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FOUR-DIMENSIONAL HAKEN COBORDISM THEORY

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Abstract. Cobordism of Haken n-manifolds is defined by a Haken $(n+1)$-manifold W whose boundary has two components, each of which is a closed Haken n-manifold. In addition, the inclusion map of the fundamental group of each boundary component to $\pi_1(W)$ is injective. In this paper, we prove that there are 4-dimensional Haken cobordisms whose boundary consists of any two closed Haken 3-manifolds. In particular, each closed Haken 3-manifold is the π_1-injective boundary of some Haken 4-manifold.

1. Introduction

The authors have defined and studied Haken n-manifolds and Haken cobordism theory in previous work [5]. These manifolds enjoy important properties for example, the universal cover of a closed Haken n-manifold is $\mathbf{R}^n$ (see Foozwell [4]). We would like to know if Haken 4-manifolds are abundant or relatively rare manifolds. We will show that they are abundant in the following sense:

For each pair of closed Haken 3-manifolds M, M', there is a Haken 4-manifold W with boundary $\partial W = M \cup M'$. In addition, the inclusion maps induce injections $\pi_1(M) \to \pi_1(W)$ and $\pi_1(M') \to \pi_1(W)$. The special case when $M' = \varnothing$ is of particular interest.

Our proof of this result will be obtained in a number of steps. The first step is to show that if M is a torus-bundle over a circle, then there is a Haken 4-manifold W with boundary $\partial W = M$. We do this in Section 3. We then show a similar result for general surface-bundles in Section 4. To show that Haken manifolds satisfy our main result, we use a result of Gabai [6] and Ni [12] in Section 5.

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It is well known that all closed 3-manifolds are null cobordant, that is, bound compact 4-manifolds. Davis, Januszkiewicz and Weinberger [2] following on from work in [1], show that if an aspherical closed n-manifold is null cobordant, then it bounds an aspherical $(n+1)$-manifold, and furthermore, the inclusion map of the boundary is π_1-injective. Haken n-manifolds satisfy the stronger property (than asphericity) that they have universal covering by $\mathbf{R}^n$, as shown in [4]. Moreover for Haken cobordism theory (see [5]), the inclusion maps of the n-manifolds into the $(n+1)$-dimensional cobordism are π_1-injective.

2. Haken n-manifolds

For simplicity, all manifolds will be assumed to be orientable throughout this paper. We work throughout in the PL category, so all manifolds and maps are assumed PL.

Let W be a compact n-manifold and let $\underline{\underline{w}}$ be a finite collection of connected $(n-1)$-dimensional submanifolds in ∂W. We say that $\underline{\underline{w}}$ is a *boundary-pattern* if whenever $A_1, \ldots, A_i$ is a collection of distinct elements of $\underline{\underline{w}}$, then $A_1 \cap \cdots \cap A_i$ is an $(n-i)$-dimensional manifold.[1] A boundary-pattern is *complete* if $\partial W = \bigcup\{A : A \in \underline{\underline{w}}\}$. The intersection complex $K = K(W, \underline{\underline{w}})$ is

$$K = \bigcup\{\partial A : A \in \underline{\underline{w}}\}.$$

A two-dimensional disk with complete boundary-pattern consisting of i elements is called an *i-faced disk.* A *small disk* is an i-faced disk for $i \leq 3$.

The empty boundary-pattern is a special case of a boundary-pattern, and thus a closed manifold is a manifold with boundary-pattern.

Boundary-patterns arise naturally in splitting situations. Suppose that M is a two-sided codimension-one submanifold of W. Let $W|M$ denote the manifold obtained by splitting W open along M. There is a surjective map $q\colon W|M \to W$, that reverses the process of splitting W open along M. We call q the *unsplitting map.* If W has a boundary-pattern $\underline{\underline{w}}$, then B is an element of the *natural boundary-pattern* of $W|M$ if either

- B is a component of $q^{-1}(A)$ for some $A \in \underline{\underline{w}}$, or
- B is a component of $q^{-1}(M)$.

A map between manifolds with boundary-patterns should relate the boundary-patterns in a reasonable way. We use the following definition. If $(W, \underline{\underline{w}})$ and $(V, \underline{\underline{v}})$ are manifolds with boundary-patterns, then an *admissible map* is a continuous function $f\colon W \to V$ that is transverse to the boundary-patterns and satisfies

$$B \in \underline{\underline{w}} \quad \Longleftrightarrow \quad B \text{ is a component of } f^{-1}(A) \text{ for some } A \in \underline{\underline{v}}.$$

[1] The only manifold of negative dimension is the empty set. The empty set is also a manifold in each non-negative dimension.

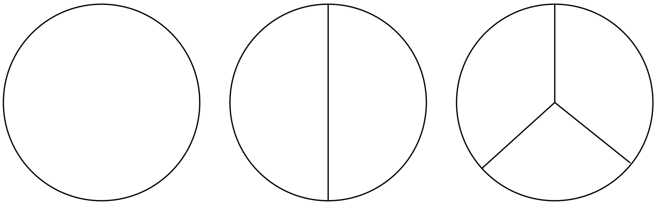

FIGURE 1. $g^{-1}(K)$ is the cone on $g^{-1}(K) \cap \partial\Delta$.

We write $f\colon (W, \underline{\underline{w}}) \to (V, \underline{\underline{v}})$ to indicate that the map f is admissible. Admissible homeomorphisms, embeddings and so on are defined in the obvious way.

DEFINITION 2.1. Let $(W, \underline{\underline{w}})$ be a manifold with boundary-pattern and let K be the intersection complex. Suppose that for each admissible map $f\colon (\Delta, \underline{\underline{\delta}}) \to (W, \underline{\underline{w}})$ of a small disk, there is a map $g\colon \Delta \to \partial W$, homotopic to $f \operatorname{rel} \partial\Delta$, such that $g^{-1}(K)$ is the cone on $g^{-1}(K) \cap \partial\Delta$. Then we say that $\underline{\underline{w}}$ is a *useful* boundary-pattern. See Figure 1.

In his solution to the word problem, Waldhausen [14] showed that the boundary-patterns that arise in splitting situations for Haken 3-manifolds can always be modified to be useful. (Note that boundary patterns were formally introduced later by Johannson in [7]—they were not explicitly mentioned in [14]).

