Illinois Journal of Mathematics
Volume 59, Number 1, Spring 2015, Pages 1–19
S 0019-2082

A NEW INTERPOLATION APPROACH TO SPACES OF TRIEBEL–LIZORKIN TYPE

PEER CHRISTIAN KUNSTMANN

ABSTRACT. We introduce in this paper new interpolation methods for closed subspaces of Banach function spaces. For $q \in [1,\infty]$, the l^q-interpolation method allows to interpolate linear operators that have bounded l^q-valued extensions. For $q = 2$ and if the Banach function spaces are r-concave for some $r < \infty$, the method coincides with the Rademacher interpolation method that has been used to characterize boundedness of the H^∞-functional calculus. As a special case, we obtain Triebel–Lizorkin spaces $F^{2\theta}_{p,q}(\mathbb{R}^d)$ by l^q-interpolation between $L^p(\mathbb{R}^d)$ and $W^2_p(\mathbb{R}^d)$ where $p \in (1,\infty)$. A similar result holds for the recently introduced generalized Triebel–Lizorkin spaces associated with R_q-sectorial operators in Banach function spaces. So, roughly speaking, for the scale of Triebel–Lizorkin spaces our method thus plays the role the real interpolation method plays in the theory of Besov spaces.

1. Introduction

Many of the classical function spaces on $\mathbb{R}^d$ can be subsumed in the scales of Besov and Triebel–Lizorkin spaces (see, e.g., [20]). These two types of spaces are usually defined via Littlewood–Paley decomposition and this common feature leads to many parallels in their theory. One important difference, however, is that Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ arise as real interpolation spaces between Lebesgue spaces $L^p(\mathbb{R}^d)$ and Sobolev spaces $W^m_p(\mathbb{R}^d)$ where $m \in \mathbb{N}$. On the one hand, this means that one can use the powerful machinery of real interpolation for their study. On the other hand, one can easily define "abstract" Besov type spaces by real interpolation between a Banach space X and the domain of a sectorial operator A in X. These spaces allow for natural

Received October 6, 2014; received in final form February 13, 2015.
2010 *Mathematics Subject Classification.* 46B70, 47A60, 42B25.

descriptions of Littlewood–Paley type where the decomposition operators are not defined via Fourier transform but, for example, via a suitable functional calculus for the operator A.

To be more precise, we recall that a linear operator A in a Banach space X is called *sectorial of type* $\omega \in [0,\pi)$ if its spectrum $\sigma(A)$ is contained in $\{z \in \mathbb{C}\setminus\{0\} : |\arg z| \le \omega\} \cup \{0\}$ and, for all $\sigma \in (\omega,\pi)$, the sets of operators $\{\lambda(\lambda+A)^{-1} : \lambda \in \mathbb{C}\setminus\{0\}, |\arg\lambda| < \pi-\sigma\}$ are bounded in $L(X)$. The infimum of all such angles (which actually is a minimum) is denoted by $\omega(A)$. If we denote $X_1(A)$, the domain $D(A)$ equipped with the graph norm then, for $\theta \in (0,1)$ and $q \in [1,\infty]$,

$$\big(X, X_1(A)\big)_{\theta,q} = \left\{ x \in X : \|x\|^R_{\theta,q} := \left(\int_0^\infty \big\| t^{1-\theta} A(1+tA)^{-1} x \big\|_X^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

or, in case $\omega(A) < \pi/2$,

$$\big(X, X_1(A)\big)_{\theta,q} = \left\{ x \in X : \|x\|^T_{\theta,q} := \left(\int_0^\infty \big\| t^{1-\theta} A e^{-tA} x \big\|_X^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

and $\|\cdot\|_X + \|\cdot\|^R_{\theta,q}$ or $\|\cdot\|_X + \|\cdot\|^T_{\theta,q}$ are equivalent to the abstract "real interpolation norms" obtained, for example, via the K-method or the J-method. The special case $A = -\Delta$ in $X = L^p(\mathbb{R}^d)$, $p \in (1,\infty)$, gives back classical Besov spaces $B^{2\theta}_{p,q}(\mathbb{R}^d)$. The functional calculus point of view on abstract Besov spaces has been extensively developed in [6] where these spaces are called McIntosh–Yagi spaces. Remarkable is G. Dore's result that a sectorial operator always has a bounded H^∞-functional calculus in its associated abstract Besov spaces [2].

For Triebel–Lizorkin spaces $F^{2\theta}_{p,q}$ on $\mathbb{R}^d$ where $p \in (1,\infty)$, $q \in [1,\infty]$ and $\theta \in (0,1)$, one has as an equivalent norm, cf. [19],

$$\|f\|_{L^p} + \left\| \left(\int_0^\infty \big| t^{1-\theta} (-\Delta) e^{t\Delta} f \big|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p}. \tag{1}$$

Recently, a generalization of Triebel–Lizorkin spaces has been introduced, cf. [12], replacing in (1) the space L^p by a Banach function space X and the operator $-\Delta$ in $L^p(\mathbb{R}^d)$ by a sectorial operator A in X. Provided the operator A is R_q*-sectorial* in X (cf. Definition 2.7 below), it is shown in [12] that the scale $X^\theta_{q,A}$ of *generalized Triebel–Lizorkin spaces* thus obtained has a nice theory analogous to the one for the scale of abstract Besov spaces associated with A. In particular, it was shown in [12] that an R_q-sectorial operator A in a Banach space X always has a bounded H^∞-calculus in its associated generalized Triebel–Lizorkin spaces $X^\theta_{q,A}$, that is, the analog of Dore's result holds for the scale of generalized Triebel–Lizorkin spaces. A key issue in [12] had been an adapted version ([12, Proposition 3.9]) of the norm equivalence result for square functions due to C. Le Merdy [15]. For $X = L^p$, these square functions had been introduced in [1] to give a characterization for the boundedness of the H^∞-calculus of a sectorial operator. Later, these characterizations had been extended to general classes of Banach spaces via Rademacher and Gaussian random sums [9], [10], [11], [8], and corresponding interpolation methods have been constructed, cf. [8], [17].

It is, however, clear that there cannot be a general interpolation method, meaning an interpolation functor from the category of interpolation couples of Banach spaces into the category of Banach spaces in the sense of [18, Definition 1.2.2], underlying generalized Triebel–Lizorkin spaces in a way real interpolation is "underlying" abstract Besov spaces. This has two reasons. The first one is obvious: one cannot make sense of expressions like $\|(\sum_j |x|^q)^{1/q}\|_X$ in arbitrary Banach spaces X (although $q=2$ is an exception due to random sums, cf. Section 3 below). For this purpose one would need, for example, a Banach function space. The second reason is less obvious: if X and Y are Banach function spaces then a bounded linear operator $X \to Y$ need not have a bounded extension $X(l^q) \to Y(l^q)$ (again, $q=2$ is an exception, cf. below). This phenomenon is responsible for the additional technical difficulties that arise in the study of F-spaces compared to B-spaces. Looking at the theory developed in [12], it seems unlikely that an arbitrary linear operator T that is bounded $X \to X$ and $X_1(A) \to X_1(A)$ acts boundedly $X^\theta_{q,A} \to X^\theta_{q,A}$. Even being certainly wrong as it stands, [12, Proposition 4.20] gives the hint that this interpolation property holds under boundedness assumptions on l^q-extensions of T.

The purpose of the present paper is to show that these two obstructions are the only ones. In other words: Taking these two aspects into account we develop an interpolation method that plays for (generalized and classical) Triebel–Lizorkin spaces the same role real interpolation does for Besov spaces. It is clear that, in order to do so, we also have to make sense of expressions like $\|(\sum_j |x|^q)^{1/q}\|_{X_1(A)}$ where $X_1(A)$ is, in general, *not* a Banach function space, even if X is, think of W^2_p and L^p where $p \in (1,\infty)$. The natural way out is to consider W^2_p as a closed subspace of another L^p-space. Thus, we are led to the class of closed subspaces of Banach function spaces. However already this simple example shows that there are several natural embeddings, for example, induced by the norms $\|f\|_{L^p} + \sum_{|\alpha|\le 2} \|\partial^\alpha f\|_{L^p}$, $\|f\|_{L^p} + \|\Delta f\|_{L^p}$, $\|(1-\Delta)f\|_{L^p}$, moreover, in the last two expressions the operator Δ can be replaced by a countless variety of other second order elliptic operators.

