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THE GLUING FORMULA OF THE ZETA-DETERMINANTS OF DIRAC LAPLACIANS FOR CERTAIN BOUNDARY CONDITIONS

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Abstract. The odd signature operator is a Dirac operator which acts on the space of differential forms of all degrees and whose square is the usual Laplacian. We extend the result see (*J. Geom. Phys.* **57** (2007) 1951–1976) to prove the gluing formula of the zeta-determinants of Laplacians acting on differential forms of all degrees with respect to the boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$, $\mathcal{P}_{+,\mathcal{L}_1}$. We next consider a double of de Rham complexes consisting of differential forms of all degrees with the absolute and relative boundary conditions. Using a similar method, we prove the gluing formula of the zeta-determinants of Laplacians acting on differential forms of all degrees with respect to the absolute and relative boundary conditions.

1. Introduction

The zeta-determinants of Laplacians are global spectral invariants on compact Riemannian manifolds with or without boundary, which play central roles in the theory of the analytic torsions and other related fields. For a global invariant, the gluing formula is very useful in various kinds of computations. The gluing formula of the zeta-determinants of Laplacians was proved by D. Burghelea, L. Friedlander and T. Kappeler in [5] by using the Dirichlet boundary condition and the Dirichlet-to-Neumann operator, which we call the BFK-gluing formula. Because of relations to topology, the relative

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and absolute boundary conditions are commonly used for Hodge Laplacians. However, the gluing formula for the zeta-determinants of Hodge Laplacians with respect to these boundary conditions is not known yet. In this paper, we discuss this problem in a weak sense. More precisely, we prove the gluing formula for the zeta-determinants of Hodge Laplacians acting on the space of differential forms of all degrees, not a single space of q-forms, with respect to the relative and absolute boundary conditions (Theorem 4.2).

K. Wojciechowski and S. Scott studied the zeta-determinants of Dirac Laplacians on compact Riemannian manifolds with boundary, acting on Clifford module bundles with respect to boundary conditions belonging to the smooth self-adjoint Grassmannian including the Atiyah–Patodi–Singer (APS) boundary condition and the Calderón projector ([18], [19], [20], [26]). Using their results and the BFK-gluing formula, P. Loya, J. Park ([16], [17]) and the second author ([15]) studied independently the gluing formula of Dirac Laplacians with respect to boundary conditions belonging to the smooth self-adjoint Grassmannian on compact Riemannian manifolds.

M. Braverman and T. Kappeler studied the refined analytic torsion on a closed odd dimensional Riemannian manifold by using the odd signature operator ([3], [4]), as an analytic analogue of the refined combinatorial torsion developed by M. Farber and V. Turaev ([6], [7], [22], [23]). The boundary problem of the refined analytic torsion was studied by B. Vertman ([24], [25]) and the authors ([9], [10], [11]) in different ways. Vertman used a double of de Rham complexes consisting of differential forms satisfying the absolute and relative boundary conditions. The authors introduced well-posed boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$, $\mathcal{P}_{+,\mathcal{L}_1}$ for the odd signature operator to define the refined analytic torsion on compact Riemannian manifolds with boundary. In [11], the authors compared these two constructions.

We note that the odd signature operator is a Dirac operator which acts on the space of differential forms of all degrees and whose square is the usual Laplacian. In this paper, we extend the result of [15] to other class of boundary conditions and discuss the gluing formula of the zeta-determinants of Laplacians acting on the space of differential forms of all degrees with respect to $\mathcal{P}_{-,\mathcal{L}_0}/\mathcal{P}_{+,\mathcal{L}_1}$ (Theorem 3.3) and the absolute/relative boundary conditions (Theorem 4.2). In case of the absolute/relative boundary conditions, we are going to use the double of De Rham complexes which was used by B. Vertman in [24].

2. Review of the gluing formula of the zeta-determinants of Dirac Laplacians

In this section, we review and extend the results in [15]. Let (M, g) be an m-dimensional compact oriented Riemannian manifold with boundary Y and $E \to M$ be a Hermitian vector bundle. Choose a collar neighborhood N of Y

which is diffeomorphic to $[0,1)\times Y$. We assume that the metric g is a product one on N and the bundle E has the product structure on N, which means that $E|_N = p^*(E|_Y)$, where $p:[0,1)\times Y \to Y$ is the canonical projection. Let $\mathcal{D}_M$ be a Dirac type operator acting on smooth sections of E and satisfying the following conditions: (1) On the collar neighborhood N of Y $\mathcal{D}_M$ has the following form

$$\mathcal{D}_M = G(\partial_u + \mathcal{A}), \tag{2.1}$$

where $G: E|_Y \to E|_Y$ is a bundle automorphism with $G^2 = -\mathrm{Id}$, ∂_u is the inward normal derivative to Y and $\mathcal{A}$ is the tangential Dirac operator. (2) G and $\mathcal{A}$ are independent of the normal coordinate u and satisfy

$$\begin{gathered} G^* = -G, \qquad G^2 = -\mathrm{Id}, \qquad \mathcal{A}^* = \mathcal{A}, \qquad G\mathcal{A} = -\mathcal{A}G \\ \dim\big(\ker(G-i)\cap\ker\mathcal{A}\big) = \dim\big(\ker(G+i)\cap\ker\mathcal{A}\big). \end{gathered} \tag{2.2}$$

Then, on N, the Dirac Laplacian $\mathcal{D}_M^2$ has the following form

$$\mathcal{D}_M^2 = -\partial_u^2 + \mathcal{A}^2. \tag{2.3}$$

We next introduce boundary conditions on Y. The Dirichlet boundary condition on Y is defined by the restriction map $\gamma_0 : C^\infty(M) \to C^\infty(Y)$, $\gamma_0(\phi) = \phi|_Y$ and the realization $\mathcal{D}^2_{M,\gamma_0}$ is defined to be the operator $\mathcal{D}_M^2$ with the following domain

$$\mathrm{Dom}\big(\mathcal{D}^2_{M,\gamma_0}\big) = \big\{\phi \in C^\infty(M) \mid \phi|_Y = 0\big\}. \tag{2.4}$$

Then $\mathcal{D}^2_{M,\gamma_0}$ is an invertible operator by the unique continuation property of $\mathcal{D}_M$ ([12], [1]).

The APS boundary condition $\Pi_>$ (or $\Pi_<$) is defined to be the orthogonal projection onto the space spanned by the positive (or negative) eigensections of $\mathcal{A}$. If $\ker\mathcal{A} \neq \{0\}$, $\ker\mathcal{A}$ is an even dimensional vector space by (2.2). We choose a unitary operator $\sigma:\ \ker\mathcal{A} \to \ker\mathcal{A}$ satisfying

$$\sigma G = -G\sigma, \qquad \sigma^2 = \mathrm{Id}_{\ker\mathcal{A}}. \tag{2.5}$$

We put $\sigma^\pm := \frac{I\pm\sigma}{2}$ and define $\Pi_{<,\sigma^-}$, $\Pi_{>,\sigma^+}$ by

$$\Pi_{<,\sigma^-} := \Pi_< + \frac{1}{2}(I-\sigma)\Big|_{\ker\mathcal{A}}, \qquad \Pi_{>,\sigma^+} := \Pi_> + \frac{1}{2}(I+\sigma)\Big|_{\ker\mathcal{A}}. \tag{2.6}$$

The realizations $\mathcal{D}_{M,\Pi_{<,\sigma^-}}$ and $\mathcal{D}^2_{M,\Pi_{<,\sigma^-}}$ are defined to be $\mathcal{D}_M$ and $\mathcal{D}_M^2$ with the following domains.

$$\begin{aligned} \mathrm{Dom}(\mathcal{D}_{M,\Pi_{<,\sigma^-}}) &= \big\{\phi \in C^\infty(M) \mid \Pi_{<,\sigma^-}(\phi|_Y) = 0\big\}, \\ \mathrm{Dom}\big(\mathcal{D}^2_{M,\Pi_{<,\sigma^-}}\big) &= \big\{\phi \in C^\infty(M) \mid \Pi_{<,\sigma^-}(\phi|_Y) = 0, \\ &\qquad \Pi_{<,\sigma^-}\big((\mathcal{D}_M\phi)|_Y\big) = 0\big\}. \end{aligned} \tag{2.7}$$

$\mathcal{D}_{M,\Pi_{>,\sigma^+}}$ and $\mathcal{D}^2_{M,\Pi_{>,\sigma^+}}$ are defined similarly. The Calderón projector $\mathcal{C}$ is defined to be the orthogonal projection from $L^2(E|_Y)$ onto the closure of $\{\phi|_Y \mid \phi \in C^\infty(M), \mathcal{D}_M\phi = 0\}$ called the Cauchy data space.

