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THE DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION IN Sol_3, WITH POSSIBLE INFINITE BOUNDARY DATA

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ABSTRACT. In this paper, we study the Dirichlet problem for the minimal surface equation in Sol_3 with possible infinite boundary data, where Sol_3 is the non-Abelian solvable 3-dimensional Lie group equipped with its usual left-invariant metric that makes it into a model space for one of the eight Thurston geometries. Our main result is a Jenkins–Serrin type theorem which establishes necessary and sufficient conditions for the existence and uniqueness of certain minimal Killing graphs with a *non-unitary* Killing vector field in Sol_3.

1. Introduction

In [10], Jenkins and Serrin considered the Dirichlet problem for the minimal surface equation in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ with possible infinite boundary data. They considered a bounded domain $\Omega \subset \mathbb{R}^2$ whose boundary contains two finite sets of open straight segments $\{A_i\}_i$ and $\{B_i\}_i$ with the property that no two segments A_i and no two segments B_i meet to form a convex corner. The remaining portion of the boundary consists of endpoints of the segments A_i and B_i and a finite number of open convex arcs $\{C_i\}_i$. They found necessary and sufficient conditions on the lengths of the sides of inscribed polygons, which guarantee the existence of a minimal solution over Ω, taking the value $+\infty$ on each A_i, $-\infty$ on each B_i and assigned continuous data on each of the open arcs C_i (see [10, Theorems 2, 3 and 4]).

Some special cases are of interest. If Ω is a quadrilateral domain with sides A_1, C_1, A_2, C_2 in that order, then the necessary and sufficient condition for a solution to exist reduces simply to $|A_1| + |A_2| < |C_1| + |C_2|$, that is, the sum

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of the lengths of the sides A_i should be less than the sum of the lengths of the sides C_i. If the sides of Ω are A_1, B_1, A_2, B_2 in that order, then the condition becomes $|A_1|+|A_2|=|B_1|+|B_2|$. This solution was found by Scherk [20] in 1835.

In recent years, there has been much activity on this Dirichlet problem in $M^2\times\mathbb{R}$ where M^2 is a two dimensional Riemannian manifold (see [3], [18], [19]) and in the Heisenberg group Nil_3 [1], in $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ [26]. Moreover, there are non-compact domains on which this problem has been solved (see [3], [6], [12], [16]). In these cases, authors considered the Killing graphs where the Killing vector field is *unitary.*

The purpose of this paper is to consider the problem of type Jenkins–Serrin on bounded domains and some unbounded domains in Sol_3 which is a three-dimensional homogeneous Riemannian manifold can be viewed as $\mathbb{R}^3$ endowed with the Riemannian metric

$$\mathsf{d}s^2 = e^{2x_3}\mathsf{d}x_1^2 + e^{-2x_3}\mathsf{d}x_2^2 + \mathsf{d}x_3^2,$$

where (x_1, x_2, x_3) are canonical coordinates of $\mathbb{R}^3$. The change of coordinates

$$x := x_2, \qquad y := e^{x_3}, \qquad t := x_1,$$

turns this model into $\mathrm{Sol}_3 = \{(x,y,t)\in\mathbb{R}^3 : y>0\}$ with the Riemannian metric

$$\mathsf{d}s^2 = \frac{\mathsf{d}x^2+\mathsf{d}y^2}{y^2} + y^2\mathsf{d}t^2.$$

By using the Poincaré half-plane model $\mathbb{H}^2$, Sol_3 has the form of a warped product $\mathrm{Sol}_3 = \mathbb{H}^2\times_y\mathbb{R}$.

For every function u of class C^2 defined on the domain $\Omega\subset\mathbb{H}^2$, we denote by $\mathrm{Gr}(u) = \{(p,t)\in\mathrm{Sol}_3 : p\in\Omega, t=u(p)\}$ a surface in Sol_3 and is called ∂_t-graph of u. $\mathrm{Gr}(u)$ is a minimal surface if and only if u satisfies the equation (see Proposition 2.5)

$$\mathfrak{M}u := \mathrm{div}\left(\frac{y^2\nabla u}{\sqrt{1+y^2\|\nabla u\|^2}}\right) = 0.$$

We will consider the case that the boundary $\partial\Omega$ is composed of the families of "convex" arcs $\{A_i\}$, $\{B_j\}$ and $\{C_k\}$. We give necessary and sufficient conditions on the geometry of the domain Ω which assure the existence of a minimal solution u defined in Ω and u assumes the value $+\infty$ on each A_i, $-\infty$ on each B_j and prescribed continuous data on each C_k.

We see that the vector field ∂_t is Killing and normal to the plane $\mathbb{H}^2$. A special point of the problem is that the vector field ∂_t is *not unitary.* The important point to note here is that when γ is a curve in $\mathbb{H}^2$, if γ is a geodesic of $\mathbb{H}^2$, the surface $\gamma\times\mathbb{R}$ is no longer minimal in this warped product Riemannian manifold Sol_3. Instead of this, $\gamma\times\mathbb{R}$ is minimal in Sol_3 if and only if γ is an Euclidean geodesic (see Corollary 2.2). Hence, these Euclidean geodesics will play an important role in our problem. Moreover, because of the non-unitary field ∂_t, we don't use the hyperbolic length to state our problem. In $M^2 \times \mathbb{R}$ the length of a compact curve $\gamma \subset M^2$ is just the area of $\gamma \times [0,1]$ in which we are interested. However, for a curve $\gamma \in \mathbb{H}^2$, the area calculated in Sol_3 of $\gamma \times [0,1]$ is the Euclidean length of γ (see Proposition 2.3).

The problem of type Jenkins–Serrin is also solved for some unbounded domains. The main idea in [3] is to approximate an unbounded domain Ω by a sequence bounded domain Ω_n by cutting Ω with horocycles.

In our case, we use the Euclidean geodesics, Euclidean length instead of the geodesics and the hyperbolic length, so we can't use the horocycle of $\mathbb{H}^2$ to consider the problem of type Jenkins–Serrin on an unbounded domain. However, we can generalize the previous result for some unbounded domains by defining the flux for the non-compact arcs instead of using the horocycles. Our main result (Jenkins–Serrin type Theorem 6.1) may be stated as follows.

THEOREM. *Let Ω be a Scherk domain in $\mathbb{H}^2$ with the families of Euclidean geodesic arcs $\{A_i\}, \{B_i\}$ and of mean convex Euclidean arcs $\{C_i\}$.*

(1) *If the family $\{C_i\}$ is nonempty, there exists a solution to the Dirichlet problem on Ω (taking the value $+\infty$ on each A_i, $-\infty$ on each B_i and prescribed continuous data on each of the open arcs C_i) if and only if*

$$2a_{\mathrm{euc}}(\mathcal{P}) < \ell_{\mathrm{euc}}(\mathcal{P}), \qquad 2b_{\mathrm{euc}}(\mathcal{P}) < \ell_{\mathrm{euc}}(\mathcal{P})$$

for every Euclidean polygonal domain $\mathcal{P}$ inscribed in Ω. Moreover, such a solution is unique if it exists.

(2) *If the family $\{C_i\}$ is empty, there exists a solution to the Dirichlet problem on Ω (taking the value $+\infty$ on each A_i, $-\infty$ on each B_i) if and only if*

$$a_{\mathrm{euc}}(\mathcal{P}) = b_{\mathrm{euc}}(\mathcal{P})$$

when $\mathcal{P} = \Omega$ and the inequalities in Assertion (1) hold for all other Euclidean polygonal domains $\mathcal{P}$ inscribed in Ω. Such a solution is unique up to an additive constant, if it exists.

In this theorem, we denote by $\ell_{\mathrm{euc}}(\mathcal{P})$ the Euclidean perimeter of $\partial\mathcal{P}$, and by $a_{\mathrm{euc}}(\mathcal{P})$ and $b_{\mathrm{euc}}(\mathcal{P})$ the sum of the Euclidean lengths of the edges A_i and B_i lying in $\partial\mathcal{P}$, respectively.

We will have similar result for the Dirichlet problem for the minimal surface equation in Sol_3 with respect to ∂_x-graph. In the case of ∂_y-graph (∂_y is not a Killing vector field), Menezes solved on some "small" squares in the (x,t)-plane with data $+\infty$ on opposite two sides and $-\infty$ on the other two sides (see [17, Theorem 2]).

We have organized the contents as follows: In Section 2, we will review some of the standard facts on Sol_3 and establish minimal surface equations. Section 3 will prove the maximum principle for the minimal surface equations, shown the existence of solutions. A local Scherk surface in Sol_3 will be constructed in Section 4. Sections 5 will be devoted to proving the monotone convergence theorem and describing the divergence set. Our main results are stated and proved in Section 6.

2. Preliminaries

2.1. A model of Sol_3**.** The three-dimensional homogeneous Riemannian manifold Sol_3 can be viewed as $\mathbb{R}^3$ endowed with the Riemannian metric

$$\mathsf{d}s^2 = e^{2x_3}\mathsf{d}x_1^2 + e^{-2x_3}\mathsf{d}x_2^2 + \mathsf{d}x_3^2,$$

where (x_1, x_2, x_3) are canonical coordinates of $\mathbb{R}^3$ (see for instance [22, §4] and the references given there for more details). The space Sol_3 has a Lie group structure with respect to which the above metric is left-invariant. The group structure is given by the multiplication

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = \big(x_1 + e^{-x_3}y_1, x_2 + e^{x_3}y_2, x_3 + y_3\big).$$

In this paper, we don't use the Lie group structure. The change of coordinates

$$x := x_2, \qquad y := e^{x_3}, \qquad t := x_1,$$

turns this model into $\mathrm{Sol}_3 = \{(x, y, t) \in \mathbb{R}^3 : y > 0\}$ with the Riemannian metric

$$\mathsf{d}s^2 = \frac{\mathsf{d}x^2 + \mathsf{d}y^2}{y^2} + y^2\mathsf{d}t^2. \tag{2.1}$$

In the present paper, the model used for the hyperbolic plane is the Poincaré half-plane, that is,

$$\mathbb{H}^2 = \big\{(x, y) \in \mathbb{R}^2 : y > 0\big\}$$

endowed with the Riemannian metric $\frac{\mathsf{d}x^2+\mathsf{d}y^2}{y^2}$. Hence, Sol_3 has the form of a warped product $\mathrm{Sol}_3 = \mathbb{H}^2 \times_y \mathbb{R}$. From (2.1), we have

$$\|\partial_x\| = \|\partial_y\| = \frac{1}{y}, \qquad \|\partial_t\| = y, \qquad \langle\partial_x, \partial_y\rangle = \langle\partial_x, \partial_t\rangle = \langle\partial_y, \partial_t\rangle = 0.$$

Hence, $\{y\partial_x, y\partial_y, \frac{1}{y}\partial_t\}$ is an orthonormal frame of Sol_3. Translations along the t-axis

$$\tau_h : \mathrm{Sol}_3 \to \mathrm{Sol}_3, \qquad (x, y, t) \mapsto (x, y, t + h)$$

are isometries. Therefore, the vertical vector field ∂_t is a Killing vector field. Note that ∂_t is not unitary.

Let us denote by $\overline{\nabla}$ the Riemannian connection of Sol_3 and by ∇ the one in $\mathbb{H}^2$. By using Koszul's formula,

$$\begin{aligned} 2\langle\overline{\nabla}_X Y, Z\rangle = {}& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \big\langle Y, [X, Z]\big\rangle - \big\langle Z, [Y, X]\big\rangle + \big\langle X, [Z, Y]\big\rangle \end{aligned} \tag{2.2}$$

for any vector field X, Y, Z of Sol_3, we obtain Table 1.

TABLE 1. Table of $\overline{\nabla}_X Y$ for $X, Y \in \{\partial_x, \partial_y, \partial_t\}$ in Sol_3

	Y		
X	∂_x	∂_y	∂_t
∂_x	$\frac{1}{y}\partial_y$	$-\frac{1}{y}\partial_x$	0
∂_y	$-\frac{1}{y}\partial_x$	$-\frac{1}{y}\partial_y$	$\frac{1}{y}\partial_t$
∂_t	0	$\frac{1}{y}\partial_t$	$-y^3\partial_y$

Hence, the surfaces $\{t = \mathrm{const}\}$ and $\{x = \mathrm{const}\}$ are the totally geodesic surfaces in Sol_3 (Note that a totally geodesic submanifold $\Sigma \subset M$ is characterized by the fact that $\overline{\nabla}_X Y$ is a tangent vector field of Σ for all tangent vector fields X, Y of Σ, where $\overline{\nabla}$ is the Riemannian connection of M). The surfaces $\{y = \mathrm{const}\}$ are minimal, are not totally geodesic surfaces and are isometric to $\mathbb{R}^2$.

2.2. Euclidean geodesic. First, we note that the vertical lines $\{p\} \times \mathbb{R} \subset \mathrm{Sol}_3$ with $p = (x, y) \in \mathbb{H}^2$ aren't geodesics in Sol_3. Indeed, let $p = (x, y)$ be a point of $\mathbb{H}^2$. A unit speed parametrization of $\{p\} \times \mathbb{R}$ is $\gamma : \mathbb{R} \to \mathrm{Sol}_3, t \mapsto (x, y, \frac{t}{y})$. One has $\gamma' = \frac{1}{y}\partial_t$. Thus, $\frac{\mathsf{d}}{\mathsf{d}t}\gamma' = \overline{\nabla}_{\frac{1}{y}\partial_t}(\frac{1}{y}\partial_t) = -y\partial_y$. Since $\frac{\mathsf{d}}{\mathsf{d}t}\gamma' \neq 0$, $\{p\} \times \mathbb{R}$ is not a geodesic in Sol_3.

PROPOSITION 2.1. *Let γ be a curve in $\mathbb{H}^2$. Then the mean curvature vector of $\gamma \times \mathbb{R}$ in Sol_3 is*

$$\vec{H}_{\gamma\times\mathbb{R}} = y^2 \vec{\kappa}_{\mathrm{euc}},$$

where $\vec{\kappa}_{\mathrm{euc}}$ is Euclidean mean curvature vector of γ in $\mathbb{H}^2$.

Proof. We first compute $\vec{H}_{\gamma\times\mathbb{R}}$. Without loss of generality, we can assume that γ is a unit speed curve. So $\{\frac{1}{y}\partial_t, \gamma'\}$ is an orthonormal frame of $\gamma \times \mathbb{R}$. The mean curvature vector of $\gamma \times \mathbb{R}$ is by definition

$$\begin{aligned}
\vec{H}_{\gamma\times\mathbb{R}} &= \left(\overline{\nabla}_{\frac{1}{y}\partial_t}\left(\frac{1}{y}\partial_t\right) + \overline{\nabla}_{\gamma'}\gamma'\right)^{\perp} \\
&= \left(-y\partial_y + \overline{\nabla}_{\gamma'}\gamma'\right)^{\perp} \\
&= -y\partial_y^{\perp} + \vec{\kappa},
\end{aligned} \tag{2.3}$$

where $\vec{\kappa}$ is the mean curvature vector of γ in $\mathbb{H}^2$.

We now compute the Euclidean mean curvature vector $\vec{\kappa}_{\mathrm{euc}}$ of γ in $\mathbb{H}^2$. By Koszul's formula (2.2), we have

$$(\nabla_{\mathrm{euc}})_X Y = \nabla_X Y + \frac{1}{y}\big((Xy)Y + (Yy)X - \langle X, Y\rangle \nabla y\big),$$

where ∇_{euc} (resp. ∇) is the Riemannian connection of $\mathbb{H}^2$ with respect to the Euclidean metric (resp. hyperbolic metric) and X, Y are tangent vector fields of $\mathbb{H}^2$. Hence

$$\big((\nabla_{\mathrm{euc}})_X Y\big)^{\perp} = (\nabla_X Y)^{\perp} - \frac{1}{y}\langle X, Y\rangle (\nabla y)^{\perp}, \tag{2.4}$$

where X, Y are tangent vector fields of γ. Since γ is a unit speed curvature, $\|\gamma'\| = 1$ and $\|\frac{\gamma'}{y}\|_{\mathrm{euc}} = 1$. By (2.4) and $\nabla y = y^2 \partial_y$, we have

$$\begin{aligned}\vec{\kappa}_{\mathrm{euc}} &= \left((\nabla_{\mathrm{euc}})_{\frac{\gamma'}{y}} \frac{\gamma'}{y}\right)^{\perp} \\ &= \left(\nabla_{\frac{\gamma'}{y}} \frac{\gamma'}{y}\right)^{\perp} - \frac{1}{y}\left\langle \frac{\gamma'}{y}, \frac{\gamma'}{y}\right\rangle (\nabla y)^{\perp} \\ &= \frac{1}{y^2}\vec{\kappa} - \frac{1}{y}\partial_y^{\perp}.\end{aligned}$$

Hence,

$$y^2 \vec{\kappa}_{\mathrm{euc}} = \vec{\kappa} - y\partial_y^{\perp}.$$

Combining this equality with (2.3), we complete the proof. □

Let us mention two important consequences of the proposition.

COROLLARY 2.2. *Let γ be a curve in $\mathbb{H}^2$ and Ω be a domain in $\mathbb{H}^2$ with $\partial\Omega \in C^2$. Then*

(1) *$\gamma \times \mathbb{R}$ is a minimal surface in Sol_3 if and only if γ is an Euclidean geodesic in $\mathbb{H}^2$. However, these Euclidean geodesics need not have constant speed parametrization.*
(2) *$\Omega \times \mathbb{R}$ is a mean convex set in Sol_3 if and only if Ω is a mean convex Euclidean in $\mathbb{H}^2$.*

PROPOSITION 2.3. *Let γ be a curve in $\mathbb{H}^2$. Then the area calculated in Sol_3 of $\gamma \times [0,1]$ is*

$$\mathcal{A}\big(\gamma \times [0,1]\big) = \ell_{\mathrm{euc}}(\gamma),$$

where $\ell_{\mathrm{euc}}(\gamma)$ is the Euclidean length of γ.

Proof. Let us first compute the area of $\gamma \times [0,1]$. The surface $\gamma \times [0,1]$ in Sol_3 is defined by

$$\gamma \times [0,1] : [0,1] \times [0,1] \to \mathrm{Sol}_3, \qquad (t_1, t_2) \mapsto \big(\gamma(t_1), t_2\big).$$

We have by definition

$$\begin{aligned}\mathcal{A}\big(\gamma \times [0,1]\big) &= \int_{[0,1]\times[0,1]} \left\| \big(\gamma \times [0,1]\big)_{t_1} \times \big(\gamma \times [0,1]\big)_{t_2} \right\| \mathrm{d}t_1 \,\mathrm{d}t_2 \\ &= \int_0^1 \int_0^1 \|\gamma'(t_1)\| y\big(\gamma(t_1)\big) \,\mathrm{d}t_1 \,\mathrm{d}t_2\end{aligned}$$

$$= \int_0^1 \|\gamma'(t_1)\| y\big(\gamma(t_1)\big)\,\mathsf{d}t_1$$
$$= \int_\gamma y\,\mathsf{d}s.$$

The Euclidean length of γ is by definition

$$\ell_{\mathrm{euc}}(\gamma) = \int_\gamma \mathsf{d}s_{\mathrm{euc}} = \int_\gamma y\,\mathsf{d}s.$$

Combining these equalities, we conclude that

$$\mathcal{A}\big(\gamma \times [0,1]\big) = \int_\gamma y\,\mathsf{d}s = \ell_{\mathrm{euc}}(\gamma).$$

This establishes the formula. □

The *ideal boundary* of $\mathbb{H}^2$ is by definition

$$\partial_\infty \mathbb{H}^2 = \big\{(x,y) \in \mathbb{R}^2 : y = 0\big\} \cup \{\infty\}.$$

The point ∞ of $\partial_\infty \mathbb{H}^2$ is specified in our model of Sol_3 and we make the distinction with points in $\{y = 0\}$.