If a properly embedded arc can be pushed into the boundary-pattern so that it is contained in no more than two boundary-pattern elements, then we say that the arc is *inessential*. We state this more precisely in the following definition.

DEFINITION 2.2. Let $(J, \underline{\underline{j}})$ be a compact connected 1-dimensional manifold with complete boundary-pattern and let $\sigma\colon (J, \underline{\underline{j}}) \to (W, \underline{\underline{w}})$ be an admissible map. We say that σ is an *inessential curve* if there is a disk Δ and an admissible map $g\colon (\Delta, \underline{\underline{\delta}}) \to (W, \underline{\underline{w}})$ such that

(1) $J = \mathrm{Cl}(\partial\Delta \setminus \bigcup\{A : A \in \underline{\underline{\delta}}\})$
(2) $\underline{\underline{\delta}}$ consists of at most two elements, and
(3) $g|_J = \sigma$.

The boundary-pattern $\underline{\underline{\delta}}$ consists of one element if both endpoints of σ are contained in the same element of $\underline{\underline{w}}$. It consists of two elements if the endpoints of σ are contained in distinct elements of $\underline{\underline{w}}$. If J is a circle, then $\underline{\underline{\delta}}$ is empty. We say that $\sigma\colon (J, \underline{\underline{j}}) \to (W, \underline{\underline{w}})$ is an *essential curve* if there is no map $g\colon (\Delta, \underline{\underline{\delta}}) \to (W, \underline{\underline{w}})$ satisfying the three properties above.

An admissible map $f\colon (M,\underline{\underline{m}}) \to (W,\underline{\underline{w}})$ is *essential* if each essential curve $\sigma\colon (J,\underline{\underline{j}}) \to (M,\underline{\underline{m}})$ defines an essential curve $f\circ\sigma\colon (J,\underline{\underline{j}}) \to (W,\underline{\underline{w}})$. Let M be a submanifold of W. We say that $(M,\underline{\underline{m}})$ is an *essential submanifold* of $(W,\underline{\underline{w}})$ if the inclusion map is admissible and essential. When we wish to prove that a submanifold $(M,\underline{\underline{m}})$ of $(W,\underline{\underline{w}})$ is essential, we will show that each curve $\sigma\colon (J,\underline{\underline{j}}) \to (M,\underline{\underline{m}})$ that is inessential in M is also inessential in W.

Let $(W,\underline{\underline{w}})$ be an n-manifold with complete and useful boundary-pattern and let $(M,\underline{\underline{m}})$ be a two-sided codimension-one submanifold of W for which the inclusion map is admissible and essential. Then we say that (W,M) is a *good pair*.

A *Haken* 1-*cell* is an arc with complete and useful boundary-pattern. If $n>1$, then a *Haken n-cell* is an n-cell with complete and useful boundary-pattern such that each element of the boundary-pattern is a Haken $(n-1)$-cell.

Let $(W_0,\underline{\underline{w}}_0)$ be an n-manifold with complete and useful boundary-pattern. A finite sequence of good pairs

$$(W_0,M_0),(W_1,M_1),\ldots,(W_k,M_k)$$

is called a *hierarchy* if

(1) W_{i+1} is obtained by splitting W_i open along M_i, and
(2) W_{k+1} is a finite disjoint union of Haken n-cells.

A manifold with a hierarchy is called a *Haken n-manifold*.

By definition, each element of the boundary-pattern of a Haken n-manifold is π_1-injective. By convention, when we say that a manifold is Haken without explicitly referring to a boundary-pattern, the boundary-pattern is simply the disjoint union of the boundary components. For example, suppose that a manifold W has two boundary components, X and Y. If we assert that W is a Haken manifold, then this is meant to imply that X and Y are π_1-injective in W and that the boundary-pattern of W is $\{X,Y\}$.

Fibre-bundles that have aspherical surfaces as base and fibre provide examples of Haken 4-manifolds. The hierarchy is obtained by lifting essential curves and arcs in the base surface to the 4-manifold. These manifolds will play an important role in this paper.

Let $\underline{\underline{w}} = \{M_1,\ldots,M_j\}$ be a finite collection of closed Haken n-manifolds. If W is a connected Haken $(n+1)$-manifold with boundary-pattern $\underline{\underline{w}}$, then we say that W is a *Haken cobordism*. If the collection $\underline{\underline{w}}$ consists of just two manifolds, then we may regard a Haken cobordism as an equivalence relation between Haken n-manifolds.

Our interest is in Haken cobordism as a relation between Haken 3-manifolds. In Section 5, we will give a condition for two connected Haken 3-manifolds to form the boundary of a Haken cobordism. We will also show that each closed Haken 3-manifold is the boundary of some Haken 4-manifold. As a first step, the following lemma was proved in Foozwell's thesis [3].

LEMMA 2.3. *If N is obtained from the Haken 3-manifold M by splitting M open along an incompressible surface F and re-gluing the boundary components, then there is a Haken 4-manifold W with $\partial W = M \sqcup N \sqcup E$, where E is a surface-bundle over the circle with fibre F.*

We first prove that a product of a Haken 3-manifold with an interval is a Haken 4-manifold. If $(M, \underline{\underline{m}})$ is a manifold with boundary-pattern, then B is an element of the *standard product boundary-pattern* $\underline{\underline{m \times i}}$ for $M \times I$ if either

- $B = M \times \{0\}$,
- $B = M \times \{1\}$, or
- $B = A \times I$ for some $A \in \underline{\underline{m}}$.

LEMMA 2.4. *Let $(M, \underline{\underline{m}})$ be an orientable Haken 3-manifold. Then $W = M \times I$ with the standard product boundary-pattern $\underline{\underline{w}}$ is a Haken 4-manifold.*

Proof. The manifold $M_1 = M$ has a hierarchy

$$(M_1, F_1), \ldots, (M_s, F_s), \ldots, (M_k, F_k),$$

where $M_k | F_k$ is a disjoint union of Haken 3-cells. We will prove that the splitting sequence

$$(W_1, F_1 \times I), \ldots, (W_s, F_s \times I), \ldots, (W_k, F_k \times I)$$

is a hierarchy for $W = W_1$, where each $W_s = M_s \times I$.