We thus prefer to make these embeddings explicit. We also take the point of view that the primary object is the Banach space X, and that this Banach space is given an additional structure by considering an embedding $J : X \to E$ into a Banach function space E. We require J to be isometric, merely for simplicity of notation, understanding that we might have to change to an equivalent norm on X (we are not interested in the isometric theory of Banach spaces here). We then call the triple (X, J, E) a *structured* Banach space and

(J, E) a *function space structure* on X. It is important, that function space structures may be "non-equivalent" (in a certain sense) even if the induced norms on X are equivalent (cf. Section 2 below). The issue as such has been noted in the context of square functions in [14] where it is less virulent (cp. the remarks in Section 3 on $q = 2$). Another point that has been essential in [14] and that we encounter here, too, is that within the class of closed subspaces X of Banach function spaces we cannot do duality arguments: since the dual X' is not a subspace of a Banach function space, in general, but a quotient space we cannot give expressions like $\|(\sum_j |x_j|^{q'})^{1/q'}\|_{X'}$ a meaning.

The paper is organized as follows: In Section 2, we introduce the l^q-interpolation method by a suitable modification of the K-method (discrete version) for real interpolation. In the case $q = 2$, there is a relation to the Rademacher interpolation method from [8] and to the γ-interpolation method from [10], [17], both working for arbitrary interpolation couples of Banach spaces. We study the relation to these methods in Section 3. This is done via a reformulation of the l^q-interpolation method in the spirit of the J-method (discrete version) for real interpoation. In Section 4, we introduce a subclass of interpolation couples for which l^q-interpolation spaces can be given a function space structure. For this structure, an interpolated linear operator is not only bounded but also has a bounded l^q-extension. Finally, we relate in Section 5 the interpolation theory presented here to the generalized Triebel–Lizorkin spaces from [12]. We restrict ourselves here to *homogeneous* generalized Triebel–Lizorkin spaces $\dot{X}^{\theta}_{q,A}$ and to $\theta \in (0,1)$, but only for simplicity of presentation. In particular, we show that all function space structures that are induced by the equivalent norms from [12] on these spaces are l^q-equivalent. We close this introduction with a few remarks on what we do *not* do in this paper.

REMARK 1.1. (a) We do not recover Triebel–Lizorkin spaces $F^s_{\infty,q}(\mathbb{R}^d)$ or $\dot{F}^s_{\infty,q}(\mathbb{R}^d)$ for $q \in [1, \infty)$ or—the case $q = 2$—$\mathrm{BMO}(\mathbb{R}^d)$. In fact, these spaces are not defined via vertical expressions in $L^\infty(L^q)$ but one has to study expressions in tent spaces. It is also possible to do this for more general sectorial operators A in L^2 but this needs more assumptions, for example, a bounded H^∞-calculus for A, a metric structure on the measure space and some off-diagonal estimates (at least of Davies–Gaffney type) for the semigroup operators e^{-tA}, see, for example, [7], [4].

(b) We do not contribute to the study of Hardy spaces associated with operators, which can be defined via conical square functions or atomic decompositions. Again this is related to tent spaces and uses suitable decay assumptions for resolvents or semigroup operators, see, for example, [7], [3].

(c) We do not go into details about sufficient conditions for the existence of l^q-bounded extensions of bounded operators $T : L^p \to L^p$ here. This is a classical topic in harmonic analysis (cf., e.g., [5]). We only want to mention that domination by a positive operator or by the Hardy–Littlewood maximal operator is sufficient, and that classical Calderón–Zygmund operators have bounded l^q-extensions in any L^p for $p, q \in (1, \infty)$.

2. l^q-interpolation for structured Banach spaces

Let X be a Banach space. We have to make sense of expressions like $\|(\sum_j |x_j|^q)^{1/q}\|$. To this end, we recall the notion of a *Banach function space* over a σ-finite measure space (Ω, μ). We fix an increasing sequence $(\Omega_n)_{n\in\mathbb{N}}$ of μ-measurable subsets of Ω of finite measure whose union is Ω, and call this a *localizing sequence.* A μ-measurable $M \subset \Omega$ is called *bounded* if $M \subset \Omega_n$ for some n. The usual choice on $\Omega = \mathbb{R}^d$ (with Lebesgue measure) will be bounded Ω_n, the usual choice on $\Omega = \mathbb{Z}$ (with counting measure) will be finite Ω_n. We will consider complex-valued function spaces here. However, this is only important in our applications to sectorial operators.

DEFINITION 2.1. Let (Ω, μ) be a σ-finite measure space with localizing sequence (Ω_n). Let $M(\mu)$ be the space of (equivalence classes of) measurable functions and $M^+(\mu) := \{f \in M(\mu) : f \geq 0\}$. A Banach space $(E, \|\cdot\|_E)$ is called a *Banach function space over* (Ω, μ) if there is a functional $\rho : M^+(\mu) \to [0, \infty]$ having the following properties for $f, g \in M^+(\mu)$, $\alpha > 0$, sequences (f_n) in $M^+(\mu)$ and μ-measurable $M \subset \Omega$:

(i) $\rho(f) = 0$ if and only if $f = 0$ μ-a.e., $\rho(\alpha f) = \alpha\rho(f)$ and $\rho(f + g) \leq \rho(f) + \rho(g)$ (*norm properties*),
(ii) $0 \leq g \leq f$ μ-a.e. implies $\rho(g) \leq \rho(f)$ (*monotonicity*),
(iii) $0 \leq f_n \nearrow f$ μ-a.e. implies $\rho(f_n) \nearrow \rho(f)$ (*Fatou property*),
(iv) if M is bounded then $\rho(1_M) < \infty$,
(v) if M is bounded then $\int_M f\, d\mu \leq C_M \rho(f)$ for a constant $C_M > 0$ independent of f,

such that $E = \{f \in M(\mu) : \rho(|f|) < \infty\}$ and $\|f\|_E = \rho(|f|)$.

REMARK 2.2. (a) If, for $\nu = 0, 1$, E_ν is a Banach function space over (Ω_ν, μ_ν) then $E_0 \times E_1$ is a Banach function space over $(\Omega_0 \dot\cup \Omega_1, \mu_0 \dot+ \mu_1)$ where $\Omega_0 \dot\cup \Omega_1$ denotes the disjoint union of Ω_0 and Ω_1 (which may be realized by $\Omega_0 \times \{0\} \cup \Omega_1 \times \{1\}$ if necessary) and $\mu_0 \dot+ \mu_1(B_0 \dot\cup B_1) = \mu_0(B_0) + \mu_1(B_1)$ for μ_ν-measurable subsets $B_\nu \subset \Omega_\nu$, $\nu = 0, 1$.

(b) If E is a Banach function spaces over (Ω, μ) and $q \in [1, \infty]$ then $E(l^q)$ is the space of all sequences $(f_j)_{j\in\mathbb{Z}}$ in E such that $\|(f_j)_j\|_{E(l^q)} := \|(\sum_{j\in\mathbb{Z}} |f_j|^q)^{1/q}\|_E < \infty$. The space $(E(l^q), \|\cdot\|_{E(l^q)})$ is a Banach function space over $(\Omega \times \mathbb{Z}, \mu \otimes \delta)$ where δ denotes the counting measure on $\mathbb{Z}$. If (Ω_n) is the localizing sequence in Ω then we consider $\Omega_n \times \{|j| \leq n\}$ as localizing sequence in $\Omega \times \mathbb{Z}$.

For a Banach function space E, one can make sense of expressions like $\|(\sum_j |f_j|^q)^{1/q}\|_E$ where $f_j \in M(\mu)$, but for later applications we have to allow for greater flexibility, so we take closed subspaces of Banach function spaces. We find it, however, helpful to keep the embedding in notation. Therefore, we define the basic objects for our interpolation method as follows.

Definition 2.3. A *structured Banach space* is a triple (X, J, E) where X is a Banach space, E is a Banach function space and $J : X \to E$ is a linear map such that $\|x\|_X = \|Jx\|_E$ for all $x \in X$, i.e. $J : X \to E$ is isometric, thus injective, but not necessarily surjective. For a given Banach space X we call a pair (J, E) a *function space structure on* X if (X, J, E) is a structured Banach space.