As a generalization of the APS boundary condition, K. Wojciekowski and B. Booss introduced the smooth self-adjoint Grassmannian $Gr^*_\infty(\mathcal{D}_M)$ ([2], [20], [26]), which is the set of all orthogonal pseudodifferential projections P such that

$$(2.8) \qquad -GPG = I - P, \quad P - \Pi_> \text{ is a classical pseudodifferential operator of order } -\infty.$$

Clearly, $\Pi_{>,\sigma^+}$ belongs to $Gr^*_\infty(\mathcal{D}_M)$. It was known by S. Scott ([18]) and G. Grubb ([8]) that $\mathcal{C}$ belongs to $Gr^*_\infty(\mathcal{D}_M)$. The realizations $\mathcal{D}_{M,P}$ and $\mathcal{D}^2_{M,P}$ are similarly defined as (2.7) by simply replacing $\Pi_{<,\sigma^-}$ with P.

Since G is a bundle automorphism on $E|_Y$ with $G^2 = -\mathrm{Id}$, $E|_Y$ splits onto $\pm i$-eigenspaces $E_Y^\pm$, say, $E|_Y = E_Y^+ \oplus E_Y^-$ and the Dirac operator $\mathcal{D}_M$ can be written, near the boundary Y, by

$$(2.9) \qquad \mathcal{D}_M = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\partial_u + \begin{pmatrix} 0 & \mathcal{A}^- \\ \mathcal{A}^+ & 0 \end{pmatrix} \right),$$

where $\mathcal{A}^\pm := \mathcal{A}|_{C^\infty(E_Y^\pm)} : C^\infty(E_Y^\pm) \to C^\infty(E_Y^\mp)$ and $(\mathcal{A}^\pm)^* = \mathcal{A}^\mp$. For $P \in Gr^*_\infty(\mathcal{D}_M)$, there exists a unitary operator $U_P : L^2(E_Y^+) \to L^2(E_Y^-)$ such that $\mathrm{graph}(U_P) = \mathrm{Im}\, P$. For simplicity, we write $U_{\mathcal{C}} = K$. By (2.8), we have

$$(2.10) \qquad U_P = K + \text{a smoothing operator.}$$

We introduce the Neumann jump operator $Q(t) : C^\infty(Y) \to C^\infty(Y)$ for $t \geq 0$ as follows. For $f \in C^\infty(Y)$, there exists a unique section $\phi \in C^\infty(E)$ satisfying $(\mathcal{D}_M^2 + t)\phi = 0$, $\phi|_Y = f$. Then we define

$$(2.11) \qquad Q(t)(f) = -(\partial_u \phi)|_Y.$$

The Green formula shows that $Q(t) - \mathcal{A}$ is a non-negative operator and $\ker(Q - \mathcal{A}) = \mathrm{Im}\,\mathcal{C}$, the Cauchy data space (Lemma 2.5 in [15]), where $Q := Q(0)$. Moreover, $Q - |\mathcal{A}|$ (Theorem 2.1 in [14]) and $P - \Pi_>$ are smoothing operators, which implies that $(I-P)(Q-\mathcal{A})(I-P)$ differs from $2\Pi_<|\mathcal{A}|$ by a smoothing operators. Hence, the zeta determinant of $(I-P)(Q-\mathcal{A})(I-P)$ is well defined even though $(I-P)(Q-\mathcal{A})(I-P)$ is not an elliptic operator. It is not difficult to show that $\ker(I-P)(Q-\mathcal{A})(I-P) = \{\psi|_Y \mid \psi \in \ker \mathcal{D}_{M,P}\}$ (Lemma 2.5 in [15]). Let $\{h_1, \ldots, h_q\}$ be an orthonormal basis for $\ker(I - P)(Q - \mathcal{A})(I - P)$, where $q = \dim\ker \mathcal{D}^2_{M,P}$. Then there exist $\psi_1, \ldots, \psi_q \in \ker \mathcal{D}^2_{M,P}$ with $\psi_i|_Y = h_i$. We define a $q \times q$ positive definite Hermitian matrix $V_{M,P}$ by

$$(2.12) \qquad V_{M,P} := (v_{ij}), \quad v_{ij} = \langle \psi_i, \psi_j \rangle_M.$$

We next define the zeta- and modified zeta-determinants of elliptic operators. Let X be a compact oriented manifold with boundary ∂X, where ∂X may be empty. If $\partial X \neq \emptyset$, we need to choose a proper boundary condition. If $\mathfrak{P}$ is an elliptic operator of order > 0 on X which has discrete spectrum $\{\lambda_j \mid j = 1, 2, 3, \dots\}$ and $\ker \mathfrak{P} = \{0\}$ on a proper domain satisfying the chosen boundary condition, we define the zeta function by $\zeta_{\mathfrak{P}}(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}$ and the zeta-determinant $\operatorname{Det} \mathfrak{P}$ by $e^{-\zeta'_{\mathfrak{P}}(0)}$. If $\mathfrak{P}$ has a non-trivial kernel, we define the modified zeta-determinant $\operatorname{Det}^* \mathfrak{P}$ by

$$\operatorname{Det}^* \mathfrak{P} := \operatorname{Det}(\mathfrak{P} + \mathrm{pr}_{\ker \mathfrak{P}}). \tag{2.13}$$

Similarly, if α is a trace class operator on X, we define the modified Fredholm determinant by

$$\det{}^*_{\mathrm{Fr}}(I + \alpha) := \det(I + \alpha + \mathrm{pr}_{\ker(I+\alpha)}). \tag{2.14}$$

Equivalently, $\operatorname{Det}^* \mathfrak{P}$ and $\det^*_{\mathrm{Fr}}(I + \alpha)$ are the determinants of $\mathfrak{P}$ and $I + \alpha$ when restricted to the orthogonal complements of $\ker \mathfrak{P}$ and $\ker(I + \alpha)$, respectively.

The following results are due to S. Scott and K. Wojciechowski ([19], [20], [26]), P. Loya and J. Park ([16], [17]) and the second author ([15]).

Theorem 2.1. *Let (M, g) be a compact oriented Riemannian manifold with boundary Y having the product structure near Y. We denote by $\mathcal{D}_M$ a Dirac type operator which has the form (2.1) and satisfies (2.2) near Y. Let $\mathcal{P}$ be a pseudodifferential projection belonging to $Gr^*_{\infty}(\mathcal{D}_M)$. Then*:

$$\begin{aligned} &\log \operatorname{Det}^* \mathcal{D}^2_{M,\mathcal{P}} - \log \operatorname{Det} \mathcal{D}^2_{M,\gamma_0} \\ &\qquad = \log \det V_{M,\mathcal{P}} + \log \operatorname{Det}^* \big((I - \mathcal{P})(Q - \mathcal{A})(I - \mathcal{P})\big), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\log \operatorname{Det}^* \mathcal{D}^2_{M,\mathcal{P}} - \log \operatorname{Det} \mathcal{D}^2_{M,\mathcal{C}} \\ &\qquad = 2 \log \det V_{M,\mathcal{P}} + 2 \log \left| \det{}^*_{\mathrm{Fr}} \left(\frac{1}{2} \big(I + U_{\mathcal{P}}^{-1} K\big) \right) \right|, \end{aligned} \tag{2.16}$$

where $(I - \mathcal{P})(Q - \mathcal{A})(I - \mathcal{P})$ is considered to be an operator defined on $\operatorname{Im}(I - \mathcal{P})$.