To do so, we will use a proof by induction on the length of the splitting sequence. We do this by proving the following three claims:

(1) W_{k+1} is a disjoint union of Haken 4-cells.
(2) If W_{s+1} has a useful boundary-pattern, then so does W_s.
(3) If W_{s+1} has useful boundary-pattern, then $F_s \times I$ is an essential submanifold of W_s.

To prove the claims (2) and (3), we use the following approach. If $f \colon \Delta \to W_s$ is a disk for which $f^{-1}(F_s \times I)$ is a subset of $\partial\Delta$, then we may regard f as a map into W_{s+1}. We use the usefulness of the boundary-pattern of W_{s+1} to homotope f into ∂W_{s+1}. We then view this as a homotopy of f in W_s. Most of the arguments then involve modifying maps of disks so that $f^{-1}(F_s \times I)$ is a subset of $\partial\Delta$.

To prove claim (1), observe that $W_k | (F_k \times I)$ is a disjoint union of 4-cells and each component is of the form $Q \times I$ where Q is a component of $M_k | F_k$. The boundary-pattern of $Q \times I$, which is induced by the splitting sequence, is the standard product boundary-pattern $\underline{\underline{q \times i}}$. Each element of the boundary-pattern of Q is a Haken 2-cell, so each element of $\underline{\underline{q \times i}}$ is a Haken 3-cell. We only need to show $\underline{\underline{q \times i}}$ is a useful boundary-pattern.

Let $f \colon (\Delta, \underline{\underline{\delta}}) \to (Q \times I, \underline{\underline{q \times i}})$ be an admissible map of a small disk. Since $Q \times I$ is a 4-cell, the map f is homotopic $\operatorname{rel} \partial\Delta$ to a map $g \colon \Delta \to \partial(Q \times I)$.

Note that $g(\partial\Delta)$ is a loop in a 3-sphere $\partial(Q\times I)$ that is subdivided into Haken 3-cells. The walls of these 3-cells are formed by the intersection complex K. The loop $g(\partial\Delta)$ is contained in at most three such 3-cells, and we can homotope g so that $g(\Delta)$ is contained in the same 3-cells that contain $g(\partial\Delta)$. Further homotopy (using standard 3-manifold techniques) allows us to simplify the map so that $g^{-1}(K)$ is the cone on $g^{-1}(K)\cap\partial\Delta$. This establishes claim (1).

We now prove claim (2): if W_{s+1} has useful boundary-pattern, then W_s has useful boundary-pattern. Suppose we have an admissible map $f\colon(\Delta,\underline{\underline{\delta}})\to(W_s,\underline{\underline{w_s}})$ of a small disk. Consider $f^{-1}(F_s\times I)$, which, since $F_s\times I$ is π_1-injective in the aspherical manifold W_s, we may assume contains no loops. If $f^{-1}(F_s\times I)$ is empty, then we may view f as a map into W_{s+1} which has useful boundary-pattern. Pushing f into the boundary in W_{s+1} is like pushing into the boundary in W_s in this case.

If $f^{-1}(F_s\times I)$ contains arcs, then choose an outermost arc that bounds a disk Δ_1. We may view $f|_{\Delta_1}$ as an admissible map of a 2-faced disk into W_{s+1}. Since W_{s+1} has useful boundary-pattern, there is a map $g\colon\Delta_1\to W_{s+1}$ homotopic to $f|_{\Delta_1}\operatorname{rel}\partial\Delta_1$ such that $g^{-1}(K(W_{s+1},\underline{\underline{w_{s+1}}}))$ is the cone on $g^{-1}(K(W_{s+1},\underline{\underline{w_{s+1}}}))\cap\partial\Delta_1$.

Now this map g can be viewed as a homotopy of the map f in W_s. The homotopy pushes the outermost disk Δ_1 into $\partial W_s\cup F_s\times I$. We may then push the disk to the other side of $F_s\times I$. The result is a map f' that is homotopic to $f\operatorname{rel}\partial$ such that ${f'}^{-1}(F_s\times I)$ has one less arc than $f^{-1}(F_s\times I)$. So we can remove all the arcs of f^{-1}. This establishes claim (2).

We now prove claim (3): if W_{s+1} has useful boundary-pattern, then $F_s\times I$ is essential in W_s.

Suppose we have a curve $\sigma\colon(J,\underline{\underline{j}})\to(F_s\times I,\underline{\underline{f_s\times i}})$ that is inessential in W_s. This means there is a map $g\colon(\Delta,\underline{\underline{\delta}})\to(W_s,\underline{\underline{w_s}})$ such that

- $J=\mathrm{Cl}(\partial\Delta\setminus\bigcup\{A:A\in\underline{\underline{\delta}}\})$,
- $\underline{\underline{\delta}}$ contains at most two elements, and
- $g|_J=\sigma$.

If $g^{-1}(F_s\times I)=J$, then we may regard g as an admissible map into W_{s+1}, which has useful boundary-pattern. Then g is homotopic $\operatorname{rel}\partial\Delta$ to a map $g_1\colon\Delta\to\partial W_{s+1}$. Observe that $\Delta=A\cup B$ where $A={g_1}^{-1}(F_s\times I)$ and $B={g_1}^{-1}(\mathrm{Cl}(\partial W_{s+1}\setminus(F_s\times I)))$, as illustrated in Figure 2. We now regard $g_1|_A$ as an admissible map of the disk A into $F_s\times I$. This is the map required to show that σ is an inessential curve in $F_s\times I$.

If $g^{-1}(F_s\times I)\neq J$, then we show how to modify the map so that the preimage is J. We remove loops from $g^{-1}(F_s\times I)$ in the usual way, and similarly we can remove arcs with both endpoints in J from $g^{-1}(F_s\times I)$.

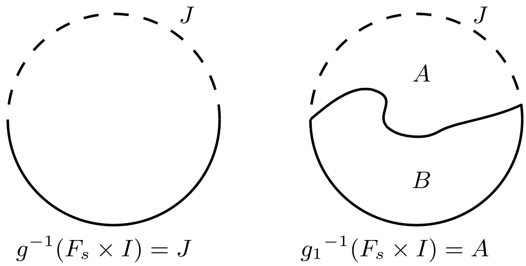

FIGURE 2. Modifying the pre-image in the disk.

