: either*

$$\left\| T_{m_i,\lambda}(\Pi) - p \right\|_{\overline{D}_{R_0}} < \left\| T_{m_i,\delta_i}(f_i) - T_{m_i,\lambda}(f_i) \right\|_{\overline{D}_{R_0}} + \frac{1}{2^{m_{i+1}-(m_i+2)}},$$

if $\lambda \in [a_0, b_0)$ or else

$$\left\| T_{m_{i_0},b_0}(\Pi) - p \right\|_{\overline{D}_{R_0}} = 0 \quad \textit{if } \lambda = b_0.$$

Proof. The case $\lambda = b_0$ is obvious. Fix $\lambda_0 \in [a_0, b_0)$ and consider the unique $i_0 \in \{1, 2, \ldots, v_0 - 1\}$ such that $\lambda_0 \in [\delta_{i_0}, \delta_{i_0+1})$.

As we have already seen,

$$T_{n,\lambda}(f) = \lambda^n \left(f^{(n)} \circ \varphi_\lambda \right) \quad \text{for every } n \in \mathbb{N}, \lambda > 0, f \in \mathcal{H}(\mathbb{C}). \tag{4}$$

By (4) and the definitions of Π, N_1 and m_i, $i = 1, 2, \ldots, v_0$ we get

$$T_{m_{i_0},\lambda_0}(\Pi) = \sum_{i=i_0}^{v_0} T_{m_{i_0},\lambda_0}(f_i). \tag{5}$$

Observe that by (5),

$$\begin{aligned} T_{m_{i_0},\lambda_0}(\Pi) - p = {} & \left(T_{m_{i_0},\lambda_0}(f_{i_0}) - T_{m_{i_0},\delta_{i_0}} - T_{m_{i_0},\delta_{i_0}}(f_{i_0}) \right) \\ & + \sum_{i=i_0+1}^{v_0} T_{m_{i_0},\lambda_0}(f_i), \end{aligned} \tag{6}$$

because f_{i_0} is the $(m_{i_0}, \delta_{i_0}, p)$ solution. Using (6) and the triangle inequality it follows that

$$\begin{aligned} \left\| T_{m_{i_0},\lambda_0}(\Pi) - p \right\|_{\overline{D}_{R_0}} \leq {} & \left\| T_{m_{i_0},\delta_{i_0}}(f_{i_0}) - T_{m_{i_0},\lambda_0}(f_{i_0}) \right\|_{\overline{D}_{R_0}} \\ & + \sum_{i=i_0+1}^{v_0} \left\| T_{m_{i_0},\lambda_0}(f_i) \right\|_{\overline{D}_{R_0}}. \end{aligned} \tag{7}$$

Now, we estimate the quantities $\|T_{m_{i_0},\lambda_0}(f_i)\|_{\overline{D}_{R_0}}$ for $i=i_0+1,\dots,v_0$. A little computation shows

$$T_{m_{i_0},\lambda_0}(f_j)(z)=\sum_{k=0}^{\ell_0}k!\beta_k\left(\frac{\lambda_0}{\delta_j}\right)^{k+m_j}\cdot\frac{1}{(k+m_j-m_{i_0})!}z^{k+m_j-m_{i_0}} \tag{8}$$

for every $j\in\{i_0+1,\dots,v_0\}$, $z\in\mathbb{C}$. Using (8) and the definition of N_1 we arrive to

$$\left\|T_{m_{i_0},\lambda_0}(f_j)\right\|_{\overline{D}_{R_0}}<\frac{1}{2^{m_j-m_{i_0}-1}}\quad\text{for every } j\in\{i_0+1,\dots,v_0\}. \tag{9}$$

By (9), we have

$$\sum_{j=i_0+1}^{v_0}\left\|T_{m_{i_0},\lambda_0}(f_j)\right\|_{\overline{D}_{R_0}}<\frac{1}{2^{m_{i_0+1}-m_{i_0}-2}}, \tag{10}$$

and thus, by (7) and (10), the result follows. □

LEMMA 11. *Let (k_n) be a subsequence of natural numbers such that $\sum_{n=1}^{+\infty}\frac{1}{k_n}=+\infty$. Then, for every positive number $M>0$ there exists a subsequence (μ_n) of (k_n) such that*

(i) $\mu_1>M$,
(ii) $\mu_{n+1}-\mu_n>M$ *for every* $n=1,2,\dots$ *and*
(iii) $\sum_{n=1}^{+\infty}\frac{1}{\mu_n}=+\infty$.