DEFINITION 2.4. A point $p \in \partial_\infty \mathbb{H}^2$ is called *removable* (resp. *essential*) if $p \in \big\{(x,y) \in \mathbb{R}^2 : y = 0\big\}$ (resp. $p = \infty$).

2.3. The minimal surface equations. Let Ω be a domain in $\mathbb{H}^2$ and u be a C^2 function on Ω. Using the previous model for Sol_3, we can consider the surface $\mathrm{Gr}(u)$ in Sol_3 parametrized by

$$(x,y) \mapsto \big(x, y, u(x,y)\big), \quad (x,y) \in \Omega.$$

Such a surface is called the vertical Killing graph of u, it is transverse to the Killing vector field ∂_t and any integral curve of ∂_t intersect at most once the surface. The upward unit normal to $\mathrm{Gr}(u)$ is given by

$$N = N_u = \frac{-y\nabla u + \frac{1}{y}\partial_t}{\sqrt{1 + y^2\|\nabla u\|^2}}, \tag{2.5}$$

where ∇ is the hyperbolic gradient operator and $\|-\|$ is the hyperbolic norm. Indeed, $\mathrm{Gr}(u) = \Phi^{-1}(0)$, where the function $\Phi : \mathrm{Sol}_3 \to \mathbb{R}$ is defined by $\Phi(x,y,t) = t - u(x,y)$. So, $\overline{\nabla}\Phi$ is a normal vector field to $\mathrm{Gr}(u)$. Moreover, since $\overline{\nabla} t = \frac{1}{y^2}\partial_t$ and $\langle \overline{\nabla} u, \partial_t \rangle = 0$, we have

$$\overline{\nabla}\Phi = \overline{\nabla} t - \overline{\nabla} u = \frac{1}{y^2}\partial_t - \nabla u, \qquad \|\overline{\nabla}\Phi\|^2 = \frac{1}{y^2} + \|\nabla u\|^2.$$

This establishes the formula (2.5). Denote

$$W = W_u := \sqrt{1 + y^2\|\nabla u\|^2}, \qquad X_u := \frac{y\nabla u}{W}.$$

It follows that

$$N = -X_u + \frac{1}{yW}\partial_t.$$

In the sequel, we will use this unit normal vector to compute the mean curvature of a Killing graph.

PROPOSITION 2.5. *Let Ω be a domain in $\mathbb{H}^2$ and u be a C^2 function on Ω. The mean curvature H of the Killing graph of u satisfies*:

$$2yH = \operatorname{div}\left(\frac{y^2\nabla u}{W}\right), \tag{2.6}$$

with div *the divergence operator in the hyperbolic metric, and after expanding all terms*:

$$2H = \frac{y^3}{W^3}\left(\left(1+y^4u_y^2\right)u_{xx} - 2y^4u_xu_yu_{xy} + \left(1+y^4u_x^2\right)u_{yy} + 2\frac{u_y}{y}\right).$$

Proof. We extend the vector field N to the whole $\Omega\times\mathbb{R}$ by using the expression given in (2.5). The mean curvature of the Killing graph $\operatorname{Gr}(u)$ of u is then given by $2H = \operatorname{div}_{\operatorname{Gr}(u)}(-N)$. Since ∂_t is a Killing vector field, we have

$$2H = \operatorname{div}_{\operatorname{Sol}_3}(-N) = \operatorname{div}_{\operatorname{Sol}_3}(X_u) - \operatorname{div}_{\operatorname{Sol}_3}\left(\frac{1}{yW}\partial_t\right).$$

Let us compute

$$\operatorname{div}_{\operatorname{Sol}_3}\left(\frac{1}{yW}\partial_t\right) = \left\langle \overline{\nabla}\frac{1}{yW}, \partial_t\right\rangle + \frac{1}{yW}\operatorname{div}_{\operatorname{Sol}_3}(\partial_t) = 0,$$

$$\operatorname{div}_{\operatorname{Sol}_3}(X_u) = \operatorname{div}(X_u) + \left\langle \overline{\nabla}_{\frac{1}{y}\partial_t}X_u, \frac{1}{y}\partial_t\right\rangle.$$

Moreover, since X_u and ∂_t are orthogonal, we see that

$$\left\langle \overline{\nabla}_{\frac{1}{y}\partial_t}X_u, \frac{1}{y}\partial_t\right\rangle = \frac{1}{y^2}\langle \overline{\nabla}_{\partial_t}X_u, \partial_t\rangle = -\frac{1}{y^2}\langle X_u, \overline{\nabla}_{\partial_t}\partial_t\rangle,$$

$$\overline{\nabla}_{\partial_t}\partial_t = -y^3\partial_y = -y\nabla y.$$

Combining these equalities, we deduce that

$$2H = \operatorname{div}(X_u) + \frac{1}{y}\langle X_u, \nabla y\rangle.$$

It follows that

$$2yH = y\operatorname{div}(X_u) + \langle X_u, \nabla y\rangle = \operatorname{div}(yX_u) = \operatorname{div}\left(\frac{y^2\nabla u}{W}\right).$$

This is the formula (2.6). Expanding (2.6) yields

$$2H = \frac{1}{y}\operatorname{div}\left(\frac{y^2\nabla u}{W}\right) = \frac{1}{y}\operatorname{div}\left(\frac{y^4u_x}{W}\partial_x + \frac{y^4u_y}{W}\partial_y\right)$$

$$= \frac{1}{y} \cdot y^2 \left(\frac{\partial}{\partial x} \left(\frac{1}{y^2} \frac{y^4 u_x}{W} \right) + \frac{\partial}{\partial_y} \left(\frac{1}{y^2} \frac{y^4 u_y}{W} \right) \right)$$
$$= \frac{y^3}{W^3} \left((1 + y^4 u_y^2) u_{xx} - 2y^4 u_x u_y u_{xy} + (1 + y^4 u_x^2) u_{yy} + 2\frac{u_y}{y} \right).$$

This completes the proof. □

By Proposition 2.5, the Killing graph of a C^2 function $u : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$ is a minimal surface in Sol_3 if and only if u satisfies the divergence form equation

$$\mathfrak{M}u := \operatorname{div}(yX_u) = 0, \tag{2.7}$$

where $X_u = \frac{y\nabla u}{\sqrt{1+y^2\|\nabla u\|^2}}$. Equation (2.7) is the divergence form of the *minimal surface equation* and can alternatively be written, by Proposition 2.5, as

$$(1 + y^4 u_y^2) u_{xx} - 2y^4 u_x u_y u_{xy} + (1 + y^4 u_x^2) u_{yy} + 2\frac{u_y}{y} = 0. \tag{2.8}$$

Definition 2.6. A C^2 function $u : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$ is said to be a *minimal solution* if u satisfies the minimal surface equation, i.e. $\mathfrak{M}u = 0$.

Example 2.7. We give some simple examples of minimal solution u.

(1) If the function u is of the form $u(x, y) = f(x)$, then (2.8) becomes $f'' = 0$. Thus, $u(x, y) = ax + b$ for $a, b \in \mathbb{R}$.
(2) If the function u is of the form $u(x, y) = f(y)$, then (2.8) becomes $f'' + 2\frac{f'}{y} = 0$. Thus $u(x, y) = \frac{a}{y} + b$ for $a, b \in \mathbb{R}$.
(3) We look for minimal solutions of the form $u(x, y) = f(\frac{x}{y})$. It follows from (2.8) that $f'' = 0$. Thus $u(x, y) = a\frac{x}{y} + b$ for $a, b \in \mathbb{R}$.

3. Maximum principle, Gradient estimate and Existence theorem

3.1. Maximum principle. A basic tool for obtaining the results of this work is the maximum principle for differences of minimal solutions. First, by applying the proof of the comparison principle [7, Theorem 10.1], we have the following theorem.

Theorem 3.1 (Maximum principle). *Let u_1, u_2 be two C^2 functions on a domain $\Omega \subset \mathbb{H}^2$. Suppose u_1 and u_2 satisfy $\mathfrak{M}u_1 \geq \mathfrak{M}u_2$. Then $u_2 - u_1$ cannot have an interior minimum unless $u_2 - u_1$ is a constant.*

It follows from this theorem that:

Proposition 3.2. *Let u_1, u_2 be two functions of class C^2 on a* bounded *domain $\Omega \subset \mathbb{H}^2$ such that $\mathfrak{M}u_1 \geq \mathfrak{M}u_2$, and* $\liminf(u_2 - u_1) \geq 0$ *for any approach to the boundary $\partial\Omega$ of Ω, then we have $u_2 \geq u_1$ in Ω.*

Proof. Assume the contrary that $\{p \in \Omega : u_2(p) < u_1(p)\}$ is not empty. Since $\liminf(u_2 - u_1) \geq 0$ for any approach to the boundary $\partial\Omega$ and Ω is bounded, $u_2 - u_1$ has an interior minimum in Ω. By Maximum principle (Theorem 3.1), $u_2 - u_1$ is constant, a contradiction. □

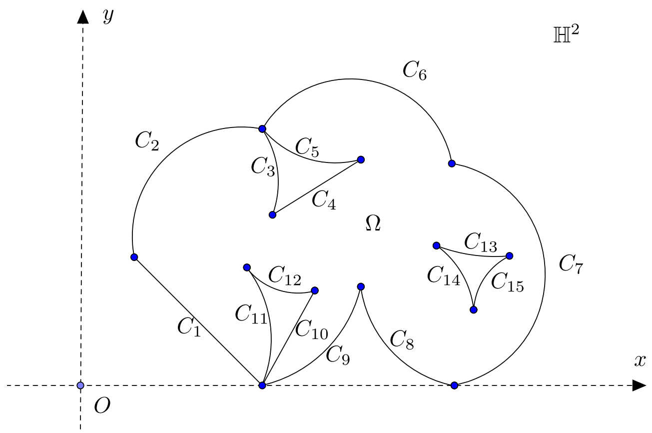

FIGURE 1. An example of admissible domain.

The following result (Theorem 3.4) is a remarkable strengthening of this situation. In what follows, for a subset Ω of $\mathbb{H}^2$, we will denote by $\partial_\infty \Omega$ the boundary of Ω in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$.

DEFINITION 3.3. A domain $\Omega \subset \mathbb{H}^2$ is called *admissible* if its boundary $\partial_\infty \Omega$ is composed of a finite number of open, mean convex Euclidean arcs C_i (of class C^2) in $\mathbb{H}^2$ together with their endpoints (see Figure 1). The endpoints of the arcs C_i are called vertices of Ω and those in $\partial_\infty \mathbb{H}^2$ are called ideal vertices of Ω. Assume in addition that, the ideal vertices of this domain are removable points (see Definition 2.4).

Let $p = (x(p), y(p)) \in \mathbb{H}^2$ and $R > 0$. Denote by $\mathbb{D}_R(p)$ the open hyperbolic disk with hyperbolic centre p and hyperbolic radius R

$$\mathbb{D}_R(p) = \left\{ q \in \mathbb{H}^2 : \mathrm{d}_{\mathbb{H}^2}(q,p) < R \right\}.$$

If $R < y(p)$, denote by $\mathbb{D}_R^{\mathrm{euc}}(p)$ the open Euclidean disk with Euclidean centre p and Euclidean radius R

$$\mathbb{D}_R^{\mathrm{euc}}(p) = \left\{ q \in \mathbb{H}^2 : \mathrm{d}_{\mathrm{euc}}(q,p) < R \right\}.$$

The closure of $\mathbb{D}_R(p)$ (resp. $\mathbb{D}_R^{\mathrm{euc}}(p)$) will be denoted by $\overline{\mathbb{D}}_R(p)$ (resp. $\overline{\mathbb{D}}_R^{\mathrm{euc}}(p)$).

THEOREM 3.4 (General maximum principle). *Let $\Omega \subset \mathbb{H}^2$ be a admissible domain. Let u_1, u_2 be two minimal solutions on Ω. Suppose that* $\limsup(u_1 - u_2) \leq 0$ *for any approach to the boundary of Ω exception of its vertices. Then $u_1 \leq u_2$.*

We should remark that this result is similar to the general maximum principle stated by Spruck [24, General Maximum Principle, p. 3] (resp. Hauswirth–Rosenberg–Spruck [8, Theorem 2.2]) for constant mean curvature surfaces in

$\mathbb{R}^2 \times \mathbb{R}$ (resp. in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$) in the case of the bounded domain Ω, and by Collin–Rosenberg [3, Theorem 2] for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ in the case of the unbounded domain Ω.

Proof of Theorem 3.4. Assume the contrary, that the set $\{p \in \Omega : u_1(p) > u_2(p)\}$ is nonempty. Let N and ε be positive constants, with N large and ε small. Define

$$\varphi = \begin{cases} 0 & \text{if } u_1 - u_2 \leq \varepsilon, \\ u_1 - u_2 - \varepsilon & \text{if } \varepsilon < u_1 - u_2 < N, \\ N - \varepsilon & \text{if } u_1 - u_2 \geq N. \end{cases}$$

Then φ is a continuous piecewise differentiable function in Ω satisfying $0 \leq \varphi < N$. Moreover, $\nabla\varphi = \nabla u_1 - \nabla u_2$ in the set where $\varepsilon < u_1 - u_2 < N$, and $\nabla\varphi = 0$ almost every where in the complement of this set.

Denote by E_1 (resp. E_2) the set of vertices in $\mathbb{H}^2$ (resp. vertices at $\partial_\infty\mathbb{H}^2$) of Ω. For each $p \in E_2$, we consider a sequence of nested ideal geodesics $H_{p,n}$, $n \geq 1$ converging to p. By nested, we mean that if $\mathcal{H}_{p,n}$ is the component of $\mathbb{H}^2 \setminus H_{p,n}$ containing p on its ideal boundary, then $\mathcal{H}_{p,n+1} \subset \mathcal{H}_{p,n}$. Assume $\overline{\mathcal{H}}_{p_1,1} \cap \overline{\mathcal{H}}_{p_2,1} = \emptyset$ for every different points $p_1, p_2 \in E_2$. For n sufficiently large satisfying $\overline{\mathbb{D}}^{\mathrm{euc}}_{\frac{1}{n}}(p_1) \cap \overline{\mathbb{D}}^{\mathrm{euc}}_{\frac{1}{n}}(p_2) = \emptyset, \forall p_1, p_2 \in E_1$ and $\overline{\mathbb{D}}^{\mathrm{euc}}_{\frac{1}{n}}(p_1) \cap \overline{\mathcal{H}}_{p_2,1} = \emptyset, \forall p_1 \in E_1, p_2 \in E_2$, we define (see Figure 2)

$$\Omega_n = \Omega \setminus \left(\left(\bigcup_{p \in E_1} \overline{\mathbb{D}}^{\mathrm{euc}}_{\frac{1}{n}}(p) \right) \cup \left(\bigcup_{p \in E_2} \overline{\mathcal{H}}_{p,n} \right) \right),$$

$$\Gamma_1 = \partial\Omega_n \cap \partial\Omega, \qquad \Gamma_2 = \partial\Omega_n \setminus \Gamma_1.$$

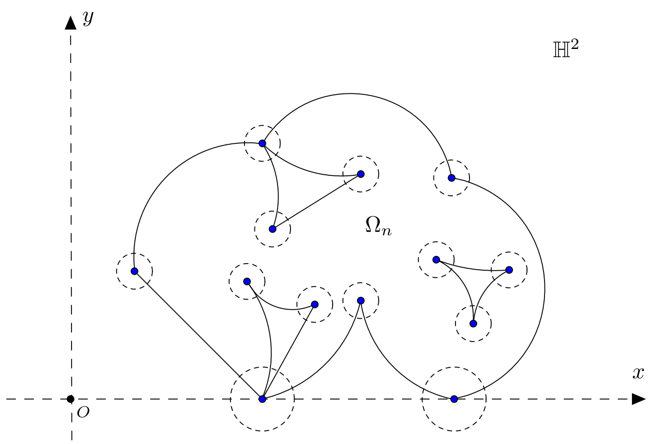

FIGURE 2. The domain Ω_n.

It follows from definition that

$$\varphi = 0 \quad \text{on a neighborhood of } \Gamma_1, \qquad \ell_{\text{euc}}(\Gamma_2) \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Define

$$J_n = \int_{\partial\Omega_n} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle \, \mathsf{d}s,$$

where ν is the exterior normal to $\partial\Omega_n$, $W_{u_i} = \sqrt{1 + y^2\|\nabla u_i\|^2}$ and $X_{u_i} = \frac{y\nabla u_i}{W_{u_i}}, i = 1, 2$.

Assertion 3.1. (1) *$J_n \geq 0$ with equality if and only if $\nabla u_1 = \nabla u_2$ on the set $\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}$.*
(2) *J_n is an increasing function of n.*

Proof. By Divergence theorem, we have

$$\begin{aligned} J_n &= \int_{\Omega_n} \operatorname{div}\big(\varphi y (X_{u_1} - X_{u_2})\big) \, \mathsf{d}\mathcal{A} \\ &= \int_{\Omega_n} \langle y\nabla\varphi, X_{u_1} - X_{u_2}\rangle \, \mathsf{d}\mathcal{A} + \int_{\Omega_n} \varphi \operatorname{div}(yX_{u_1} - yX_{u_2}) \, \mathsf{d}\mathcal{A} \\ &= \int_{\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}} \langle y\nabla\varphi, X_{u_1} - X_{u_2}\rangle \, \mathsf{d}\mathcal{A} + \int_{\Omega_n} \varphi \operatorname{div}(yX_{u_1} - yX_{u_2}) \, \mathsf{d}\mathcal{A}. \end{aligned}$$

By our assumptions,

$$\varphi \operatorname{div}(yX_{u_1} - yX_{u_2}) = \varphi(\mathfrak{M}u_1 - \mathfrak{M}u_2) = 0.$$

Moreover, on $\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}$, by formula (3.2) of Lemma 3.5, we have

$$\langle y\nabla\varphi, X_{u_1} - X_{u_2}\rangle = \left\langle y\nabla u_1 - y\nabla u_2, \frac{y\nabla u_1}{W_{u_1}} - \frac{y\nabla u_2}{W_{u_2}} \right\rangle \geq 0$$

and equality if and only if $y\nabla u_1 = y\nabla u_2$. Then

$$J_n = \int_{\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}} \langle y\nabla\varphi, X_{u_1} - X_{u_2}\rangle \, \mathsf{d}\mathcal{A} \geq 0$$

and $J_n = 0$ if and only if $\nabla u_1 = \nabla u_2$ on $\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}$. Since Ω_n is an increasing domain, i.e. $\Omega_n \subset \Omega_{n+1}$, J_n is an increasing function of n. This proves the assertion. □

Assertion 3.2. *$J_n = o(1)$ as $n \to \infty$.*

Proof. We have

$$J_n = \int_{\Gamma_1} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle \, \mathsf{d}s + \int_{\Gamma_2} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle \, \mathsf{d}s.$$

By Property (3.1), $\|X_{u^i}\| \leq 1, i = 1, 2$ and $0 \leq \varphi < N$, we have

$$\int_{\Gamma_1} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle \, \mathsf{d}s = 0$$

and

$$\left|\int_{\Gamma_2} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle \,\mathsf{d}s\right| = \left|\int_{\Gamma_2} \varphi \langle X_{u_1} - X_{u_2}, \nu\rangle \,\mathsf{d}s_{\mathrm{euc}}\right| \\ \leq 2N\ell_{\mathrm{euc}}(\Gamma_2) = o(1) \quad \text{as } n \to \infty.$$

Assertion is then proved. □

It follows from the previous assertions that $\nabla u_1 = \nabla u_2$ on the set $\{\varepsilon < u_1 - u_2 < N\}$. Since ε and N are arbitrary, $\nabla u_1 = \nabla u_2$ whenever $u_1 > u_2$. So $u_1 = u_2 + c$ $(c > 0)$ in any nontrivial component of the set $\{u_1 > u_2\}$. Then the maximum principle (Theorem 3.1) ensures $u_1 = u_2 + c$ in Ω and by assumptions of the theorem, the constant must be nonpositive, a contradiction. □

Lemma 3.5. *Let v_1, v_2 be two vectors in a finite dimensional Euclidean space. Then*

$$\left\langle v_1 - v_2, \frac{v_1}{W_1} - \frac{v_2}{W_2}\right\rangle = \frac{W_1 + W_2}{2}\left(\left\|\frac{v_1}{W_1} - \frac{v_2}{W_2}\right\|^2 + \left(\frac{1}{W_1} - \frac{1}{W_2}\right)^2\right),$$

where $W_i = \sqrt{1 + \|v_i\|^2}$. In particular,

$$\left\langle v_1 - v_2, \frac{v_1}{W_1} - \frac{v_2}{W_2}\right\rangle \geq \left\|\frac{v_1}{W_1} - \frac{v_2}{W_2}\right\|^2 \geq 0 \tag{3.2}$$

with equality at a point if and only if $v_1 = v_2$.