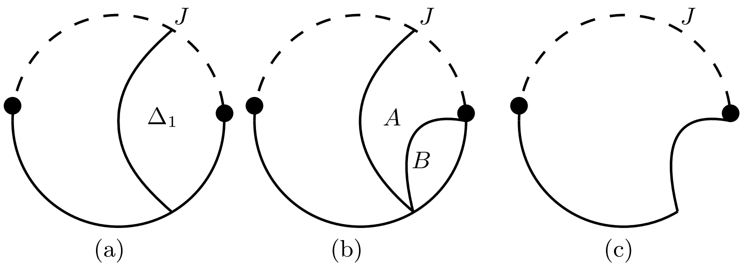

FIGURE 3. Modifying the pullback.

Suppose we have an arc with at least one endpoint not in J. An outermost such arc bounds a disk Δ_1 in Δ, as in Figure 3(a). Then $g|_{\Delta_1}$ is an admissible map in W_{s+1}, which has a useful boundary-pattern, so there is a map $g_2\colon \Delta_1 \to \partial W_{s+1}$ homotopic to $g|_{\Delta_1}$ rel $\partial\Delta_1$. Then $\Delta_1 = A \cup B$ where $A = g_2{}^{-1}(F_s \times I)$ and $B = g_2{}^{-1}(\mathrm{Cl}(\partial W_{s+1} \setminus (F_s \times I)))$. See Figure 3(b). We cut B out of Δ to obtain a new disk Δ_2 and we push g_2 to the other side of $F_s \times I$ so that we have a map with one less arc in the pullback. See Figure 3(c). Continuing in this fashion, we may assume that $g^{-1}(F_s \times I) = J$. □

Proof of Lemma 2.3. Form $M \times [0,1]$ and attach a copy of $R(F) \times [0,1]$ to a regular neighbourhood of parallel copies of $F \times \{1\}$ in $M \times \{1\}$ as indicated in Figure 4. We denote by $R(F)$ the regular neighbourhood of F. It is easy to see that the right boundary components are obtained. The first essential submanifold in the hierarchy of W is $R(F) \times \{1/2\}$. After splitting W open along $R(F) \times \{1/2\}$, we obtain a manifold W_1 equivalent to $M \times I$, but with a boundary-pattern different to the standard product boundary-pattern. To define the boundary-pattern, let $R(F_-) \times \{1\}$ and $R(F_+) \times \{1\}$ be sufficiently small regular neighbourhoods of parallel copies of F in $M \times \{1\}$. Then B is an element of the boundary-pattern $\underline{\underline{w_1}}$ of $M \times I$ if

- $B = M \times \{0\}$,

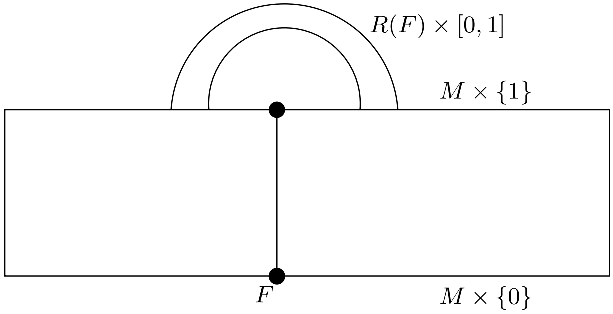

FIGURE 4. Building a Haken 4-manifold with three boundary components.

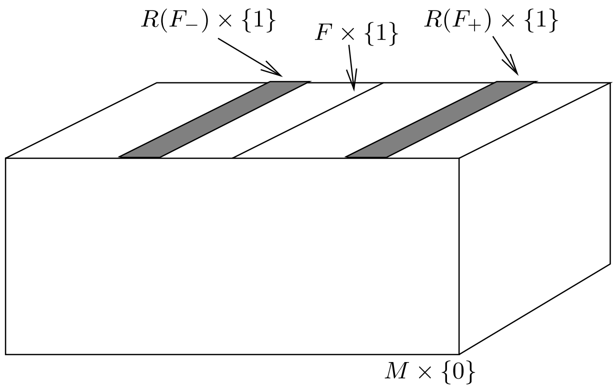

FIGURE 5. New boundary-pattern on the product $M \times I$.

- $B = R(F_-) \times \{1\}$
- $B = R(F_+) \times \{1\}$
- B is a component of $\mathrm{Cl}(M \times \{1\} \setminus R(F_\pm) \times \{1\})$.

The boundary-pattern $\underline{\underline{w_1}}$, which is illustrated in Figure 5, is useful because we can homotope admissible disks away from $F_\pm \times I$ as in earlier parts of the proof. The splitting sequence for $W_1 = M_1 \times I$ is

$$(W_1, F_1 \times I), \ldots, (W_s, F_s \times I), \ldots, (W_k, F_k \times I)$$

as in the proof of Lemma 2.4. This sequence is a hierarchy for $(W_1, \underline{\underline{w_1}})$ because we choose the regular neighbourhoods of $F_\pm \times I$ to be sufficiently small. $\square$

After dealing with bundles in the next two sections, we will see how to improve upon Lemma 2.3.

3. Torus-bundles

In this section, we show that each torus-bundle over the circle is the boundary of some Haken 4-manifold.

We first introduce some conventions of notation and orientation that will be used throughout this paper.

If $g\colon S \to S$ is a homeomorphism of a surface S, then $S(g)$ is the surface-bundle over the circle with fibre S and monodromy g. More concretely,

$$S(g) = S \times [0,1]/(x,0) \sim \big(g(x),1\big). \tag{3.1}$$

We will use the above notation for fibre-bundles throughout this paper.

The following conventions regarding orientations on manifolds and their boundaries will be used. If S is an orientable surface, then an orientation for S can be specified by an ordered linearly independent pair of vectors (w,x) at a single point $p \in S$. The standard orientation for $S(g)$ is then (w,x,y) where y is a non-zero vector based at $(p,0)$ tangent to $\{p\} \times [0,1]$ and directed towards 1. The standard orientation of the 4-manifold $S(g) \times [0,1]$ is (w,x,y,z) where z is a non-zero vector based at $(p,0,0)$ tangent to $\{(p,0)\} \times [0,1]$ and directed towards 1. We write the boundary of $S(g) \times [0,1]$ as

$$\partial\big(S(g) \times [0,1]\big) = S\big(g^{-1}\big) \sqcup S(g). \tag{3.2}$$

Since $S(g^{-1})$ is homeomorphic to $S(g)$, with a reversal of orientation, we use the term $S(g^{-1})$ in expression (3.2) to represent the manifold $S(g) \times \{0\}$ with the orientation induced by the outward normal convention. The term $S(g)$ in expression (3.2) represents the manifold $S(g) \times \{1\}$, also with the outward normal convention.