It will not be essential that J is isometric, it would be sufficient that $\|Jx\|_E$ is equivalent to the norm in X, but things are easier written down this way. For a structured Banach space (X, J, E), we can thus make sense of expressions like $\|(\sum_{j=1}^n |Jx_j|^q)^{1/q}\|_E$, and—via the Fatou property—we can take limits

$$\left\|\left(\sum_{j=1}^{\infty} |Jx_j|^q\right)^{1/q}\right\|_E = \lim_{n\to\infty}\left\|\left(\sum_{j=1}^{n} |Jx_j|^q\right)^{1/q}\right\|_E .$$

In this paper, we always understand that $(\sum_j |f_j|^q)^{1/q}$ means $\sup_j |f_j|$ in case $q = \infty$. We extend the notion of R_q-bounded (sets of) operators to our setting.

Definition 2.4. Let $\mathcal{X} = (X, J, E)$ and $\mathcal{Y} = (Y, K, F)$ be structured Banach spaces and $q \in [1, \infty]$. A set $\mathscr{T}$ of linear operators $X \to Y$ is called *l^q-bounded* or *R_q-bounded* (w.r.t. the function space structures (J, E) on X and (K, F) on Y) if there exists a constant C such that, for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $T_1, \dots, T_n \in \mathscr{T}$,

$$\left\|\left(\sum_{j=1}^{n} |KT_jx_j|^q\right)^{1/q}\right\|_F \le C\left\|\left(\sum_{j=1}^{n} |Jx_j|^q\right)^{1/q}\right\|_E .$$

The least constant C is denoted $R_q(\mathscr{T})$ and called *the R_q-bound of $\mathscr{T}$*. A single linear operator $T : X \to Y$ is called l^q-bounded if the set $\{T\}$ is l^q-bounded. The least constant is denoted $R_q(T)$ in this case. Occasionally we shall say that $\mathscr{T}$ or T is R_q-bounded $\mathcal{X} \to \mathcal{Y}$ which is a more precise notation.

Denoting the set of R_q-bounded operators $T : X \to Y$ by $R_qL(X, Y)$ it can be shown that $R_qL(X, Y)$ is a Banach space for the norm $R_q(\cdot)$ (cf. [12, Proposition 2.6]).

Remark 2.5. The notion has been called R_q-boundedness in [21] in the context of R-boundedness of sets of operators in general Banach spaces. For the purpose of this paper, it seems more natural to call it l^q-boundedness here. However, we shall use the (somehow established) notion of R_q-sectorial operators below. Hence, we use both terms l^q-boundedness and R_q-boundedness in this paper, the choice depending on the context.

A fact we have to accept is that a single operator $T:\mathcal{X}\to\mathcal{Y}$ need not be l^q-bounded in general if $q\neq 2$. Related is the phenomenon that the notion of l^q-boundedness depends on the function space structures on X and Y.

Example 2.6. The Rademacher sequence (r_k) is an orthonormal system in $L^2[0,1]$. Let X denote their closed span in $E=L^2[0,1]$. Then $\mathcal{X}=(X,J,E)$ is a structured Banach space where J denotes the inclusion map $X\to E$. Now let (h_k) be the normalized characteristic functions on intervals $(2^{-k},2^{1-k}]$ which also form an orthonormal sequence in $E=L^2[0,1]$ and define $\tilde{J}:X\to E$ by $\sum_k a_k r_k\mapsto\sum_k a_k h_k$. Then also $\tilde{\mathcal{X}}=(X,\tilde{J},E)$ is a structured Banach space. We have (cf., e.g., [12, Example 2.16]):

$$\left\|\left(\sum_{j=1}^n |r_j|^q\right)^{1/q}\right\|_{L^2}=n^{1/q},\qquad \left\|\left(\sum_{j=1}^n |h_j|^q\right)^{1/q}\right\|_{L^2}=n^{1/2}.$$

Hence, the identity is not l^q-bounded $\mathcal{X}\to\tilde{\mathcal{X}}$ for $1\le q<2$ and not l^q-bounded $\tilde{\mathcal{X}}\to\mathcal{X}$ for $q>2$. The example can be made to work on $L^2[0,1]$ as well.

Therefore, on a Banach space X, we call function space structures (J,E) and $(\tilde{J},\tilde{E})$ that give rise to equivalent norms *l^q-equivalent* if there exists a constant C such that, for all $n\in\mathbb{N}$ and $x_1,\dots,x_n\in X$,

$$C^{-1}\left\|\left(\sum_{j=1}^n |Jx_j|^q\right)^{1/q}\right\|_E\le\left\|\left(\sum_{j=1}^n |\tilde{J}x_j|^q\right)^{1/q}\right\|_{\tilde{E}}\le C\left\|\left(\sum_{j=1}^n |Jx_j|^q\right)^{1/q}\right\|_E,$$

that is, if the identity operator $(X,J,E)\to(X,\tilde{J},\tilde{E})$ is l^q-bounded in both directions. It is clear that l^q-boundedness is preserved (with equivalent l^q-bounds) if we change to l^q-equivalent function space structures.

As an example that shall become important lateron, we extend the notion of R_q-sectorial operators (cf. [21], [12]) from Banach function spaces to structured Banach spaces.

Definition 2.7. Let $\mathcal{X}=(X,J,E)$ be a structured Banach space and $q\in[1,\infty]$. A sectorial operator A in X is called *R_q-sectorial of type $\omega\in[0,\pi)$* (w.r.t. to the function space structure (J,E) on X) if it is sectorial of type ω and for all $\sigma\in(\omega,\pi)$ the set

$$\left\{\lambda(\lambda+A)^{-1}:\lambda\in\mathbb{C}\setminus\{0\},|\arg\lambda|<\pi-\sigma\right\}$$

is R_q-bounded in $\mathcal{X}$. The infimum of all such angles ω is denoted $\omega_q(A)$.

Definition 2.8. We call a pair $(\mathcal{X}_0,\mathcal{X}_1)=((X_0,J_0,E_0),(X_1,J_1,E_1))$ an interpolation couple if $X_0\hookrightarrow Z$ and $X_1\hookrightarrow Z$ with continuous injections for some Hausdorff topological vector space Z, that is, if (X_0,X_1) is an interpolation couple in the usual sense (cf. [18]).

We now introduce the l^q-interpolation method for interpolation couples of structured Banach spaces.

DEFINITION 2.9. For $q \in [1, \infty]$, an interpolation couple $(\mathcal{X}_0, \mathcal{X}_1) = ((X_0, J_0, E_0), (X_1, J_1, E_1))$, and $\theta \in (0,1)$ we let

$$\|x\|_{\theta,l^q} := \inf\Big\{ \Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{-j\theta} J_0 x_0(j)\big|^q \Big)^{1/q} \Big\|_{E_0} + \Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{j(1-\theta)} J_1 x_1(j)\big|^q \Big)^{1/q} \Big\|_{E_1} : \forall j \in \mathbb{Z} : x = x_0(j) + x_1(j), x_0(j) \in X_0, x_1(j) \in X_1 \Big\}$$

for $x \in X_0 + X_1$ and define

$$(\mathcal{X}_0, \mathcal{X}_1)_{\theta,l^q} := X_{\theta,l^q} := \big\{ x \in X_0 + X_1 : \|x\|_{\theta,l^q} < \infty \big\}$$

with norm $\|\cdot\|_{\theta,l^q}$.

PROPOSITION 2.10. *For $\theta \in (0,1)$ and $q \in [1,\infty]$ the normed space $(X_{\theta,l^q}, \|\cdot\|_{\theta,l^q})$ is a Banach space.*

Proof. One has to show essentially that, for $\nu = 0, 1$,

$$U_\nu := \big\{ \big(x_\nu(j)\big)_{j\in\mathbb{Z}} \subset X_\nu : \big(2^{j(\nu-\theta)} J_\nu x_\nu(j)\big)_j \in E_\nu\big(l^q(\mathbb{Z})\big) \big\}$$

is complete for the associated norm and that

$$D := \big\{ \big(x_0(j), x_1(j)\big) \in U_0 \times U_1 : \forall j \in \mathbb{Z} : x_0(j) = -x_1(j) \big\}$$

is closed in $U_0 \times U_1$. This is easy. □

The following is the basic interpolation property.

THEOREM 2.11. *Let $(\mathcal{X}_0, \mathcal{X}_1) = ((X_0, J_0, E_0), (X_1, J_1, E_1))$ and $(\mathcal{Y}_0, \mathcal{Y}_1) = ((Y_0, K_0, F_0), (Y_1, K_1, F_1))$ be interpolation couples of structured Banach spaces. Let $q \in [1, \infty]$ and let $T : X_0 + X_1 \to Y_0 + Y_1$ be a linear operator such that $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ are R_q-bounded with R_q-bounds M_0 and M_1, respectively. Then for any $\theta \in (0,1)$ the operator T acts as a bounded linear operator $X_{\theta,l^q} \to Y_{\theta,l^q}$ with norm $\le c_\theta M_0^{1-\theta} M_1^\theta$ where $c_\theta = 2^\theta$.*

As will become apparent in the proof, the constant c_θ is the price to pay for the use of discrete l^q-norms in the definition of our method.