Proof. Let us compute

$$\begin{aligned}\left\langle v_1 - v_2, \frac{v_1}{W_1} - \frac{v_2}{W_2}\right\rangle &= \frac{\|v_1\|^2}{W_1} + \frac{\|v_2\|^2}{W_2} - \langle v_1, v_2\rangle\left(\frac{1}{W_1} + \frac{1}{W_2}\right) \\ &= W_1 - \frac{1}{W_1} + W_2 - \frac{1}{W_2} - \langle v_1, v_2\rangle\left(\frac{1}{W_1} + \frac{1}{W_2}\right) \\ &= (W_1 + W_2)\left(1 - \frac{\langle v_1, v_2\rangle}{W_1W_2} - \frac{1}{W_1W_2}\right) \\ &= (W_1 + W_2)\left(\frac{1}{2}\left\|\frac{v_1}{W_1} - \frac{v_2}{W_2}\right\|^2 + \frac{1}{2W_1^2}\right. \\ &\qquad \left. + \frac{1}{2W_2^2} - \frac{1}{W_1W_2}\right) \\ &= \frac{W_1 + W_2}{2}\left(\left\|\frac{v_1}{W_1} - \frac{v_2}{W_2}\right\|^2 + \left(\frac{1}{W_1} - \frac{1}{W_2}\right)^2\right).\end{aligned}$$

This proves the lemma. □

3.2. Gradient estimate. An important result concerning minimal solutions is a gradient estimate.

THEOREM 3.6 (Interior gradient estimate). *Let u be a nonnegative minimal solution on a disk $\mathbb{D}_R(p) \subset \mathbb{H}^2$. Then there exists a constant $C = C(R,p)$ that depends only on R and p (C doesn't depend on the function u) such that*

$$\|\nabla u(p)\| \leq f\left(\frac{u(p)}{R}\right),$$

where $f(t) = e^{C(1+t^2)}$. Moreover, if $\mathbb{D}_{R_1}(p_1) \subset \mathbb{D}_{R_2}(p_2)$ then $C(R_1,p_1) \leq C(R_2,p_2)$.

The proof of this result is similar to the one of the gradient estimate proved by Spruck [25, Theorem 1.1] and Mazet [13, Proposition 16]. Before beginning the proof, let us make some preliminary computation.

In this subsection, let us denote by Σ the Killing graph of u. The subscript Σ in $\nabla_\Sigma, \mathrm{div}_\Sigma, \Delta_\Sigma$ signifies that we compute the object in the Riemannian metric of the surface Σ. If f is a function on Ω, then we also denote by f the composition $\Omega \times \mathbb{R} \to \Omega \to \mathbb{R}$, $(x,y,t) \mapsto f(x,y)$.

LEMMA 3.7. *Let u be a minimal solution on a domain $\Omega \subset \mathbb{H}^2$. Then*

$$\nabla_\Sigma u = \frac{1}{y^2}\partial_t^\top, \qquad \|\nabla_\Sigma u\|^2 = \frac{1}{y^2}\left(1 - \frac{1}{W^2}\right) \quad \text{and} \quad \Delta_\Sigma u = \frac{2\langle \partial_y, N\rangle}{W},$$

where $W = \sqrt{1 + y^2\|\nabla u\|^2}$.

Proof. Since $u|_\Sigma$ is the restriction of t to Σ, we have

$$\nabla_\Sigma u = \nabla_\Sigma t = (\overline{\nabla} t)^\top = \frac{1}{y^2}\partial_t^\top. \tag{3.3}$$

It follows that

$$\|\nabla_\Sigma u\|^2 = \frac{1}{y^4}\left(\|\partial_t\|^2 - \langle \partial_t, N\rangle^2\right) = \frac{1}{y^2}\left(1 - \frac{1}{W^2}\right).$$

We continue to compute $\Delta_\Sigma u$. Since Σ is minimal and ∂_t is a Killing vector field, Equality (3.3) gives

$$\Delta_\Sigma u = \mathrm{div}_\Sigma\left(\frac{1}{y^2}\partial_t^\top\right) = \mathrm{div}_\Sigma\left(\frac{1}{y^2}\partial_t\right) = \left\langle \nabla_\Sigma \frac{1}{y^2}, \partial_t \right\rangle = -\frac{2}{y^3}\langle \nabla_\Sigma y, \partial_t\rangle.$$

Furthermore, we have

$$\nabla_\Sigma y = \nabla y - \langle \nabla y, N\rangle N = \nabla y - y^2\langle \partial_y, N\rangle N.$$

Combining these equalities with Equality $\langle N, \partial_t\rangle = \frac{y}{W}$, we obtain

$$\Delta_\Sigma u = -\frac{2}{y^3}\left(-y^2\right)\langle \partial_y, N\rangle\langle N, \partial_t\rangle = \frac{2\langle \partial_y, N\rangle}{W}.$$

This completes the proof of the lemma. □

Since ∂_t is a Killing vector field and $\frac{y}{W} = \langle \partial_t, N\rangle$, then by the formula [2, (1.147) p. 41] (see also [23, Theorem 3.2.2]) we have

$$\Delta_\Sigma \frac{y}{W} = -\big(\|A\|^2 + \mathrm{Ric}_{\mathrm{Sol}_3}(N,N)\big)\frac{y}{W}, \tag{3.4}$$

where $\mathrm{Ric}_{\mathrm{Sol}_3}$ is the Ricci tensor of Sol_3 and $\|A\|^2$ is the square of the norm of the second fundamental form.

LEMMA 3.8. *Let u be a minimal solution on a domain $\Omega \subset \mathbb{H}^2$. For each C^2 function $\varphi : \Omega \to \mathbb{R}$, the Laplacian of φ on Σ is given by*

$$\Delta_\Sigma \varphi = \Delta\varphi - \frac{y^2}{W^2}\langle \nabla_{\nabla u}\nabla\varphi, \nabla u\rangle + \frac{1}{y}\left(1 - \frac{1}{W^2}\right)\langle \nabla\varphi, \nabla y\rangle.$$

Proof. Since the surface Σ is minimal, we have

$$\Delta_\Sigma\varphi = \mathrm{div}_\Sigma \nabla_\Sigma \varphi = \mathrm{div}_\Sigma \nabla\varphi = \mathrm{div}_{\mathrm{Sol}_3}\nabla\varphi - \langle \overline{\nabla}_N \nabla\varphi, N\rangle.$$

Since $\frac{1}{y}\partial_t$ is a unit normal vector field to $\mathbb{H}^2$ in Sol_3, we deduce that

$$\begin{aligned}\mathrm{div}_{\mathrm{Sol}_3}\nabla\varphi &= \mathrm{div}\,\nabla\varphi + \left\langle \overline{\nabla}_{\frac{1}{y}\partial_t}\nabla\varphi, \frac{1}{y}\partial_t\right\rangle = \Delta\varphi + \frac{1}{y^2}\langle \overline{\nabla}_{\partial_t}\nabla\varphi, \partial_t\rangle \\ &= \Delta\varphi - \frac{1}{y^2}\langle \overline{\nabla}_{\partial_t}\partial_t, \nabla\varphi\rangle = \Delta\varphi + \frac{1}{y}\langle\nabla\varphi, \nabla y\rangle.\end{aligned}$$

Equality $N = -\frac{y\nabla u}{W} + \frac{\partial_t}{yW}$ yields

$$\begin{aligned}\langle \overline{\nabla}_N\nabla\varphi, N\rangle &= \left\langle \overline{\nabla}_{-\frac{y\nabla u}{W}}\nabla\varphi, -\frac{y\nabla u}{W}\right\rangle + \left\langle \overline{\nabla}_{\frac{\partial_t}{yW}}\nabla\varphi, \frac{\partial_t}{yW}\right\rangle \\ &= \frac{y^2}{W^2}\langle \nabla_{\nabla u}\nabla\varphi, \nabla u\rangle + \frac{1}{y^2W^2}\langle \overline{\nabla}_{\partial_t}\nabla\varphi, \partial_t\rangle \\ &= \frac{y^2}{W^2}\langle \nabla_{\nabla u}\nabla\varphi, \nabla u\rangle + \frac{1}{yW^2}\langle\nabla\varphi, \nabla y\rangle.\end{aligned}$$

Combining these equalities, we conclude that

$$\Delta_\Sigma \varphi = \Delta\varphi - \frac{y^2}{W^2}\langle \nabla_{\nabla u}\nabla\varphi, \nabla u\rangle + \frac{1}{y}\left(1 - \frac{1}{W^2}\right)\langle \nabla\varphi, \nabla y\rangle,$$

which completes the proof. □

Let us mention an important consequence of the lemma.

COROLLARY 3.9. *Let $\Omega \subset \mathbb{H}^2$ be a bounded domain and p be a point of Ω. Denote by $d = \mathrm{d}_{\mathbb{H}^2}(-, p)$ the hyperbolic distance to p. There exists a constant $C = C_\Omega$ depending only on Ω such that*

$$\sup_\Omega \left|\Delta_\Sigma d^2\right| \le C,$$

where Σ is the graph of a minimal solution u on Ω. Moreover, if $\Omega_1 \subset \Omega_2$ are bounded domains then $C_{\Omega_1} \le C_{\Omega_2}$.

Proof. It follows from Lemma 3.8 and Equalities $\nabla y = y^2\partial_y$, $W^2 = 1 + y^2\|\nabla u\|^2$ that

$$\begin{aligned}\left|\Delta_\Sigma d^2\right| &\leq \left|\Delta d^2\right| + \frac{y^2}{W^2}\left|\left\langle \nabla_{\nabla u}\nabla d^2, \nabla u\right\rangle\right| + \frac{1}{y}\left\|\nabla d^2\right\|\|\nabla y\| \\ &\leq \left|\Delta d^2\right| + \frac{y^2}{1+y^2\|\nabla u\|^2}\left\|\nabla_{\nabla u}\nabla d^2\right\|\|\nabla u\| + \left\|\nabla d^2\right\|.\end{aligned}$$

Moreover, we have $\|\nabla_{\nabla u}\nabla d^2\| \leq \|\nabla(\nabla d^2)\|\|\nabla u\|$ where $\|\nabla(\nabla d^2)\|$ is the operator norm of $(1,1)$-tensor field $\nabla(\nabla d^2)$. Combining these inequalities, we obtain

$$\left|\Delta_\Sigma d^2\right| \leq \left|\Delta d^2\right| + \left\|\nabla\left(\nabla d^2\right)\right\| + \left\|\nabla d^2\right\|.$$

Define $C = C_\Omega = \sup_{p\in\Omega}\sup_\Omega(|\Delta d^2| + \|\nabla(\nabla d^2)\| + \|\nabla d^2\|)$ and the proof is complete. $\square$

Using Lemma 3.7, Formula (3.4) and Corollary 3.9, we are ready to write the proof of Interior gradient estimate.

Proof of Theorem 3.6. We first consider the case $u(p) > 0$. Let us denote $\upsilon := \frac{y}{W} = \langle\partial_t, N\rangle$. By definition, $\partial_t = \partial_t^\top + \upsilon N$. We define an operator L on Σ by

$$Lf := \Delta_\Sigma f - 2\upsilon\left\langle \nabla_\Sigma \frac{1}{\upsilon}, \nabla_\Sigma f\right\rangle.$$

We remark that the maximum principle is true for L. By Formula (3.4), we have

$$\begin{aligned}\Delta_\Sigma\frac{1}{\upsilon} &= -\frac{1}{\upsilon^2}\Delta_\Sigma\upsilon + \frac{2}{\upsilon^3}\|\nabla_\Sigma\upsilon\|^2 \\ &= -\frac{1}{\upsilon^2}\left(-\left(\mathrm{Ric}_{\mathrm{Sol}_3}(N,N) + \|A\|^2\right)\upsilon\right) + \frac{2}{\upsilon^3}\left\|-\upsilon^2\nabla_\Sigma\frac{1}{\upsilon}\right\|^2 \\ &= \left(\mathrm{Ric}_{\mathrm{Sol}_3}(N,N) + \|A\|^2\right)\frac{1}{\upsilon} + 2\upsilon\left\|\nabla_\Sigma\frac{1}{\upsilon}\right\|^2.\end{aligned}$$

From this and Inequality $\mathrm{Ric}_{\mathrm{Sol}_3} \geq -2$ (see, for instance, [5]), we deduce that

$$L\frac{1}{\upsilon} = \Delta_\Sigma\frac{1}{\upsilon} - 2\upsilon\left\langle\nabla_\Sigma\frac{1}{\upsilon}, \nabla_\Sigma\frac{1}{\upsilon}\right\rangle = \left(\mathrm{Ric}_{\mathrm{Sol}_3}(N,N) + \|A\|^2\right)\frac{1}{\upsilon} \geq -\frac{2}{\upsilon}.$$

Let us define $h = \eta\frac{1}{\upsilon}$ where η is a nonnegative function. Let us compute

$$\begin{aligned}Lh = L\left(\eta\frac{1}{\upsilon}\right) &= \Delta_\Sigma\left(\eta\frac{1}{\upsilon}\right) - 2\upsilon\left\langle\nabla_\Sigma\frac{1}{\upsilon}, \nabla_\Sigma\left(\eta\frac{1}{\upsilon}\right)\right\rangle \\ &= \left(\eta\Delta_\Sigma\frac{1}{\upsilon} + 2\left\langle\nabla_\Sigma\eta, \nabla_\Sigma\frac{1}{\upsilon}\right\rangle + \frac{1}{\upsilon}\Delta_\Sigma\eta\right) \\ &\quad - 2\upsilon\left\langle\nabla_\Sigma\frac{1}{\upsilon}, \eta\nabla_\Sigma\frac{1}{\upsilon} + \frac{1}{\upsilon}\nabla_\Sigma\eta\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \eta L \frac{1}{\upsilon} + \frac{1}{\upsilon}\Delta_\Sigma \eta \\
&\geq (\Delta_\Sigma \eta - 2\eta)\frac{1}{\upsilon}.
\end{aligned}
$$

Fix $\varepsilon \in (0, \frac{1}{2})$. We define on Σ the function

$$
\varphi(q) = \max\left\{ -\frac{u(q)}{2u(p)} + 1 - \varepsilon - \frac{d(q)^2}{R^2}, 0 \right\},
$$

where $d = \mathrm{d}_{\mathbb{H}^2}(-, p)$. By definition,

$$
\varphi(p) = \frac{1}{2} - \varepsilon, \qquad 0 \leq \varphi \leq 1 - \varepsilon, \qquad \mathrm{supp}(\varphi) \subset\subset \Sigma.
$$

We define $\eta = e^{K\varphi} - 1$ with K a positive constant that will be chosen later. We calculate $\eta'(\varphi) = Ke^{K\varphi}$, $\eta''(\varphi) = K^2 e^{K\varphi}$. We then have $\sup_\Sigma h > 0$ and it is reached at q inside the support of φ. At the point q, we have

$$
\begin{aligned}
\Delta_\Sigma \eta - 2\eta &= \big(\eta'(\varphi)\Delta_\Sigma \varphi + \eta''(\varphi)\|\nabla_\Sigma \varphi\|^2\big) - 2\big(e^{K\varphi} - 1\big) \\
&= e^{K\varphi}\big(K^2\|\nabla_\Sigma \varphi\|^2 + K\Delta_\Sigma \varphi - 2\big) + 2 \\
&\geq e^{K\varphi}\big(K^2\|\nabla_\Sigma \varphi\|^2 + K\Delta_\Sigma \varphi - 2\big).
\end{aligned}
$$

The vector field ∂_d is well defined in Sol_3 outside $\{(p,t) : t \in \mathbb{R}\}$ and has unit length; $d\partial_d$ is well defined everywhere. The definition of φ, Lemma 3.7 and Inequality $d(q) \leq R$ yield

$$
\begin{aligned}
\|\nabla_\Sigma \varphi\|^2 &= \left\| -\frac{\nabla_\Sigma u}{2u(p)} - \frac{\nabla_\Sigma d^2}{R^2} \right\|^2 = \left\| \frac{\partial_t^\top}{2u(p)y^2} + \frac{2d\partial_d^\top}{R^2} \right\|^2 \\
&= \frac{1}{4u(p)^2y^2}\left(1 - \frac{1}{W^2}\right) + \frac{4d^2}{R^4}\left\|\partial_d^\top\right\|^2 + \frac{2d}{u(p)R^2y^2}\langle \partial_t^\top, \partial_d^\top \rangle \\
&\geq \frac{1}{4u(p)^2y^2}\left(1 - \frac{1}{W^2}\right) + 0 - \frac{2d}{u(p)R^2y^2}\upsilon\langle \partial_d, N\rangle \\
&= \frac{1}{4u(p)^2y^2}\left(1 - \frac{1}{W^2} - \frac{8yu(p)}{R}\frac{d}{R}\langle \partial_d, N\rangle \frac{1}{W}\right) \\
&\geq \frac{1}{4u(p)^2y^2}\left(1 - \frac{1}{W^2} - \frac{8yu(p)}{R}\frac{1}{W}\right).
\end{aligned} \tag{3.5}
$$

Hence, if $\frac{1}{W} \leq \min\{\frac{1}{2}, \frac{R}{32yu(p)}\}$ at q, then $\|\nabla_\Sigma \varphi\|^2 \geq \frac{1}{8u(p)^2y^2}$. Define $C_1 = M = \sup_{\mathbb{D}_R(p)} y$ and $C_2 = M^2 C_{\mathbb{D}_R(p)}$ where $C_{\mathbb{D}_R(p)}$ is the constant defined in Corollary 3.9. Moreover, Corollary 3.9 gives

$$
\begin{aligned}
\Delta_\Sigma \varphi &= -\frac{\Delta_\Sigma u}{2u(p)} - \frac{\Delta_\Sigma d^2}{R^2} \\
&= -\frac{1}{2u(p)}\left(\frac{2}{Wy^2}\langle \nabla y, N\rangle\right) - \frac{\Delta_\Sigma d^2}{R^2}
\end{aligned} \tag{3.6}
$$

$$= -\frac{1}{y^2u(p)^2}\left(\frac{\langle \nabla y, N\rangle}{W}u(p) + \frac{y^2\Delta_\Sigma d^2}{R^2}u(p)^2\right)$$
$$\geq -\frac{1}{y^2u(p)^2}\left(C_1u(p) + \frac{C_2}{R^2}u(p)^2\right).$$