Example 3.1. Let $\mathrm{T}^2(\varphi)$ be the torus-bundle over a circle with monodromy φ a single Dehn twist. We represent the torus-bundle $\mathrm{T}^2(\varphi)$ by considering the torus as the square $[0,1] \times [0,1]$ in the plane with sides identified in the usual way. The monodromy φ is represented by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. We represent $\mathrm{T}^2(\varphi)$ visually in Figure 6, regarding $\mathrm{T}^2(\varphi)$ as the quotient space $(\mathrm{T}^2 \times [0,1])/(x,0) \sim (\varphi(x),1)$.

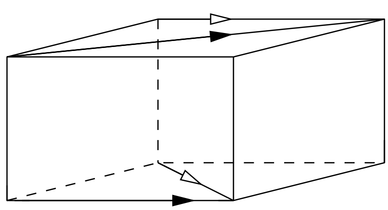

Figure 6. Torus bundle with single Dehn twist.

We consider a special case of Lemma 2.3 that we will use subsequently. Let $W_1 = \mathrm{T}^2(\varphi) \times [0,1]$, which is a torus-bundle over an annulus. The boundary of W_1 is $(\mathrm{T}^2(\varphi) \times \{0\}) \sqcup (\mathrm{T}^2(\varphi) \times \{1\})$. Let us pick out two disjoint parallel torus fibres in $\mathrm{T}^2(\varphi) \times \{1\}$. These are: $T_i = \mathrm{T}^2 \times \{i/3\} \times \{1\}$ for $i = 1$ or 2. Let ε be a sufficiently small positive number,[2] and consider the ε-neighbourhoods of these tori: $T_i(\varepsilon) = \mathrm{T}^2 \times [i/3 - \varepsilon, i/3 + \varepsilon] \times \{1\}$. Attach a copy of $\mathrm{T}^2 \times [-\varepsilon, \varepsilon] \times [0,1]$ to $\mathrm{T}^2(\varphi) \times \{1\}$ so that $\mathrm{T}^2 \times [-\varepsilon, \varepsilon] \times \{0\}$ meets $T_1(\varepsilon)$ and $\mathrm{T}^2 \times [-\varepsilon, \varepsilon] \times \{1\}$ meets $T_2(\varepsilon)$. We choose the attachment so that the boundary of the resulting manifold W is

$$\mathrm{T}^2\left(\varphi^{-1}\right) \sqcup \mathrm{T}^2\left(\psi^{-1}\right) \sqcup \mathrm{T}^2(\varphi \circ \psi)$$

where $\psi \in \mathrm{SL}(2, \mathbf{Z})$. The manifold W is an orientable Haken 4-manifold with three boundary components. The orientations on the boundary components is based on the orientation convention in expression (3.2). If we regard ψ as a product of k Dehn twists, then this example shows how to construct a Haken cobordism between torus-bundles with $k+1$ Dehn twists, k Dehn twists and a single Dehn twist.

Theorem 3.2. *If M is a torus-bundle over a circle, then there is a Haken 4-manifold W with boundary $\partial W = M$.*

We will prove Theorem 3.2 via a sequence of lemmas. The first of these is a simple observation that is probably well-known.

Lemma 3.3. *Let F and G be closed orientable incompressible surfaces in a closed orientable 3-manifold M. Suppose that $F \cap G$ is a simple closed curve α. The manifold obtained by splitting M open along F and regluing via a Dehn twist along α is homeomorphic to the manifold obtained by splitting M open along G and regluing via a Dehn twist along α.*

Proof. The result of either operation is simply Dehn surgery on the curve α. □

Lemma 3.4. *If M_φ is a torus-bundle over a circle with monodromy φ a single Dehn twist, then there is a Haken 4-manifold W with boundary $\partial W = M_\varphi$.*

Proof. Let Σ be a closed orientable surface of genus three. We may regard Σ as the double of the thrice-punctured disk, which is shown in Figure 7. Three of the four boundary components of the thrice-punctured disk are labelled in Figure 7. Let ϵ_i be a curve parallel to the boundary component labelled i in Figure 7. The curve ϵ_4 is parallel to the unlabelled boundary component. Let α, β and γ be the curves shown in Figure 7. Up to isotopy, the identity mapping $\mathrm{id}\colon \Sigma \to \Sigma$ can be written as a product of three positive Dehn twists and four negative Dehn twists. This observation is a consequence of the lantern relation [8] of the mapping class group. Let f_α be the right-handed

[2] The number ε is sufficiently small in the sense that $T_1(\varepsilon) \cap T_2(\varepsilon) = \varnothing$.

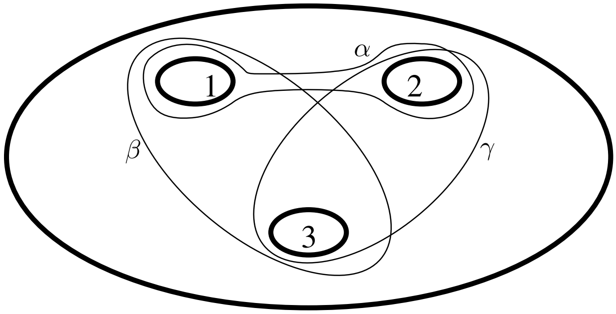

FIGURE 7. Lantern relation.