For a proof of the previous lemma, Lemma 11, see [21]. After the above preparation, we are now ready to prove Lemma 7.

Proof of Lemma 7. We fix $n_0,j_0,s_0>2$, $\rho_0>2$ and we will prove that the set $\bigcup_{m=1}^{+\infty}E(n_0,\rho_0,j_0,s_0,m)$ is dense in $(\mathcal{H}(\mathbb{C}),\mathcal{T}_u)$. Let some non-zero polynomial Q and $\varepsilon_1>0$. We consider the neighborhood of Q, $\mathcal{U}(Q,K,\varepsilon_1):=\{h\in\mathcal{H}(\mathbb{C})\mid\|Q-h\|_K<\varepsilon_1\}$ for some fixed compact set $K\subseteq\mathbb{C}$ and we set $E:=\bigcup_{m=1}^{+\infty}E(n_0,\rho_0,j_0,s_0,m)$ for simplicity reasons. It suffices to prove that $E\cap\mathcal{U}(Q,K,\varepsilon_1)\neq\emptyset$. Fix some positive number R_0 such that for the set $\overline{D}_{R_0}:=\{z\in\mathbb{C}\mid|z|\leq R_0\}$ we have: $K\cup C_{n_0}\subset\overline{D}_{R_0}$. Of course $R_0>1$. It suffices to find some polynomial f and some natural number m_0 such that:

(a) $\|f-Q\|_{\overline{D}_{R_0}}<\varepsilon_0$ and
(b) $f\in E(n_0,\rho_0,j_0,s_0,m_0)$, where $\varepsilon_0:=\min\{\varepsilon_1,\frac{1}{s_0}\}$.

Let $p_{j_0}(z)=\sum_{j=0}^{\ell_0}\beta_j z^j$, $\beta_j\in\mathbb{C}$, for $j=0,1,\dots,\ell_0$ and for some $\ell_0\in\{0,1,2,\dots\}$. We now set $M_0:=\max\{|\beta_j|,j=0,1,\dots,\ell_0\}$ and $M_1:=M_0\cdot\sum_{j=0}^{\ell_0}R_0^j$. Of course $M_1\geq M_0>0$. We remind that p_{j_0} is a non-zero polynomial. Let us fix some $\delta_0\in(0,\frac{1}{\rho_0}\log(1+\frac{\varepsilon_0}{4M_1}))$ (for example $\delta_0=$

$\frac{1}{2\rho_0}\log(1+\frac{\varepsilon_0}{4M_1})$). Then we have:

$$e^{\rho_0\delta_0} < 1 + \frac{\varepsilon_0}{4M_1}. \tag{11}$$

Recall the very useful limit

$$\lim_{n\to+\infty}\left(1+\frac{\rho_0\delta_0}{n}\right)^n = e^{\rho_0\delta_0}. \tag{12}$$

By (11) and (12) there exists some natural number $v_1 \in \mathbb{N}$ such that:

$$\left(1+\frac{\rho_0\delta_0}{k_n}\right)^{k_n} < 1 + \frac{\varepsilon_0}{4M_1} \quad \text{for every } v\in\mathbb{N}, v\geq v_1.$$

Thus,

$$\frac{\delta_0}{k_v} < \frac{1}{\rho_0}\left(\sqrt[k_v]{1+\frac{\varepsilon_0}{4M_1}} - 1\right) \quad \text{for every } v\in\mathbb{N}, v\geq v_1. \tag{13}$$

Let us consider the sequence

$$\gamma_n' := \frac{(2\rho_0 R_0)^n}{n!}\cdot \ell_0!\cdot M_0 \quad \text{for } n = 1,2,\ldots.$$