Proof of Theorem 2.11. Let $x \in X_{\theta,l^q}$ and $\varepsilon > 0$. For each $j \in \mathbb{Z}$, we choose a decomposition $x = x_0(j) + x_1(j)$ with $x_\nu(j) \in X_\nu$, $\nu = 0, 1$, such that

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{-j\theta} J_0 x_0(j)\big|^q \Big)^{1/q} \Big\|_{E_0} + \Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{j(1-\theta)} J_1 x_1(j)\big|^q \Big)^{1/q} \Big\|_{E_1} \le \|x\|_{\theta,l^q} + \varepsilon.$$

We choose an integer m such that $2^{m-1} < M_1/M_0 \le 2^m$. Letting $\tilde{x}_\nu(j) := x_\nu(j+m)$ for $j \in \mathbb{Z}$ and $\nu = 0,1$, we have $Tx = T\tilde{x}_0(j) + T\tilde{x}_1(j)$ with $T\tilde{x}_\nu(j) \in Y_\nu$ for $\nu = 0,1$. Hence,

$$\|Tx\|_{\theta,l^q}$$
$$\le \Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{-j\theta} K_0 T\tilde{x}_0(j)\big|^q \Big)^{1/q} \Big\|_{F_0} + \Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{j(1-\theta)} K_1 T\tilde{x}_1(j)\big|^q \Big)^{1/q} \Big\|_{F_1}$$

which is

$$\le \Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{-j\theta} M_0 J_0 \tilde{x}_0(j)\big|^q \Big)^{1/q} \Big\|_{E_0} + \Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{j(1-\theta)} M_1 J_1 \tilde{x}_1(j)\big|^q \Big)^{1/q} \Big\|_{E_1}$$

by assumption. Now we introduce m and the last line becomes

$$= 2^{m\theta} M_0 \Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{-(j+m)\theta} J_0 x_0(j+m)\big|^q \Big)^{1/q} \Big\|_{E_0}$$
$$+ 2^{m(\theta-1)} M_1 \Big\| \Big(\sum_{j\in\mathbb{Z}} \big|2^{(j+m)(1-\theta)} J_1 x_1(j+m)\big|^q \Big)^{1/q} \Big\|_{E_1}.$$

We shift the summation index by m and obtain

$$\le \max\big\{2^{m\theta} M_0, 2^{-m(1-\theta)} M_1\big\} \big(\|x\|_{\theta,l^q} + \varepsilon\big).$$

We let $\varepsilon \to 0$ and observe

$$2^{m\theta} M_0 \le 2^\theta M_0^{1-\theta} M_1^\theta, \qquad 2^{-m(1-\theta)} M_1 \le M_0^{1-\theta} M_1^\theta.$$

We thus have shown the claim with $c_\theta = 2^\theta$. □

A short comment on the functor property of our interpolation method seems to be in order.

Remark 2.12. Let $q \in [1,\infty]$. We consider the category $\mathfrak{C}$ of all Banach spaces with bounded linear operators as morphisms (cf. [18, Definition 1.2.2]). Now let the category $\mathfrak{C}^q$ be given by taking as objects interpolation couples $(\mathcal{X}_0, \mathcal{X}_1)$ of structured Banach spaces and as morphisms between two couples $(\mathcal{X}_0, \mathcal{X}_1)$ and $(\mathcal{Y}_0, \mathcal{Y}_1)$ linear operators $T : X_0 + X_1 \to Y_0 + Y_1$ such that $T : \mathcal{X}_0 \to \mathcal{Y}_0$ and $T : \mathcal{X}_1 \to \mathcal{Y}_1$ are l^q-bounded. Then l^q-interpolation is a covariant functor from $\mathfrak{C}^q$ to $\mathfrak{C}$ which is of type θ.

We note several simple properties.

Proposition 2.13. *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be an interpolation couple of structured Banach spaces. For $q \in [1,\infty]$ and $\theta \in (0,1)$, we have*

(a) $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,l^q} = (\mathcal{X}_1, \mathcal{X}_0)_{1-\theta,l^q}$,
(b) $(\mathcal{X}_0, \mathcal{X}_0)_{\theta,l^q} = X_0$,
(c) *if* $q < \tilde{q} \le \infty$ *then* $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,l^q} \hookrightarrow (\mathcal{X}_0, \mathcal{X}_1)_{\theta,l^{\tilde{q}}}$,

(d) $(X_0,X_1)_{\theta,1} \hookrightarrow (\mathcal{X}_0,\mathcal{X}_1)_{\theta,l^q} \hookrightarrow (X_0,X_1)_{\theta,\infty}$ *where* $(X_0,X_1)_{\theta,r}$ *denote real interpolation spaces.*

Proof. (a) is obvious, and (c) follows from $l^q \hookrightarrow l^{\tilde q}$. For the proof of (b), we decompose x by taking $x_0(j)=x$ for $j\ge 0$, $x_0(j)=0$ for $j<0$ and $x_1(j)=0$ for $j\ge 0$, $x_1(j)=x$ for $j<0$. This gives $X_0 \hookrightarrow (\mathcal{X}_0,\mathcal{X}_0)_{\theta,l^q}$, but the reverse embedding is clear since always $(\mathcal{X}_0,\mathcal{X}_1)_{\theta,l^q} \hookrightarrow X_0+X_1$. For the proof of (d), we notice

$$\begin{aligned}
\sup_j \big\|2^{j(\nu-\theta)}x_\nu(j)\big\|_{X_\nu} &\le \Big\|\sup_j \big|2^{j(\nu-\theta)}J_\nu x_\nu(j)\big|\Big\|_{E_\nu} \\
&\le \Big\|\Big(\sum_j \big|2^{j(\nu-\theta)}J_\nu x_\nu(j)\big|^q\Big)^{1/q}\Big\|_{E_\nu} \\
&\le \Big\|\sum_j \big|2^{j(\nu-\theta)}J_\nu x_\nu(j)\big|\Big\|_{E_\nu} \\
&\le \sum_j \big\|2^{j(\nu-\theta)}x_\nu(j)\big\|_{X_\nu}.
\end{aligned}$$

□

Remark 2.14. Proposition 2.13(d) implies for $0<\theta_0<\theta_1<1$, $q_0,q_1,q\in[1,\infty]$ and $\lambda\in(0,1)$ by reiteration

$$\big((\mathcal{X}_0,\mathcal{X}_1)_{\theta_0,l^{q_0}},(\mathcal{X}_0,\mathcal{X}_1)_{\theta_1,l^{q_1}}\big)_{\lambda,q} = (X_0,X_1)_{(1-\lambda)\theta_0+\lambda\theta_1,q}.$$

3. The case $q=2$ and Rademacher interpolation

The case $q=2$ in l^q-interpolation is a special one: If in $\mathcal{X}=(X,J,E)$ the Banach function space E is q_E-concave for some $q_E<\infty$, then (cf. [16, Thm 1.d.6(i)]) we have equivalence of expressions

$$\Big\|\Big(\sum_{j=1}^n |Jx_j|^2\Big)^{1/2}\Big\|_E \quad\text{and}\quad \int_0^1 \Big\|\sum_{j=1}^n r_j(u)Jx_j\Big\|_E\,du = \mathbb{E}\Big\|\sum_{j=1}^n r_jx_j\Big\|_X$$

uniformly in n where the r_j denote Rademacher functions on $[0,1]$. Moreover, such a space E is of finite cotype and we have equivalence of expressions

$$\mathbb{E}\Big\|\sum_{j=1}^n r_jx_j\Big\|_X, \qquad \mathbb{E}\Big\|\sum_{j=1}^n \gamma_jx_j\Big\|_X$$

uniformly in n where the γ_j are independent Gaussian variables.

This has two consequences. The first one is well known: If, in addition, in $\mathcal{Y}=(Y,K,F)$ the space F is q_F-concave for some $q_F<\infty$, then a set of operators $\mathscr{T}$ from $X\to Y$ is R_2-bounded $\mathcal{X}\to\mathcal{Y}$ if and only if it is R-*bounded*, that is, if and only if there exists a constant C such that, for all

$n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, and $T_1, \dots, T_n \in \mathscr{T}$, we have

$$\mathbb{E}\left\|\sum_{j=1}^{n} r_j T_j x_j\right\|_Y \le C\mathbb{E}\left\|\sum_{j=1}^{n} r_j x_j\right\|_X .$$

Since singletons $\{T\}$ are always R-bounded, we obtain under these assumptions that each bounded operator $T : X \to Y$ is R_2-bounded $\mathcal{X} \to \mathcal{Y}$. The same holds for the related notion of γ-boundedness.

