

ON THE KÄHLER STRUCTURES OVER QUOT SCHEMES, II

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ABSTRACT. Let X be a compact connected Riemann surface of genus g , with $g \geq 2$, and let \mathcal{O}_X denote the sheaf of holomorphic functions on X . Fix positive integers r and d and let $\mathcal{Q}(r, d)$ be the Quot scheme parametrizing all torsion coherent quotients of $\mathcal{O}_X^{\oplus r}$ of degree d . We prove that $\mathcal{Q}(r, d)$ does not admit a Kähler metric whose holomorphic bisectional curvatures are all nonnegative.

1. Introduction

Let X be a compact connected Riemann surface of genus g , with $g \geq 2$. For any positive integer d , let $S^d(X)$ denote the d -fold symmetric product of X . The main theorem of [BoR] says the following (see [BoR, Theorem 1.1]): If $d \leq 2(g - 1)$, then $S^d(X)$ does not admit any Kähler metric for which all the holomorphic bisectional curvatures are nonnegative.

A natural generalization of the symmetric product $S^d(X)$ is the Quot scheme $\mathcal{Q}(r, d)$ that parametrizes all torsion coherent quotients of $\mathcal{O}_X^{\oplus r}$ of degree d . So, $\mathcal{Q}(1, d) = S^d(X)$. These Quot schemes arise naturally in the study of vector bundles on curves (see [BGL]). They also appear in mathematical physics as moduli spaces of vortices (cf. [Ba], [BiR], [BoR]).

Our aim here is to prove the following (see Theorem 3.1):

The complex manifold $\mathcal{Q}(r, d)$ does not admit any Kähler metric such that all the holomorphic bisectional curvatures are nonnegative.

In [BS], this was proved under the assumption that $d \leq 2(g - 1)$.

2. The Albanese map for \mathcal{Q}

Let X be a compact connected Riemann surface of genus g , with $g \geq 2$. The sheaf of germs of holomorphic functions on X will be denoted by \mathcal{O}_X .

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Fix positive integers r and d . Let

$$\mathcal{Q} := \mathcal{Q}(r, d)$$

be the Quot scheme that parametrizes all torsion coherent quotients of $\mathcal{O}_X^{\oplus r}$ of degree d . In other words, \mathcal{Q} parametrizes all coherent subsheaves of $\mathcal{O}_X^{\oplus r}$ of rank r and degree $-d$.

The group of permutations of $\{1, \dots, d\}$ will be denoted by P_d . Let $S^d(X) := X^d/P_d$ be the symmetric product of X . The Quot scheme $\mathcal{Q}(1, d)$ is identified with $S^d(X)$ by sending any quotient $Q' \in \mathcal{Q}(1, d)$ of \mathcal{O}_X to the scheme-theoretic support of Q' . Take any quotient $Q \in \mathcal{Q} = \mathcal{Q}(r, d)$. Consider the corresponding short exact sequence

$$0 \rightarrow \mathcal{K}_Q \rightarrow \mathcal{O}_X^{\oplus r} \rightarrow Q \rightarrow 0$$

on X . Let

$$0 \rightarrow \bigwedge^r \mathcal{K}_Q \rightarrow \bigwedge^r \mathcal{O}_X^{\oplus r} = \mathcal{O}_X \rightarrow \left(\bigwedge^r \mathcal{O}_X^{\oplus r} \right) / \left(\bigwedge^r \mathcal{K}_Q \right) := Q' \rightarrow 0$$

be the short exact sequence obtained from it by taking r th exterior product. Sending any $Q \in \mathcal{Q}$ to $Q' \in \mathcal{Q}(1, d)$ constructed as above from Q we get a morphism

$$(2.1) \quad f : \mathcal{Q} \rightarrow S^d(X) = \mathcal{Q}(1, d).$$

LEMMA 2.1. *Take any point $z \in S^d(X)$. There is a sequence of holomorphic maps ending at the point z ,*

$$Y_d^z \xrightarrow{p_d} Y_{d-1}^z \xrightarrow{p_{d-1}} \dots \xrightarrow{p_2} Y_1^z \xrightarrow{p_1} Y_0^z = \{z\},$$

such that

- all the fibers of each p_i are isomorphic to $\mathbb{C}\mathbb{P}^{r-1}$, and
- there is a surjective holomorphic map from Y_d^z to the fiber $f^{-1}(z)$.