Then, there is some natural number $v_2 \in \mathbb{N}$ such that:

$$\gamma_n' < 1 \quad \text{for every } n\in\mathbb{N}, n\geq v_2. \tag{14}$$

Fix some $v_0 \in \mathbb{N}$ so that:

$$\left(1+\frac{\varepsilon_0}{2M_1}\right)^{\frac{v_0}{v_0+\ell_0}} > 1 + \frac{\varepsilon_0}{4M_1} \tag{15}$$

and set

$$v_3 := \max\left\{v_0, v_1, v_2, \ell_0, \deg Q, 3 + \frac{1}{\log 2}\cdot\log\left(\frac{1}{\varepsilon_0}\right)\right\} + 1.$$

Applying Lemma 11, we conclude the existence of some subsequence (μ_n), $n = 1,2,\ldots$ of (k_n) such that

(i) $\mu_1 > v_3$,
(ii) $\mu_{n+1} - \mu_n > v_3$ for $n = 1,2,\ldots,$
(iii) $\sum_{n=1}^{+\infty}\frac{1}{\mu_n} = +\infty$.

Of course $\sum_{n=1}^{+\infty}\frac{\delta_0}{\mu_n} = +\infty$.

Let N_0 be the minimum natural number such that

$$\sum_{n=1}^{N_0+1}\frac{\delta_0}{\mu_n} > \rho_0 - \frac{1}{\rho_0}. \tag{16}$$

Having fixed the above notation and terminology we now define a partition $\mathcal{P}$ of the closed interval $[\frac{1}{\rho_0}, \rho_0]$ as follows: $a_1 := \frac{1}{\rho_0}$, $a_2 := \frac{1}{\rho_0} + \frac{\delta_0}{\mu_1}$, $a_3 := a_2 + \frac{\delta_0}{\mu_2}, \ldots,$ $a_{i+1} = a_i + \frac{\delta_0}{\mu_i}$, for $i = 1,2,\ldots,N_0-1$; if $a_{N_0} + \frac{\delta_0}{\mu_{N_0}} < \rho_0$ define

$a_{N_0+1} := a_{N_0} + \frac{\delta_0}{\mu_{N_0}}$ and $a_{N_0+2} := \rho_0$, otherwise, that is if $a_{N_0} + \frac{\delta_0}{\mu_{N_0}} \geq \rho_0$, define $a_{N_0+1} := \rho_0$.

Hence, our partition is: either

$$\mathcal{P} := \left\{ a_1 = \frac{1}{\rho_0} < a_2 < \cdots < a_{N_0} < a_{N_0+1} < a_{N_0+2} = \rho_0 \right\}$$

if $a_{N_0} + \frac{\delta_0}{\mu_{N_0}} < \rho_0$, or

$$\mathcal{P} := \left\{ a_1 = \frac{1}{\rho_0} < a_2 < \cdots < a_{N_0+1} = \rho_0 \right\}$$

if $a_{N_0} + \frac{\delta_0}{\mu_{N_0}} \geq \rho_0$.

After the above preparation, we are now ready to define the function $f \in \mathcal{H}(\mathbb{C})$ which satisfies the above properties (a), (b). Consider the solutions (μ_i, a_i, p_{j_0}) for $i = 1, 2, \dots, N_0$ if $a_{N_0+1} < \rho_0$ and the solution $(\mu_{N_0+1}, \rho_0, p_{j_0})$ if $a_{N_0+1} = \rho_0$. Let us denote the solution (μ_i, a_i, p_{j_0}) as f_i, for $i = 1, 2, \dots, N_0$ and let f_{N_0+1} be the solution $(\mu_{N_0+1}, \rho_0, p_{j_0})$ if $a_{N_0+1} = \rho_0$. Finally, set $f := Q + \sum_{i=1}^{N_0} f_i$ if $a_{N_0+1} < \rho_0$ and $f := Q + \sum_{i=1}^{N_0+1} f_i$ if $a_{N_0+1} = \rho_0$. Since f is a polynomial, $f \in \mathcal{H}(\mathbb{C})$. We will show that f satisfies property (a). Suppose that $a_{N_0+1} < \rho_0$. The other case follows in a similar manner. We have:

$$\|f - Q\|_{\overline{D}_{R_0}} \leq \sum_{i=1}^{N_0} \|f_i\|_{\overline{D}_{R_0}}. \tag{17}$$