The second consequence is that l^2-interpolation spaces for an interpolation couple $(\mathcal{X}_0, \mathcal{X}_1)$, for which E_ν is q_ν-concave for some $q_\nu < \infty$ and $\nu = 0, 1$, coincide with the spaces obtained for the couple (X_0, X_1) by Rademacher interpolation or by γ-interpolation (which are equivalent in this case). In order to see this, we present the following reformulation of our method for general q. For $q = 2$, the relation to the Rademacher interpolation spaces from [8, Definition 7.1] is then obvious.

Theorem 3.1. *Let $q \in [1, \infty]$ and let $(\mathcal{X}_0, \mathcal{X}_1)$ be an interpolation couple. For $\theta \in (0, 1)$ and $x \in X_0 + X_1$ we let*

$$\|x\|^J_{\theta,l^q} := \inf\Bigg\{\left\|\left(\sum_{j\in\mathbb{Z}}\left|2^{-j\theta} J_0 x_j\right|^q\right)^{1/q}\right\|_{E_0} + \left\|\left(\sum_{j\in\mathbb{Z}}\left|2^{j(1-\theta)} J_1 x_j\right|^q\right)^{1/q}\right\|_{E_1} : \\ \forall j \in \mathbb{Z} : x_j \in X_0 \cap X_1 \text{ and } x = \sum_{j\in\mathbb{Z}} x_j \text{ in } X_0 + X_1\Bigg\}.$$

Then $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,l^q} = \{x \in X_0 + X_1 : \|x\|^J_{\theta,l^q} < \infty\}$ and $\|x\|^J_{\theta,l^q}$ is an equivalent norm on $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,l^q}$.

Corollary 3.2. *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be an interpolation couple such that, for $\nu = 0, 1$, the space E_ν is q_ν-concave for some $q_\nu < \infty$. Then for $\theta \in (0, 1)$ the space $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,l^2}$ coincides with the Rademacher interpolation space $\langle X_0, X_1\rangle_\theta$ and the norms are equivalent.*

Proof. Note that, for $\nu = 0, 1$, the expressions

$$\left\|\left(\sum_{j\in\mathbb{Z}}\left|2^{j(\nu-\theta)} J_\nu x_j\right|^2\right)^{1/2}\right\|_{E_\nu} = \sup_N\left\|\left(\sum_{|j|\le N}\left|2^{j(\nu-\theta)} J_\nu x_j\right|^2\right)^{1/2}\right\|_{E_\nu}$$

and

$$\sup_N \mathbb{E}\left\|\sum_{|j|\le N} 2^{j(\nu-\theta)} r_j x_j\right\|_{X_\nu}$$

are equivalent and take a look at [8, Definition 7.1]. ☐

Proof of Theorem 3.1. Let $x \in X_0 + X_1$. We have to prove two inequalities. First suppose that $(x_0(j))$, $(x_1(j))$ are sequences in X_0, X_1, respectively, such that $x = x_0(j) + x_1(j)$ for all $j \in \mathbb{Z}$ and

$$C := \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{-j\theta} J_0 x_0(j) \right|^q \right)^{1/q} \right\|_{E_0} + \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{j(1-\theta)} J_1 x_1(j) \right|^q \right)^{1/q} \right\|_{E_1} < \infty.$$

Then $y_j := x_0(j+1) - x_0(j) = -(x_1(j+1) - x_1(j)) \in X_0 \cap X_1$ for $j \in \mathbb{Z}$, and

$$\sum_{j=-l}^{k} y_j = x_0(k+1) - x_0(-l) = x_1(-l) - x_1(k+1) \quad \text{for } k, l > 0.$$

The assumption implies $\|2^{-k\theta} J_0 x_0(k)\|_{E_0} \le C$, that is,

$$\left\| x_0(k) \right\|_{X_0} \le C 2^{k\theta} \to 0 \quad (k \to -\infty),$$

and, similarly,

$$\left\| x_1(k) \right\|_{X_1} \le C 2^{-k(1-\theta)} \to 0 \quad (k \to \infty).$$

We conclude that $x = \sum_{j\in\mathbb{Z}} y_j = \lim_{k,l\to\infty} \sum_{j=-l}^{k} y_j$ in $X_0 + X_1$. Finally, we have, for $\nu = 0, 1$,

$$\begin{aligned} \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{j(\nu-\theta)} J_\nu y_j \right|^q \right)^{1/q} \right\|_{E_\nu} &\le 2^{\theta-\nu} \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{(j+1)(\nu-\theta)} J_\nu x_\nu(j+1) \right|^q \right)^{1/q} \right\|_{E_\nu} \\ &\quad + \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{j(\nu-\theta)} J_\nu x_\nu(j) \right|^q \right)^{1/q} \right\|_{E_\nu}, \end{aligned}$$

which yields $\|x\|_{\theta,l^q} \le (1 + 2^\theta) C$.

Now we suppose that (y_j) is a sequence in $X_0 \cap X_1$ such that $x = \sum_{j\in\mathbb{Z}} y_j$ (convergence in $X_0 + X_1$) and

$$C := \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{-j\theta} J_0 y_j \right|^q \right)^{1/q} \right\|_{E_0} + \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{j(1-\theta)} J_1 y_j \right|^q \right)^{1/q} \right\|_{E_1} < \infty.$$

For $k \in \mathbb{Z}$ we now let $x_0(k) := \sum_{j=-\infty}^{k} y_j$ and $x_1(k) := \sum_{j=k+1}^{\infty} y_j$. It is clear that the series converge in $X_0 + X_1$ and that $x_0(k) + x_1(k) = x$ for each $k \in \mathbb{Z}$. For $\nu = 0, 1$ we have to show that the series for $x_\nu(k)$ converges in X_ν. For $n, m \in \mathbb{Z}$ with $n < m$, we have by Hölder and assumption

$$\begin{aligned} \left\| \sum_{j=n}^{m} y_j \right\|_{X_\nu} &= \left\| \sum_{j=n}^{m} J_\nu y_j \right\|_{E_\nu} \\ &\le \left\| \left(\sum_{j=n}^{m} 2^{j(\theta-\nu)q'} \right)^{1/q'} \left(\sum_{j=n}^{m} \left| 2^{j(\nu-\theta)} J_\nu y_j \right|^q \right)^{1/q} \right\|_{E_\nu} \\ &\le C \left(\sum_{j=n}^{m} 2^{j(\theta-\nu)q'} \right)^{1/q'}. \end{aligned}$$

For $\nu=0$, this tends to 0 for $n,m\to-\infty$, and for $\nu=1$ this tends to 0 for $n,m\to\infty$. We conclude that $x_\nu(k)\in X_\nu$ for $\nu=0,1$. Now we write

$$\begin{aligned}N_0 &:= \Big\| \Big(\sum_{k\in\mathbb{Z}} \big| 2^{-k\theta} J_0 x_0(k)\big|^q \Big)^{1/q} \Big\|_{E_0} \\ &= \Big\| \Big(\sum_{k\in\mathbb{Z}} \Big| \sum_{j=-\infty}^{k} 2^{(j-k)\theta} J_0\big(2^{-j\theta} y_j\big)\Big|^q \Big)^{1/q} \Big\|_{E_0} \\ &= \Big\| \Big(\sum_{k\in\mathbb{Z}} \Big| \sum_{j\in\mathbb{Z}} a_{kj} z_j \Big|^q \Big)^{1/q} \Big\|_{E_0}\end{aligned}$$

where $a_{kj}=2^{(j-k)\theta}$ for $j\le k$, $a_{kj}=0$ for $j>k$, and $z_j=J_0(2^{-j\theta}y_j)$. It is easily checked that $\sum_k a_{kj}=\sum_j a_{kj}=\frac{2^\theta}{2^\theta-1}$, hence $(\eta_j)\mapsto(\sum_j a_{kj}\eta_j)_k$ defines a bounded operator $l^q(\mathbb{Z})\to l^q(\mathbb{Z})$ with norm $\le\frac{2^\theta}{2^\theta-1}$. We thus obtain

$$N_0 \le \frac{2^\theta}{2^\theta-1} \Big\| \Big(\sum_j \big| 2^{-j\theta} J_0 y_j\big|^q \Big)^{1/q} \Big\|_{E_0}.$$

Similarly, we prove

$$\begin{aligned}N_1 &:= \Big\| \Big(\sum_{k\in\mathbb{Z}} \big| 2^{k(1-\theta)} J_1 x_1(k)\big|^q \Big)^{1/q} \Big\|_{E_1} \\ &\le \frac{1}{2^{1-\theta}-1} \Big\| \Big(\sum_j \big| 2^{j(1-\theta)} J_1 y_j\big|^q \Big)^{1/q} \Big\|_{E_1},\end{aligned}$$

and obtain finally $N_0+N_1\le\max\{\frac{2^\theta}{2^\theta-1},\frac{1}{2^{1-\theta}-1}\}C$. This ends the proof. ☐

Another consequence of Theorem 3.1 is the following denseness property (cp. [18, Theorem 1.6.2] for the real method).