Next, we extend Theorem 2.1 to a certain pseudodifferential projection $\mathcal{P}$ satisfying the following conditions.

Condition A. (1) $\mathcal{P} : L^2(Y, E|_Y) \to L^2(Y, E|_Y)$ is a pseudodifferential projection which gives a well-posed boundary condition with respect to $\mathcal{D}_M$ in the sense of Seeley ([8], [21]). (2) $\operatorname{Im} \mathcal{P} = \operatorname{graph}(U_{\mathcal{P}})$, where $U_{\mathcal{P}} : L^2(E_Y^+) \to L^2(E_Y^-)$ is a unitary operator. (3) $U^*_{\mathcal{P}} U_{\Pi_{>,\sigma^+}} + U^*_{\Pi_{>,\sigma^+}} U_{\mathcal{P}}$ is a trace class operator and a ΨDO of order at most -1. (4) The zeta-determinants of $(I - \mathcal{P})(Q(t) - \mathcal{A})(I - \mathcal{P})$ and $\mathcal{P}(Q(t) - \mathcal{A})\mathcal{P}$ for $t \geq 0$ are well defined and have asymptotic expansions for $t \to \infty$ with zero constant term.

Remark. A pseudodifferential projection belonging to $Gr^*_\infty(\mathcal{D}_M)$ satisfies the items (1), (2) and (4) but not (3) in the Condition A above.

The following lemma is straightforward by (2.10).

Lemma 2.2. *If $\mathcal{P}$ satisfies the Condition A, then $U_{\mathcal{P}}^{-1}K + K^{-1}U_{\mathcal{P}}$ is a trace class operator on $L^2(E_Y^+)$.*

The proof of the following result is a verbatim repetition of the proof of Theorem 1.1 in [15], which is an analogue of (2.15).

Theorem 2.3. *Let (M,g) be a compact oriented Riemannian manifold with boundary Y having the product structure near Y. We denote by $\mathcal{D}_M$ a Dirac type operator which has the form (2.1) and satisfies (2.2) near Y. Let $\mathcal{P}$ be a well-posed boundary condition with respect to $\mathcal{D}_M$ satisfying the Condition A. Then the following equality holds.*

$$\begin{aligned}\log \operatorname{Det}^* \mathcal{D}^2_{M,\mathcal{P}} &- \log \operatorname{Det} \mathcal{D}^2_{M,\gamma_0}\\ &= \log\det V_{M,\mathcal{P}} + \log \operatorname{Det}^*\big((I-\mathcal{P})(Q-\mathcal{A})(I-\mathcal{P})\big)\\ &= \log\det V_{M,\mathcal{P}} + \log \operatorname{Det}^*\big(2(I-\mathcal{P})(Q-\mathcal{A})(I-\mathcal{P})\big)\\ &\quad - \log 2\cdot \zeta_{(I-\mathcal{P})(Q-\mathcal{A})(I-\mathcal{P})}(0),\end{aligned}$$

where $(I-\mathcal{P})(Q-\mathcal{A})(I-\mathcal{P})$ is considered to be an operator defined on $\operatorname{Im}(I-\mathcal{P})$.

Theorem 2.3 and (2.15) in Theorem 2.1 lead to the following result, which is an analogue of (2.16).

Theorem 2.4. *We assume the same assumptions and notations as in Theorem 2.3. Then*:

$$\frac{\operatorname{Det}^* \mathcal{D}^2_{M,\mathcal{P}}}{\operatorname{Det} \mathcal{D}^2_{M,\mathcal{C}}} = (\det V_{M,\mathcal{P}})^2 \cdot \det{}^*_{\mathrm{Fr}}\left(I + \frac{1}{2}\big(U_{\mathcal{P}}^{-1}K + K^{-1}U_{\mathcal{P}}\big)\right) \cdot 2^{-\zeta_{(I-\mathcal{P})(Q-\mathcal{A})(I-\mathcal{P})}(0)}.$$

Proof. The proof is almost verbatim repetition of the proof of Theorem 1.2 in [15]. We here present the proof very briefly and refer to [15] for details. We first define U, L by

$$\begin{aligned}U &= \operatorname{Im}(I-\mathcal{P}) \cap \operatorname{Im}\mathcal{C}\\ &= \ker(I-\mathcal{P})(Q-\mathcal{A})(I-\mathcal{P}) = \big\{\phi|_Y \mid \mathcal{D}_M\phi = 0, \phi|_Y \in \operatorname{Im}(I-\mathcal{P})\big\},\\ L &= (I-U_{\mathcal{P}})^{-1}(U) = (I+K)^{-1}(U) = \big\{x \in L^2\big(E_Y^+\big) \mid U_P x = -Kx\big\}.\end{aligned}$$

We denote by $\operatorname{Im}(I-\mathcal{P})^*$ and $L^2(E_Y^+)^*$ the orthogonal complements of U, L so that

$$\operatorname{Im}(I-\mathcal{P}) = \operatorname{Im}(I-\mathcal{P})^* \oplus U, \qquad L^2\big(E_Y^+\big) = L^2\big(E_Y^+\big)^* \oplus L.$$

The item (3) in the Condition A implies that $(I+K^{-1}U_{\mathcal{P}})|_{L^2(E_Y^+)^*}: L^2(E_Y^+)^* \to L^2(E_Y^+)^*$ is an invertible operator. For simplicity, we write $((I+K^{-1}U_{\mathcal{P}})|_{L^2(E_Y^+)^*})^{-1}$ by $(I+K^{-1}U_{\mathcal{P}})^{-1}$. We proceed as (3.5) in the proof of Theorem 1.2 in [15]. Then:

$$\begin{aligned}
&\log \operatorname{Det}^*\big(2(I-\mathcal{P})(Q-\mathcal{A})(I-\mathcal{P})\big)\\
&\quad= \log \operatorname{Det}\big(2(I-\mathcal{P})(Q-\mathcal{A})(I-\mathcal{P}) + \operatorname{pr}_U\big)\\
&\quad= \log \det\nolimits_{\mathrm{Fr}}\bigg(\frac{1}{2}\big(I+K^{-1}U_{\mathcal{P}}\big)\big(I+U_{\mathcal{P}}^{-1}K\big)\\
&\qquad + \operatorname{pr}_L(I-K)^{-1}(Q-\mathcal{A})^{-1}(I-K)\operatorname{pr}_L\bigg)\\
&\qquad + \log \operatorname{Det}\big((I-K)^{-1}(Q-\mathcal{A})(I-K)\big)\\
&\quad= \log \det\nolimits^*_{\mathrm{Fr}}\bigg(I+\frac{1}{2}\big(K^{-1}U_{\mathcal{P}}+U_{\mathcal{P}}^{-1}K\big)\bigg)\\
&\qquad + \log \det\big(\operatorname{pr}_L(I-K)^{-1}(Q-\mathcal{A})^{-1}(I-K)\operatorname{pr}_L\big)\\
&\qquad + \log \operatorname{Det}\big((I-\mathcal{C})(Q-\mathcal{A})(I-\mathcal{C})\big).
\end{aligned}$$