The polynomials f_i, for $i = 1, 2, \dots, N_0$ are the solutions (μ_i, a_i, p_{j_0}), that is

$$f_i(z) = \sum_{k=0}^{\ell_0} \frac{k!}{(k+\mu_i)!} \frac{\beta_k}{a_i^{k+\mu_i}} z^{k+\mu_i} \quad \text{for } i = 1, 2, \dots, N_0, z \in \mathbb{C}.$$

So, for $i = 1, 2, \dots, N_0$, $z \in \overline{D}_{R_0}$:

$$\left|f_i(z)\right| \leq \ell_0! M_0 \cdot \sum_{k=0}^{\ell_0} \frac{(\rho_0 R_0)^{k+\mu_i}}{(k+\mu_i)!}. \tag{18}$$

By (14) and the definition of v_3 we get

$$\sum_{k=0}^{\ell_0} \ell_0! \cdot M_0 \frac{(R_0 \rho_0)^{k+\mu_i}}{(k+\mu_i)!} < \frac{1}{2^{\mu_i - 1}} \tag{19}$$

and (18), (19) imply:

$$\sum_{i=1}^{N_0} \|f_i\|_{\overline{D}_{R_0}} < \frac{1}{2^{\mu_1 - 2}}. \tag{20}$$

By (17), (20) and the definition of v_3 we conclude that $\|f - Q\|_{\overline{D}_{R_0}} < \varepsilon_0$ and this shows that f satisfies property (a). It remains to prove (b) and this will be done for $m_0 = \mu_{N_0}$. So we need to prove that $f \in E(n_0, \rho_0, j_0, s_0, \mu_{N_0})$. Fix $\lambda_0 \in \Delta_{\rho_0} = [\frac{1}{\rho_0}, \rho_0]$. We consider the unique $i_0 \in \{1, 2, \dots, N_0 + 1\}$ such that $\lambda_0 \in [a_{i_0}, a_{i_0+1})$ for $1 \le i_0 \le N_0$ and set $i_0 = N_0 + 1$ if $\lambda_0 \in [a_{N_0+1}, \rho_0]$. We shall show that:

$$\left\|T_{\mu_{i_0},\lambda_0}(f) - p_{j_0}\right\|_{\overline{D}_{R_0}} < \frac{1}{s_0}. \tag{21}$$

Observe that all the assumptions of Lemma 10 are satisfied, so we get:

$$\begin{aligned}\left\|T_{\mu_{i_0},\lambda_0}(f) - p_{j_0}\right\|_{\overline{D}_{R_0}} < {} & \left\|T_{\mu_{i_0},a_{i_0}}(f_{i_0}) - T_{\mu_{i_0},\lambda_0}(f_{i_0})\right\|_{\overline{D}_{R_0}} \\ & + \frac{1}{2^{\mu_{i_0+1}-(\mu_{i_0}+2)}}.\end{aligned} \tag{22}$$

Let us show that $\lambda_0 \in [a_{i_0}, a_{i_0} \cdot \sqrt[\mu_{i_0}+\ell_0]{1 + \frac{\varepsilon_0}{2M_1}})$. By (15) and the definition of the sequence (μ_n),

$$1 + \frac{\varepsilon_0}{4M_1} < \left(1 + \frac{\varepsilon_0}{2M_1}\right)^{\frac{\mu_{i_0}}{\mu_{i_0}+\ell_0}}. \tag{23}$$

Relation (23) implies the following

$$\frac{1}{\rho_0}\left(\sqrt[\mu_{i_0}]{1 + \frac{\varepsilon_0}{4M_1}} - 1\right) < \frac{1}{\rho_0}\left(\sqrt[\mu_{i_0}+\ell_0]{1 + \frac{\varepsilon_0}{2M_1}} - 1\right). \tag{24}$$