Dehn twist about α, and define f_β and f_γ similarly. Let f_i be the right-handed Dehn twist about ϵ_i. The lantern relation is

$$f_\gamma f_\beta f_\alpha = f_1 f_2 f_3 f_4.$$

We may write this relation in a number of ways; each ϵ_i is disjoint from the other curves, so for example, f_i commutes with the other Dehn twists. Thus, up to isotopy, we may write the identity mapping as

$$f_1 f_\gamma^{-1} f_2 f_\alpha^{-1} f_3 f_\beta^{-1} f_4.$$

We define the following maps:

$$\theta_7 = f_1 f_\gamma^{-1} f_2 f_\alpha^{-1} f_3 f_\beta^{-1} f_4, \qquad \theta_3 = \theta_4 f_\alpha,$$
$$\theta_6 = \theta_7 f_4^{-1}, \qquad \theta_2 = \theta_3 f_2^{-1},$$
$$\theta_5 = \theta_6 f_\beta, \qquad \theta_1 = \theta_2 f_\gamma,$$
$$\theta_4 = \theta_5 f_3^{-1}, \qquad \theta_0 = \theta_1 f_1^{-1}.$$

We now show how to construct a Haken 4-manifold W_7 with three surface-bundle boundary components. Specifically,

$$\partial W_7 = \Sigma\big(\theta_7^{-1}\big) \sqcup \Sigma(\theta_6) \sqcup \mathrm{T}^2(\varphi).$$

The boundary components are written using representatives from the appropriate orientation-preserving homeomorphism class. The orientations of the boundary components are in accordance with the convention from expression (3.2).

To see how to build W_7, first note that, by the lantern relation, θ_7 is isotopic to the identity, so $\Sigma(\theta_7) = \Sigma \times \mathrm{S}^1$. Then observe that $\Sigma \times \mathrm{S}^1$ is related to $\Sigma(\theta_6)$ by splitting open along a fibre and regluing by a Dehn twist along the curve ϵ_4 in the fibre. There is an incompressible torus T in $\Sigma \times \mathrm{S}^1$ that intersects the fibre in the curve ϵ_4. By Lemma 3.3, we can also obtain $\Sigma(\theta_6)$ by splitting $\Sigma \times \mathrm{S}^1$ open along T and regluing with a Dehn twist. Then Lemma 2.3 tells us how to construct W_7; we attach a manifold of the form $\mathrm{T}^2 \times [0,1] \times [0,1]$ to a boundary-component of $(\Sigma \times \mathrm{S}^1) \times [0,1]$.

Observe that in $\Sigma(\theta_6)$ there is an incompressible torus that intersects the fibre in the curve β. By attaching a manifold of the form $\mathrm{T}^2 \times [0,1] \times [0,1]$ to a boundary-component of $\Sigma(\theta_6) \times [0,1]$ we obtain a Haken 4-manifold W_6 with boundary

$$\partial W_6 = \Sigma\big(\theta_6^{-1}\big) \sqcup \Sigma(\theta_5) \sqcup \mathrm{T}^2\big(\varphi^{-1}\big).$$

Similarly, there is an incompressible torus in $\Sigma(\theta_5)$ that intersects the fibre in the curve ϵ_3. We then construct a 4-manifold W_5 with boundary

$$\partial W_5 = \Sigma\big(\theta_5^{-1}\big) \sqcup \Sigma(\theta_4) \sqcup \mathrm{T}^2(\varphi).$$

We continue creating Haken cobordisms with three boundary components. However, we no longer need to find incompressible tori that intersect the lantern curves. Instead, all the boundary components will be Σ-bundles over the circle. Lemma 2.3 produces Haken 4-manifolds W_4, W_3, W_2, and W_1 with boundaries as follows:

$$\partial W_4 = \Sigma\big(\theta_4^{-1}\big) \sqcup \Sigma(\theta_3) \sqcup \Sigma\big(f_\alpha^{-1}\big), \qquad \partial W_2 = \Sigma\big(\theta_2^{-1}\big) \sqcup \Sigma(\theta_1) \sqcup \Sigma\big(f_\gamma^{-1}\big),$$
$$\partial W_3 = \Sigma\big(\theta_3^{-1}\big) \sqcup \Sigma(\theta_2) \sqcup \Sigma(f_2), \qquad \partial W_1 = \Sigma\big(\theta_1^{-1}\big) \sqcup \Sigma(\theta_0) \sqcup \Sigma(f_1).$$

Note that θ_0 is the identity mapping so $\Sigma(\theta_0) = \Sigma \times \mathrm{S}^1$.

So we have seven orientable Haken 4-manifolds each with three boundary components. We can glue these seven manifolds together to form a connected manifold W' with boundary

$$\partial W' = \Sigma\big(\theta_7^{-1}\big) \sqcup \mathrm{T}^2(\varphi) \sqcup \mathrm{T}^2\big(\varphi^{-1}\big) \sqcup \mathrm{T}^2(\varphi) \sqcup \Sigma\big(f_\alpha^{-1}\big)$$
$$\sqcup\, \Sigma(f_2) \sqcup \Sigma\big(f_\gamma^{-1}\big) \sqcup \Sigma(f_1) \sqcup \Sigma(\theta_0).$$

The idea is illustrated in Figure 8, which schematically shows the manifolds W_7 and W_6 being joined together.

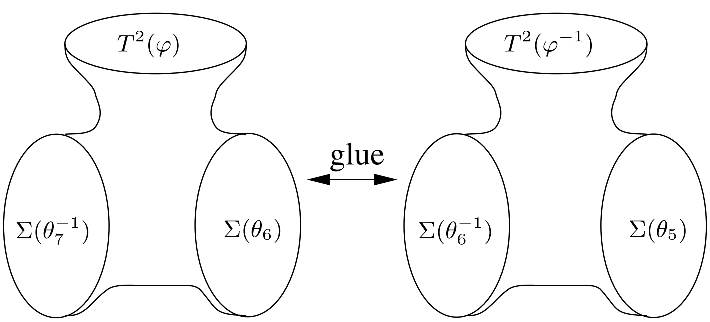

FIGURE 8. Identifying the $\Sigma(\theta_6)$ boundary-components of W_7 and W_6 to produce a connected 4-manifold.

We glue eight of these boundary components in pairs, leaving just one boundary component $\mathrm{T}^2(\varphi)$. That is, we glue $\mathrm{T}^2(\varphi^{-1}) \subset W_6$ to $\mathrm{T}^2(\varphi) \subset W_5$, glue $\Sigma(f_\alpha^{-1})$ to $\Sigma(f_2)$ and glue $\Sigma(f_\gamma^{-1})$ to $\Sigma(f_1)$. We also glue $\Sigma(\theta_7^{-1})$ to $\Sigma(\theta_0)$. This can all be done so that the result is orientable. Hence, there is an orientable Haken 4-manifold W with boundary $\partial W = \mathrm{T}^2(\varphi)$. □

LEMMA 3.5. *If M_ψ is a torus-bundle over a circle with monodromy ψ a product of a finite number of Dehn twists, then there is a Haken 4-manifold W with boundary $\partial W = M_\psi$.*

Proof. The construction is similar to that of Example 3.1 and is by induction on the number of Dehn twists, say k. Write the monodromy as $\psi = \tau \circ \sigma$ where τ is a product of $k-1$ Dehn twists and σ is a Dehn twist. We modify the torus-bundle $M_\psi \times [0,1]$ by attaching a copy of $\mathrm{T}^2 \times [-\varepsilon, \varepsilon] \times [0,1]$ to ε-neighbourhoods of disjoint torus fibres in $M_\psi \times \{0\}$ as in Example 3.1, except we choose the gluing so that the boundary components are M_ψ, M_τ and M_σ.