Lemma 3.1 in [15] shows that $\det(\operatorname{pr}_L(I-K)^{-1}(Q-\mathcal{A})^{-1}(I-K)\operatorname{pr}_L) = \det V_{M,\mathcal{P}}$, from which together with Theorem 2.3 the result follows. □

Remark. The kernel of $(I+\frac{1}{2}(K^{-1}U_{\mathcal{P}}+U_{\mathcal{P}}^{-1}K))$ is L and hence we may write

$$\begin{aligned}
&\det\nolimits^*_{\mathrm{Fr}}\bigg(I+\frac{1}{2}\big(K^{-1}U_{\mathcal{P}}+U_{\mathcal{P}}^{-1}K\big)\bigg)\\
&\qquad = \det\nolimits_{\mathrm{Fr}}\bigg(I+\frac{1}{2}\big(K^{-1}U_{\mathcal{P}}+U_{\mathcal{P}}^{-1}K\big)\bigg)\Big|_{L^2(E_Y^+)^*}.
\end{aligned}$$

We next discuss the gluing formula of the zeta-determinants of Dirac Laplacians. Let $(\widehat{M},\widehat{g})$ be a closed Riemannian manifold and Y be a hypersurface of $\widehat{M}$ such that $\widehat{M}-Y$ has two components. We denote by M_1, M_2 the closure of each component, *that is*, $\widehat{M}=M_1\cup_Y M_2$. We assume that $\widehat{g}$ is a product metric on a collar neighborhood N of Y and N is isometric to $(-1,1)\times Y$. Let $\widehat{E}\to\widehat{M}$ be a Hermitian vector bundle having the product structure on N and $\mathcal{D}_{\widehat{M}}$ be a Dirac type operator acting on smooth sections of $\widehat{E}$ which has the form, on N, $\mathcal{D}_{\widehat{M}}=G(\partial_u+\mathcal{A})$ and satisfies (2.2) as before. Without loss of generality, we assume that ∂_u points outward on the boundary of M_1 and points inward on the boundary of M_2. We denote by $\mathcal{D}_{M_1}$, $\mathcal{D}_{M_2}$ the restriction of $\mathcal{D}_{\widehat{M}}$ to M_1, M_2 and denote by γ_0 the restriction map to Y. Suppose that $\{h_1,\dots,h_q\}$ is an orthonormal basis for $(\ker\mathcal{D}^2_{\widehat{M}})|_Y := \{\Phi|_Y \mid \mathcal{D}^2_{\widehat{M}}\Phi=0\}$, where $q=\dim\ker\mathcal{D}_{\widehat{M}}$. Then there exist $\Phi_1,\dots,\Phi_q$ in $\ker\mathcal{D}^2_{\widehat{M}}$ with $\Phi_i|_Y=h_i$.

We define a positive definite Hermitian matrix A_0 by

$$A_0 = (a_{ij}), \quad \text{where } a_{ij} = \langle \Phi_i, \Phi_j \rangle_{\widehat{M}}. \tag{2.17}$$

Let $\mathcal{C}_1$, $\mathcal{C}_2$ be Calderón projectors for $\mathcal{D}_{M_1}$, $\mathcal{D}_{M_2}$ and K_1, $K_2 : C^\infty(E_Y^+) \to C^\infty(E_Y^-)$ be unitary operators such that $\operatorname{graph}(K_i) = \operatorname{Im} \mathcal{C}_i$, $i = 1, 2$. The following result is due to P. Loya, J. Park ([16], [17]) and the second author ([15]), independently.

$$\begin{aligned}\log \operatorname{Det}^* \mathcal{D}^2_{\widehat{M}} - \log \operatorname{Det} \mathcal{D}^2_{M_1, \mathcal{C}_1} - \log \operatorname{Det} \mathcal{D}^2_{M_2, \mathcal{C}_2} \\ = -\log 2 \cdot \big(\zeta_{\mathcal{A}^2}(0) + l\big) + 2 \log \det A_0 \\ + 2 \log \left| \operatorname{det}^*_{\mathrm{Fr}} \left(\frac{1}{2} \big(I - K_1^{-1} K_2 \big) \right) \right|,\end{aligned} \tag{2.18}$$

where $l = \dim \ker \mathcal{A}$.

Remark. We note that $\mathcal{D}_{\widehat{M}} = G(\partial_u + \mathcal{A}) = -G(-\partial_u - \mathcal{A})$ near Y. We use the form $G(\partial_u + \mathcal{A})$ on M_2 so that $K_2 = U_{\Pi_>} + \mathfrak{F}_2$ for some smoothing operator $\mathfrak{F}_2$ by (2.10). Similarly, We use the form $-G(-\partial_u - \mathcal{A})$ on M_1 so that $K_1 = U_{\Pi_<} + \mathfrak{F}_1$ for some smoothing operator $\mathfrak{F}_1$. Since $U_{\Pi_<} = -U_{\Pi_>}$, $K_2 = -K_1 + \mathfrak{F}$ for some smoothing operator $\mathfrak{F}$ and hence $\frac{1}{2}(I - K_1^{-1} K_2)$ is of the form $I + \alpha$ for some trace class operator α. Moreover, The kernel of $I - K_1^{-1} K_2$ consists of $x \in L^2(E_Y^+)$ such that $x + K_1 x$ $(= x + K_2 x)$ can be extended to a harmonic section of $\widehat{\mathcal{D}}$ on $\widehat{M}$.

Theorem 2.4 and (2.18) lead to the following result, which is an analogue of Theorem 1.3 in [15].

Theorem 2.5. *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be orthogonal pseudodifferential projections satisfying the Condition A with respect to M_1 and M_2, respectively. Suppose that for $i = 1, 2$, $U_{\mathcal{P}_i} : C^\infty(E_Y^+) \to C^\infty(E_Y^-)$ is a unitary operator such that* $\operatorname{graph}(U_{\mathcal{P}_i}) = \operatorname{Im} \mathcal{P}_i$. *We also denote by $\mathcal{A}_i$ the tangential Dirac operator of $\mathcal{D}_{M_i}$ and by Q_i the Neumann jump operator with respect to $\mathcal{D}^2_{M_i}$ on M_i. Then the following equality holds.*

$$\begin{aligned}\log \operatorname{Det}^* \mathcal{D}^2_{\widehat{M}} - \log \operatorname{Det}^* \mathcal{D}^2_{M_1, \mathcal{P}_1} - \log \operatorname{Det}^* \mathcal{D}^2_{M_2, \mathcal{P}_2} \\ = -\log 2 \cdot \big(\zeta_{\mathcal{A}^2}(0) + l\big) + 2 \log \det A_0 \\ - 2 \sum_{i=1}^{2} \log \det V_{M_i, \mathcal{P}_i} + 2 \log \left| \operatorname{det}^*_{\mathrm{Fr}} \left(\frac{1}{2} \big(I - K_1^{-1} K_2 \big) \right) \right| \\ - \sum_{i=1}^{2} \log \operatorname{det}^*_{\mathrm{Fr}} \left(I + \frac{1}{2} \big(U_{\mathcal{P}_i}^{-1} K_i + K_i^{-1} U_{\mathcal{P}_i} \big) \right) \\ + \log 2 \sum_{i=1}^{2} \zeta_{(I - \mathcal{P}_i)(Q_i - \mathcal{A}_i)(I - \mathcal{P}_i)}(0).\end{aligned}$$

In the next two sections, we are going to apply Theorem 2.5 to some boundary conditions satisfying the Condition A.