By (13), (24) and the definition of the sequence (μ_n), we get

$$\frac{\delta_0}{\mu_{i_0}} < \frac{1}{\rho_0} \cdot \left(\sqrt[\mu_{i_0}+\ell_0]{1 + \frac{\varepsilon_0}{2M_1}} - 1\right). \tag{25}$$

Relation (25) gives

$$a_{i_0+1} < a_{i_0} \sqrt[\mu_{i_0}+\ell_0]{1 + \frac{\varepsilon_0}{2M_1}} \tag{26}$$

and since $\lambda_0 \in [a_{i_0}, a_{i_0+1})$, we arrive to

$$\lambda_0 \in \left[a_{i_0}, a_{i_0} \cdot \sqrt[\mu_{i_0}+\ell_0]{1 + \frac{\varepsilon_0}{2M_1}}\right)$$

as we wanted. Lemma 9 yields

$$\left\|T_{\mu_{i_0},a_{i_0}}(f_{i_0}) - T_{\mu_{i_0},\lambda_0}(f_{i_0})\right\|_{\overline{D}_{R_0}} \le \frac{\varepsilon_0}{2} \tag{27}$$

and the hypothesis $\mu_1 > v_3$ implies

$$\frac{1}{2^{\mu_{i_0}+1-(\mu_{i_0}+2)}} < \frac{\varepsilon_0}{2}. \tag{28}$$

Combining (22), (27) and (28), we conclude that

$$\left\|T_{\mu_{i_0},\lambda_0}(f) - p_{j_0}\right\|_{\overline{D}_{R_0}} < \varepsilon_0 \le \frac{1}{s_0},$$

which is nothing but the desired inequality (21). The remaining case $i_0 = N_0 + 1$ can be proved similarly. This completes the proof of the lemma. $\square$

Now, by Lemmas 5, 6, 7, the fact that the space $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$ is a complete metric space and Baire's Category theorem the proof of Proposition 1 follows.

3. The general case: Proof of Theorem 4

The notation introduced in the previous section will be used in the present section as well. For instance, the sequence (p_j) denotes an enumeration of all non-zero polynomials the polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. We introduce some extra terminology and a non-trivial result from the book [14]. For a real number x, let $[x]$ denote the integer part of x, that is, the greatest integer $\leq x$; let $\{x\} = x - [x]$ be the fractional part of x, or the residue of x modulo 1. We note that the fractional part of any real number is contained in the unit interval $I = [0, 1)$. Let $\omega = (x_n)$ $n = 1, 2, \ldots$, be a given sequence of real numbers. For a positive integer N and a subset E of I, let the counting function $A(E; N; \omega)$ be defined as the number of terms x_n, $1 \leq n \leq N$, for which $\{x_n\} \in E$. If there is no risk of confusion, we shall often write $A(E; N)$ instead of $A(E; N; \omega)$. Here is our basic definition.

Definition 12. The sequence $\omega = (x_n)$, $n = 1, 2, \ldots$, of real numbers is said to be uniformly distributed modulo 1 (abbreviated u.d. mod 1) if for every pair a, b of real numbers with $0 \leq a < b \leq 1$ we have

$$\lim_{N \to +\infty} \frac{A([a, b); N; \omega)}{N} = b - a.$$

Thus, in simple terms, the sequence (x_n) is u.d. mod 1 if every half-open subinterval of I eventually gets its “proper share” of fractional parts.

Let $(\mathcal{U}_n(x))$, $n = 1, 2, \ldots$, for every x lying in some given bounded or unbounded interval J, be a sequence of real numbers. The sequence $(\mathcal{U}_n(x))$ is said to be u.d. mod 1 for almost all x, if for every $x \in J$, apart of a set which has Lebesgue measure 0, the sequence $(\mathcal{U}_n(x))$ is u.d. mod 1.