Since σ is a single Dehn twist, we can glue on the compact 4-manifold found in Lemma 3.4 to fill in the boundary component M_ψ. We obtain a manifold W with two boundary components M_ψ, M_τ. It is easy to see that W is a Haken 4-manifold. The proof now follows by induction since τ is a product of $k-1$ Dehn twists. So we can find a Haken 4-manifold with boundary M_τ and glue this onto W to build the required Haken 4-manifold with boundary M_ψ.

Note that the case $k=1$ follows from Lemma 3.4. □

Putting the results of the lemmas in this section together constitutes a proof of Theorem 3.2.

4. Higher genus surface-bundles

We will use Lemma 2.3 in our proof of the main theorem of this section.

THEOREM 4.1. *If M is a closed surface-bundle over a circle, then there is a Haken 4-manifold W with $\partial W = M$.*

Proof. As before, the proof is by induction on the number of Dehn twists needed to represent the monodromy. To prove Theorem 4.1, we must construct a Haken 4-manifold whose boundary is a surface-bundle with given monodromy.

To start the induction, let F be a closed orientable surface of genus at least two, and let M_φ be the surface bundle $F(\varphi)$,where φ is a Dehn twist along an essential curve α in F. It is clear that we can construct M_φ from the product $F \times \mathrm{S}^1$ by cutting $F \times \mathrm{S}^1$ open along the fibre $F \times \{p\}$ and regluing with a Dehn twist. By Lemma 3.3, we can construct M_φ by splitting $F \times \mathrm{S}^1$ along an incompressible torus containing α and regluing with a Dehn twist.

The manifold M_φ is related to the product manifold $F \times \mathrm{S}^1$ by a change in homeomorphism along an incompressible torus. Hence, there is a Haken

4-manifold W_1 with boundary $\partial W_1 = M_\varphi \sqcup (F \times \mathrm{S}^1) \sqcup E$ where E is the total space of a torus-bundle over a circle. In Section 3, we showed that E is the boundary of a Haken 4-manifold, W_2. The product $F \times \mathrm{S}^1$ is also the boundary of a Haken 4-manifold. For example, take a Haken 3-manifold N with boundary $\partial N = F$. Then $N \times \mathrm{S}^1$ will suffice. We attach W_2 and $N \times \mathrm{S}^1$ to the appropriate boundary components of W_1 to obtain a Haken 4-manifold with boundary M_φ.

To prove the general case, we proceed exactly as in Lemma 3.5. Assume that M_φ is a surface bundle over a circle whose monodromy φ is a product of k Dehn twists. Write $\varphi = \tau \circ \psi$ where ψ is a single Dehn twist and τ is a product of $k-1$ Dehn twists. Using Lemma 2.3 and the case above of a surface bundle with monodromy consisting of a single Dehn twist, we can construct a Haken 4-manifold with boundary consisting of the disjoint union of $M_\varphi, M_\tau, M_\psi$ and then glue on a Haken 4-manifold with boundary M_ψ, since ψ is a single Dehn twist. By induction on the number k of Dehn twists, there is another Haken 4-manifold with boundary M_τ since τ is a product of $k-1$ Dehn twists. Gluing this on completes the proof of the theorem. □

5. Other Haken manifolds

We first prove an extension of Lemma 2.3, which gives a sufficient condition for two Haken 3-manifolds to be Haken cobordant.

THEOREM 5.1. *If N is obtained from the closed connected Haken 3-manifold M by splitting M open along an incompressible surface F and re-gluing the boundary components, then there is a Haken 4-manifold W with $\partial W = M \sqcup N$, and boundary-pattern $\underline{\underline{w}} = \{M, N\}$.*

Proof. We use the construction in the proof of Lemma 2.3 to obtain a Haken 4-manifold X with boundary $\partial X = M \sqcup N \sqcup E$ and boundary-pattern $\underline{\underline{x}} = \{M, N, E\}$, where E is a a surface-bundle over a circle with fibre F. By Theorems 3.2 and 4.1, there is another Haken 4-manifold Y with boundary $\partial Y = E$ and boundary-pattern $\underline{\underline{y}} = \{E\}$. We form a quotient space of $X \sqcup Y$ by gluing the E boundary components together via a homeomorphism to obtain the required Haken 4-manifold W. □

Gabai [6] announced the following result in 1983 with an outline of the proof, and recently Ni [12] has provided the details of the proof.

THEOREM 5.2. *Let M_1 be a closed Haken 3-manifold. There is a sequence*

$$M_1, M_2, M_3, \ldots, M_n$$

such that M_{i+1} is obtained from M_i by splitting M_i open along an incompressible surface and re-gluing the boundary components, and M_n is a product $\Sigma \times \mathrm{S}^1$, where Σ is a closed surface.

Using Theorem 5.2, we can show that every pair of closed Haken 3-manifolds is the boundary of some Haken 4-manifold.