3. Gluing formula of Dirac Laplacians with respect to $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$

Let (M, g) be an m-dimensional compact oriented Riemannian manifold with boundary Y and $E \to M$ be a Hermitian flat vector bundle with a flat connection ∇ which is compatible to the Hermitian structure on E. We extend ∇ to the de Rham operator acting on E-valued differential forms $\Omega^*(M, E)$, which we denote by ∇ again. We assume that near Y g is a product metric and E has a product structure. Using the Hodge star operator $*_M$, we define an involution $\Gamma : \Omega^q(M, E) \to \Omega^{m-q}(M, E)$ by

$$\Gamma\omega := i^{[\frac{m+1}{2}]}(-1)^{\frac{q(q+1)}{2}} *_M \omega, \quad \omega \in \Omega^q(M, E), \tag{3.1}$$

where $[\frac{m+1}{2}] = \frac{m}{2}$ for m even and $\frac{m+1}{2}$ for m odd. Then $\Gamma^2 = \mathrm{Id}$. The odd signature operator $\mathcal{B}$ acting on $\Omega^\bullet(M, E)$ is defined by

$$\mathcal{B} = \nabla\Gamma + \Gamma\nabla : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E). \tag{3.2}$$

Let u be the normal coordinate to Y. A differential form ω is expressed near Y by $\omega = \omega_{\tan} + du \wedge \omega_{\mathrm{nor}}$, where $\omega_{\tan}$ and ω_{nor} are called the tangential and normal parts of ω, respectively. Using the product structure, we can induce a flat connection $\nabla^Y : \Omega^\bullet(Y, E|_Y) \to \Omega^{\bullet+1}(Y, E|_Y)$ from ∇ and a Hodge star operator $*_Y : \Omega^\bullet(Y, E|_Y) \to \Omega^{m-1-\bullet}(Y, E|_Y)$ from $*_M$. We define two involutions β and Γ^Y by

$$\begin{aligned} \beta : \Omega^q(Y, E|_Y) \to \Omega^q(Y, E|_Y), &\qquad \beta(\omega) = (-1)^q\omega \\ \Gamma^Y : \Omega^q(Y, E|_Y) \to \Omega^{m-1-q}(Y, E|_Y), &\qquad \Gamma^Y(\omega) = i^{[\frac{m}{2}]}(-1)^{\frac{q(q+1)}{2}} *_Y \omega. \end{aligned} \tag{3.3}$$

Then $\beta^2 = (\Gamma^Y)^2 = \mathrm{Id}$. If we write $\phi_{\tan} + du \wedge \phi_{\mathrm{nor}}$ by $\binom{\phi_{\tan}}{\phi_{\mathrm{nor}}}$ near the boundary Y, $\mathcal{B}$ is written by

$$\begin{aligned} \mathcal{B} = \frac{1}{\sqrt{(-1)^m}}\beta\Gamma^Y \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big\{ \partial_u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ + \left(-\nabla^Y - \Gamma^Y\nabla^Y\Gamma^Y\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Big\}. \end{aligned} \tag{3.4}$$

Comparing (3.4) with (2.1), we have

$$G = \frac{1}{\sqrt{(-1)^m}}\beta\Gamma^Y \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{A} = -\left(\nabla^Y + \Gamma^Y\nabla^Y\Gamma^Y\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.5}$$

which satisfy the relations (2.2).

We next describe the boundary conditions $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$. We put $\mathcal{B}_Y := \Gamma^Y\nabla^Y + \nabla^Y\Gamma^Y$. Then $\mathcal{H}^\bullet(Y, E|_Y) := \ker \mathcal{B}_Y^2$ is a finite dimensional vector space and we can decompose

$$\Omega^\bullet(Y, E|_Y) = \operatorname{Im}\nabla^Y \oplus \operatorname{Im}\Gamma^Y\nabla^Y\Gamma^Y \oplus \mathcal{H}^\bullet(Y, E|_Y).$$

If $\nabla\phi = \Gamma\nabla\Gamma\phi = 0$ for $\phi \in \Omega^\bullet(M, E)$, simple computation shows that ϕ is expressed on Y by

$$\begin{aligned} \phi|_Y = \nabla^Y\varphi_1 + \varphi_2 + du \wedge \big(\Gamma^Y\nabla^Y\Gamma^Y\psi_1 + \psi_2\big), \\ \varphi_1, \psi_1 \in \Omega^\bullet(Y, E|_Y), \varphi_2, \psi_2 \in \mathcal{H}^\bullet(Y, E|_Y). \end{aligned} \tag{3.6}$$

Here φ_2 and ψ_2 are harmonic parts of $\iota^*\phi$ and $*_Y\iota^*(*_M\phi)$ up to sign, where $\iota : Y \to M$ is the natural inclusion. We define $\mathcal{K}$ by

$$\mathcal{K} := \big\{\varphi_2 \in \mathcal{H}^\bullet(Y, E|_Y) \mid \nabla\phi = \Gamma\nabla\Gamma\phi = 0\big\}, \tag{3.7}$$

where ϕ has the form (3.6). If ϕ satisfies $\nabla\phi = \Gamma\nabla\Gamma\phi = 0$, so is $\Gamma\phi$ and hence

$$\Gamma^Y\mathcal{K} = \big\{\psi_2 \in \mathcal{H}^\bullet(Y, E|_Y) \mid \nabla\phi = \Gamma\nabla\Gamma\phi = 0\big\}, \tag{3.8}$$

where ϕ has the form (3.6). Green formula (Corollary 2.3 in [9]) shows that $\mathcal{K}$ is perpendicular to $\Gamma^Y\mathcal{K}$. We then have the following decomposition (cf. Corollary 8.4 in [13], Lemma 2.4 in [9]).

$$\mathcal{K} \oplus \Gamma^Y\mathcal{K} = \mathcal{H}^\bullet(Y, E|_Y), \tag{3.9}$$

which shows that $(\mathcal{H}^\bullet(Y, E|_Y), \langle , \rangle_Y, \frac{1}{\sqrt{(-1)^m}}\beta\Gamma^Y)$ is a symplectic vector space with Lagrangian subspaces $\mathcal{K}$ and $\Gamma^Y\mathcal{K}$. We denote by

$$\mathcal{L}_0 = \begin{pmatrix} \mathcal{K} \\ \mathcal{K} \end{pmatrix}, \qquad \mathcal{L}_1 = \begin{pmatrix} \Gamma^Y\mathcal{K} \\ \Gamma^Y\mathcal{K} \end{pmatrix}. \tag{3.10}$$

We next define the orthogonal projections $\mathcal{P}_{-,\mathcal{L}_0}$, $\mathcal{P}_{+,\mathcal{L}_1} : \begin{pmatrix} \Omega^\bullet(Y,E|_Y) \\ \oplus \\ \Omega^\bullet(Y,E|_Y) \end{pmatrix} \to \begin{pmatrix} \Omega^\bullet(Y,E|_Y) \\ \oplus \\ \Omega^\bullet(Y,E|_Y) \end{pmatrix}$ by

$$\begin{aligned} \operatorname{Im}\mathcal{P}_{-,\mathcal{L}_0} &= \begin{pmatrix} \bigoplus_{q=0}^{m-1} \Omega^{q,-}(Y, E|_Y) \\ \bigoplus_{q=0}^{m-1} \Omega^{q,-}(Y, E|_Y) \end{pmatrix} \oplus \mathcal{L}_0, \\ \operatorname{Im}\mathcal{P}_{+,\mathcal{L}_1} &= \begin{pmatrix} \bigoplus_{q=0}^{m-1} \Omega^{q,+}(Y, E|_Y) \\ \bigoplus_{q=0}^{m-1} \Omega^{q,+}(Y, E|_Y) \end{pmatrix} \oplus \mathcal{L}_1, \end{aligned} \tag{3.11}$$

where $\Omega^{\bullet,-}(Y, E|_Y) := \operatorname{Im}\nabla^Y$ and $\Omega^{\bullet,+}(Y, E|_Y) := \operatorname{Im}\Gamma^Y\nabla^Y\Gamma^Y$. Then $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ are pseudodifferential operators and give well-posed boundary conditions for $\mathcal{B}$ and the refined analytic torsion (Lemma 2.15 in [9]). The authors discussed the boundary problem of the refined analytic torsion on compact manifolds with boundary with these boundary conditions in [9], [10], [11]. We denote by $\mathcal{B}_{\mathcal{P}_{-,\mathcal{L}_0}}$ and $\mathcal{B}^2_{q,\mathcal{P}_{-,\mathcal{L}_0}}$ the realizations of $\mathcal{B}$ and $\mathcal{B}^2_q$ with respect to $\mathcal{P}_{-,\mathcal{L}_0}$, i.e.