Proof. Fix a point

$$(x_1, \dots, x_d) \in X^d$$

that projects to z under the quotient map $X^d \rightarrow X^d/P_d = S^d(X)$. For any integer $i \in [1, d]$, let Y_i^z be the space of all filtrations of coherent analytic subsheaves

$$(2.2) \quad W_i \subset W_{i-1} \subset \dots \subset W_1 \subset W_0 = \mathcal{O}_X^{\oplus r},$$

where W_j/W_{j+1} , $0 \leq j \leq i-1$, is a torsion sheaf supported at x_{j+1} such that the dimension of the stalk of W_j/W_{j+1} at x_{j+1} is one. Therefore, W_{j+1} is a holomorphic vector bundle of rank r and degree $-j-1$. In particular, W_i is a holomorphic vector bundle of rank r and degree $-i$. We have the natural forgetful map

$$p_i : Y_i^z \rightarrow Y_{i-1}^z$$

that forgets the smallest subsheaf in the filtration (it forgets W_i in (2.2)). The inverse image, under the projection p_i , of a point

$$W_{i-1} \subset \cdots \subset W_1 \subset W_0 = \mathcal{O}_X^{\oplus r}$$

of Y_{i-1}^z is the projective space $\mathbb{P}(W_{i-1})_{x_i}$ that parametrizes all hyperplanes in the fiber of the vector bundle W_{i-1} over the point x_i . In particular, the map p_i makes Y_i^z a projective bundle over Y_{i-1}^z of relative dimension $r - 1$.

Sending any point

$$W_d \subset W_{d-1} \subset \cdots \subset W_1 \subset W_0 = \mathcal{O}_X^{\oplus r},$$

of Y_d^z to the subsheaf $W_d \subset \mathcal{O}_X^{\oplus r}$ we get a holomorphic map

$$Y_d^z \longrightarrow f^{-1}(z).$$

This map is clearly surjective. □

Let

$$(2.3) \quad \varphi : S^d(X) \longrightarrow \text{Pic}^d(X)$$

be the morphism that sends a divisor to the corresponding holomorphic line bundle on X .

COROLLARY 2.2. *The composition $\varphi \circ f : \mathcal{Q} \longrightarrow \text{Pic}^d(X)$ is the Albanese morphism for the variety \mathcal{Q} .*

Proof. Since there is no nonconstant morphism from a rational curve to an Abelian variety, from Lemma 2.1 it follows immediately that the Albanese morphism for \mathcal{Q} factors through f : Take any holomorphic line bundle $L \in \text{Pic}^d(X)$. Note that the fiber $\varphi^{-1}(L)$ is the projective space $\mathbb{P}H^0(X, L)^*$. Therefore, the Albanese morphism for \mathcal{Q} factors through $\varphi \circ f$. □

3. Holomorphic bisectonal curvatures

We continue with the notation of Section 2.

THEOREM 3.1. *The complex manifold \mathcal{Q} does not admit any Kähler metric such that all the holomorphic bisectonal curvatures are nonnegative.*

Proof. Assume that \mathcal{Q} admits a Kähler metric $g_{\mathcal{Q}}$ with nonnegative holomorphic bisectonal curvatures. Let ω denote the Kähler form of $g_{\mathcal{Q}}$.

Let

$$(3.1) \quad \rho : \tilde{\mathcal{Q}} \longrightarrow \mathcal{Q}$$

be the universal cover. Then, by Mok’s uniformization theorem for compact Kähler manifolds with nonnegative bisectional curvature [Mo, p. 179, Main Theorem], the Kähler manifold $(\tilde{\mathcal{Q}}, \rho^*g_Q)$ decomposes as a Cartesian product

$$(\tilde{\mathcal{Q}}, \rho^*g) = (\mathbb{C}^N, g_0) \times \left(\prod_{i=1}^k (\mathbb{C}P^{r_i}, g_i) \right) \times \left(\prod_{j=1}^\ell (H_j, h_j) \right)$$