Combining (3.5) with (3.6) yields

$$K^2\|\nabla_\Sigma\varphi\|^2 + K\Delta_\Sigma\varphi - 2$$
$$\geq \frac{1}{8u(p)^2y^2}K^2 - \frac{1}{y^2u(p)^2}\left(C_1u(p) + \frac{C_2}{R^2}u(p)^2\right)K - 2$$
$$\geq \frac{1}{8u(p)^2y^2}\left(K^2 - 8\left(C_1u(p) + \frac{C_2}{R^2}u(p)^2\right)K - 8C_3u(p)^2\right),$$

where $C_3 = 2M^2$. It follows that, if

$$K = \left(8C_1 + \frac{C_3}{C_1}\right)u(p) + 8\frac{C_2}{R^2}u(p)^2 = 10Mu(p) + 8M^2C_{\mathbb{D}_R(p)}\left(\frac{u(p)}{R}\right)^2$$

we obtain $K^2\|\nabla_\Sigma\varphi\|^2 + K\Delta_\Sigma\varphi - 2 > 0$, then, $Lh > 0$. By Maximum principle applied to L, it implies that the maximum of h can only be attained at a point q where $\frac{1}{W(q)} \geq \min\{\frac{1}{2}, \frac{R}{32y(q)u(p)}\}$. Thus,

$$\left(e^{K(\frac{1}{2}-\varepsilon)} - 1\right)\frac{1}{\upsilon(p)} = h(p) \leq h(q) = \left(e^{K\varphi(q)} - 1\right)\frac{1}{\upsilon(q)}$$
$$\leq \frac{e^K - 1}{\min\{\frac{y(q)}{2}, \frac{R}{32u(p)}\}}.$$

Letting ε tending to 0 we get $\upsilon(p) \geq \min\{\frac{y(q)}{4}, \frac{R}{64u(p)}\}e^{-\frac{K}{2}}$. Hence,

$$\|\nabla u(p)\| \leq \max\left\{\frac{4}{\inf_{\mathbb{D}_R(p)} y}, 64\frac{u(p)}{R}\right\}e^{\frac{1}{2}(10Mu(p)+8M^2C_{\mathbb{D}_R(p)}(\frac{u(p)}{R})^2)}.$$

Combining this with Inequalities $\ln(t) \leq t$ and $2t \leq 1 + t^2$, we obtain

$$\|\nabla u(p)\| \leq e^{C(1+(\frac{u(p)}{R})^2)}, \tag{3.7}$$

where $C = C(R,p) = 32 + \frac{5}{2}MR + \max\{\frac{4}{\inf_{\mathbb{D}_R(p)} y}, 4M^2C_{\mathbb{D}_R(p)}\}$. In the case $u(p) = 0$, Maximum principle (Theorem 3.1) yields $u = 0$ on $\mathbb{D}_R(p)$. The inequality (3.7) is still true. This completes the proof. □

3.3. Existence theorem. In this subsection, we give a result concerning the existence of a solution of the Dirichlet problem for the minimal surface equation. By using interior gradient estimate (Theorem 3.6), elliptic estimate, and Arzelà–Ascoli theorem, we obtain the compactness theorem as follows.

THEOREM 3.10 (Compactness theorem). *Let $\{u_n\}_n$ be a sequence of minimal solutions on a domain $\Omega \subset \mathbb{H}^2$. Suppose that $\{u_n\}_n$ is uniformly bounded* *on compact subsets of* Ω*. Then there exists a subsequence of* $\{u_n\}_n$ *converging on compact subsets of* Ω *to a minimal solution on* Ω*.*

THEOREM 3.11. *Let* $\Omega \subset \mathbb{H}^2$ *be a bounded mean convex Euclidean domain with* $\partial\Omega \in C^2$*. Let* $f \in C^0(\partial\Omega)$ *be a continuous function. Then there exists a unique minimal solution* u *on* Ω *such that* $u = f$ *on* $\partial\Omega$*.*

Proof. The uniqueness is deduced by General maximum principle (Theorem 3.4).

`Existence:` Let α, β be two real numbers such that $\alpha < f(x) < \beta$ for all $x \in \partial\Omega$. Since $\Omega \subset \mathbb{H}^2$ is a bounded mean convex Euclidean domain, by Corollary 2.2, $M^3 := \overline{\Omega} \times [\alpha, \beta]$ is a manifold of dimension 3, compact, and mean convex. Define a Jordan curve $\sigma \subset \partial M^3$ by

$$\sigma = \big\{ \big(x, f(x)\big) : x \in \partial\Omega \big\}.$$

By Geometric Dehn's lemma (see [15, Theorem 1], [2, Theorem 6.28]), the Jordan curve σ is the boundary of a least-area compact disk $\overline{\Sigma}$ in M^3, and $\Sigma := \overline{\Sigma} \setminus \sigma$ is embedded. By the maximum principle, Σ is a subset of $\Omega \times \mathbb{R}$.

Then, it is sufficient to show that Σ is a graph. Since $\overline{\Sigma} \subset \overline{\Omega} \times [\alpha, \beta]$, for h sufficiently large, $\tau_h(\overline{\Sigma}) \cap \overline{\Sigma} = \emptyset$ where τ_h is the translation along the t-axis. So letting h decrease from $+\infty$ to 0, since σ is a graph on $\partial\Omega$, we get by the maximum principle that $\tau_h(\overline{\Sigma})$ and $\overline{\Sigma}$ do not intersect until $h = 0$. This implies that Σ is a minimal graph. □

In order to prove General existence theorem, Theorem 3.14, we shall make use of Theorem 3.11, together with the classical Perron technique [4] (see also [7, Section 2.8]).

A function $u \in C^0(\Omega)$ will be called *subsolution* (resp. *supersolution*) in Ω if for every hyperbolic disk $D \subset\subset \Omega$ and every minimal solution h in D satisfying $u \le h$ (resp. $u \ge h$) on ∂D, we also have $u \le h$ (resp. $u \ge h$) in D. We will have the following properties of $C^0(\Omega)$ subsolution.

REMARK 3.12. (1) A function $u \in C^2(\Omega)$ is a subsolution if and only if $\mathfrak{M}u \ge 0$. Indeed, the sufficient condition follows from Proposition 3.2. To prove the necessary condition, assume the contrary that there exists a subsolution u in Ω satisfying $\mathfrak{M}u < 0$ on some hyperbolic disk $D \subset\subset \Omega$. By Theorem 3.11, there exists a minimal solution h in D such that $h = u$ on ∂D. Then $u \ge h$ on D by Proposition 3.2. Since u is a subsolution, $u \le h$. Thus, $u = h$ in D. This implies that $\mathfrak{M}u = \mathfrak{M}h = 0$ in D, a contradiction.

(2) If u is a subsolution and v is a supersolution in the same bounded domain Ω and $v \ge u$ on $\partial\Omega$, then $v \ge u$ on Ω. To prove this assertion, we suppose the contrary. Then at some point $p_0 \in \Omega$ we have

$$(u - v)(p_0) = \sup_{\Omega}(u - v) = M > 0$$

and we may assume there is a disk $D = \mathbb{D}_r(p_0) \subset\subset \Omega$ such that $u - v \not\equiv M$ on ∂D. Denote by $\overline{u}, \overline{v}$ the minimal solutions respectively equal to u, v on ∂D by Theorem 3.11, one sees that

$$M \geq \sup_{\partial D}(\overline{u} - \overline{v}) \geq (\overline{u} - \overline{v})(p_0) \geq (u - v)(p_0) = M$$

and hence the equality holds throughout. By the maximum principle for minimal solution (Theorem 3.1), it follows that $\overline{u} - \overline{v} \equiv M$ in D and hence $u - v = M$ on ∂D, which contradicts the choice of D.

(3) Let u be subsolution in Ω and D be a hyperbolic disk strictly contained in Ω. Denote by $\overline{u}$ the minimal solution in D satisfying $\overline{u} = u$ on ∂D. We define in Ω the minimal lifting of u (in D) by

$$U(p) = \begin{cases} \overline{u}(p), & p \in D, \\ u(p), & p \in \Omega \setminus D. \end{cases}$$

Then the function U is also subsolution in Ω. Indeed, consider an arbitrary hyperbolic disk $D' \subset\subset \Omega$ and let h be a minimal solution in D' satisfying $h \geq U$ on $\partial D'$. Since $u \leq U$ in D' we have $u \leq h$ in D' and hence $U \leq h$ in $D' \setminus D$. Since U is minimal solution in D, we have by the maximum principle $U \leq h$ in $D \cap D'$. Consequently $U \leq h$ in D' and U is subsolution in Ω.

(4) Let $u_1, u_2, \ldots, u_N$ be subsolution in Ω. Then the function $u(p) = \max\{u_1(p), \ldots, u_N(p)\}$ is also subsolution in Ω. This is a trivial consequence of the definition of subsolution.

Corresponding results for supersolution functions are obtained by replacing u by $-u$ in properties (1), (2), (3) and (4).

Now let Ω be bounded domain and f be a bounded function on $\partial\Omega$. A function $u \in C^0(\overline{\Omega})$ will be called a subfunction (resp. superfunction) relative to f if u is a subsolution (resp. supersolution) in Ω and $u \leq f$ (resp. $u \geq f$) on $\partial\Omega$. By Remark 3.12(2), every subfunction is less than or equal to every superfunction. In particular, constant functions $\leq \inf_\Omega f$ (resp. $\geq \sup_\Omega f$) are subfunctions (resp. superfunctions). Denote by S_f the set of subfunctions relative to f. The basic result of the Perron method is contained in the following proposition.

Proposition 3.13. *The function* $u(p) = \sup_{v \in S_f} v(p)$ *is a minimal solution in* Ω. *Furthermore,* $\inf_{\partial\Omega} f \leq u \leq \sup_{\partial\Omega} f$.

Proof. By Remark 3.12(2), any function $v \in S_f$ satisfies $v \leq \sup_{\partial\Omega} f$. Since the constant function $v = \inf_{\partial\Omega} f$ belongs to S_f, this set is nonempty, so that u is well defined. Let q be an arbitrary fixed point of Ω. By the definition of u, there exists a sequence $\{v_n\}_n \subset S_f$ such that $v_n(q) \to u(q)$. By replacing v_n with $\max\{v_n, \inf_{\partial\Omega} f\}$, we may assume that the sequence $\{v_n\}_n$ is bounded. Now choose R so that the disk $D = \mathbb{D}_R(q) \subset\subset \Omega$ and define V_n

to be the minimal lifting of v_n in D according to Remark 3.12(3). Then $V_n \in S_f, V_n(q) \to u(q)$ and by Compactness theorem (Theorem 3.10) the sequence $\{V_n\}_n$ contains a subsequence $\{V_{n_k}\}_k$ converging uniformly in any disk $\mathbb{D}_\rho(q)$ with $\rho < R$ to a function v that is minimal solution in D. Clearly $v \le u$ in D and $v(q) = u(q)$.

We claim now that in fact $v = u$ in D. For suppose $v(\overline{q}) < u(\overline{q})$ at some $\overline{q} \in D$. Then there exists a function $\overline{u} \in S_f$ such that $v(\overline{q}) < \overline{u}(\overline{q})$. Defining $w_k = \max\{\overline{u}, V_{n_k}\}$ and also the minimal liftings W_k as in Remark 3.12(3), we obtain as before a subsequence of the sequence $\{W_k\}_k$ converging to a minimal solution function w satisfying $v \le w \le u$ in D and $v(q) = w(q) = u(q)$. But then by the maximum principle (Theorem 3.1) we must have $v = w$ in D. This contradicts the definition of $\overline{u}$ and hence u is minimal solution in Ω. □

We will show the solution that we obtained (called the *Perron solution*) will be the solution of the Dirichlet problem as follows.

THEOREM 3.14. *Let Ω be a bounded admissible domain with $\{C_i\}_i$ the open arcs of $\partial\Omega$. Let $f_i \in C^0(C_i)$ be* bounded *functions. Assume C_i are mean convex Euclidean to Ω then there exists a unique minimal solution u on Ω such that $u = f_i$ on C_i for all i.*

Proof. The uniqueness of the solution is deduced from General maximum principle (Theorem 3.4). Let a function f defined on $\partial\Omega$ such that $f(p) = f_i(p)$ if $p \in C_i$. Denote by u the Perron solution relative to $\mathfrak{M}$ and f. We prove that the minimal solution u satisfies the boundary conditions $u = f_i$ on C_i. Fix $\xi \in C_i$, for some i. We must prove that

$$\lim_{p\in\Omega, p\to\xi} u(p) = f(\xi). \tag{3.8}$$

We construct the local barrier at ξ as follows. For $r > 0$ small enough, consider the domain $\Omega \cap \mathbb{D}_r(\xi)$. We approximate $\Omega \cap \mathbb{D}_r(\xi)$ by C^2 mean convex Euclidean domain $\Omega_\xi \subset \Omega \cap \mathbb{D}_r(\xi)$ by rounding each corner point of $\Omega \cap \mathbb{D}_r(\xi)$. By Theorem 3.11, there exist minimal solutions $w_\pm \in C^2(\Omega_\xi) \cap C^0(\overline{\Omega_\xi})$ on Ω_ξ such that $w_\pm(\xi) = f(\xi)$ and

$$\begin{cases} w_- \le f \le w_+ & \text{on } \partial\Omega_\xi \cap \partial\Omega, \\ w_- \le \inf_{\partial\Omega} f \le \sup_{\partial\Omega} f \le w_+ & \text{on } \partial\Omega_\xi \cap \Omega. \end{cases}$$

From the definition of u and the fact that every subfunction is dominated by every superfunction, we have

$$w_- \le u \le w_+, \quad \text{on } \Omega_\xi,$$

we obtain (3.8). □

4. A local Scherk surface in Sol_3 and Flux formula

4.1. A local Scherk surface in Sol_3.

THEOREM 4.1. *Let $\Omega \subset \mathbb{H}^2$ be a convex Euclidean quadrilateral domain whose boundary $\partial\Omega$ is composed of open Euclidean geodesic arcs A_1, C_1, A_2 and C_2 in that order together with their endpoints. Suppose that*

$$\ell_{\mathrm{euc}}(A_1) + \ell_{\mathrm{euc}}(A_2) < \ell_{\mathrm{euc}}(C_1) + \ell_{\mathrm{euc}}(C_2). \tag{4.1}$$

Let f_i be a nonnegative *continuous function on C_i, $i = 1, 2$. Then there exists a minimal solution u in Ω taking $+\infty$ on A_i and f_i on C_i for $i = 1, 2$.*

This result is an important case of Jenkins–Serrin type theorem. The graph of the minimal solution in Theorem 4.1 is said to be a *local Scherk surface* in Sol_3. This construction was motivated by [18, Theorem 2].

Proof of Theorem 4.1. This proof is divided into two cases.

CASE 4.1. *Case $f_1 = 0$ and $f_2 = 0$.*

Proof. Let n be a fixed natural number. By Theorem 3.14, there exists a minimal solution u_n in Ω taking n on A_i and 0 on C_i for $i = 1, 2$. By General maximum principle (Theorem 3.4), $0 \le u_n \le u_{n+1}$. We will prove that the sequence $\{u_n\}_n$ is uniformly bounded on compact subsets K of $\Omega \cup C_1 \cup C_2$. We first construct minimal annulus.

Let $h > 0$ be fixed. Let Γ_i be the curves that are the boundary of $C_i \times [0, h]$ and let Σ_i be a minimal disk with boundary Γ_i. Using the maximum principle, $\Sigma_i = C_i \times [0, h]$. By Proposition 2.3, the area calculated in Sol_3 of Σ_i is

$$\mathcal{A}(\Sigma_i) = \mathcal{A}\big(C_i \times [0, h]\big) = h \cdot \ell_{\mathrm{euc}}(C_i).$$

Consider the annulus $\mathfrak{A}$ with boundary $\Gamma_1 \cup \Gamma_2$ (see Figure 3):

$$\mathfrak{A} = \Omega \cup \tau_h(\Omega) \cup \bigcup_{i=1}^{2} \big(A_i \times [0, h]\big),$$

where τ_h is the translation along the t-axis. By Proposition 2.3 and the fact that the translations along the t-axis are isometries, the area calculated in Sol_3 of $\mathfrak{A}$ is

$$\mathcal{A}(\mathfrak{A}) = 2\mathcal{A}(\Omega) + h\big(\ell_{\mathrm{euc}}(A_1) + \ell_{\mathrm{euc}}(A_2)\big).$$

Therefore,

$$\mathcal{A}(\mathfrak{A}) - \big(\mathcal{A}(\Sigma_1) + \mathcal{A}(\Sigma_2)\big) = 2\mathcal{A}(\Omega) + h\big(\ell_{\mathrm{euc}}(A_1) + \ell_{\mathrm{euc}}(A_2) - \ell_{\mathrm{euc}}(C_1) - \ell_{\mathrm{euc}}(C_2)\big).$$

By the hypothesis (4.1), $\mathcal{A}(\mathfrak{A}) - (\mathcal{A}(\Sigma_1) + \mathcal{A}(\Sigma_2)) < 0$ if $h \ge h_0$ where h_0 is sufficiently large. Hence, $\mathcal{A}(\mathfrak{A})$ is strictly less than the sum of the areas of the disks Σ_i, and by the Douglas criteria [11] (see also [14, Theorem 1]), there exists a least area minimal annulus $\mathfrak{A}(h)$ with boundary $\Gamma_1 \cup \Gamma_2$ for all $h \ge h_0$.

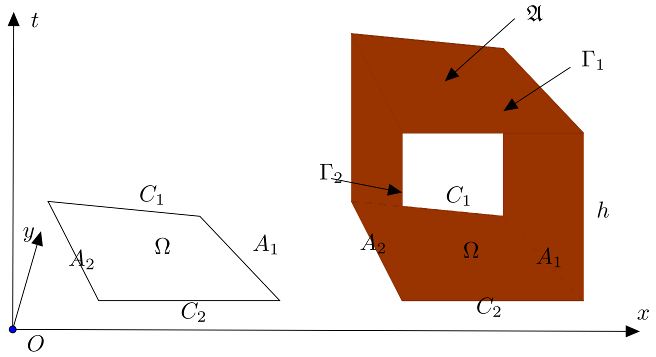

FIGURE 3. Annulus $\mathfrak{A}$.

ASSERTION 4.1. *For all $h \geq h_0$, the annulus $\mathfrak{A}(h)$ is an upper barrier for the Killing graphs of the minimal solution u_n. Moreover, the vertical projections of the annulus $\mathfrak{A}(h), h \geq h_0$ is an exhaustion for $\Omega \cup C_1 \cup C_2$.*

Proof. For the proof, we refer the reader to [18, p. 271, 272] and [19, p. 126, 127]. □

By this assertion, we conclude that the sequence $\{u_n\}_n$ is uniformly bounded on compact subsets of $\Omega \cup C_1 \cup C_2$. By Compactness theorem (Theorem 3.10), the sequence $\{u_n\}_n$ converges on compact subsets of Ω to a minimal solution u on Ω which assumes the above prescribed boundary values on $\partial\Omega$. □

CASE 4.2. *General case.*

Proof. For every natural number n, by applying Theorem 3.14, there exists a minimal solution u_n on Ω with boundary values

$$u_n|_{A_i} = n \quad \text{and} \quad u_n|_{C_i} = \min\{n, f_i\} \quad \text{for } i = 1, 2.$$

By General maximum principle (Theorem 3.4), $u_n \leq u_{n+1}$.