$$
\begin{aligned}
\operatorname{Dom}(\mathcal{B}_{\mathcal{P}_{-,\mathcal{L}_0}}) &= \big\{\psi \in \Omega^\bullet(M,E) \mid \mathcal{P}_{-,\mathcal{L}_0}(\psi|_Y) = 0\big\},\\
\operatorname{Dom}\big(\mathcal{B}^2_{q,\mathcal{P}_{-,\mathcal{L}_0}}\big) &= \big\{\psi \in \Omega^q(M,E) \mid \mathcal{P}_{-,\mathcal{L}_0}(\psi|_Y) = 0,\\
&\qquad \mathcal{P}_{-,\mathcal{L}_0}\big((\mathcal{B}\psi)|_Y\big) = 0\big\}.
\end{aligned} \tag{3.12}
$$

We define $\mathcal{B}_{\mathcal{P}_{+,\mathcal{L}_1}}$, $\mathcal{B}^2_{q,\mathcal{P}_{+,\mathcal{L}_1}}$ in the same way. For $\psi = \psi_{\tan} + du \wedge \psi_{\mathrm{nor}} \in \Omega^q(M,E)$, we define $\mathcal{B}^2_{q,\mathrm{rel}}$ and $\mathcal{B}^2_{q,\mathrm{abs}}$ by

$$
\begin{aligned}
\operatorname{Dom}\big(\mathcal{B}^2_{q,\mathrm{rel}}\big) &= \big\{\psi \in \Omega^q(M,E) \mid \psi_{\tan}|_Y = 0, (\partial_u \psi_{\mathrm{nor}})|_Y = 0\big\},\\
\operatorname{Dom}\big(\mathcal{B}^2_{q,\mathrm{abs}}\big) &= \big\{\psi \in \Omega^q(M,E) \mid (\partial_u \psi_{\tan})|_Y = 0, \psi_{\mathrm{nor}}|_Y = 0\big\}.
\end{aligned} \tag{3.13}
$$

The following result is straightforward (Lemma 2.11 in [9]).

Lemma 3.1.

$$
\begin{aligned}
\ker \mathcal{B}^2_{q,\mathcal{P}_{-,\mathcal{L}_0}} &= \ker \mathcal{B}^2_{q,\mathrm{rel}} = H^q(M,Y;E),\\
\ker \mathcal{B}^2_{q,\mathcal{P}_{+,\mathcal{L}_1}} &= \ker \mathcal{B}^2_{q,\mathrm{abs}} = H^q(M;E).
\end{aligned}
$$

We denote by $(\Omega^\bullet(M,E)|_Y)^*$ the orthogonal complement of $\left(\begin{smallmatrix}\mathcal{H}^\bullet(Y,E|_Y)\\ \mathcal{H}^\bullet(Y,E|_Y)\end{smallmatrix}\right)$ in $(\Omega^\bullet(M,E)|_Y)$. Then the action of the unitary operator G splits according to the following decomposition.

$$
\begin{aligned}
G : \big(\Omega^\bullet(M,E)|_Y\big)^* \oplus \begin{pmatrix}\mathcal{H}^\bullet(Y,E|_Y)\\ \mathcal{H}^\bullet(Y,E|_Y)\end{pmatrix}\\
\to \big(\Omega^\bullet(M,E)|_Y\big)^* \oplus \begin{pmatrix}\mathcal{H}^\bullet(Y,E|_Y)\\ \mathcal{H}^\bullet(Y,E|_Y)\end{pmatrix}.
\end{aligned} \tag{3.14}
$$

We define unitary maps $U_{\mathcal{P}_-}$, $U_{\Pi_>} : (\Omega^\bullet(M,E)|_Y)^* \to (\Omega^\bullet(M,E)|_Y)^*$ by

$$
\begin{aligned}
U_{\mathcal{P}_-} &= \big(\mathcal{B}^2_Y\big)^{-1}\big(\big(\mathcal{B}^2_Y\big)^- - \big(\mathcal{B}^2_Y\big)^+\big)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\\
U_{\Pi_>} &= \big(\mathcal{B}^2_Y\big)^{-\frac{1}{2}}\big(\nabla^Y + \Gamma^Y \nabla^Y \Gamma^Y\big)\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},
\end{aligned} \tag{3.15}
$$

where $(\mathcal{B}^2_Y)^- := \nabla^Y \Gamma^Y \nabla^Y \Gamma^Y$, $(\mathcal{B}^2_Y)^+ := \Gamma^Y \nabla^Y \Gamma^Y \nabla^Y$ and $\mathcal{B}^2_Y$ is understood to be defined on $(\Omega^\bullet(M,E)|_Y)^*$. We denote the $\pm i$-eigenspace of G in $(\Omega^\bullet(M,E)|_Y)^*$, $\left(\begin{smallmatrix}\mathcal{H}^\bullet(Y,E|_Y)\\ \mathcal{H}^\bullet(Y,E|_Y)\end{smallmatrix}\right)$ and $\Omega^\bullet(M,E)|_Y$ by

$$
\begin{aligned}
\big(\Omega^\bullet(M,E)|_Y\big)^*_{\pm i} &:= \frac{1}{2}(I \mp iG)\big(\Omega^\bullet(M,E)|_Y\big)^*,\\
(\ker \mathcal{A})_{\pm i} &:= \frac{1}{2}(I \mp iG)\begin{pmatrix}\mathcal{H}^\bullet(Y,E|_Y)\\ \mathcal{H}^\bullet(Y,E|_Y)\end{pmatrix},\\
\big(\Omega^\bullet(M,E)|_Y\big)_{\pm i} &:= \big(\Omega^\bullet(M,E)\big)^*_{\pm i} \oplus (\ker \mathcal{A})_{\pm i}.
\end{aligned} \tag{3.16}
$$

The following lemma is straightforward (cf. (3.2)–(3.5) and Lemma 3.1 in [10]).

LEMMA 3.2. (1) $U_{\mathcal{P}_-}$ *and* $U_{\Pi_>}$ *map* $(\Omega^\bullet(M,E)|_Y)^*_{\pm i}$ *onto* $(\Omega^\bullet(M,E)|_Y)^*_{\mp i}$.
(2) $U_{\mathcal{P}_-}U_{\mathcal{P}_-} = U_{\Pi_>}U_{\Pi_>} = \mathrm{Id}$. *Hence,* $U^*_{\mathcal{P}_-} = U_{\mathcal{P}_-}$ *and* $U^*_{\Pi_>} = U_{\Pi_>}$.
(3) $U^*_{\Pi_>}U_{\mathcal{P}_-} + U^*_{\mathcal{P}_-}U_{\Pi_>} = 0$.
(4) $\mathrm{Im}\,\mathcal{P}_- = \{\omega + U_{\mathcal{P}_-}\omega \mid \omega \in (\Omega^\bullet(M,E)|_Y)^*_{+i}\}$, $\mathrm{Im}\,\Pi_> = \{\omega + U_{\Pi_>}\omega \mid \omega \in (\Omega^\bullet(M,E)|_Y)^*_{+i}\}$.