of Kähler manifolds for some nonnegative integers r_1, \dots, r_k, ℓ ; here (\mathbb{C}^N, g_0) is the Euclidean space with its standard complex structure and metric, g_i are Kähler metrics of nonnegative bisectional curvature on $\mathbb{C}P^{r_i}$, and h_j are the standard symmetric Kähler metrics on certain Hermitian symmetric spaces H_j of compact type. In what follows, we do not need the specific information on the non-Euclidean factors in the decomposition. We write

$$(\tilde{\mathcal{Q}}, \rho^*g_Q) = (\mathbb{C}^N, g_0) \times (H, g'),$$

where $(H, g') = (\prod_{i=1}^k (\mathbb{C}P^{r_i}, g_i)) \times (\prod_{j=1}^\ell (H_j, h_j))$.

Let

$$q_1 : \mathbb{C}^N \times H \longrightarrow \mathbb{C}^N$$

be the projection to the first factor. Let

$$\gamma \in \text{Gal}(\rho)$$

be a deck transformation for the covering ρ in (3.1). Since there is no non-constant holomorphic map from H to \mathbb{C}^N , there is a holomorphic isometry $\gamma' \in \text{Aut}(\mathbb{C}^N)$ such that

$$q_1 \circ \gamma = \gamma' \circ q_1.$$

Therefore, q_1 descends to a holomorphic submersion of the form

$$(3.2) \quad \beta : \mathcal{Q} \longrightarrow Y$$

such that Y is (regularly) covered by \mathbb{C}^N ; so \mathbb{C}^N is a universal cover of Y . More precisely, there is a Kähler form preserving covering map

$$\rho' : \mathbb{C}^N \longrightarrow Y$$

such that $\rho' \circ q_1 = \beta \circ \rho$. We note that β makes \mathcal{Q} a holomorphic fiber bundle over Y with fibers isomorphic to H .

Consider the C^∞ distribution

$$\mathcal{S} \subset T\mathcal{Q}$$

on \mathcal{Q} obtained by taking the orthogonal complement, with respect to ω , of the relative tangent bundle for the projection β in (3.2). The pulled back distribution

$$\rho^*\mathcal{S} = (d\rho)^{-1}(\mathcal{S}) \subset T\tilde{\mathcal{Q}}$$

coincides with the relative tangent bundle for the projection $q_2 : \mathbb{C}^N \times H \rightarrow H$ to the second factor; here $d\rho : T\tilde{\mathcal{Q}} \rightarrow T\mathcal{Q}$ is the differential of ρ . In particular, $\rho^*\mathcal{S}$ is an integrable holomorphic distribution. In other words, $\rho^*\mathcal{S}$ defines a flat holomorphic connection on the fiber bundle

$$q_1 : \mathbb{C}^N \times H \rightarrow \mathbb{C}^N.$$

This implies that \mathcal{S} is a flat holomorphic connection on the fiber bundle $\beta : \mathcal{Q} \rightarrow Y$. Let

$$(3.3) \quad \nabla$$

denote this flat holomorphic connection on $\beta : \mathcal{Q} \rightarrow Y$.

We now need the following proposition.

PROPOSITION 3.2. *The fibers of the map β coincides with the fibers of the map $\varphi \circ f$. In particular, there is a biholomorphism*

$$\alpha : Y \rightarrow \text{image}(\varphi \circ f)$$

such that $\alpha \circ \beta = \varphi \circ f$.

Proof. Take a point $y \in Y$. Consider the restriction of the map $\varphi \circ f$ to $\beta^{-1}(y)$. Since $\beta^{-1}(y) = H$ is simply connected, this map

$$(\varphi \circ f)|_{\beta^{-1}(y)} : \beta^{-1}(y) \rightarrow \text{Pic}^d(X)$$

lifts to a map

$$\beta^{-1}(y) \rightarrow \widetilde{\text{Pic}}^d(X),$$

where $\widetilde{\text{Pic}}^d(X)$ is the universal cover of $\text{Pic}^d(X)$. But $\widetilde{\text{Pic}}^d(X)$ is isomorphic to \mathbb{C}^g , and hence any holomorphic map to it from $\beta^{-1}(y)$ is a constant map.