ASSERTION 4.2. *The sequence u_n is uniformly bounded on every compact subset K of $\Omega \cup C_1 \cup C_2$.*

Proof. Denote by K a compact subset of $\Omega \cup C_1 \cup C_2$. Let $\Omega' \subset \Omega$ be a convex Euclidean quadrilateral domain whose boundary $\partial\Omega'$ is composed of open Euclidean geodesic arcs A'_1, C'_1, A'_2 and C'_2 in that order together with their endpoints, moreover, C'_i is a relatively compact subset of $C_i, i = 1, 2$. By Condition (4.1) and the compactness of the set K, we can choose Ω' large enough such that $K \subset \Omega'$ and $\ell_{\text{euc}}(A'_1) + \ell_{\text{euc}}(A'_2) < \ell_{\text{euc}}(C'_1) + \ell_{\text{euc}}(C'_2)$ (see Figure 4). There is, by the previous case, a minimal solution w on Ω' which obtain the values $+\infty$ on A'_i and 0 on $C'_i, i = 1, 2$.

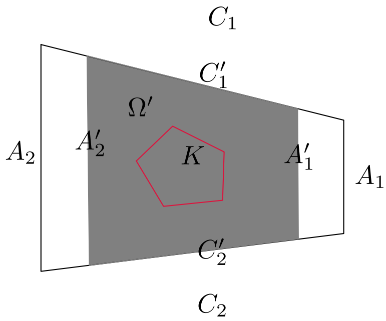

FIGURE 4. The quadrilateral domain $\Omega' \subset \Omega$.

Since C'_i is a relatively compact subset of C_i and f_i is a continuous function on C_i, f_i is bounded on C'_i for $i = 1, 2$. By General maximum principle (Theorem 3.4), we have $0 \le u_n \le w + \max\{\sup_{C'_1} f_1, \sup_{C'_2} f_2\}$ on $\Omega' \cup C'_1 \cup C'_2$. Since K is a compact subset of $\Omega' \cup C'_1 \cup C'_2$, $\{u_n\}_n$ is uniformly bounded on K. □

It follows from Assertion 4.2 and the compactness theorem (Theorem 3.10) that, the sequence $\{u_n\}_n$ converges on each compact subset of $\Omega \cup C_1 \cup C_2$ to a minimal solution u on Ω. Moreover, we have $u|_{C_i} = \lim_{n\to\infty} u_n|_{C_i} = f_i$ and $u|_{A_i} = \lim_{n\to\infty} u_n|_{A_i} = +\infty$. This completes the proof. □

□

PROPOSITION 4.2. *Let $\Omega \subset \mathbb{H}^2$ be a bounded convex Euclidean domain whose boundary $\partial\Omega$ is composed of an open Euclidean geodesic arc A and an open mean convex Euclidean arc C with their endpoints. Let f be a bounded continuous function on C. Then, there exists a minimal solution u in Ω taking $+\infty$ on A and f on C.*

Proof. For every natural number n, by applying Theorem 3.14, there is a minimal solution u_n on Ω taking n on A and f on C.

ASSERTION 4.3. *There exists an Euclidean triangular domain $T' \subset \mathbb{H}^2$ whose boundary $\partial T'$ is composed of open Euclidean geodesic arcs A', B', C' together with their endpoints, moreover $A \subset A'$ and $\overline{\Omega} \setminus A' \subset T'$.*

Proof. Let d be the intersection of $\mathbb{H}^2$ with the Euclidean line in $\mathbb{R}^2$ containing A. Since the domain Ω is bounded, there exist two real numbers x_1, y_1 with $y_1 > 0$ such that $\Omega \subset \{(x, y) \in \mathbb{H}^2 : x > x_1, y > y_1\}$. Let d' be the line $\{(x, y) \in \mathbb{H}^2 : x = x_1 - 1\}$ if d is of the form $\{(x, y) \in \mathbb{H}^2 : y = y_0\}$ and the line $\{(x, y) \in \mathbb{H}^2 : y = \frac{y_1}{2}\}$ otherwise. Let p be the point of intersection of d and d'. For $q \in d \setminus \{p\}$ and $q' \in d' \setminus \{p\}$, denote by $\triangle(p, q, q')$ the Euclidean

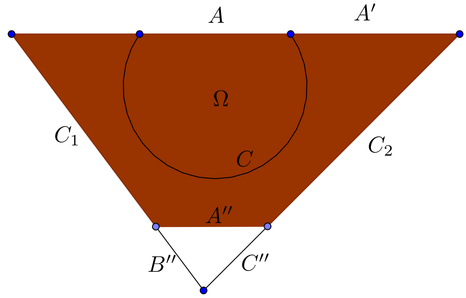

FIGURE 5. The Euclidean quadrilateral domain $T' \setminus \overline{T''}$.

triangular domain in $\mathbb{H}^2$ with vertices p, q and q'. Since one of the lines d or d' is of the form $\{(x,y) \in \mathbb{H}^2 : y = y_0\}$ for some $y_0 > 0$, we see that

$$\bigcup_{q,q'} \triangle\big(p,q,q'\big) = \mathbb{H}^2 \setminus \big(d \cup d'\big). \tag{4.2}$$

Since the domain Ω is convex Euclidean, by the definitions of d and d', the domain Ω is contained in a component of $\mathbb{H}^2 \setminus (d \cup d')$. It follows from the boundedness of Ω and Formula (4.2) that there exists an Euclidean triangle T' of a form $\triangle(p,q,q')$ satisfying the assertion. □

We define δ to be $\ell_{\text{euc}}(B') + \ell_{\text{euc}}(C') - \ell_{\text{euc}}(A'), \delta > 0$. Taking an Euclidean triangular subdomain T'' of T' with sides A'', B'', C'' where $B'' \subset B', C'' \subset C'$ such that $\ell_{\text{euc}}(A'') + \ell_{\text{euc}}(B'') + \ell_{\text{euc}}(C'') < \delta$ and $\overline{\Omega} \cap \overline{T''} = \emptyset$ (see Figure 5). Hence, $T' \setminus \overline{T''}$ is a convex Euclidean quadrilateral domain whose boundary is composed of four open Euclidean geodesic arcs A_1, C_1, A_2, C_2 in that order with their endpoints, where $A_1 = A', A_2 = A'', C_1 \subset B'$ and $C_2 \subset C'$. By definition, we have $\Omega \cup C \subset T' \setminus \overline{T''}$ and $\ell_{\text{euc}}(A_1) + \ell_{\text{euc}}(A_2) < \ell_{\text{euc}}(C_1) + \ell_{\text{euc}}(C_2)$.

It follows from Theorem 4.1 that there exists a minimal solution w defined on the Euclidean quadrilateral domain $T' \setminus \overline{T''}$ taking the value $+\infty$ on A_i and 0 on C_i, $i = 1, 2$. By General maximum principle (Theorem 3.4), we have

$$\min\Big\{0, \inf_C f\Big\} \le u_n \le u_{n+1} \le w + \sup_C f \quad \text{on } \Omega \cup C.$$

It follows from Compactness theorem (Theorem 3.10) that the sequence $\{u_n\}_n$ converges on every compact subset of $\Omega \cup C$ to a minimal solution u on Ω. Moreover, we have $u|_C = f$ and $u|_A = \lim_{n\to\infty} u_n|_A = +\infty$. This completes the proof. □

LEMMA 4.3. *Let $\Omega \subset \mathbb{H}^2$ be a bounded convex Euclidean domain whose boundary $\partial\Omega$ is composed of an open Euclidean geodesic arc A and an open mean convex Euclidean arc C with their endpoints. Let K be a compact subset* *of* $\Omega \cup C$. *There exists a real number* M *such that if* u *is a minimal solution on* Ω *and*

(1) *if* $\liminf u \geq c$ *for any approach to* C *within* Ω *and if* $\liminf u > -\infty$ *for any approach to* A *within* Ω *then* $u \geq c - M$ *on* K;
(2) *if* $\limsup u \leq c$ *for any approach to* C *within* Ω *and if* $\limsup u < +\infty$ *for any approach to* A *within* Ω *then* $u \leq c + M$ *on* K.

Proof. Suppose that $\liminf u \geq c$ for any approach to C within Ω and $\liminf u > -\infty$ for any approach to A within Ω (otherwise let $u := -u$). It follows from Proposition 4.2 that there exists a minimal solution w on Ω such that $w|_A = +\infty$ and $w|_C = 0$. Define $M = \sup_K w \in \mathbb{R}$, by the general maximum principle (Theorem 3.4), we have $u \geq c - w$ on Ω. From this, we conclude that $u \geq c - M$ on K. This completes the proof. □

COROLLARY 4.4 (Straight line lemma). *Let* $\Omega \subset \mathbb{H}^2$ *be a domain, let* $C \subset \partial\Omega$ *be an open mean convex Euclidean arc (convex towards* Ω*) and* u *be a minimal solution in* Ω. *If* u *diverges to* $+\infty$ *or* $-\infty$ *as one approaches* C *within* Ω, *then* C *is an Euclidean geodesic arc.*

Proof. Assume the contrary, that there exists a minimal solution u over Ω that takes the value $+\infty$ on C where C is not an Euclidean geodesic arc. Let $\Gamma(C)$ be the open Euclidean geodesic arc of $\mathbb{H}^2$ joining the endpoints of C. Denote by $\Omega(C)$ the domain delimited by $C \cup \Gamma(C)$. After shrinking C if necessary, we may assume that $\Omega(C) \cup \Gamma(C) \subset \Omega$ (see Figure 6).

Let q be a point in Ω. It follows from the lemma 4.3 that there exists a real number M depending only on q such that $u(q) \geq c - M$ for all real number c, a contradiction. □

THEOREM 4.5 (Boundary values lemma, [3, p. 1882]). *Let* $\Omega \subset \mathbb{H}^2$ *be a domain and let* C *be an open mean convex Euclidean arc in* $\partial\Omega$. *Suppose* $\{u_n\}_n$ *is a sequence of minimal solutions in* Ω *that converges uniformly on every*

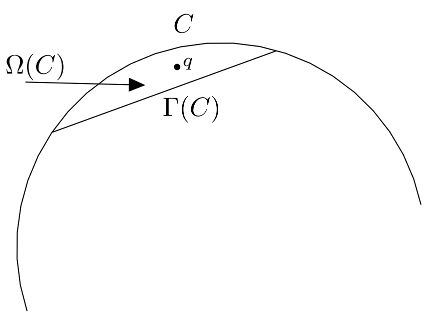

FIGURE 6. The domain $\Omega(C)$.

compact subset of Ω to a minimal solution u. Suppose each u_n is continuous on $\Omega \cup C$.

(1) *If $\{u_n|_C\}_n$ converges uniformly on every compact subset of C to a continuous function f on C then u is continuous on $\Omega \cup C$ and $u|_C = f$.*
(2) *If $\{u_n|_C\}_n$ diverges uniformly on every compact subset of C to $+\infty$ (resp. $-\infty$), then u diverges to $+\infty$ (resp. $-\infty$) when we approach C within Ω.*

Proof. For $p \in C$, define $f(p) = \lim_{n\to\infty} u_n(p)$. It is sufficient to show that, for $p \in C$ and $M \in \mathbb{R}$ such that $f(p) > M$, there exists a neighborhood U of p in $\Omega \cup C$ that satisfies $u > M$ on $U \cap \Omega$.

Let M' such that $M < M' < f(p)$. Since f is continuous (or $f \equiv +\infty$) and $u_n|_C$ converges uniformly on every compact subset of C to f, there is a neighborhood C' of p in C and $N_0 \in \mathbb{N}$ such that $u_n(x) > M'$ for every $x \in C'$ and for every $n \geq N_0$. Consider two cases as follows.

(i) *If C' is not an Euclidean geodesic arc.* Denote by $\Gamma(C')$ the open Euclidean geodesic arc of $\mathbb{H}^2$ joining the endpoints of C' and by Ω' the domain of $\mathbb{H}^2$ delimited by $C' \cup \Gamma(C')$ (see Figure 7). After shrinking C' if necessary, we may assume that $\Omega' \cup \Gamma(C')$ is contained in Ω.

By Proposition 4.2, there exists a minimal solution w on Ω' such that $w|_{C'} = M'$ and $w|_{\Gamma(C')} = -\infty$. It follows from the general maximum principle (Theorem 3.4), that $u_n \geq w$ on Ω' for every $n \geq N_0$. Hence, we have $u \geq w$ on Ω'. Since w is continuous on $\Omega' \cup C'$ and $w(p) = M' > M$, there is a neighborhood U of p in $\Omega' \cup C'$ such that $w > M$ on U. Therefore, $u > M$ on $U \cap \Omega$.

(ii) *If C' is an Euclidean geodesic arc.* Consider a convex Euclidean quadrilateral domain $\mathcal{P} \subset \Omega$ such that $\partial\mathcal{P}$ is composed of four open Euclidean geodesic arcs B_1, C_1, B_2, C_2 in that order with their endpoints, where $p \in C_1 \subset C'$, $\partial\mathcal{P} \setminus \overline{C'} \subset \Omega$ and $\ell_{\mathrm{euc}}(B_1) + \ell_{\mathrm{euc}}(B_2) < \ell_{\mathrm{euc}}(C_1) + \ell_{\mathrm{euc}}(C_2)$ (see Figure 8).

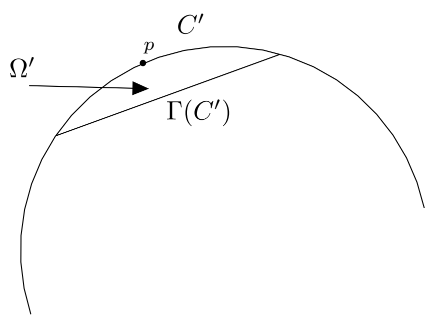

FIGURE 7. The domain Ω'.

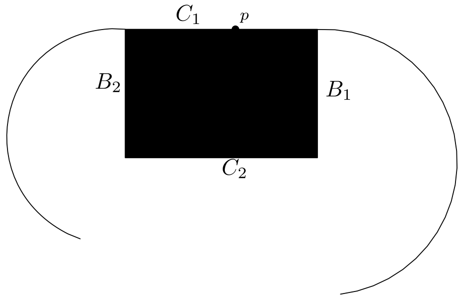

FIGURE 8. The domain Ω' when C' is Euclidean geodesic.

Since u_n converges uniformly on each compact subset of Ω to u, $M'' := \inf_{x\in C_2, n\in\mathbb{N}} u_n(x) > -\infty$. By Theorem 4.1, there is a minimal solution w on $\mathcal{P}$ such that $w|_{C_1} = M', w|_{C_2} = M''$ and $w = -\infty$ on $B_1 \cup B_2$. It follows from the general maximum principle (Theorem 3.4), that $u_n \geq w$ on $\mathcal{P}$ for every $n \geq N_0$. Hence, we have $u \geq w$ on $\mathcal{P}$. Since w is continuous on $\mathcal{P} \cup C_1$ and $w(p) = M' > M$, there exists a neighborhood U of p in $\mathcal{P} \cup C_1$ such that $w > M$ on U. Then $u > M$ on $U \cap \Omega$. This completes the proof. □

4.2. Flux formula. Fix a minimal solution u on a domain $\Omega \subset \mathbb{H}^2$. By definition, we have that $\operatorname{div}(yX_u) = 0$ where $X_u = \frac{y\nabla u}{\sqrt{1+y^2\|\nabla u\|^2}}$ is a vector field on Ω, $\|X_u\| < 1$.

Let γ be an arc in $\overline{\Omega} \cap \mathbb{H}^2$ such that its Euclidean length $\ell_{\text{euc}}(\gamma)$ is finite. Denote by ν a unit normal to γ in $\mathbb{H}^2$. Then, we define the flux $F_u(\gamma)$ of u across γ by

$$F_u(\gamma) = \int_\gamma \langle yX_u, \nu\rangle \,\mathsf{d}s,$$

if $\gamma \subset \Omega$, if not, we define $F_u(\gamma) = F_u(\Gamma)$, where Γ is an arc in Ω joining the end-points of γ such that $\ell_{\text{euc}}(\Gamma) < \infty$ and the domain in $\mathbb{H}^2$ delimited by γ and Γ is simply connected. Clearly, $F_u(\gamma)$ changes sign if we choose $-\nu$ in place of ν. In the case $\gamma \subset \partial\Omega$, ν will always be chosen to be the outer normal to $\partial\Omega$.

The following result gives geometric interpretation of flux. Let $\gamma : (0,1) \to \Omega$ be an arc in Ω. Denote by ν a unit normal to γ. Denote by $\widehat{\gamma}$ the arc on the graph $\operatorname{Gr}(u)$ defined by $(0,1) \to \mathrm{Sol}_3, \widehat{\gamma}(t) = (\gamma(t), u(\gamma(t)))$. Let $\widehat{\nu}$ be the unit normal to $\widehat{\gamma}$ in $\operatorname{Gr}(u)$ such that the frame $(N, \widehat{\gamma}', \widehat{\nu})$ is positively oriented at a point $\widehat{\gamma}(t)$ if and only if the frame $(\partial_t, \gamma', \nu)$ is positively oriented at $\gamma(t)$.

PROPOSITION 4.6. *Let u be a minimal solution in a domain $\Omega \subset \mathbb{H}^2$ and let $\gamma : (0,1) \to \Omega$ be an arc in Ω. Then, we have*

$$F_u(\gamma) = \int_{\widehat{\gamma}} \langle \partial_t, \widehat{\nu}\rangle \,\mathsf{d}s.$$

Proof. Without loss of generality, we assume that the frames $(\partial_t, \gamma', \nu)$ and $(N, \widehat{\gamma}', \widehat{\nu})$ are positively oriented. Then $N \times \widehat{\gamma}' = \|\widehat{\gamma}'\|\widehat{\nu}$ and $\partial_t \times \gamma' = y\|\gamma'\|\nu$. Since $\widehat{\gamma}(t) = (\gamma(t), u(\gamma(t)))$, we have $\widehat{\gamma}' = \gamma' + (u \circ \gamma)'\partial_t$. It follows that

$$\|\widehat{\gamma}'\|\widehat{\nu} = N \times \widehat{\gamma}' = \left(-X_u + \frac{1}{yW}\partial_t\right) \times \big(\gamma' + (u \circ \gamma)'\partial_t\big).$$

From this, we deduce that

$$\|\widehat{\gamma}'\|\langle \partial_t, \widehat{\nu}\rangle = \big\langle \partial_t, -X_u \times \gamma'\big\rangle = \big\langle X_u, -\gamma' \times \partial_t\big\rangle = \|\gamma'\|y\langle X_u, \nu\rangle.$$

Integrating this on $(0,1)$, we see that

$$\int_{\widehat{\gamma}} \langle \partial_t, \widehat{\nu}\rangle \, \mathrm{d}s = \int_0^1 \|\widehat{\gamma}'\|\langle \partial_t, \widehat{\nu}\rangle \, \mathrm{d}t = \int_0^1 \|\gamma'\|y\langle X_u, \nu\rangle \, \mathrm{d}t = \int_\gamma \langle yX_u, \nu\rangle \, \mathrm{d}s,$$

which proves the proposition. □

PROPOSITION 4.7 (Flux theorem). *Let u be a minimal solution on a domain $\Omega \subset \mathbb{H}^2$.*

(1) *For every curve γ in $\overline{\Omega} \cap \mathbb{H}^2$ that $\ell_{\mathrm{euc}}(\gamma) < \infty$ we have $|F_u(\gamma)| \le \ell_{\mathrm{euc}}(\gamma)$.*
(2) *For every admissible domain Ω' of Ω such that $\ell_{\mathrm{euc}}(\partial\Omega') < \infty$, we have $F_u(\partial\Omega') = 0$.*
(3) *Let γ be an open arc in Ω or an open mean convex Euclidean arc in $\partial\Omega$ on which u is continuous, obtains the finite value and $\ell_{\mathrm{euc}}(\gamma) < \infty$. Then $|F_u(\gamma)| < \ell_{\mathrm{euc}}(\gamma)$.*
(4) *Let $\gamma \subset \partial\Omega$ be an open Euclidean geodesic arc ($\ell_{\mathrm{euc}}(\gamma) < \infty$) such that u diverges to $+\infty$ (resp. $-\infty$) as one approaches γ within Ω, then $F_u(\gamma) = \ell_{\mathrm{euc}}(\gamma)$ (resp. $F_u(\gamma) = -\ell_{\mathrm{euc}}(\gamma)$).*

Proof. (1) - Case $\gamma \subset \Omega$. Since $\|X_u\| < 1$ we have

$$\big|F_u(\gamma)\big| \le \int_\gamma \big|\langle yX_u, \nu\rangle\big| \, \mathrm{d}s \le \int_\gamma y \, \mathrm{d}s = \ell_{\mathrm{euc}}(\gamma).$$

- Case $\gamma \not\subset \Omega$. By definition, for every $\varepsilon > 0$, there is an arc $\Gamma \subset \Omega$ joining the endpoints of γ such that $\ell_{\mathrm{euc}}(\Gamma) \le \ell_{\mathrm{euc}}(\gamma) + \varepsilon$ and $F_u(\gamma) = F_u(\Gamma)$. Moreover, the previous case yields $|F_u(\Gamma)| \le \ell_{\mathrm{euc}}(\Gamma)$. Then $|F_u(\gamma)| \le \ell_{\mathrm{euc}}(\gamma) + \varepsilon$. This proved the result.