We prove the following lemma. Its proof is a simple variation of the proof of Proposition 5.2 in [3].

Proposition 13. *Let $(X, \mathcal{T})$ be an arbitrary second countable and T_1 topological space (i.e. every singleton is closed) without isolated points. Let $T_n : X \to X$ be a hypercyclic sequence of maps, not necessarily continuous. Let $x_0 \in \mathcal{HC}(T_n)$ and $(f(n))$ be a sequence of real numbers such that for some $\delta > 0$, $|f(n) - f(m)| > \delta$ for every $n, m \in \mathbb{N}$, $n \neq m$. Then there exists a subset $\mathcal{A}$ of $\mathbb{R}$ which is G_δ and dense in $\mathbb{R}$, it has full 1-dimensional Lebesque measure and for every $\theta \in \mathcal{A}$ the set $\{(T_n(x_0), e^{2\pi i f(n)\theta}),\ n \in \mathbb{N}\}$ is dense in $X \times C(0,1)$ where $C(0,1) := \{z \in \mathbb{C} \mid |z| = 1\}$.*

Proof. Let (U_m), $m=1,2,\dots$ be a denumerable basis of $(X,\mathcal{T})$ and each U_m is non-empty. For every $m=1,2,\dots$ we consider the set

$$\mathcal{N}_m := \big\{ n\in\mathbb{N} \mid T_n(x_0)\in U_m \big\}.$$

Because $x_0\in\mathcal{HC}(T_n)$ we have that $\mathcal{N}_m\neq\emptyset$ for every $m=1,2,\dots$ and because the space $(X,\mathcal{T})$ is T_1, without isolated points we can see easily that the set $\mathcal{N}_m$ is infinite. Let $\mathcal{N}_m := \{v_{m,1}, v_{m,2},\dots,v_{m,n},\dots\}$ where the sequence $(v_{m,j})$, $j=1,2,\dots$ is the unique enumeration of $\mathcal{N}_m$ by its natural order, that is $v_{m,j}<v_{m,j+1}$ for every $j=1,2,\dots$ and $T_{v_{m,\lambda}}(x_0)\neq T_{v_{m,\rho}}(x_0)$ for every $m\in\mathbb{N}$ and $\lambda,\rho\in\mathbb{N}$, $\lambda\neq\rho$. Now let (t_ℓ), $\ell=1,2,\dots$ be a dense subset of $C(0,1):=\{z\in\mathbb{C}\mid |z|=1\}$. For every $\ell,s,m\in\mathbb{N}$ and $n\in\mathcal{N}_m$ we define the set

$$\mathcal{A}(\ell,s,n) := \left\{ \theta\in\mathbb{R} \mid \left| e^{2\pi i\theta f(n)} - t_\ell \right| < \frac{1}{s} \right\}.$$

It is easy to see that $\mathcal{A}(\ell,s,n)$ is open in $\mathbb{R}$ for every $\ell,s,\in\mathbb{N}$, $n\in\mathcal{N}_m$. We shall show that for every $\ell,s,m\in\mathbb{N}$ the set

$$\bigcup_{n\in\mathcal{N}_m} \mathcal{A}(\ell,s,n) \text{ is dense in } \mathbb{R}.$$

We fix $m_0,\ell_0,s_0\in\mathbb{N}$. Let some $\theta_0\in\mathbb{R}$ and $\varepsilon_0>0$. We will prove that

$$\bigcup_{n\in\mathcal{N}_{m_0}} \mathcal{A}(\ell_0,s_0,n)\cap B(\theta_0,\varepsilon_0)\neq\emptyset,$$

where $B(\theta_0,\varepsilon_0):=\{\theta\in\mathbb{R}\mid |\theta-\theta_0|<\varepsilon_0\}$. The sequence $(f(v_{m_0,j}))$, $j=1,2,\dots$ is a subsequence of $(f(n))$, so we have

$$\big| f(v_{m_0,n}) - f(v_{m_0,m}) \big| > \delta \quad \text{for every } n,m\in\mathbb{N}, n\neq m.$$

At this point we apply the following proposition (Corollary 4.3, page 35 in [14]), that is due to Koksma and generalizes a classical result of Weyl.