Conversely, take any point $L \in \varphi \circ f(\mathcal{Q})$. We will show that the restriction of the map β to the fiber $(\varphi \circ f)^{-1}(L)$ is a constant map. For this, we first show that there is no nonconstant holomorphic map from $\mathbb{C}\mathbb{P}^1$ to Y . To prove this note that $\mathbb{C}\mathbb{P}^1$ is simply connected and the universal cover of Y is \mathbb{C}^N . Therefore, any holomorphic map from $\mathbb{C}\mathbb{P}^1$ to Y lifts to a holomorphic map from $\mathbb{C}\mathbb{P}^1$ to \mathbb{C}^N , but there are no such nonconstant holomorphic maps.

Since there is no nonconstant holomorphic map from $\mathbb{C}\mathbb{P}^1$ to Y , from Lemma 2.1 it follows that the map

$$\beta|_{(\varphi \circ f)^{-1}(L)} : (\varphi \circ f)^{-1}(L) \rightarrow Y$$

factors through $(\varphi \circ f)^{-1}(L) \xrightarrow{f} \varphi^{-1}(L)$. As noted in the proof of Corollary 2.2, the fiber $\varphi^{-1}(L)$ is a projective space. Hence, the restriction of β to the fiber $(\varphi \circ f)^{-1}(L)$ is a constant map. This completes the proof of the proposition. □

Continuing with the proof of the theorem, since $g \geq 2$, we know that $\mathcal{Q}(r, d)$ does not admit any Kähler metric with nonnegative holomorphic bisectional curvature if $d < g$ [BS, Corollary 4.1]. Therefore, we assume that $d \geq g$. Consequently, the map φ is surjective. Since f is surjective, this implies that $\varphi \circ f$ is surjective. Now from Proposition 3.2 we know that Y is an Abelian variety.

Choose a point of Y to make it a group. The universal covering map $\mathbb{C}^N \rightarrow Y$ is chosen to be a homomorphism of groups. For any point v in the universal cover \mathbb{C}^N of Y , let

$$\tau_v : Y \rightarrow Y$$

be the automorphism given by the automorphism $w \mapsto w + v$ of \mathbb{C}^N . We have a one-parameter family of automorphisms Y given by τ_{tv} , $t \in [0, 1]$, that connects τ_v with the identity map of Y .

Consider the flat holomorphic connection ∇ in (3.3). Given a point $y \in Y$, we may take parallel translation of the fiber of β over y along the path $\tau_{tv}(y)$, $t \in [0, 1]$. Taking parallel translations for ∇ along such paths τ_{tv} , $t \in [0, 1]$, we get holomorphic automorphisms of \mathcal{Q} . If v is sufficiently close to zero, then the automorphism τ_v of Y clearly does not have any fixed-point. Hence for v sufficiently close to zero, we get holomorphic automorphisms of \mathcal{Q} that do not have any fixed point.

Let $\text{Aut}(\mathcal{Q})$ denote the group of holomorphic automorphisms of \mathcal{Q} . Let $\text{Aut}^0(\mathcal{Q}) \subset \text{Aut}(\mathcal{Q})$ be the connected component of it containing the identity element.

The standard action of $\text{GL}(r, \mathbb{C})$ on \mathbb{C}^r produces an action of $\text{GL}(r, \mathcal{C})$ on the trivial vector bundle $\mathcal{O}_X^{\oplus r} = X \times \mathbb{C}^r$. The corresponding action of $\text{GL}(r, \mathbb{C})$ on \mathcal{Q} factors through the quotient group $\text{PGL}(r, \mathbb{C}) = \text{GL}(r, \mathbb{C})/\mathbb{C}^*$. Since this action of $\text{PGL}(r, \mathbb{C})$ on \mathcal{Q} is effective, we have

$$\text{PGL}(r, \mathbb{C}) \subset \text{Aut}^0(\mathcal{Q}).$$

It is known that $\text{PGL}(r, \mathbb{C}) = \text{Aut}^0(\mathcal{Q})$ [BDH, Theorem 1.1]. Since the standard action on $\mathbb{C}\mathbb{P}^{r-1}$ of any element $A \in \text{PGL}(r, \mathbb{C})$ has a fixed point, the action of A on \mathcal{Q} has a fixed point.

This contradicts our earlier construction of automorphisms of \mathcal{Q} without fixed points. In view of this contradiction, we conclude that \mathcal{Q} does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonnegative. \square

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