(2) - Case Ω' is bounded. By divergence theorem, we have

$$F_u\big(\partial\Omega'\big) = \int_{\partial\Omega'} \langle yX_u, \nu\rangle \, \mathrm{d}s = \int_{\Omega'} \operatorname{div}(yX_u) \, \mathrm{d}\mathcal{A} = 0.$$

- Case Ω' is unbounded. Denote by E the set of ideal vertices of Ω'. For each $p \in E$, we take a net of the geodesics $H_{p,n}$ that converges to p (see Figure 9). Let us denote by $\mathcal{H}_{p,n}$ the component of $\mathbb{H}^2 \setminus H_{p,n}$ containing p on its ideal

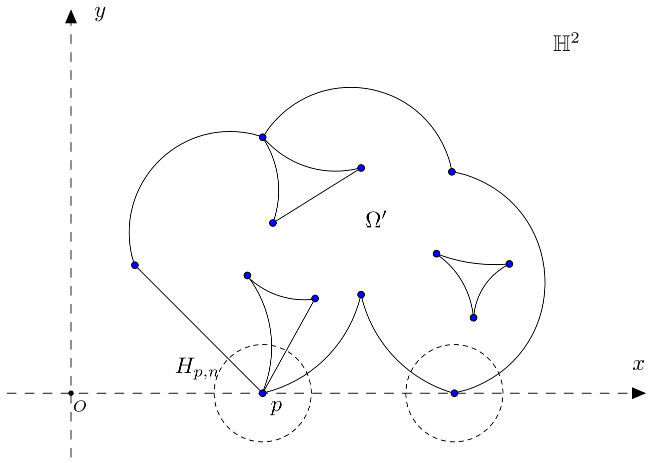

FIGURE 9. The domain Ω' and $H_{p,n}$.

boundary. Assume $\overline{\mathcal{H}}_{p_1,1} \cap \overline{\mathcal{H}}_{p_2,1} = \emptyset$ for every different ideal vertices p_1, p_2 of Ω'. We define

$$\Omega'_n = \Omega' \Big\backslash \Big(\bigcup_{p \in E} \overline{\mathcal{H}}_{p,n} \Big).$$

These subdomains of Ω' are bounded. It follows from the previous case that $F_u(\partial\Omega'_n) = 0$. Thus, we have

$$F_u\big(\partial\Omega'\big) = F_u\big(\partial\Omega'\big) - F_u\big(\partial\Omega'_n\big) = \sum_{p\in E} F_u\big(\partial\Omega' \cap \mathcal{H}_{p,n}\big) - F_u\big(\partial\Omega'_n \setminus \partial\Omega'\big).$$

Since $\ell_{\text{euc}}(\partial\Omega') < \infty$, by (1) we have

$$\sum_{p\in E} \big|F_u\big(\partial\Omega' \cap \mathcal{H}_{p,n}\big)\big| \le \sum_{p\in E} \ell_{\text{euc}}\big(\partial\Omega' \cap \mathcal{H}_{p,n}\big) \to 0 \quad \text{as } n \to \infty.$$

Moreover, applying (1) again yields

$$\big|F_u\big(\partial\Omega'_n \setminus \partial\Omega'\big)\big| \le \ell_{\text{euc}}\big(\partial\Omega'_n \setminus \partial\Omega'\big) \le \sum_{p\in E} \ell_{\text{euc}}(H_{p,n}) \to 0 \quad \text{as } n \to \infty.$$

This completes the proof.

(3) - Case $\gamma \subset \Omega$. Since $\|X_u\| < 1$ we have $|\langle yX_u, \nu\rangle| < y$, then

$$\big|F_u(\gamma)\big| \le \int_\gamma \big|\langle yX_u, \nu\rangle\big| \, \mathsf{d}s < \int_\gamma y \, \mathsf{d}s = \ell_{\text{euc}}(\gamma).$$

- Case $\gamma \subset \partial\Omega$. It is sufficient to show that $|F_u(\gamma)| < \ell_{\text{euc}}(\gamma)$ for a small arc γ. Fix $p \in \gamma$. Let $\varepsilon > 0$ such that $\Omega_\varepsilon(p) := \Omega \cap \mathbb{D}_\varepsilon(p)$ is a domain whose boundary

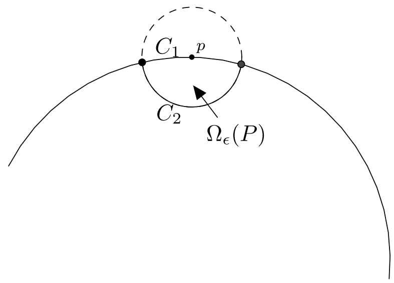

FIGURE 10. The domain $\Omega_\varepsilon(p)$.

is composed of two open arcs C_1, C_2 and their endpoints, moreover $C_1 \subset \gamma$ and $C_2 \subset \Omega \cap \partial \mathbb{D}_\varepsilon(p)$ (see Figure 10).

By the general existence theorem (Theorem 3.14), for $\delta \in \{-1, 1\}$, there is a minimal solution v_δ on $\Omega_\varepsilon(p)$ with $v_\delta = u + \delta$ on C_1 and $v_\delta = u$ on C_2. It follows from the Lemma 3.5, that

$$\int_{\Omega_\varepsilon(p)} \langle \nabla v_\delta - \nabla u, yX_{v_\delta} - yX_u \rangle \, \mathsf{d}\mathcal{A} > 0.$$

Since u, v_δ are the minimal solutions

$$\langle \nabla v_\delta - \nabla u, yX_{v_\delta} - yX_u \rangle = \operatorname{div}\big((v_\delta - u)(yX_{v_\delta} - yX_u)\big).$$

By the divergence theorem and the fact that $v_\delta - u$ takes the value δ on C_1 and 0 on C_2, we have

$$0 < \int_{\partial\Omega_\varepsilon(p)} \big\langle (v_\delta - u)(yX_{v_\delta} - yX_u), \nu \big\rangle \, \mathsf{d}s = \delta\big(F_{v_\delta}(C_1) - F_u(C_1)\big).$$

Combining these inequalities and Assertion (1), we obtain

$$\begin{cases} F_u(C_1) < F_{v_1}(C_1) \le \ell_{\text{euc}}(C_1), \\ F_u(C_1) > F_{v_{-1}}(C_1) \ge -\ell_{\text{euc}}(C_1), \end{cases}$$

which completes the proof.

(4) We show for the case u diverges to $+\infty$ as one approaches γ within Ω. For each $q \in \Omega$, denote by $N(q)$ the unit upward pointing normal vector to the graph of $u - u(q)$ at the point $(q, 0)$. We first prove that

$$\lim_{q \in \Omega, q \to p} N(q) = -\nu(p), \quad \forall p \in \gamma. \tag{4.3}$$

Assume the contrary that there exists a sequence $q_n \in \Omega, q_n \to p$ such that $\lim_{n\to\infty} N(q_n) = v \neq -\nu(p)$. Let Σ be the Killing graph of u. Define $Q_n = (q_n, u(q_n))$. Since $u|_\gamma = +\infty$, $\mathrm{d}_\Sigma(Q_n, \partial\Sigma) \ge \mathrm{d}_{\mathbb{H}^2}(q_n, \partial\Omega \setminus \gamma)$. Moreover, $\lim_{n\to\infty} q_n = p$, there exists $R > 0$ such that $\mathrm{d}_\Sigma(Q_n, \partial\Sigma) > R$ for n large

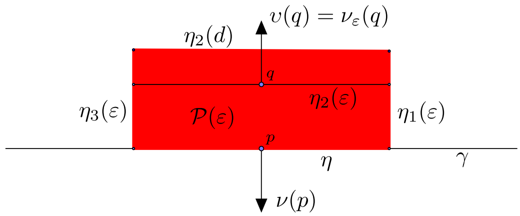

FIGURE 11. The domain $\mathcal{P}(\varepsilon)$.

enough. Since Σ is stable, we deduce from Schoen's curvature estimate [21] (see also [2, Theorem 2.10]) that

$$\sup_{q\in\mathbb{D}^{\Sigma}_{R/2}(Q_n)} \big\|A(q)\big\| \le \kappa,$$

where A is the second fundamental form of Σ and κ is an absolute constant.

Hence, by [2, Lemma 2.4], around each Q_n the surface Σ is a graph over a disk $\mathbb{D}_r(Q_n)$ of the tangent plane at Q_n of Σ and the graph has bounded distance from the disk $\mathbb{D}_r(Q_n)$. The radius of the disk depends only on R, hence it is independent of n. So, if q_n is close enough to γ, then the horizontal projection of $\mathbb{D}_r(Q_n)$ and thus of the surface Σ is not contained in Ω, contradiction. Thus, (4.3) is proved.

Let η be a compact subarc of γ. Define $d := \frac{1}{2}\mathrm{d}_{\mathrm{euc}}(\eta, \partial\Omega\setminus\gamma)$. For each $0<\varepsilon\le d$, let $\mathcal{P}(\varepsilon)\subset\Omega$ be the rectangular domain with sides $\eta, \eta_1(\varepsilon), \eta_2(\varepsilon)$ and $\eta_3(\varepsilon)$ in that order such that $\ell_{\mathrm{euc}}(\eta_1(\varepsilon)) = \ell_{\mathrm{euc}}(\eta_3(\varepsilon)) = \varepsilon$ (see Figure 11). Denote by ν_ε the unit outer normal to $\partial\mathcal{P}(\varepsilon)$. By definition, $\nu_\varepsilon(p) = \nu(p)$ for $p\in\eta$. For each $q\in\mathcal{P}(d)$, define $\upsilon(q) = \nu_\varepsilon(q)$ where ε is the unique real number satisfying $q\in\eta_2(\varepsilon)$. For each $p\in\eta$, we have $\lim_{q\in\Omega, q\to p}\upsilon(q) = -\nu(p)$. Combining with (4.3), we obtain

$$\lim_{q\in\Omega, q\to p}\big\langle X_u(q), \upsilon(q)\big\rangle = -1, \quad \forall p\in\eta.$$

We deduce that

$$F_u\big(\eta_2(\varepsilon)\big) = \int_{\eta_2(\varepsilon)} \langle yX_u, \nu_\varepsilon\rangle\,\mathsf{d}s \xrightarrow{\varepsilon\to 0} -\int_\eta y\,\mathsf{d}s = -\ell_{\mathrm{euc}}(\eta). \tag{4.4}$$

Now applying Assertions (1) and (2) for $\partial\mathcal{P}(\varepsilon)$, we have

$$0 = F_u\big(\partial\mathcal{P}(\varepsilon)\big) = F_u(\eta) + \sum_{i=1}^{3} F_u\big(\eta_i(\varepsilon)\big),$$
$$F_u\big(\eta_i(\varepsilon)\big) \le \ell_{\mathrm{euc}}\big(\eta_i(\varepsilon)\big) = \varepsilon, \quad \forall i\in\{1,3\}.$$

Combining with (4.4) and Assertion (1), we have $F_u(\eta) = \ell_{\mathrm{euc}}(\eta)$. It follows that $F_u(\gamma) = \ell_{\mathrm{euc}}(\gamma)$. □

PROPOSITION 4.8. *Let $\{u_n\}_n$ be a sequence of minimal solutions on a fixed domain $\Omega \subset \mathbb{H}^2$ which extends continuously to $\partial\Omega$ and let A be an Euclidean geodesic arc in $\partial\Omega$ such that $\ell_{\mathrm{euc}}(A) < \infty$. Then*

(1) *If $\{u_n\}_n$ diverges uniformly to $+\infty$ on compact sets of A and while remaining uniformly bounded on compact sets of Ω, then*

$$\lim_{n\to\infty} F_{u_n}(A) = \ell_{\mathrm{euc}}(A).$$

(2) *If $\{u_n\}_n$ diverges uniformly to $+\infty$ on compact sets of Ω while remaining uniformly bounded on compact sets of and A, then*

$$\lim_{n\to\infty} F_{u_n}(A) = -\ell_{\mathrm{euc}}(A).$$

5. Monotone convergence theorem and Divergence set theorem

In this section, we will state Monotone convergence theorem and Divergence set theorem for minimal solutions. The results are adapted from [10], [18].

5.1. Monotone convergence theorem. This subsection will be devoted to the proof of Monotone convergence theorem (Theorem 5.2). Interior gradient estimate (Theorem 3.6) implies a version of the Harnack inequality for minimal solutions, which is crucial for this proof (see [9, Theorem 3] for a similar result for minimal solutions in $\mathbb{R}^3$).

THEOREM 5.1 (Local Harnack inequality). *Let $\mathbb{D}_R(p)$ be a disk in $\mathbb{H}^2$. There exists a continuous function $r : [0,\infty) \to (0,\infty)$ and a function $\Phi(t,s)$ defined on $t \in [0,\infty), s \in [0, r(t))$ such that*

$$u(q) \le \Phi\big(u(p), \mathrm{d}_{\mathbb{H}^2}(p,q)\big) \tag{5.1}$$

for every nonnegative minimal solution u on $\mathbb{D}_R(p)$ and every point $q \in \mathbb{D}_R(p)$ satisfying $\mathrm{d}_{\mathbb{H}^2}(p,q) < r(u(p))$. Moreover,

(1) *the function r is a strictly decreasing function tending to zero as t tends to infinity;*
(2) *for each fixed t, $\Phi(t,-)$ is a continuous strictly increasing function with $\Phi(t,0) = t$ and $\lim_{s\to r(t)^-} \Phi(t,s) = \infty$;*
(3) *for $t_1, t_2 \in [0,\infty), t_1 < t_2$ and $s \in [0, r(t_2))$, we have $\Phi(t_1,s) < \Phi(t_2,s)$.*

Proof. Let u be a nonnegative minimal solution on $\mathbb{D}_R(p)$ and let q be a point of $\mathbb{D}_R(p)$. Denote by $\gamma : [0,R) \to \mathbb{D}_R(p)$ the unique unit speed hyperbolic geodesic passing through q with initial point p. Let $f(t)$ be the function $e^{C(1+t^2)}$ as in Interior gradient estimate (Theorem 3.6), where $C = C(R,p)$. For $s \in [0,R)$, since u is defined on the disk $\mathbb{D}_{R-s}(\gamma(s))$, we have by Interior gradient estimate (Theorem 3.6):

$$(u\circ\gamma)'(s) \le \big\|\nabla u\big(\gamma(s)\big)\big\| \le f\left(\frac{u(\gamma(s))}{R-s}\right).$$

For each $t \geq 0$, we define a function $s \mapsto \Phi(t,s)$ by the conditions

$$\frac{\mathsf{d}\Phi}{\mathsf{d}s}(t,s) = f\left(\frac{\Phi(t,s)}{R-s}\right), \qquad \Phi(t,0) = t. \tag{5.2}$$

Define the function $\upsilon(t,s) = \frac{\Phi(t,s)}{R-s}$. The conditions (5.2) yield

$$\frac{\frac{\mathsf{d}\upsilon}{\mathsf{d}s}(t,s)}{f(\upsilon(t,s)) + \upsilon(t,s)} = \frac{1}{R-s}, \qquad \upsilon(t,0) = \frac{t}{R}.$$

These conditions give

$$\int_{\frac{t}{R}}^{\upsilon(t,s)} \frac{\mathsf{d}\bar{s}}{f(\bar{s}) + \bar{s}} = \log\frac{R}{R-s}. \tag{5.3}$$

By a similar argument, we have

$$\int_{\frac{u(p)}{R}}^{\frac{u(\gamma(s))}{R-s}} \frac{\mathsf{d}\bar{s}}{f(\bar{s}) + \bar{s}} \leq \log\frac{R}{R-s}. \tag{5.4}$$

It follows from (5.3) and (5.4) that $u(\gamma(s)) \leq \Phi(u(p),s)$ whenever $\Phi(u(p),-)$ is well defined on $[0,s]$. This proves (5.1).

Since the right-hand side of (5.3) is a strictly increasing function on $s \in [0,R)$ and tending to $+\infty$ as $s \to R$ and the integral $\int_{\frac{t}{R}}^{\infty} \frac{\mathsf{d}\bar{s}}{f(\bar{s})+\bar{s}}$ is convergent, the functions $\upsilon(t,-), \Phi(t,-)$ are defined on $[0, r(t))$ where

$$r(t) = R - \exp\left(\log(R) - \int_{\frac{t}{R}}^{\infty} \frac{\mathsf{d}\bar{s}}{f(\bar{s}) + \bar{s}}\right) \tag{5.5}$$

and $\lim_{s\to r(t)^-} \upsilon(t,s) = \infty$. Since $r(t) < R$, we have $\lim_{s\to r(t)^-} \Phi(t,s) = \lim_{s\to r(t)^-} (R-s)\upsilon(t,s) = \infty$. From this and (5.3), we obtain (2). Assertion (1) follows from (5.5). And finally, (5.3) gives $\upsilon(t_1,s) < \upsilon(t_2,s)$ if $t_1 < t_2$ and $0 \leq s < r(t_2)$, which proves Assertion (3). □

THEOREM 5.2 (Monotone convergence theorem). *Let $\{u_n\}_n$ be a monotone sequence of minimal solutions on a domain $\Omega \subset \mathbb{H}^2$. We define the subsets $\mathcal{U} = \mathcal{U}(\{u_n\}_n)$ and $\mathcal{V} = \mathcal{V}(\{u_n\}_n)$ of Ω by the formulas*

$$\mathcal{U} = \left\{p \in \Omega : \sup_{n\in\mathbb{N}} \left|u_n(p)\right| < \infty\right\}, \qquad \mathcal{V} = \Omega \setminus \mathcal{U}.$$

Then, $\mathcal{U}$ is an open set. Moreover, $\{u_n\}_n$ converges uniformly to a minimal solution on compact subsets of $\mathcal{U}$ and diverges uniformly to $+\infty$ or $-\infty$ on compact subsets of $\mathcal{V}$.