THEOREM 5.3. *Let M, M' be a pair of closed Haken 3-manifolds. Then there is a Haken 4-manifold W with $\partial W = M \sqcup M'$ and boundary-pattern $\underline{\underline{w}} = \{M, M'\}$.*

Proof. Write $M = M_1$ and using the notation of Theorem 5.2 we have a sequence

$$M_1, M_2, M_3, \ldots, M_n$$

such that M_{i+1} is obtained from M_i by splitting M_i open along an incompressible surface and re-gluing the boundary components, and $M_n = \Sigma \times \mathrm{S}^1$, for some closed orientable aspherical surface Σ. Using induction on the number of terms in the sequence, we use Theorem 5.1 to obtain a Haken 4-manifold X with boundary $\partial X = M_1 \sqcup M_n$. Similarly (with obvious notation) there is a Haken 4-manifold Y with boundary $\partial Y = M_1' \sqcup M_p'$, where $M_p' = \Sigma' \times \mathrm{S}^1$ and Σ' is a closed orientable aspherical surface. If Σ' is homeomorphic to Σ, we can glue X to Y along the product boundary components to obtain the required Haken cobordism. Otherwise, take a Haken 3-manifold N with boundary $\partial N = \Sigma \sqcup \Sigma'$. Then $N \times \mathrm{S}^1$ is a Haken 4-manifold with boundary $(\Sigma \times \mathrm{S}^1) \sqcup (\Sigma' \times \mathrm{S}^1)$. We can then glue X and Y to the appropriate boundary components of $N \times \mathrm{S}^1$ to obtain the required Haken cobordism. □

COROLLARY 5.4. *If M is a closed Haken 3-manifold, then there is a Haken 4-manifold W with $\partial W = M$ and boundary-pattern $\underline{\underline{w}} = \{M\}$.*

6. Hyperbolic case

In Long and Reid [9], it is shown that if a closed hyperbolic 3-manifold M is the totally geodesic boundary of a compact hyperbolic 4-manifold W, then $\eta(M)$ takes an integer value. In [9], M is said to *geometrically bound* W. On the other hand, Meyerhoff and Neumann [11], show that $\eta(N_\alpha)$ takes a dense set of values in $\mathbf{R}$ for the set $\{N_\alpha\}$ of Dehn surgeries on a hyperbolic knot in S^3. So this implies that 'generically' hyperbolic 3-manifolds do not geometrically bound hyperbolic 4-manifolds.

The existence of π_1-injective 2-tori in the Haken 4-manifolds constructed in Corollary 5.4 gives an obvious obstruction to these 4-manifolds admitting hyperbolic or even strictly negatively curved metrics.

In [10], Long and Reid give examples of n-dimensional hyperbolic manifolds which geometrically bound hyperbolic $(n+1)$-dimensional hyperbolic manifolds, for all n.

7. Some questions

The Haken 4-manifolds that we have constructed in this paper fall into a special class. In a sense, they are analogues of the graph manifolds of Waldhausen. Other examples of Haken 4-manifolds exist. For example, the hyperbolic 4-manifolds of Ratcliffe and Tschantz [13] are all finitely covered by Haken 4-manifolds (see [5] for a proof of this). In [5], examples of Haken 4-manifolds which admit metrics of strictly negative curvature but which do not admit hyperbolic metrics are given.

Question 7.1. If M is a closed Haken 3-manifold, does there exist a Haken 4-manifold W with $\partial W = M$ and which contains only non-separating submanifolds in its hierarchy? (Note that then the complement of the hierarchy is a single 4-cell.)

Question 7.2. Which closed hyperbolic Haken 3-manifolds M geometrically bound hyperbolic Haken 4-manifolds? Are there other obstructions than that in [9] that the eta invariant of M must be an integer? What about the situation if the Haken 4-manifold admits a metric of strictly negative or non-positive curvature? Is it still true that the eta invariant of M must be an integer in this case?

Question 7.3. For $n > 3$, what are the Haken cobordism classes for Haken n-manifolds? We say that Haken n-manifolds N and N' belong to the same Haken cobordism class if there is a Haken $(n+1)$-manifold W for which $\partial W = N \sqcup N'$ so that N, N' are essential in W. In private communication, Allan Edmonds has constructed a Haken 4-manifold with odd Euler characteristic, so we know, for example, that Haken 4-manifolds need not be null cobordant.

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References

[1] M. Davis and T. Januszkiewicz, *Hyperbolization of polyhedra*, J. Differential Geom. **34** (1991), 347–388. MR 1131435

[2] M. Davis, T. Januszkiewicz and S. Weinberger, *Relative hyperbolization and aspherical bordisms; An addendum to 'Hyperbolization of polyhedra'*, J. Differential Geom. **58** (2001), 535–541. MR 1906785

[3] B. Foozwell, *Haken n-manifolds*, Ph.D. Thesis, University of Melbourne, 2007, available at `https://sites.google.com/site/bellfoozwell/`.

[4] B. Foozwell, *The universal covering space of a Haken n-manifold*, preprint, 2011, available at `arXiv:1108.0474`.

[5] B. Foozwell and H. Rubinstein, *Introduction to the theory of Haken n-manifolds*, Topology and geometry in dimension three: Triangulations, invariants, and geometric

structures, Contemp. Math., vol. 560, American Mathematical Society, Providence, R.I., 2011, pp. 71–84. MR 2866924

[6] D. Gabai, *An internal hierarchy for* 3*-manifolds*, Knot theory and manifolds, Lecture Notes in Math., vol. 1144, Springer, Providence, R.I., 1983, pp. 14–18. MR 0823279

[7] K. Johannson, *Homotopy equivalences of* 3*-manifolds with boundaries*, Lecture Notes in Math., vol. 761, Springer, Berlin, 1979. MR 0551744

[8] D. L. Johnson, *Homeomorphisms of a surface which act trivially on homology*, Proc. Amer. Math. Soc. **75** (1979), 119–125. MR 0529227

[9] D. Long and A. Reid, *On the geometric boundaries of hyperbolic* 4*-manifolds*, Geom. Topol. **4** (2000), 171–178. MR 1769269

[10] D. Long and A. Reid, *Constructing hyperbolic manifolds which bound geometrically*, Math. Res. Lett. **8** (2001), 443–456. MR 1849261

[11] R. Meyerhoff and W. Neumann, *An asymptotic formula for the eta invariant for hyperbolic* 3*-manifolds*, Comment. Math. Helv. **67** (1992), 28–46. MR 1144612

[12] Y. Ni, *Some applications of Gabai's internal hierarchies*, Adv. Math. **250** (2014), 467–495. MR 3122174

[13] J. Ratcliffe and S. Tschantz, *The volume spectrum of hyperbolic* 4*-manifolds*, Exp. Math. **9** (2000), 101–125. MR 1758804

[14] F. Waldhausen, *The word problem in fundamental groups of sufficiently large irreducible* 3*-manifolds*, Ann. of Math. (2) **88** (1968), 272–280. MR 0240822

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