Corollary 3.3. *Let $(\mathcal{X}_0,\mathcal{X}_1)$ be an interpolation couple of structured Banach spaces and $1\le q<\infty$, $\theta\in(0,1)$. Then $X_0\cap X_1$ is dense in $(\mathcal{X}_0,\mathcal{X}_1)_{\theta,l^q}$ and*

$$(\mathcal{X}_0,\mathcal{X}_1)_{\theta,l^q} = \big(\mathcal{X}_0^0,\mathcal{X}_1\big)_{\theta,l^q} = \big(\mathcal{X}_0,\mathcal{X}_1^0\big)_{\theta,l^q} = \big(\mathcal{X}_0^0,\mathcal{X}_1^0\big)_{\theta,l^q},$$

where, for $\nu=0,1$, $\mathcal{X}_\nu^0=(X_\nu^0, J_\nu|_{X_\nu^0}, E_\nu)$ and X_ν^0 denotes the closure of $X_0\cap X_1$ in X_ν.

Proof. Take $x\in(\mathcal{X}_0,\mathcal{X}_1)_{\theta,l^q}$ and a representation $x=\sum_{j\in\mathbb{Z}}y_j$ as in Theorem 3.1. We just have to note that the sequence $(\sum_{|j|\le n}y_j)_n$ converges to x in $(\mathcal{X}_0,\mathcal{X})_{\theta,l^q}$, which is implied by

$$\Big\| \Big(\sum_{|j|>n} \big| 2^{j(\nu-j)} J_\nu y_j\big|^q \Big)^{1/q} \Big\|_{E_\nu} \to 0 \quad (n\to\infty)$$

(use $q<\infty$ and the Fatou property). □

The assertion of Corollary 3.3 does certainly not hold for $q=\infty$.

4. l^q-quasilinearizable interpolation couples

The point that may seem unsatisfactory in the theory presented so far is that we do not obtain R_q-bounded operators between the l^q-interpolation spaces although we start with R_q-bounded operators. On the other hand, we do not know what l^q-boundedness would mean since we do not even have function space structures on l^q-interpolation spaces. Indeed, the norm in l^q-interpolation spaces comes as a quotient norm. However, we do obtain a natural function space structure for an equivalent norm if a linear selection of representatives is possible in a suitable way. The following modification of quasi-linearizability (cf. [18, Definition 1.8.4]) is a formalization of this idea.

Definition 4.1. Let $q\in[1,\infty]$. The interpolation couple $(\mathcal{X}_0,\mathcal{X}_1)$ is said to be l^q-quasilinearizable if there are families $(V_\nu(t))_{t>0}$ of linear operators $X_0+X_1\to X_\nu$ for $\nu=0,1$ such that

(i) for all $t>0$: $V_0(t)+V_1(t)=I_{X_0+X_1}$,
(ii) for $\nu,\rho=0,1$ the sets $\{t^{\nu-\rho}V_\nu(t):t>0\}$ are R_q-bounded $\mathcal{X}_\rho\to\mathcal{X}_\nu$.

In this case, we shall write $\mathcal{V}=(V_0(t),V_1(t))_{t>0}$.

Strictly speaking, we only need the operators $V_\nu(t)$ for $t=2^j$, $j\in\mathbb{Z}$, in the following. Note, however, that a corresponding modification of the definition would not lead to greater generality since we can have the operators $V_\nu(t)$ constant on dyadic t-intervals.

Proposition 4.2. *Let $q\in[1,\infty]$ and let the interpolation couple $(\mathcal{X}_0,\mathcal{X}_1)=((X_0,J_0,E_0),(X_1,J_1,E_1))$ be l^q-quasilinearizable with corresponding operator family $\mathcal{V}=(V_0(t),V_1(t))_{t>0}$. For $\theta\in(0,1)$ and $x\in X_0+X_1$ we define*

$$\|x\|_{\theta,l^q}^{\mathcal{V}}:=\Big\|\Big(\sum_{j\in\mathbb{Z}}\big|2^{-j\theta}J_0V_0(2^j)x\big|^q\Big)^{1/q}\Big\|_{E_0}+\Big\|\Big(\sum_{j\in\mathbb{Z}}\big|2^{j(1-\theta)}J_1V_1(2^j)x\big|^q\Big)^{1/q}\Big\|_{E_1}.$$

Then $(\mathcal{X}_0,\mathcal{X}_1)_{\theta,l^q}=\{x\in X_0+X_1:\|x\|_{\theta,l^q}^{\mathcal{V}}<\infty\}$ and $\|\cdot\|_{\theta,l^q}^{\mathcal{V}}$ is an equivalent norm on $(\mathcal{X}_0,\mathcal{X}_1)_{\theta,l^q}$.

Proof. We denote by C the maximum of the R_q-bounds of the sets in Definition 4.1. For $x\in X_0+X_1$ we clearly have $\|x\|_{\theta,l^q}\le\|x\|_{\theta,l^q}^{\mathcal{V}}$. If, on the other hand, $x = x_0(j) + x_1(j)$ for $j \in \mathbb{Z}$ then

$$\begin{aligned}
&\sum_{\nu=0}^{1} \Big\| \Big(\sum_{j\in\mathbb{Z}} \big| 2^{j(\nu-\theta)} J_\nu V_\nu\big(2^j\big)x \big|^q \Big)^{1/q} \Big\|_{E_\nu} \\
&\quad \le \sum_{\nu=0}^{1}\sum_{\rho=0}^{1} \Big\| \Big(\sum_{j\in\mathbb{Z}} \big| 2^{j(\nu-\theta)} J_\nu V_\nu\big(2^j\big)x_\rho(j) \big|^q \Big)^{1/q} \Big\|_{E_\nu} \\
&\quad \le \sum_{\nu=0}^{1}\sum_{\rho=0}^{1} \Big\| \Big(\sum_{j\in\mathbb{Z}} \big| 2^{j(\rho-\theta)} J_\nu \big(2^{j(\nu-\rho)} V_\nu\big(2^j\big)\big)x_\rho(j) \big|^q \Big)^{1/q} \Big\|_{E_\nu} \\
&\quad \le 2C \sum_{\rho=0}^{1} \Big\| \Big(\sum_{j\in\mathbb{Z}} \big| 2^{j(\rho-\theta)} J_\rho x_\rho(j) \big|^q \Big)^{1/q} \Big\|_{E_\rho}
\end{aligned}$$

by the R_q-boundedness properties of $V_0(t)$, $V_1(t)$. Hence $\|x\|^{\mathcal{V}}_{\theta,l^q} \le 2C\|x\|_{\theta,l^q}$. □

Example 4.3. As an example, we mention the special case of a sectorial operator A in a structured Banach space $\mathcal{X} = \mathcal{X}_0 = (X, J, E)$ with $0 \in \rho(A)$. We let $\mathcal{X}_1 := (X_1, J_1, E)$ where $X_1 = D(A)$ with norm $\|x\|_{X_1} = \|Ax\|_X$ and $J_1 x := JAx$. If A is R_q-sectorial then taking $V_0(t) = tA(1+tA)^{-1}$ and $V_1(t) = (1+tA)^{-1}$ shows that the couple $(\mathcal{X}_0, \mathcal{X}_1)$ is l^q-quasilinearizable. Moreover, the R_q-bounds of the four sets in Definition 4.1(ii) are all equal to the R_q-bound $\mathcal{X} \to \mathcal{X}$ of $\{\lambda(\lambda+A)^{-1} : \lambda > 0\}$. This gives the link to (generalized) Triebel–Lizorkin spaces in the next section.