The set $\mathcal{U}$ (resp. $\mathcal{V}$) in Monotone convergence theorem (Theorem 5.2) is called to be *convergence set* (resp. *divergence set*) of the sequence of minimal solutions $\{u_n\}_n$.

Proof of Theorem 5.2. Suppose that $\{u_n\}_n$ be an increasing sequence (otherwise let $u_n := -u_n$). Let p be a point of $\mathcal{U}$. There is a positive real number R such that

$$\mathbb{D}_R(p) \subset \Omega, \qquad C := \inf_{q \in \mathbb{D}_R(p)} u_1(q) > -\infty.$$

Define $\mu = \sup_{n\in\mathbb{N}} u_n(p) - C \in \mathbb{R}_{\geq 0}$. The function $\Phi(t,s)$ in Local Harnack inequality (Theorem 5.1) is well defined on $[0,\mu] \times [0, r(\mu))$. Define $\varepsilon = \frac{1}{2}\min\{r(\mu), R\}$. For each $q \in \mathbb{D}_\varepsilon(p)$, by using the local Harnack inequality (Theorem 5.1), we have

$$u_n(q) - C \leq \Phi\big(u_n(p) - C, \mathrm{d}_{\mathbb{H}^2}(p,q)\big) \leq \Phi\left(\mu, \frac{r(\mu)}{2}\right). \tag{5.6}$$

By the definition of $\mathcal{U}$, $\mathbb{D}_\varepsilon(P) \subset \mathcal{U}$. Then $\mathcal{U}$ is open.

Since $\{u_n\}_n$ is monotonically increasing, by (5.6), the sequence $\{u_n\}_n$ is uniformly bounded on compact subsets of $\mathcal{U}$. It follows from the monotonicity and Compactness theorem (Theorem 3.10) that $\{u_n\}_n$ converges uniformly to a minimal solution on compact subsets of $\mathcal{U}$.

Finally, by the Dini's monotone convergence theorem, $\{u_n\}_n$ diverges uniformly to $+\infty$ on compact subsets of $\mathcal{V}$. We include a proof for completeness. Let K be a compact subset of $\mathcal{V}$ and let $N \in \mathbb{R}$ be given. For each n let $V_n = \{p \in \mathcal{V} : u_n(p) > N\}$. Each u_n is continuous so V_n is an open subset of $\mathcal{V}$. Since $u_n \leq u_{n+1}$, we have $V_n \subset V_{n+1}$. Since u_n converges pointwise to $+\infty$ on $\mathcal{V}$, the sequence $\{V_n\}_n$ is an open cover of $\mathcal{V}$. Moreover, since K is compact, there is some $\bar{n} \in \mathbb{N}$ depending on N such that $K \subset V_n$ for all $n \geq \bar{n}$. That is, if $n \geq \bar{n}$ and $p \in K$, then $u_n(p) > N$. This completes the proof. □

5.2. Divergence set theorem. We now show that the boundary $\partial\mathcal{V}$ of the divergence set $\mathcal{V}$ has a very special structure, when $\mathcal{V}$ is not empty.

THEOREM 5.3 (Divergence set theorem). *Let $\Omega \subset \mathbb{H}^2$ be a admissible domain whose boundary is composed with finitely open mean convex Euclidean arcs C_i. Let $\{u_n\}_n$ be an increasing or decreasing sequence of minimal solutions on Ω. Then, for each open arc C_i, we assume that, for every n, u_n extends continuously on C_i and either $\{u_n|_{C_i}\}_n$ converges uniformly on every compact subset of C_i to a continuous function or $\{u_n|_C\}_n$ diverges uniformly on every compact subset of C_i to $+\infty$ or $-\infty$. Let $\mathcal{V} = \mathcal{V}(\{u_n\})$ be the divergence set associated to $\{u_n\}_n$.*

(1) *The boundary of $\mathcal{V}$ consists of the union of a set of non-intersecting interior Euclidean geodesic chords in Ω joining two points of $\partial\Omega$, together with arcs in $\partial\Omega$. Moreover, a component of $\mathcal{V}$ cannot be an isolated point.*
(2) *A component of $\mathcal{V}$ cannot be an interior chord.*
(3) *No two interior chords in $\partial\mathcal{V}$ can have a common endpoint at a convex corner of $\mathcal{V}$.*

(4) *The endpoints of interior Euclidean geodesic chords are among the vertices of $\partial\Omega$. So the boundary of $\mathcal{V}$ has a finite set of interior Euclidean geodesic chords in Ω joining two vertices of $\partial\Omega$.*

Proof. Without loss of generality, assume that the sequence $\{u_n\}_n$ is increasing and the divergence set is not empty.

(1) It is clear by Lemma 4.3 and Straight line lemma (Corollary 4.4) that each arc of $\partial\mathcal{V}$ must be Euclidean geodesic and that no vertex of $\partial\mathcal{V}$ lies in Ω, then Assertion (1) follows (see [9, Theorem 6.2] for more details).

(3) Assume the contrary that (3) does not hold. Let γ_1, γ_2 be two arcs of $\partial\mathcal{V}$ having a common endpoint $p \in \partial\Omega$ at a convex corner of $\mathcal{V}$. Choose two points $q_i \in \gamma_i$, $i = 1, 2$ such that the triangle $\triangle$ with vertices p, q_1, q_2 lies in Ω. We can always assume that the Euclidean triangle $\triangle$ is either in $\mathcal{U}$ or in $\mathcal{V}$. Indeed, if $\triangle \not\subset \mathcal{V}$, we take a component $\triangle'$ of $\mathcal{U} \cap \triangle$. Let γ_1', γ_2' be two Euclidean geodesic chords in Ω having a common endpoint p such that the domain delimited by them is the smallest domain containing $\triangle'$. Then $\gamma_1', \gamma_2' \subset \partial\mathcal{V}$ and $\triangle'$ is the triangle delimited by γ_1', γ_2' and $\overline{q_1q_2}$ and $\triangle' \subset \mathcal{U}$. We can choose γ_1', γ_2' in place of γ_1, γ_2. By Proposition 4.8, we have

$$0 = F_{u_n}(\partial\triangle) = F_{u_n}(\overline{pq_1}) + F_{u_n}(\overline{pq_2}) + F_{u_n}(\overline{q_1q_2}),$$

$$\lim_{n\to\infty} F_{u_n}(\overline{pq_i}) = \begin{cases} \ell_{\text{euc}}(\overline{pq_i}) & \text{if } \triangle \subset \mathcal{U}, \\ -\ell_{\text{euc}}(\overline{pq_i}) & \text{if } \triangle \subset \mathcal{V}, \end{cases} \quad i = 1, 2.$$

On the other hand $\lim_{n\to\infty} |F_{u_n}(\overline{q_1q_2})| \le \ell_{\text{euc}}(\overline{q_1q_2})$. Hence

$$\ell_{\text{euc}}(\overline{q_1q_2}) \ge \ell_{\text{euc}}(\overline{pq_1}) + \ell_{\text{euc}}(\overline{pq_2}),$$

a contradiction.

(2) and (3) are proved with analogous arguments, using Lemma 4.3 and Straight line lemma (Corollary 4.4). The details are left to the reader (see, for instance, [10, pp. 329–331]). ☐

6. Jenkins–Serrin type theorem

Let $\Omega \subset \mathbb{H}^2$ be a domain whose boundary $\partial_\infty\Omega$ consists of a finite number of open Euclidean geodesic arcs A_i, B_i, a finite number of open, mean convex Euclidean arcs C_i (convex towards Ω) together with their endpoints, which are called the vertices of Ω and those in $\partial_\infty\mathbb{H}^2$ are called ideal vertices of Ω. We mark the A_i edges by $+\infty$ and the B_i edges by $-\infty$, and assign arbitrary continuous data f_i on the arcs C_i, respectively. Assume that no two A_i edges and no two B_i edges meet at a convex corner. We call such a domain Ω *Scherk domain* (see Figure 12). Assume in addition that, the ideal vertices of Scherk domain are the removable points. A solution to the Dirichlet problem on Ω is by definition a minimal solution on Ω assuming the above prescribed boundary values on the arcs A_i, B_i and C_i.

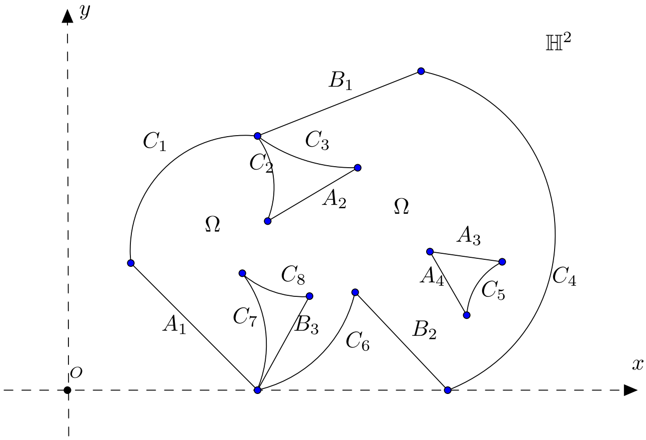

FIGURE 12. An example of Scherk domain.

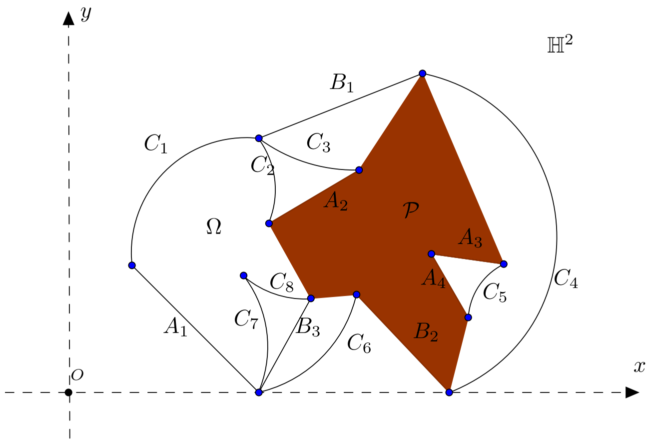

FIGURE 13. A polygonal domain $\mathcal{P}$ inscribed in Ω.

It is worth noting that if Ω is a Scherk domain, the Euclidean length of boundary $\partial_\infty\Omega$ is finite.

An *Euclidean polygonal domain* $\mathcal{P}$ in $\mathbb{H}^2$ is a domain whose boundary $\partial_\infty\mathcal{P}$ is composed of finitely many open Euclidean geodesic arcs in $\mathbb{H}^2$ together with their endpoints, which are called the vertices of $\mathcal{P}$. An Euclidean polygonal domain $\mathcal{P}$ is said to be inscribed in a Scherk domain Ω if $\mathcal{P} \subset \Omega$ and its vertices are among the vertices of Ω. We notice that a vertex may be in $\partial_\infty\mathbb{H}^2$ and an edge may be one of the A_i or B_i (see Figure 13).

Given an Euclidean polygonal domain $\mathcal{P}$ inscribed in Ω, we denote by $\ell_{\mathrm{euc}}(\mathcal{P})$ the Euclidean perimeter of $\partial\mathcal{P}$, and by $a_{\mathrm{euc}}(\mathcal{P})$ and $b_{\mathrm{euc}}(\mathcal{P})$ the sum of the Euclidean lengths of the edges A_i and B_i lying in $\partial\mathcal{P}$, respectively.

Now is a good time to state and to prove the main theorem of this paper. This theorem is similar in spirit to that of [10], [18], [3], [19], [12].

THEOREM 6.1 (Jenkins–Serrin type theorem). *Let Ω be a Scherk domain in $\mathbb{H}^2$ with the families $\{A_i\}, \{B_i\}, \{C_i\}$.*

(1) *If the family $\{C_i\}$ is nonempty, there exists a solution to the Dirichlet problem on Ω if and only if*

$$2a_{\mathrm{euc}}(\mathcal{P}) < \ell_{\mathrm{euc}}(\mathcal{P}), \qquad 2b_{\mathrm{euc}}(\mathcal{P}) < \ell_{\mathrm{euc}}(\mathcal{P}) \tag{6.1}$$

for every Euclidean polygonal domain $\mathcal{P}$ inscribed in Ω. Moreover, such a solution is unique if it exists.

(2) *If the family $\{C_i\}$ is empty, there exists a solution to the Dirichlet problem on Ω if and only if*

$$a_{\mathrm{euc}}(\mathcal{P}) = b_{\mathrm{euc}}(\mathcal{P}) \tag{6.2}$$

when $\mathcal{P} = \Omega$ and the inequalities in (6.1) hold for all other Euclidean polygonal domains $\mathcal{P}$ inscribed in Ω. Such a solution is unique up to an additive constant, if it exists.

Proof. The *uniqueness* of the solution is deduced from Theorem 6.2.

Let us now prove that the conditions of Jenkins–Serrin type theorem (Theorem 6.1) are necessary for the existence. Let u be a solution to the Dirichlet problem on Ω. When the family $\{C_i\}$ is empty and $\mathcal{P} = \Omega$, using Flux theorem (Proposition 4.7), we have

$$\begin{aligned} 0 = F_u(\partial\mathcal{P}) &= \sum_{A_i \subset \partial\mathcal{P}} F_u(A_i) + \sum_{B_i \subset \partial\mathcal{P}} F_u(B_i) \\ &= \sum_{A_i \subset \partial\mathcal{P}} \ell_{\mathrm{euc}}(A_i) + \sum_{B_i \subset \partial\mathcal{P}} -\ell_{\mathrm{euc}}(B_i) \\ &= a_{\mathrm{euc}}(\mathcal{P}) - b_{\mathrm{euc}}(\mathcal{P}), \end{aligned}$$

as the condition (6.2).

In the other case, $\partial\mathcal{P} \setminus ((\bigcup_{A_i \subset \partial\mathcal{P}} A_i) \cup (\bigcup_{B_i \subset \partial\mathcal{P}} B_i))$ is nonempty and u is continuous on this set. By Flux theorem (Proposition 4.7), we have

$$\begin{aligned} 0 &= F_u(\partial\mathcal{P}) \\ &= \sum_{A_i \subset \partial\mathcal{P}} F_u(A_i) + \sum_{B_i \subset \partial\mathcal{P}} F_u(B_i) \\ &\quad + F_u\Big(\partial\mathcal{P} \setminus \Big(\Big(\bigcup_{A_i \subset \partial\mathcal{P}} A_i\Big) \cup \Big(\bigcup_{B_i \subset \partial\mathcal{P}} B_i\Big)\Big)\Big), \end{aligned}$$

$$\sum_{A_i\subset\partial\mathcal{P}} F_u(A_i) = \sum_{A_i\subset\partial\mathcal{P}} \ell_{\mathrm{euc}}(A_i) = a_{\mathrm{euc}}(\mathcal{P}),$$
$$\sum_{B_i\subset\partial\mathcal{P}} F_u(B_i) = \sum_{B_i\subset\partial\mathcal{P}} -\ell_{\mathrm{euc}}(B_i) = -b_{\mathrm{euc}}(\mathcal{P})$$

and

$$\begin{aligned}\Big|F_u\Big(\partial\mathcal{P}\setminus\Big(\Big(\bigcup_{A_i\subset\partial\mathcal{P}} A_i\Big)\cup\Big(\bigcup_{B_i\subset\partial\mathcal{P}} B_i\Big)\Big)\Big)\Big| \\ < \ell_{\mathrm{euc}}\Big(\partial\mathcal{P}\setminus\Big(\Big(\bigcup_{A_i\subset\partial\mathcal{P}} A_i\Big)\cup\Big(\bigcup_{B_i\subset\partial\mathcal{P}} B_i\Big)\Big)\Big) \\ = \ell_{\mathrm{euc}}(\mathcal{P}) - a_{\mathrm{euc}}(\mathcal{P}) - b_{\mathrm{euc}}(\mathcal{P}).\end{aligned}$$

We obtain $|a_{\mathrm{euc}}(\mathcal{P}) - b_{\mathrm{euc}}(\mathcal{P})| < \ell_{\mathrm{euc}}(\mathcal{P}) - a_{\mathrm{euc}}(\mathcal{P}) - b_{\mathrm{euc}}(\mathcal{P})$. It follows the conditions (6.1).

Finally, we prove that the conditions of Jenkins–Serrin type theorem (Theorem 6.1) are sufficient. We distinguish the following cases:

CASE 6.1. *Assume that the families $\{A_i\}$ and $\{B_i\}$ are both empty and the continuous functions f_i are bounded.*

Proof. For any ideal vertex p of Ω, we take a net of geodesics $H_{p,n}$ which converges to p. Denote by $\mathcal{H}_{p,n}$ the component of $\mathbb{H}^2\setminus H_{p,n}$ containing p on its ideal boundary. Assume $\overline{\mathcal{H}}_{p_1,1}\cap\overline{\mathcal{H}}_{p_2,1}=\emptyset$ for every different ideal vertices p_1,p_2 of Ω and assume that $\overline{\mathcal{H}}_{p,1}$ doesn't contain the vertices of Ω in $\mathbb{H}^2$ where p is an ideal vertex. Let us define Ω_n a mean convex Euclidean subdomain of Ω delimited by $\partial\Omega\setminus\bigcup_{p\in E}\mathcal{H}_{p,n}$ and by the Euclidean geodesics in $\Omega\cap(\bigcup_{p\in E}\mathcal{H}_{p,n})$ joining the points of $\partial\Omega\cap(\bigcup_{p\in E}H_{p,n})$ where E is the set of ideal vertices of Ω. By definition, the boundary of Ω_n is composed of open, mean convex Euclidean arcs $C'_{i,n}\subset C_i$ and open Euclidean geodesic arcs $C_{p,n}\subset\mathcal{H}_{p,n}, p\in E$ together with their endpoints.

By Theorem 3.14, for each $n\in\mathbb{N}$, there exists a minimal solution u_n on an Euclidean polygonal domain of Ω_n such that $u_n=f_i$ on $C'_{i,n}$ and $u_n=0$ on $\bigcup_{p\in E}C_{p,n}$. By General maximum principle (Theorem 3.4) the sequence $\{u_n\}_n$ is uniformly bounded on Ω. By Compactness theorem (Theorem 3.10) there exists a subsequence of the sequence $\{u_n\}_n$ converges uniformly on every compact set of Ω to a minimal solution $u:\Omega\to\mathbb{R}$ that obtains the values f_i on C_i. □

CASE 6.2. *The family $\{B_i\}$ is empty and the functions f_i are non-negative.*

Proof. There exists, by the previous step 6.1, for each n, a minimal solution u_n on Ω taking the value n on A_i and $\min\{n,f_i\}$ on C_i. It follows from the general maximum principle (Theorem 3.4) that $0\le u_n\le u_{n+1}$ for each n. Hence, we can apply Divergence set theorem (Theorem 5.3).