We next choose a unitary map $U_{\mathcal{L}_0} : (\ker\mathcal{A})_{+i} \to (\ker\mathcal{A})_{-i}$ so that $\mathrm{graph}(U_{\mathcal{L}_0}) = \mathrm{Im}\,\mathcal{L}_0$ and define $U_{\mathcal{P}_-,\mathcal{L}_0}$, $U_{\Pi_>,\mathcal{L}_0} : (\Omega^\bullet(M,E)|_Y)_{+i} \to (\Omega^\bullet(M,E)|_Y)_{-i}$ by

$$U_{\mathcal{P}_-,\mathcal{L}_0} = U_{\mathcal{P}_-}|_{(\Omega^\bullet(M,E)|_Y)^*_{+i}} + U_{\mathcal{L}_0}, \qquad U_{\Pi_>,\mathcal{L}_0} = U_{\Pi_>}|_{(\Omega^\bullet(M,E)|_Y)^*_{+i}} + U_{\mathcal{L}_0}. \tag{3.17}$$

Then $\mathrm{graph}(U_{\mathcal{P}_-,\mathcal{L}_0}) = \mathrm{Im}\,\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathrm{graph}(U_{\Pi_>,\mathcal{L}_0}) = \mathrm{Im}\,\Pi_{>,\mathcal{L}_0}$. By (3.5) we have

$$\mathcal{P}_{-,\mathcal{L}_0}\mathcal{A}\mathcal{P}_{-,\mathcal{L}_0} = \mathcal{P}_{+,\mathcal{L}_1}\mathcal{A}\mathcal{P}_{+,\mathcal{L}_1} = 0. \tag{3.18}$$

Moreover, Theorem 2.1 in [14] shows that for $t \geq 0$,

$$Q(t) = \sqrt{\mathcal{A}^2 + t} + \text{a smoothing operator}, \tag{3.19}$$

which together with (3.18) shows that

$$\begin{aligned}\mathcal{P}_{-,\mathcal{L}_0}\big(Q(t) - \mathcal{A}\big)\mathcal{P}_{-,\mathcal{L}_0} &= \mathcal{P}_{-,\mathcal{L}_0}Q(t)\mathcal{P}_{-,\mathcal{L}_0}\\ &= \mathcal{P}_{-,\mathcal{L}_0}\sqrt{\mathcal{A}^2 + t}\,\mathcal{P}_{-,\mathcal{L}_0} + \text{a smoothing operator.}\end{aligned}$$

The same equality holds for $\mathcal{P}_{+,\mathcal{L}_1}$. This shows that $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ satisfy the item (4) in the Condition A. Since $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ are orthogonal pseudodifferential projections and $U_{\mathcal{P}_{+,\mathcal{L}_1}} = -U_{\mathcal{P}_{-,\mathcal{L}_0}}$, the assertion (3) in Lemma 3.2 shows that $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ satisfy the item (2), (3) in the Condition A and hence satisfy the Condition A.

Let $(\widehat{M}, \widehat{g})$ be a closed Riemannian manifold and Y be a hypersurface of $\widehat{M}$ such that $\widehat{M} - Y$ has two components, whose closures are denoted by M_1, M_2, i.e. $\widehat{M} = M_1 \cup_Y M_2$. We assume that $\widehat{g}$ is a product metric near Y. We denote the odd signature operator on $\widehat{M}$ by $\mathcal{B}_{\widehat{M}}$ and its restriction to M_1 and M_2 by $\mathcal{B}_{M_1}$ and $\mathcal{B}_{M_2}$. We now apply Theorem 2.5 with $\mathcal{P}_1 = \mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_2 = I - \mathcal{P}_{-,\mathcal{L}_0} = \mathcal{P}_{+,\mathcal{L}_1}$. Then we have the following equality.

$$\begin{aligned}&\log\mathrm{Det}^*\mathcal{B}^2_{\widehat{M}} - \log\mathrm{Det}^*\mathcal{B}^2_{M_1,\mathcal{P}_{-,\mathcal{L}_0}} - \log\mathrm{Det}^*\mathcal{B}^2_{M_2,\mathcal{P}_{+,\mathcal{L}_1}}\\ &\quad= -\log 2\cdot\big(\zeta_{\mathcal{A}^2}(0) + l\big) + 2\log\det A_0\\ &\qquad - 2(\log\det V_{M_1,\mathcal{P}_{-,\mathcal{L}_0}} + \log\det V_{M_2,\mathcal{P}_{+,\mathcal{L}_1}})\end{aligned} \tag{3.20}$$

$$
\begin{aligned}
&+ 2\log\left|\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\big(I - K_1^{-1}K_2\big)\right)\right| \\
&- \left\{\log\det{}^*_{\mathrm{Fr}}\left(I + \frac{1}{2}\big(U^{-1}_{\mathcal{P}_{-,\mathcal{L}_0}}K_1 + K_1^{-1}U_{\mathcal{P}_{-,\mathcal{L}_0}}\big)\right)\right. \\
&\left. + \log\det{}^*_{\mathrm{Fr}}\left(I - \frac{1}{2}\big(U^{-1}_{\mathcal{P}_{-,\mathcal{L}_0}}K_2 + K_2^{-1}U_{\mathcal{P}_{-,\mathcal{L}_0}}\big)\right)\right\} \\
&+ \log 2\big(\zeta_{(\mathcal{P}_{-,\mathcal{L}_0}(Q_1-\mathcal{A}_1)\mathcal{P}_{-,\mathcal{L}_0})}(0) + \zeta_{(\mathcal{P}_{+,\mathcal{L}_1}(Q_2-\mathcal{A}_2)\mathcal{P}_{+,\mathcal{L}_1})}(0)\big),
\end{aligned}
$$

where $K_i : (\Omega^\bullet(M_i,E)|_Y)_{+i} \to (\Omega^\bullet(M_i,E)|_Y)_{-i}$ is a unitary operator such that $\mathrm{graph}(K_i) = \mathrm{Im}\,\mathcal{C}_i$, the Cauchy data space with respect to $\mathcal{B}_{M_i}$. By Lemma 3.1, we have

$$
\begin{aligned}
\log\det A_0 &= \sum_{q=0}^{m} \log\det A_{0,q}, \\
\log\det V_{M_1,\mathcal{P}_{-,\mathcal{L}_0}} &= \sum_{q=0}^{m} \log\det V_{M_1,q,\mathrm{rel}}, \qquad (3.21)\\
\log\det V_{M_2,\mathcal{P}_{+,\mathcal{L}_1}} &= \sum_{q=0}^{m} \log\det V_{M_2,q,\mathrm{abs}},
\end{aligned}
$$

where $A_{0,q}$ is the Hermitian matrix obtained by simply replacing $\mathcal{D}^2_{\widehat{M}}$ in (2.17) with $\mathcal{B}^2_{\widehat{M},q}$ acting on $\Omega^q(\widehat{M},\widehat{E})$. Similarly, $V_{M_i,q,\mathrm{rel}\,/\,\mathrm{abs}}$ is the Hermitian matrix obtained by replacing $\mathcal{D}^2_{M,P}$ in (2.12) with $\mathcal{B}^2_{M_i,q,\mathrm{rel}\,/\,\mathrm{abs}}$ acting on $\Omega^q(M_i,E)$ satisfying the relative/absolute boundary conditions. Lemma 2.5 in [15] and Lemma 3.1 show that

$$
\begin{aligned}
\dim\ker\big(\mathcal{P}_{-,\mathcal{L}_0}(Q_1-\mathcal{A}_1)\mathcal{P}_{-,\mathcal{L}_0}\big) &= \sum_{q=0}^{m}\beta_q(M_1) = \sum_{q=0}^{m}\beta_q(M_1,Y), \qquad (3.22)\\
\dim\ker\big(\mathcal{P}_{+,\mathcal{L}_1}(Q_2-\mathcal{A}_2)\mathcal{P}_{+,\mathcal{L}_1}\big) &= \sum_{q=0}^{m}\beta_q(M_2,Y) = \sum_{q=0}^{m}\beta_q(M_2),
\end{aligned}
$$

