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MODEL THEORY AND THE QWEP CONJECTURE

ISAAC GOLDBRING

ABSTRACT. We observe that Kirchberg's QWEP conjecture is equivalent to the statement that $C^*(\mathbb{F})$ is elementarily equivalent to a QWEP C* algebra. We also make a few other model-theoretic remarks about WEP and LLP C* algebras.

For the sake of simplicity, all C* algebras in this note are assumed to be unital.

Suppose that B is a C* algebra and A is a subalgebra. We say that A is *relatively weakly injective* in B if there is a u.c.p. map $\phi : B \to A^{**}$ such that $\phi|B = \mathrm{id}_A$; such a map is referred to as a *weak conditional expectation.* (We view A as canonically embedded in its double dual.) A C* algebra A is said to have the *weak expectation property* (or be WEP) if it is relatively weakly injective in every extension and A is said to be *QWEP* if it is the quotient of a WEP algebra.

Kirchberg's QWEP conjecture states that every separable C* algebra is QWEP. In the seminal paper [10], Kirchberbg proved that the QWEP conjecture is equivalent to the Connes Embedding Problem (CEP), namely that every finite von Neumann algebra embeds into an ultrapower of the hyperfinite II_1 factor.

If $\mathbb{F}$ is the free group on countably many generators, then using the fact that $C^*(\mathbb{F})$ is surjectively universal, we see that the QWEP conjecture is equivalent to the statement that $C^*(\mathbb{F})$ is QWEP. The main point of this note is to point to an a priori weaker equivalent statement of the QWEP conjecture:

THEOREM 1. *The QWEP conjecture is equivalent to the statement that $C^*(\mathbb{F})$ is elementarily equivalent to a QWEP C* algebra.*

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Here, two C* algebras A and B are *elementarily equivalent* if they have the same first-order theories in the sense of model theory. (Here, we work in the signature for *unital* C* algebras.)

The next lemma is probably well known to the experts but since we could not locate it in the literature we include a proof here.

PROPOSITION 2. *Let A be a C* algebra and ω a nonprincipal ultrafilter on some (possibly uncountable) index set.*

(1) *Suppose that A is a subalgebra of B and that B admits a u.c.p. map into A^ω that restricts to the diagonal embedding of A. Then A is relatively weakly injective in B.*
(2) *A is relatively weakly injective in A^ω.*

Proof. Part (1) is almost identical to the easy direction of [10, Corollary 3.2(ii)] except there he works with the corona algebra instead of the ultrapower. In any event, the proof is easy so we give it here: Suppose that $\phi : B \to A^\omega$ is a u.c.p. map restricting to the diagonal embedding of A. Let $\theta : A^\omega \to A^{**}$ be the u.c.p. ultralimit map, that is $\theta((a_n)^\bullet) := \lim_\omega a_n$ (ultraweak limit). Then the desired weak expectation $\psi : B \to A^{**}$ is given by $\psi := \theta \circ \phi$. (2) follows immediately from (1) by taking a nonprincipal ultrafilter ω' on a big enough index set so as to allow for an embedding A^ω into $A^{\omega'}$ that restricts to the diagonal embedding of A into $A^{\omega'}$. $\square$

Theorem 1 follows immediately from the following proposition.

PROPOSITION 3. *The set of C* algebras with QWEP forms an axiomatizable class.*

Proof. We use the semantic test for axiomatizability, namely we show that the class of QWEP algebras is closed under isomorphism, ultraproduct, and ultraroot. (See [1, Proposition 5.14].) Clearly the class of QWEP algebras is preserved under isomorphism. To see that it is closed under ultraproducts, it suffices to note that it is closed under products [10, Corollary 3.3(i)] and (obviously) closed under quotients. To see that it is closed under ultraroots, we use the fact that A is relatively weakly injective in its ultrapower (Proposition 2(2)) together with the fact that QWEP passes to relatively weakly injective unital subalgebras [10, Corollary 3.3(iii)]. $\square$

REMARK 4. Proposition 3 is false with QWEP replaced by WEP: in [7] the authors show that the ultrapower of $\mathcal{B}(H)$ does not have WEP.

REMARK 5. Inductive limits of QWEP algebras are again QWEP (see [2, Lemma 13.3.6]), so the class of QWEP algebras is $\forall\exists$-axiomatizable.