Corollary 4.4. *In the situation of Proposition 4.2, we let*

$$X^{\mathcal{V}}_{\theta,l^q} := \big(X_{\theta,l^q}, \|\cdot\|^{\mathcal{V}}_{\theta,l^q}\big), \qquad E_{l^q} := E_0\big(l^q(\mathbb{Z})\big) \times E_1\big(l^q(\mathbb{Z})\big).$$

Then $\mathcal{X}^{\mathcal{V}}_{\theta,l^q} := (X^{\mathcal{V}}_{\theta,l^q}, J^{\mathcal{V}}_{\theta,l^q}, E_{l^q})$ *is a structured Banach space where*

$$J^{\mathcal{V}}_{\theta,l^q} : X^{\mathcal{V}}_{\theta,l^q} \to E_{l^q}, \qquad J^{\mathcal{V}}_{\theta,l^q} x := \big(2^{-j\theta} J_0 V_0\big(2^j\big)x, 2^{j(1-\theta)} J_1 V_1\big(2^j\big)x\big)_{j\in\mathbb{Z}}.$$

The following is the announced improvement of the assertion of Theorem 2.11.

Theorem 4.5. *Let $q \in [1,\infty]$ and let $(\mathcal{X}_0, \mathcal{X}_1)$ and $(\mathcal{Y}_0, \mathcal{Y}_1)$ be l^q-quasilinearizable interpolation couples with corresponding families $\mathcal{V}$, $\mathcal{W}$, respectively. Let $\mathscr{T}$ be a set of linear operators $T : X_0 + X_1 \to Y_0 + Y_1$ such that $\mathscr{T}$ is R_q-bounded $\mathcal{X}_0 \to \mathcal{Y}_0$ and $\mathcal{X}_1 \to \mathcal{Y}_1$ with R_q-bounds M_0 and M_1, respectively. Then, for $\theta \in (0,1)$, the set $\mathscr{T}$ is R_q-bounded $\mathcal{X}^{\mathcal{V}}_{\theta,l^q} \to \mathcal{Y}^{\mathcal{W}}_{\theta,l^q}$ with R_q-bound $\le c_{\theta,\mathcal{V},\mathcal{W}} M_0^{1-\theta} M_1^{\theta}$.*

Proof. This is done by a suitable modification of the proof of Theorem 2.11. We choose an integer m such that $2^{m-1} < M_1/M_0 \le 2^m$. We have to bound expressions

$$\Big\|\Big(\sum_{j,k}\big|2^{j(\nu-\theta)}K_\nu W_\nu\big(2^j\big)T_k x_k\big|^q\Big)^{1/q}\Big\|_{F_\nu}$$

where $\nu=0,1$ and $\sum_{j,k}=\sum_{j\in\mathbb{Z}}\sum_{k=1}^n$. To this end, we write

$$x_k=V_0\big(2^{j+m}\big)x_k+V_1\big(2^{j+m}\big)x_k$$

and estimate

$$\Big\|\Big(\sum_{j,k}\big|2^{j(\nu-\theta)}K_\nu W_\nu\big(2^j\big)T_k V_\rho\big(2^{j+m}\big)x_k\big|^q\Big)^{1/q}\Big\|_{F_\nu}$$

for $\nu,\rho=0,1$. By Definition 4.1(ii) for W_ν and the assumption on $\mathscr{T}$ we get that this is

$$\begin{aligned}&\le c_{\mathcal{W}}M_\rho\Big\|\Big(\sum_{j,k}\big|2^{j(\nu-\theta)}2^{j(\rho-\nu)}J_\rho V_\rho\big(2^{j+m}\big)x_k\big|^q\Big)^{1/q}\Big\|_{F_\rho}\\&=c_{\mathcal{W}}M_\rho 2^{m(\theta-\rho)}\Big\|\Big(\sum_{j,k}\big|2^{(j+m)(\rho-\theta)}J_\rho V_\rho\big(2^{j+m}\big)x_k\big|^q\Big)^{1/q}\Big\|_{F_\rho}.\end{aligned}$$

Now we can proceed as before. $\square$

5. Generalized Triebel–Lizorkin spaces revisited

In this section, let X be a Banach function space with absolute continuous norm and $q\in[1,\infty]$. Let A be a sectorial operator in X with dense domain $D(A)$ and range $R(A)$ that is R_q-sectorial of type $\omega_q(A)$. Then A is injective. We first recall the construction of generalized Triebel–Lizorkin spaces associated with A from [12]. For the sake of a simple presentation, we restrict to the case $\theta\in(0,1)$ and *homogeneous* generalized Triebel–Lizorkin spaces. To this end, we define $\dot X_1(A)$ as the completion of the normed space $(D(A),\|A\cdot\|_X)$. Then $(X,\dot X_1(A))$ is an interpolation couple of Banach spaces and A has an extension to an isometry $\dot X_1(A)\to X$ which we denote again by A. We refer to, for example, [13], [8], [6] for more background. We denote $\mathcal{X}_0:=(X,I,X)$ and $\mathcal{X}_1:=(\dot X_1(A),A,X)$ and obtain an interpolation couple in the sense of Definition 2.8. As in Example 4.3, this couple is l^q-quasilinearizable with $V_0(t)=tA(1+tA)^{-1}$ and $V_1(t)=(1+tA)^{-1}$.

For $\omega\in(0,\pi)$, we denote the open sector $\{z\in\mathbb{C}\setminus\{0\}:|\arg z|<\omega\}$ by Σ_ω and let $\Sigma_0:=(0,\infty)$. Then $\overline{\Sigma_\omega}=\{z\in\mathbb{C}\setminus\{0\}:|\arg z|\le\omega\}\cup\{0\}$ for any $\omega\in[0,\pi)$. For an angle $\omega\in(0,\pi)$, we denote by $H^\infty(\Sigma_\omega)$ the set of all bounded homolorphic functions on Σ_ω and by $H_0^\infty(\Sigma_\omega)$ the subset of those functions $f\in H^\infty(\Sigma_\omega)$ that satisfy, for some $\varepsilon>0$, $|f(z)|=O(|z|^\varepsilon)$ as $z\to0$ and $|f(z)|=O(|z|^{-\varepsilon})$ as $z\to\infty$. We denote the extended Dunford–Riesz class $H_0^\infty(\Sigma_\omega)+\langle z\mapsto(1+z)^{-1},z\mapsto1\rangle$ by $\mathcal{E}(\Sigma_\omega)$. Thus $\mathcal{E}(\Sigma_\omega)$ consists of all $f \in H^\infty(\Sigma_\omega)$ that have limits $f(0)$ and $f(\infty)$ such that $f(z) - f(0) = O(|z|^\varepsilon)$ $(z \to 0)$ and $f(z) - f(\infty) = O(|z|^{-\varepsilon})$ $(z \to \infty)$ for some $\varepsilon > 0$. Any sectorial operator of type $\omega(A) \in [0, \pi)$ has an $\mathcal{E}(\Sigma_\omega)$ functional calculus for $\omega \in (\omega(A), \pi)$, cf. [12] or [6].

We combine [12, Definition 4.1] and [12, Proposition 4.10]: For $\omega > \omega_q(A)$ and $\varphi \in \mathcal{E}(\Sigma_\omega) \setminus \{0\}$ such that $z \mapsto z^{-\theta}\varphi(z) \in H_0^\infty(\Sigma_\omega)$ and $x \in X + \dot{X}_1(A)$ let

$$\|x\|_{\theta,q,\varphi} := \left\| \left(\int_0^\infty \left| t^{-\theta}\varphi(tA)x \right|^q \frac{dt}{t} \right)^{1/q} \right\|_X$$

and $\dot{X}^\theta_{q,A,\varphi} := \{x \in X + \dot{X}_1(A) : \|x\|_{\theta,q,\varphi} < \infty\}$. This is a Banach space. By [12, Proposition 4.5] the space $\dot{X}^\theta_{q,A,\varphi}$ is independent of φ and all the norms are equivalent. Hence this space is denoted $\dot{X}^\theta_{q,A}$. There are discrete analogs of these norm expressions but only for a restricted class of functions φ, cf. [12, Section 3.3]. Here we only need that $\psi(z) = z(1+z)^{-1}$ belongs to this class, i.e. for $\theta \in (0,1)$, the function $\psi_\theta(z) = z^{1-\theta}(1+z)^{-1}$ satisfies property (UE) from [12, Definition 3.11], which can be seen as in [12, Example 3.13] and is even simpler. In fact, the conclusion of [12, Lemma 3.12] can be checked directly here: We have, for $t \in [1,2]$ and $j \in \mathbb{Z}$,

$$S_j(t) = \psi_\theta\big(2^j tA\big)^{-1}\psi_\theta\big(2^j A\big) = t^{\theta-1}\big(1 + 2^j tA\big)\big(1 + 2^j A\big)^{-1},$$
$$\overline{S}_j(t) = \psi_\theta\big(2^j A\big)^{-1}\psi_\theta\big(2^j tA\big) = t^{1-\theta}\big(1 + 2^j A\big)\big(1 + 2^j tA\big)^{-1}.$$