Proposition. *Let δ be a positive constant, and let (λ_n), $n=1,2,\dots$ be a sequence of real numbers with $|\lambda_m-\lambda_n|\geq\delta$ for $m\neq n$. Then the sequence $(\lambda_n x)$, $n=1,2,\dots$ is u.d. mod 1 for almost all real numbers x.*

We easily see that under the assumptions of the above proposition, the sequence $e^{2\pi i\lambda_n x}$ is dense in $C(0,1)=\{z\in\mathbb{C}\mid |z|=1\}$ for almost all real numbers x. We apply now this fact for the sequence $\lambda_n := f(v_{m_0,n})$, $n = 1,2,\dots$ and we conclude that there exists some real number $\theta_1\in B(\theta_0,\varepsilon_0)$ such that the sequence $e^{2\pi i f(v_{m_0,n})\theta_1}$ is dense in $C(0,1)$. So, there exists some $n_0\in\mathbb{N}$ such that $|e^{2\pi i f(v_{m_0,n_0})\theta_1} - t_{\ell_0}| < \frac{1}{s_0}$. Thus, $\theta_1\in\mathcal{A}$ for $v_{m_0,n_0}\in\mathcal{N}_{m_0}$, that is $\theta_1\in\mathcal{A}(\ell_0,s_0,v_{m_0,n_0})$. Hence, $\bigcup_{n\in\mathcal{N}_m}\mathcal{A}(\ell,s,n)$ is dense for every $\ell,s,m\in\mathbb{N}$. Defining

$$\mathcal{B}_{m_0} := \big\{ x\in\mathbb{R} \mid \big(f(v_{m_0,n})x\big), n=1,2,\dots \text{is u.d. mod } 1 \big\},$$

it is easy to see that $\mathcal{B}_{m_0} \subset \bigcup_{n\in\mathcal{N}_{m_0}} \mathcal{A}(\ell_0, s_0, n)$ for every $\ell_0, s_0, m_0 \in \mathbb{N}$ and that the set $\mathcal{B}_{m_0}$ has full 1-dimensional Lebesque measure. Then, we set

$$\mathcal{A} := \bigcap_{\ell=1}^{+\infty} \bigcap_{s=1}^{+\infty} \bigcap_{m=1}^{+\infty} \Bigg(\bigcup_{n\in\mathcal{N}_m} \mathcal{A}(\ell, s, n) \Bigg) \quad \text{and} \quad \mathcal{B} := \bigcap_{m=1}^{+\infty} \mathcal{B}_m.$$

From the above we easily conclude the following: the set $\mathcal{A}$ is G_δ and dense in $\mathbb{R}$, the set $\mathcal{B}$ has full 1-dimensional Lebesque measure and $\mathcal{B} \subseteq \mathcal{A}$. Finally by the definition of $\mathcal{A}$ and the hypercyclicity of the sequence (T_n) we further conclude that for every $\theta \in \mathcal{A}$ the set $\{(T_n(x_0), e^{2\pi i f(n)\theta}), n \in \mathbb{N}\}$ is dense in $X \times C(0,1)$ and this completes the proof. $\square$

After the previous proposition we are now ready to prove Theorem 4.

Proof of Theorem 4. Fix $n_0, j_0, v_0 \in \mathbb{N}$, $\lambda_0 > 0$ and $\theta_0 \in \mathbb{R}$. We also fix $f \in \mathcal{H}(\mathbb{C})$ and set $g(z) = p_{j_0}(e^{2\pi i\theta_0} z)$, $z \in \mathbb{C}$. Of course $g \in \mathcal{H}(\mathbb{C})$. It is easy to show that:

$$\begin{aligned} \tag{29} &\big\| T_{k_{v_0}, \lambda_0 e^{2\pi i\theta_0}}(f) - g \big\|_{\overline{D}_{n_0}} \\ &\quad \le \big| e^{2\pi i\theta_0 k_{v_0}} - 1 \big| \cdot \big(\big\| T_{k_{v_0},\lambda_0}(f) - p_{j_0} \big\|_{\overline{D}_{n_0}} + \|p_{j_0}\|_{\overline{D}_{n_0}} \big) \\ &\qquad + \big\| T_{k_{v_0},\lambda_0}(f) - p_{j_0} \big\|_{\overline{D}_{n_0}}. \end{aligned}$$