ASSERTION 6.1. *The divergence set $\mathcal{V}=\mathcal{V}(\{u_n\}_n)$ is empty.*

Proof. Assume the contrary, that $\mathcal{V}$ is not empty. By Straight line lemma (Corollary 4.4) and Divergence set theorem (Theorem 5.3), $\mathcal{V}$ consists of a finite number of Euclidean polygonal domains inscribed in Ω. Let $\mathcal{P}$ be a component of $\mathcal{V}$. By Flux theorem (Proposition 4.7) and Proposition 4.8, we have:

$$0=F_{u_n}(\partial\mathcal{P})=\sum_{A_i\subset\partial\mathcal{P}}F_{u_n}(A_i)+F_{u_n}\left(\partial\mathcal{P}\setminus\left(\bigcup_i A_i\right)\right),$$

$$\left|\sum_{A_i\subset\partial\mathcal{P}}F_{u_n}(A_i)\right|\le\sum_{A_i\subset\partial\mathcal{P}}\left|F_{u_n}(A_i)\right|\le\sum_{A_i\subset\partial\mathcal{P}}\ell_{\mathrm{euc}}(A_i)=a_{\mathrm{euc}}(\mathcal{P}),$$

and

$$\begin{aligned}\lim_{n\to\infty}F_{u_n}\left(\partial\mathcal{P}\setminus\left(\bigcup_i A_i\right)\right)&=-\ell_{\mathrm{euc}}\left(\partial\mathcal{P}\setminus\left(\bigcup_i A_i\right)\right)\\&=-\big(\ell_{\mathrm{euc}}(\mathcal{P})-a_{\mathrm{euc}}(\mathcal{P})\big).\end{aligned}$$

We conclude that $\ell_{\mathrm{euc}}(\mathcal{P})-a_{\mathrm{euc}}(\mathcal{P})\le a_{\mathrm{euc}}(\mathcal{P})$, which contradicts with the condition (6.1). □

By Assertion 6.1, we have $\mathcal{U}(\{u_n\}_n)=\Omega$. Thus $\{u_n\}_n$ converges uniformly on the compact sets of Ω to a minimal solution u. By Boundary values lemma (Theorem 4.5), u takes the values $+\infty$ on A_i and f_i on C_i. □

CASE 6.3. *The family $\{C_i\}$ is nonempty.*

Proof. By the previous steps, 6.1 and 6.2, there exists the minimal solutions u^+,u^- and u_n on Ω with the following boundary values

	A_i	B_i	C_i
u^+	$+\infty$	0	$\max\{f_i,0\}$
u_n	n	$-n$	$[f_i]_{-n}^{n}$
u^-	0	$-\infty$	$\min\{f_i,0\}$

where $[f_i]_{-n}^{n}$ is defined by

$$[f_i]_{-n}^{n}(p)=\begin{cases}-n & \text{if } f_i(p)\le -n,\\ f_i(p) & \text{if } -n<f_i(p)<n,\\ n & \text{if } f_i(p)\ge n.\end{cases}$$

It follows from General maximum principle (Theorem 3.4) that $u^-\le u_n\le u^+$ for each n. Then there exists, by Compactness theorem (Theorem 3.10) a subsequence of $\{u_n\}_n$ converging on compact subsets of Ω to a minimal solution u on Ω. Moreover, by Boundary values lemma (Theorem 4.5), u takes the desired boundary conditions. □

CASE 6.4. *The family $\{C_i\}$ is empty.*

Proof. We fix a number $n \in \mathbb{N}$. There exists, by Case 6.1, a minimal solution v_n on Ω that obtains the values n on A_i and 0 on B_i. It follows from General maximum principle (Theorem 3.4) that $0 \le v_n \le n$. For each $c \in (0, n)$, we define

$$E_c = \{v_n > c\}, \qquad F_c = \{v_n < c\}.$$

Since $v_n = n$ on A_i, there exists a component E_c^i of E_c satisfying $A_i \subset \partial E_c^i$. Moreover, by the general maximum principle (Theorem 3.4), $E_c = \bigcup_i E_c^i$. Similarly, there exists, for each i, a component F_c^i of F_c satisfying $B_i \subset \partial F_c^i$, and, we have $F_c = \bigcup_i F_c^i$. A detailed proof can be found in [3, Proof of Theorem 1]. The set F_c is disconnected (resp. connected) for $c = \varepsilon$ (resp. $c = n - \varepsilon$) with $\varepsilon > 0$ small enough. We define

$$\mu_n = \inf\bigl\{c \in (0, n) : \text{the set } F_c \text{ is connected}\bigr\}, \qquad u_n = v_n - \mu_n.$$

By definition, u_n is a minimal solution on Ω which assumes the values $n - \mu_n$ on A_i and $-\mu_n$ on B_i.

ASSERTION 6.2. *There exist two piecewise minimal solutions u^+, u^- on Ω such that $u^- \le u_n \le u^+$ for every n.*

Proof. There exist, by the case 6.2, the minimal solutions $u_i^\pm$ on Ω such that

$$u_i^+ = \begin{cases} \infty & \text{on } \bigcup_{i' \ne i} A_{i'}, \\ 0 & \text{on } A_i \cup (\bigcup_j B_j), \end{cases} \qquad u_i^- = \begin{cases} -\infty & \text{on } \bigcup_{i' \ne i} B_{i'}, \\ 0 & \text{on } B_i \cup (\bigcup_j A_j). \end{cases}$$

Define

$$u^+ = \max_i u_i^+, \qquad u^- = \min_i u_i^-.$$

Observe that, by definition of μ_n, both E_{μ_n} and F_{μ_n} are disconnected. In particular, for every i_1, there exists an i_2 such that $E_{\mu_n}^{i_1} \cap E_{\mu_n}^{i_2} = \emptyset$ and we obtain, applying the maximum principle,

$$0 \le u_n|_{E_{\mu_n}^{i_1}} \le u_{i_2}^+|_{E_{\mu_n}^{i_1}}.$$

Similarly, for every j_1, there exists an j_2 such that $F_{\mu_n}^{j_1} \cap F_{\mu_n}^{j_2} = \emptyset$ and we obtain, applying the maximum principle,

$$u_{j_2}^-|_{F_{\mu_n}^{j_1}} \le u_n|_{F_{\mu_n}^{j_1}} \le 0.$$

It follows that $u^- \le u_n \le u^+$ for every n. □

By the previous assertion and the compactness theorem (Theorem 3.10), there exists a subsequence $\{u_{\sigma(n)}\}_n$ of $\{u_n\}_n$ that converges on compact sets of Ω to a minimal solution u.

ASSERTION 6.3. *We have*:

$$\lim_{n\to\infty}\mu_{\sigma(n)}=\infty,\qquad \lim_{n\to\infty}(n-\mu_{\sigma(n)})=\infty.$$

Proof. Assume the contrary, that there exists a subsequence $\{\mu_{\sigma'(n)}\}_n$ of $\{\mu_{\sigma(n)}\}_n$ that converges to some μ_∞. Then, by definition of u, that u takes the values ∞ on A_i and $-\mu_\infty$ on B_i. So, by the proof of necessity, $2a_{\mathrm{euc}}(\Omega)<\ell_{\mathrm{euc}}(\Omega)$, which contradicts with Hypothesis (6.1). Then $\lim_{n\to\infty}\mu_{\sigma(n)}=\infty$. In the same way, we can show that $\lim_{n\to\infty}(n-\mu_{\sigma(n)})=\infty$. □

So, by the previous assertion, we conclude u takes $+\infty$ on A_i and $-\infty$ on B_i. □

This completes the proof of the existence part of the theorem. □

The remainder of this section will be devoted to prove a maximum principle that is valid for solutions with possible infinite boundary data. This result immediately proves the uniqueness of Jenkins–Serrin type theorem (Theorem 6.1). The proof we give is a modification of the proof of the corresponding result of Jenkins–Serrin [10], Spruck [24], Nelli–Rosenberg [18].

THEOREM 6.2 (Maximum principle for unbounded domains with possible infinite boundary data). *Let $\Omega\subset\mathbb{H}^2$ be a domain whose boundary $\partial_\infty\Omega$ consists of a finite number of Euclidean geodesic arcs A_i,B_i, a finite number of mean convex Euclidean arcs* (*convex towards Ω*) *C_i in $\mathbb{H}^2$ together with their endpoints*, *which are called the vertices of Ω. Let u_1,u_2 be two minimal solutions on Ω taking the value $+\infty$ on A_i and $-\infty$ on B_i.*

(1) *If the family $\{C_i\}$ is nonempty*, *assume that* $\limsup(u_1-u_2)\le 0$ *when ones approach to $\bigcup_i C_i$.*
(2) *If $\{C_i\}$ is empty*, *suppose that $u_1\le u_2$ at some point $p\in\Omega$.*

Then in either case $u_1\le u_2$ on Ω.

Proof. Assume the contrary, that the set $\{p\in\Omega: u_1(p)>u_2(p)\}$ is nonempty. Let N and ε be two positive constants with N large and ε small. Define

$$\varphi=\begin{cases}0 & \text{if } u_1-u_2\le\varepsilon,\\ u_1-u_2-\varepsilon & \text{if } \varepsilon<u_1-u_2<N,\\ N-\varepsilon & \text{if } u_1-u_2\ge N.\end{cases}$$

Then φ is a continuous piecewise differentiable function in Ω satisfying $0\le\varphi<N$. Moreover, $\nabla\varphi=\nabla u_1-\nabla u_2$ in the set where $\varepsilon<u_1-u_2<N$, and $\nabla\varphi=0$ almost every where in the complement of this set.

Denote by E_1 (resp. E_2) the set of vertices in $\mathbb{H}^2$ (resp. vertices at $\partial_\infty\mathbb{H}^2$) of Ω. For each $p\in E_2$, we consider a sequence of nested ideal geodesics $H_{p,n}$, $n\ge 1$ converging to p. By nested, we mean that if $\mathcal{H}_{p,n}$ is the component of $\mathbb{H}^2\setminus H_{p,n}$ containing p on its ideal boundary, then $\mathcal{H}_{p,n+1}\subset\mathcal{H}_{p,n}$. Assume

$\overline{\mathcal{H}}_{p_1,1} \cap \overline{\mathcal{H}}_{p_2,1} = \emptyset$ for every different points $p_1, p_2 \in E_2$. For n sufficiently large satisfying $\overline{\mathbb{D}}^{\mathrm{euc}}_{\frac{1}{n}}(p_1) \cap \overline{\mathbb{D}}^{\mathrm{euc}}_{\frac{1}{n}}(p_2) = \emptyset, \forall p_1, p_2 \in E_1$ and $\overline{\mathbb{D}}^{\mathrm{euc}}_{\frac{1}{n}}(p_1) \cap \overline{\mathcal{H}}_{p_2,1} = \emptyset, \forall p_1 \in E_1, p_2 \in E_2$, we define

$$\Omega_n = \Omega \setminus \left(\left(\bigcup_{p \in E_1} \overline{\mathbb{D}}^{\mathrm{euc}}_{\frac{1}{n}}(p) \right) \cup \left(\bigcup_{p \in E_2} \overline{\mathcal{H}}_{p,n} \right) \right), \qquad \Gamma = \Omega \cap \partial \Omega_n$$

and

$$\Gamma_X = (\partial \Omega_n) \cap \left(\bigcup_i X_i \right) \quad \text{for } X \in \{A, B, C\}.$$

It follows from definition that

$$\varphi = 0 \quad \text{on a neighborhood of } \Gamma_C, \qquad \ell_{\mathrm{euc}}(\Gamma) \to 0 \quad \text{as } n \to \infty. \tag{6.3}$$

Define

$$J_n = \int_{\Omega_n} \mathrm{div}\big(\varphi y (X_{u_1} - X_{u_2})\big) \, \mathsf{d}\mathcal{A}.$$

ASSERTION 6.4. (1) *$J_n \geq 0$ with equality if and only if $\nabla u_1 = \nabla u_2$ on the set $\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}$.*
(2) *J_n is an increasing function of n.*

Proof. We have

$$\begin{aligned} J_n &= \int_{\Omega_n} \langle y \nabla \varphi, X_{u_1} - X_{u_2} \rangle \, \mathsf{d}\mathcal{A} + \int_{\Omega_n} \varphi \, \mathrm{div}(y X_{u_1} - y X_{u_2}) \, \mathsf{d}\mathcal{A} \\ &= \int_{\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}} \langle y \nabla \varphi, X_{u_1} - X_{u_2} \rangle \, \mathsf{d}\mathcal{A} + \int_{\Omega_n} \varphi \, \mathrm{div}(y X_{u_1} - y X_{u_2}) \, \mathsf{d}\mathcal{A}. \end{aligned}$$

By our assumptions,

$$\varphi \, \mathrm{div}(y X_{u_1} - y X_{u_2}) = \varphi (\mathfrak{M} u_1 - \mathfrak{M} u_2) = 0.$$

Moreover, on $\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}$, by formula (3.2) of Lemma 3.5, we have

$$\langle y \nabla \varphi, X_{u_1} - X_{u_2} \rangle = \left\langle y \nabla u_1 - y \nabla u_2, \frac{y \nabla u_1}{W_{u_1}} - \frac{y \nabla u_2}{W_{u_2}} \right\rangle \geq 0$$

and equality if and only if $y \nabla u_1 = y \nabla u_2$. Then

$$J_n = \int_{\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}} \langle y \nabla \varphi, X_{u_1} - X_{u_2} \rangle \, \mathsf{d}\mathcal{A} \geq 0$$

and $J_n = 0$ if and only if $\nabla u_1 = \nabla u_2$ on $\Omega_n \cap \{\varepsilon < u_1 - u_2 < N\}$. Since Ω_n is an increasing domain, i.e. $\Omega_n \subset \Omega_{n+1}$, J_n is an increasing function of n. This proves the assertion. ☐

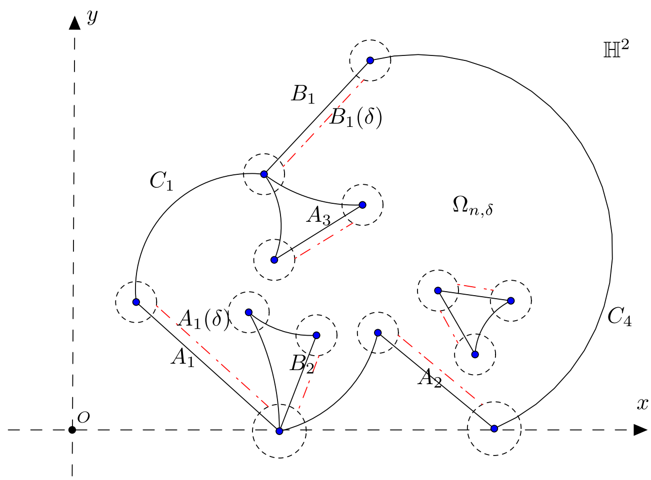

FIGURE 14. The domain $\Omega_{n,\delta}$.

ASSERTION 6.5. *$J_n = o(1)$ as $n \to \infty$.*

Proof. For $\delta > 0$ sufficiently small, define

$$\Omega_{n,\delta} = \Omega_n \Big\backslash \Big(\bigcup_{p \in \Gamma_A \cup \Gamma_B} \overline{\mathbb{D}}_\delta^{\text{euc}}(p) \Big).$$

As δ decreases to zero, the set $\Omega_{n,\delta}$ are expanding and $\bigcup_\delta \Omega_{n,\delta} = \Omega_n$. Then $J_n = \lim_{\delta \to 0} J_n(\delta)$ where $J_n(\delta) := \int_{\Omega_{n,\delta}} \operatorname{div}(\varphi y(X_{u_1} - X_{u_2}))\, \mathsf{d}\mathcal{A}$. By Divergence theorem $J_n(\delta) = \int_{\partial\Omega_{n,\delta}} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle\, \mathsf{d}s$ where ν is the exterior normal to $\partial\Omega_{n,\delta}$. The boundary $\partial\Omega_{n,\delta}$ of $\Omega_{n,\delta}$ consists of arcs $A_i(\delta)$ parallel to A_i, arcs $B_i(\delta)$ parallel to B_i, $\Gamma(\delta) := \Gamma \cap \partial\Omega_{n,\delta}$ and Γ_C (see Figure 14).

Thus

$$\begin{aligned} J_n(\delta) = &\sum_i \int_{A_i(\delta)} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle\, \mathsf{d}s \\ &+ \sum_i \int_{B_i(\delta)} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle\, \mathsf{d}s \\ &+ \int_{\Gamma(\delta)} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle\, \mathsf{d}s + \int_{\Gamma_C} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle\, \mathsf{d}s. \end{aligned}$$

By Property (6.3), $\|X_{u^i}\| \le 1, i = 1, 2$ and $0 \le \varphi < N$, we have

$$\int_{\Gamma_C} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle\, \mathsf{d}s = 0$$

and

$$\begin{aligned}\left|\int_{\Gamma(\delta)} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle \, \mathsf{d}s\right| &= \left|\int_{\Gamma(\delta)} \varphi \langle X_{u_1} - X_{u_2}, \nu\rangle \, \mathsf{d}s_{\mathrm{euc}}\right| \\ &\leq 2N\ell_{\mathrm{euc}}\big(\Gamma(\delta)\big) \\ &\leq 2N\ell_{\mathrm{euc}}(\Gamma) = o(1) \quad \text{as } n \to \infty.\end{aligned}$$

Otherwise, we have

$$\begin{aligned}&\int_{A_i(\delta)} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle \, \mathsf{d}s \\ &\quad = \int_{A_i(\delta)} \varphi y \big(1 - \langle X_{u_2}, \nu\rangle\big) \, \mathsf{d}s - \int_{A_i(\delta)} \varphi y \big(1 - \langle X_{u_1}, \nu\rangle\big) \, \mathsf{d}s \\ &\quad \leq N \int_{A_i(\delta)} y \big(1 - \langle X_{u_2}, \nu\rangle\big) \, \mathsf{d}s\end{aligned}$$

and

$$\begin{aligned}&\int_{B_i(\delta)} \varphi y \langle X_{u_1} - X_{u_2}, \nu\rangle \, \mathsf{d}s \\ &\quad = \int_{B_i(\delta)} \varphi y \big(1 + \langle X_{u_1}, \nu\rangle\big) \, \mathsf{d}s - \int_{B_i(\delta)} \varphi y \big(1 + \langle X_{u_2}, \nu\rangle\big) \, \mathsf{d}s \\ &\quad \leq N \int_{B_i(\delta)} y \big(1 + \langle X_{u_1}, \nu\rangle\big) \, \mathsf{d}s.\end{aligned}$$

Now applying Flux theorem (Proposition 4.7) for the component of $\Omega_n \setminus \Omega_{n,\delta}$ containing $A_i(\delta)$ on its boundary, we obtain

$$\int_{A_i(\delta)} y \langle X_{u_2}, \nu\rangle \, \mathsf{d}s = \ell_{\mathrm{euc}}(A_i \cap \Gamma_A) + o(1) = \ell_{\mathrm{euc}}\big(A_i(\delta)\big) + o(1)$$

as $\delta \to 0$. Equivalently, we have $\int_{A_i(\delta)} y(1 - \langle X_{u_2}, \nu\rangle) \, \mathsf{d}s = o(1)$ as $\delta \to 0$. Similarly, applying Flux theorem (Proposition 4.7) for the component of $\Omega_n \setminus \Omega_{n,\delta}$ containing $B_i(\delta)$ on its boundary, we obtain $\int_{B_i(\delta)} y(1 + \langle X_{u_1}, \nu\rangle) \, \mathsf{d}s = o(1)$ as $\delta \to 0$.

Combining these estimates, the assertion is then proved. □

It follows from the previous assertions that $\nabla u_1 = \nabla u_2$ on the set $\{\varepsilon < u_1 - u_2 < N\}$. Since ε and N are arbitrary, $\nabla u_1 = \nabla u_2$ whenever $u_1 > u_2$. So $u_1 = u_2 + c$ $(c > 0)$ in any nontrivial component of the set $\{u_1 > u_2\}$. Then the maximum principle (Theorem 3.1) ensures $u_1 = u_2 + c$ in Ω and by assumptions of the theorem, the constant must be nonpositive, a contradiction. □

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