Hence R_q-boundedness of the set $\{S_j(t), \overline{S}_j(t) : j \in \mathbb{Z}, t \in [1,2]\}$ in X follows directly from the definition of R_q-sectoriality of A. By [12, Proposition 4.5] we have, for $\theta \in (0,1)$, that

$$\|x\|^\Sigma_{\theta,q,A} := \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{-j\theta}\psi\big(2^j A\big)x \right|^q \right)^{1/q} \right\|_X$$

is an equivalent norm on $\dot{X}^\theta_{q,A}$. But $\psi(2^j A) = V_0(2^j) = 2^j A V_1(2^j)$, which means

$$\|x\|^{\mathcal{V}}_{\theta,l^q} = 2\|x\|^\Sigma_{\theta,q,A},$$

and we have proved the following.

Proposition 5.1. *Under the assumptions of this section we have, for $\theta \in (0,1)$ and φ as above,*

$$\dot{X}^\theta_{q,A,\varphi} = (\mathcal{X}_0, \mathcal{X}_1)_{\theta,l^q}$$

with equivalent norms.

Actually, the norm expressions mentioned so far give rise to l^q-equivalent function space structures. We let $L^q_* := L^q(0,\infty,\frac{dt}{t})$ and use the notation $\psi(z) := z(1+z)^{-1}$ from above. We define

$$J_\theta : \dot{X}^\theta_{q,A} \to X\big(l^q(\mathbb{Z})\big), \qquad J_\theta x := \big(2^{-j\theta}\psi\big(2^j A\big)x\big)_{j\in\mathbb{Z}}.$$

Then $(J_\theta, X(l^q(\mathbb{Z}))$ is clearly l^q-equivalent to the function space structure we have on $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,l^q}$ by Corollary 4.4.

For $\varphi \in \mathcal{E}(\Sigma_\omega) \setminus \{0\}$ with $z^{-\theta}\varphi(z) \in H_0^\infty(\Sigma_\omega)$ where $\omega > \omega_q(A)$ we define

$$J_{\varphi,\theta} : \dot{X}^\theta_{q,A,\varphi} \to X\big(L^q_*\big), \qquad (J_{\theta,\varphi}x)(t) = t^{-\theta}\varphi(tA)x.$$

Then $(\dot{X}^\theta_{q,A,\varphi}, J_{\theta,\varphi}, X(L^q_*))$ is a structured Banach space.

THEOREM 5.2. *Under the assumptions of this section, for $\theta \in (0,1)$, and φ as above, the function space structures $(J_{\theta,\varphi}, X(L^q_*))$ and $(J_\theta, X(l^q))$ are l^q-equivalent on $\dot{X}^\theta_{q,A}$.*

Proof. This is essentially a Fubini argument. We have to show equivalence of expressions

$$\Big\| \Big(\int_0^\infty \sum_{k=1}^n \big|t^{-\theta}\varphi(tA)x_k\big|^q \frac{dt}{t}\Big)^{1/q} \Big\|_X \quad \text{and}$$

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \sum_{k=1}^n \big|2^{-j\theta}\psi\big(2^jA\big)x_k\big|^q \Big)^{1/q} \Big\|_X$$

uniformly in n. To this end, we use the operator $\widetilde{A}_q$ defined in [12, Definition 3.2] in $X(l^q)$ by $\widetilde{A}_q(x_j) := (Ax_j)$ with domain all sequences (x_j) in $D(A)$ such that $(x_j), (Ax_j) \in X(l^q)$ which had been shown to be sectorial in $X(l^q)$. We take $l^q = l^q_n$ here, but estimates will be uniform in n. The operator $\widetilde{A}_q$ is even R_q-sectorial in $X(l^q)$ as is easily checked (same R_q-bound as for A in X). Now we can apply Proposition 5.1 to the operator $\widetilde{A}_q$, and this gives what we need. An inspection of the arguments in [12] shows that the equivalence is indeed uniform in n. □

We close with a remark on the H^∞-calculus of the part of A in $\dot{X}^\theta_{q,A}$.

REMARK 5.3. Under the assumptions of this section it can be proved—via a suitable modification of [12, Proposition 3.9]—that, for $\theta \in (0,1)$, the part of A in $\dot{X}^\theta_{q,A}$ even has an R_q-bounded H^∞-calculus in $\dot{X}^\theta_{q,A}$.

REFERENCES

[1] M. G. Cowling, I. Doust, A. McIntosh and A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, J. Aust. Math. Soc. A **60** (1996), no. 1, 51–89. MR 1364554

[2] G. Dore, *H^∞-functional calculus in real interpolation spaces*, Studia Math. **137** (1999), no. 2, 161–167. MR 1734394

[3] X. T. Duong and J. Li, *Hardy spaces associated with operators satisfying Davies–Gaffney estimates and bounded holomorphic functional calculus*, J. Funct. Anal. **264** (2013), no. 6, 1409–1437. MR 3017269

[4] D. Frey and P. C. Kunstmann, *A $T(1)$-theorem for non-integral operators*, Math. Ann. **357** (2013), no. 1, 215–278. MR 3084347

[5] J. Garcìa-Cuerva and J. L. Rubio di Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, Notas de Matemática [Mathematical Notes], vol. 104, North-Holland, Amsterdam, 1985. MR 0807149

[6] M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser, Basel, 2006. MR 2244037

[7] S. Hofmann and S. Mayboroda, *Hardy and BMO spaces associated with divergence form elliptic operators*, Math. Ann. **344** (2009), no. 1, 37–116. MR 2481054

[8] N. J. Kalton, P. C. Kunstmann and L. Weis, *Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators*, Math. Ann. **336** (2006), no. 4, 747–801. MR 2255174

[9] N. J. Kalton and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann. **321** (2001), no. 2, 319–345. MR 1866491

[10] N. J. Kalton and L. Weis, *The H^∞-functional calculus and square function estimates*, manuscript, 2004.

[11] N. J. Kalton and L. Weis, *Euclidean structures*, manuscript, 2004.

[12] P. C. Kunstmann and A. Ullmann, *R_s-sectorial operators and generalized Triebel–Lizorkin spaces*, J. Fourier Anal. Appl. **20** (2014), no. 1, 135–185. MR 3180892

[13] P. C. Kunstmann and L. Weis, *Maximal L^p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, Functional analytic methods for evolution equations, Lecture Notes in Math., vol. 1855, Springer, Berlin, 2004, pp. 65–311. MR 2108959

[14] F. Lancien and C. Le Merdy, *Square functions and H^∞ calculus on subspaces of L^p and on Hardy spaces*, Math. Z. **251** (2005), 101–115. MR 2176466

[15] C. Le Merdy, *On square functions associated to sectorial operators*, Bull. Soc. Math. France **132** (2004), no. 1, 137–156. MR 2075919

[16] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, Springer, Berlin, 1996. Reprint of the 1st edn. MR 0500056

[17] J. Suárez and L. Weis, *Interpolation of Banach spaces by the γ-method*, Methods in Banach space theory, London Math. Soc. Lecture Note Ser., vol. 337, Cambridge University Press, Cambridge, 2006, pp. 293–306. MR 2326391

[18] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, North-Holland, Amsterdam, 1978. MR 0503903

[19] H. Triebel, *Characterizations of Besov–Hardy–Sobolev spaces via harmonic functions, temperatures, and related means*, J. Approx. Theory **35** (1982), no. 3, 275–297. MR 0663673

[20] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser, Basel, 1983. MR 0781540

[21] L. Weis, *A new approach to maximal L^p-regularity*, Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), Lecture Notes in Pure and Appl. Math., vol. 215, Dekker, New York, 2001, pp. 195–214. MR 1818002

Peer Christian Kunstmann, Karlsruhe Institute of Technologie (KIT), Institute for Analysis, Kaiserstr. 89, D-76128 Karlsruhe, Germany

E-mail address: peer.kunstmann@kit.edu