By Proposition 1 there exists $f \in \bigcap_{\lambda\in(0,+\infty)} \mathcal{HC}(T_{k_v,\lambda})$. Of course $f \in \mathcal{HC}(T_{k_v,\lambda_0})$. Apply now the Proposition 13 for $(X, \mathcal{T}) = (\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$, $x_0 = f$, $T_n := T_{k_n,\lambda_0}$, $f(n) = k_n$, $n = 1, 2, \ldots$. Then, there exists a G_δ and dense subset $\mathcal{A}$ of $\mathbb{R}$ with full 1-dimensional Lebesque measure such that for every $\theta \in \mathcal{A}$ the set $\{(T_{k_v,\lambda_0}(f), e^{2\pi i k_v\theta}), n \in \mathbb{N}\}$ is dense in $\mathcal{H}(\mathbb{C}) X C(0,1)$. Of course the space $\mathcal{H}(\mathbb{C}) X C(0,1)$ is a metric space with the product topology, where we suppose that the space $\mathcal{H}(\mathbb{C})$ is endowed with the topology of local uniform convergence and the space $C(0,1)$ is endowed with the relative topology of the usual topology of $\mathbb{C}$. We fix $\theta_0 \in \mathcal{A} \cap [0,1)$, $j_0 \in \mathbb{N}$ and the polynomial p_{j_0}. Then for the element $(p_{j_0}, 1)$ of $(\mathcal{H}(\mathbb{C}) \times C(0,1))$ there exists a subsequence (μ_n) of (k_n) such that $(T_{\mu_n,\lambda_0}(f), e^{2\pi i\mu_n\theta_0}) \to (p_{j_0}, 1)$ as $n \to +\infty$. By this convergence and (29) we get

$$T_{\mu_n,\lambda_0} e^{2\pi i\theta_0}(f) \to p_{j_0}\big(e^{2\pi i\theta_0} z\big) \quad \text{as } n \to +\infty.$$

Because the set of polynomials $\Psi := \{p_1, p_2, \ldots\}$ is dense in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$, the set of polynomials $\Psi_{\theta_0} := \{p_1(e^{2\pi i\theta_0} z), p_2(e^{2\pi i\theta_0} z), \ldots\}$ is also dense in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$. Therefore the set $\{T_{k_v,\lambda_0 e^{2\pi i\theta_0}}(f),\ v = 1, 2, \ldots\}$ is dense in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$. Of course, the set $\{T_{k_v,\lambda_0 e^{2\pi i\theta}}(f),\ v = 1, 2, \ldots\}$ is dense in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$ for every $\theta \in \mathcal{A} \cap [0,1)$. Finally, from the above we conclude that for every $\lambda > 0$ there exists a subset $\mathcal{A}_\lambda \in [0,1)$ such that $m_1(\mathcal{A}_\lambda) = 1$ and the sequence $\{T_{k_v,\lambda e^{2\pi i\theta}}(f),$ $v = 1, 2, \ldots\}$ is dense in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$ for every $\theta \in \mathcal{A}_\lambda$. That is, for every $\lambda > 0$

there exists a subset $\mathcal{A}_\lambda \subset [0,1]$ such that $m_1(\mathcal{A}_\lambda) = 1$ and $f \in \mathcal{HC}(T_{k_v,\lambda e^{2\pi i\theta}})$ for every $\theta \in \mathcal{A}_\lambda$. Setting now

$$J := \{z \in \mathbb{C} \mid f \in \mathcal{HC}(T_{k_v,z})\}$$

it is easy to check that the set J has radially full measure (according to the relevant definition stated in the Introduction). This completes the proof of Theorem 4. $\square$

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