There is one other model-theoretic way to settle the QWEP conjecture; we refer the reader to [6] for the definition of existential embeddings.

Proposition 6. *The QWEP conjecture is equivalent to the statement that there is a QWEP C* algebra A containing $C^*(\mathbb{F})$ as a subalgebra such that the inclusion is an existential embedding of operator systems.*

Proof. Suppose that A is as in the conclusion of the proposition. Then there is a u.c.p. embedding $A \hookrightarrow C^*(\mathbb{F})^\omega$ whose restriction to $C^*(\mathbb{F})$ is the diagonal embedding. It follows that $C^*(\mathbb{F})$ is relatively weakly injective in A, whence it is itself QWEP by the aforementioned result of Kirchberg. ☐

The previous proposition appeared as [6, Corollary 2.24] but with QWEP replaced by WEP.

In [3], the authors ask whether or not every C* algebra is elementarily equivalent to a nuclear C* algebra. It seems that the authors there were unaware of the fact that if their question had a positive answer, then the QWEP conjecture (and hence CEP) would also be settled. Nevertheless, in the forthcoming manuscript [5], the authors settle this question in the negative by showing that neither $C_r^*(\mathbb{F})$ nor $\prod_\omega M_n$ have nuclear models.

A question of Kirchberg, first appearing in print in [11], asks something seemingly more modest than the QWEP conjecture: is there an example of a non-nuclear C* algebra that has both WEP and the *local lifting property* (LLP)? Indeed, Kirchberg showed that the QWEP conjecture is equivalent to the statement that the LLP implies WEP. Let us call the statement that there exists a non-nuclear C* algebra that has both WEP and LLP the *weak QWEP conjecture.*

Proposition 7. *Let A be either $C_r^*(\mathbb{F})$ or $\prod_\omega M_n$. If A is elementarily equivalent to a C* algebra B with LLP, then B yields a positive solution to the weak QWEP conjecture.*

Proof. Since A is QWEP (for the case of $C_r^*(\mathbb{F})$, see [2, Proposition 3.3.8]), we have that B is also QWEP by Proposition 3; since B has LLP, we see that B also has WEP [10, Corollary 2.6(ii)]. B is not nuclear from the aforementioned result in [5]. ☐

We end this note with something only tangentially related. First, a preparatory result.

Proposition 8. *Suppose that A is a nonseparable C* algebra with a cofinal collection of separable subalgebras that have WEP. Then A has WEP.*

Proof. Suppose that $A \subseteq B$; we must show that A is relatively weakly injective in B. To see this, for each separable $C \subseteq A$ with WEP, there is a weak conditional expectation $\phi_C : B \to C^{**} \subseteq A^{**}$. By taking a pointwise-ultraweak limit of ϕ_C as C ranges over a cofinal family of separable subalgebras with WEP, we get a witness to the fact that A is relatively weakly injective in B. ☐

The following first appeared as Theorem 3.1 of [4].

Corollary 9. *The theory of unital C* algebras does not have a model companion, meaning that the class of existentially closed unital C* algebras is not axiomatizable.*

Proof. Suppose that T is the model companion of the theory of C* algebras, so the models of T are precisely the existentially closed C* algebras. By [6, Corollary 2.4], all models of T have WEP. Let A be a model of T. Then A^ω has a cofinal collection of WEP separable subalgebras, namely the separable elementary substructures of A^ω. By Proposition 8, A^ω has WEP, whence A is subhomogeneous by [7, Corollary 4.14]. In particular, A is finite. Since existentially closed C* algebras are purely infinite by [6, Corollary 2.7], we have a contradiction. □

The advantage of the previous proof over the one in [4] is that the latter proof invokes a serious result of Haagerup and Thorbjornsen [8], while the above proof ultimately relies instead on the (fundamental but more elementary) work of Junge and Pisier [9].

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Isaac Goldbring, Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Science and Engineering Offices M/C 249, 851 S. Morgan St., Chicago, IL 60607-7045, USA and Department of Mathematics University of California, Irvine 340 Rowland Hall (Bldg.# 400) Irvine, CA 92697-3875, USA

E-mail address: isaac@math.uci.edu; *URL*: http://www.math.uci.edu/~isaac