where $\beta_q(M_i,Y) := \dim H^q(M_i,Y;E)$ and $\beta_q(M_i) := \dim H^q(M_i;E)$. By (3.18) and (3.22), we have

$$
\begin{aligned}
&\zeta_{(\mathcal{P}_{-,\mathcal{L}_0}Q_1\mathcal{P}_{-,\mathcal{L}_0})}(0) + \dim\ker(\mathcal{P}_{-,\mathcal{L}_0}Q_1\mathcal{P}_{-,\mathcal{L}_0}) \qquad (3.23)\\
&\quad = \zeta_{\mathcal{P}_{-,\mathcal{L}_0}\sqrt{\mathcal{A}_1^2}\mathcal{P}_{-,\mathcal{L}_0}}(0) + \dim\ker\mathcal{P}_{-,\mathcal{L}_0}\sqrt{\mathcal{A}_1^2}\mathcal{P}_{-,\mathcal{L}_0} \\
&\quad = \sum_{q=0}^{m-1}\big(\zeta_{\mathcal{B}^2_{Y,q}}(0) + \beta_q(Y)\big),
\end{aligned}
$$

where $\beta_q(Y) := \dim\ker H^q(Y; E|_Y)$. Similarly, we have

$$\begin{aligned}\zeta_{(\mathcal{P}_{+,\mathcal{L}_1} Q_2 \mathcal{P}_{+,\mathcal{L}_1})}(0) &+ \dim\ker(\mathcal{P}_{+,\mathcal{L}_1} Q_2 \mathcal{P}_{+,\mathcal{L}_1}) \\ &= \sum_{q=0}^{m-1} \big(\zeta_{\mathcal{B}^2_{Y,q}}(0) + \beta_q(Y)\big).\end{aligned} \tag{3.24}$$

On the other hand,

$$\zeta_{\mathcal{A}^2}(0) + l = 2\sum_{q=0}^{m-1} \big(\zeta_{\mathcal{B}^2_{Y,q}}(0) + \beta_q(Y)\big). \tag{3.25}$$

Summarizing the above argument, we have the following result, which is the main result of this section.

THEOREM 3.3. *Let $(\widehat{M}, \widehat{g})$ be a closed Riemannian manifold and Y be a hypersurface of $\widehat{M}$ with $\widehat{M} = M_1 \cup_Y M_2$. We assume that $\widehat{g}$ is a product metric near Y. Then*:

$$\begin{aligned}&\sum_{q=0}^{m} \big(\log \mathrm{Det}^* \mathcal{B}^2_{\widehat{M},q} - \log \mathrm{Det}^* \mathcal{B}^2_{M_1,q,\mathcal{P}_{-,\mathcal{L}_0}} - \log \mathrm{Det}^* \mathcal{B}^2_{M_2,q,\mathcal{P}_{+,\mathcal{L}_1}}\big) \\ &\quad = -\log 2 \cdot \sum_{q=0}^{m} \big(\beta_q(M_1) + \beta_q(M_2)\big) + 2\sum_{q=0}^{m} \log\det A_{0,q} \\ &\qquad - 2\sum_{q=0}^{m} (\log\det V_{M_{1,q,\mathrm{rel}}} + \log\det V_{M_{2,q,\mathrm{abs}}}) \\ &\qquad + 2\log\left|\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\big(I - K_1^{-1}K_2\big)\right)\right| \\ &\qquad - \log\det{}^*_{\mathrm{Fr}}\left(I + \frac{1}{2}\big(U^{-1}_{\mathcal{P}_{-,\mathcal{L}_0}} K_1 + K_1^{-1} U_{\mathcal{P}_{-,\mathcal{L}_0}}\big)\right) \\ &\qquad - \log\det{}^*_{\mathrm{Fr}}\left(I - \frac{1}{2}\big(U^{-1}_{\mathcal{P}_{-,\mathcal{L}_0}} K_2 + K_2^{-1} U_{\mathcal{P}_{-,\mathcal{L}_0}}\big)\right).\end{aligned}$$

REMARK. The kernel of $(I + \frac{1}{2}(U^{-1}_{\mathcal{P}_{-,\mathcal{L}_0}} K_1 + K_1^{-1} U_{\mathcal{P}_{-,\mathcal{L}_0}}))$ consists of $\omega \in (\Omega^\bullet(M_1, E)|_Y)_{+i}$ such that $\omega - U_{\mathcal{P}_{-,\mathcal{L}_0}}\omega (= \omega + K_1\omega)$ can be extended to a solution of $\mathcal{B}_{M_1,\mathcal{P}_{-,\mathcal{L}_0}}$. The same result holds for $(I - \frac{1}{2}(U^{-1}_{\mathcal{P}_{-,\mathcal{L}_0}} K_2 + K_2^{-1} U_{\mathcal{P}_{-,\mathcal{L}_0}}))$.

4. Gluing formula of Dirac Laplacians with respect to the absolute and relative boundary conditions

We continue to use the same notations as in the previous section. In this section, we consider a double of de Rham complexes $\Omega^\bullet(M, E \oplus E) :=$

$\Omega^\bullet(M,E)\oplus\Omega^\bullet(M,E)$, which was used in [24]. We define the odd signature operator $\widetilde{\mathcal{B}}$ and a boundary condition $\widetilde{\mathcal{P}}$ in this context as follows.

$$\begin{aligned}\widetilde{\mathcal{B}} &= \begin{pmatrix} 0 & \mathcal{B} \\ \mathcal{B} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \Gamma\nabla+\nabla\Gamma \\ \Gamma\nabla+\nabla\Gamma & 0 \end{pmatrix} : \Omega^\bullet(M,E\oplus E)\to\Omega^\bullet(M,E\oplus E) \\ \widetilde{\mathcal{P}} &= \begin{pmatrix} \mathcal{P}_{\mathrm{rel}} & 0 \\ 0 & \mathcal{P}_{\mathrm{abs}} \end{pmatrix} : \Omega^\bullet(M,E\oplus E)|_Y\to\Omega^\bullet(M,E\oplus E)|_Y,\end{aligned} \tag{4.1}$$

where $\mathcal{P}_{\mathrm{rel}}$ and $\mathcal{P}_{\mathrm{abs}}$ are orthogonal projections defined by

$$\begin{aligned}\mathcal{P}_{\mathrm{rel}}(\omega_{\tan}|_Y + du\wedge\omega_{\mathrm{nor}}|_Y) &= \omega_{\tan}|_Y, \\ \mathcal{P}_{\mathrm{abs}}(\omega_{\tan}|_Y + du\wedge\omega_{\mathrm{nor}}|_Y) &= \omega_{\mathrm{nor}}|_Y.\end{aligned} \tag{4.2}$$

Then the realization $\widetilde{\mathcal{B}}^2_{\widetilde{\mathcal{P}}}$ with respect to the boundary condition $\widetilde{\mathcal{P}}$ is given as follows.

$$\begin{aligned}\mathrm{Dom}\big(\widetilde{\mathcal{B}}^2_{\widetilde{\mathcal{P}}}\big) &= \left\{ \begin{pmatrix}\phi\\ \psi\end{pmatrix}\in\Omega^\bullet(M,E\oplus E) \;\middle|\; \widetilde{\mathcal{P}}\begin{pmatrix}\phi|_Y\\ \psi|_Y\end{pmatrix}=0, \right. \\ &\qquad \left. \widetilde{\mathcal{P}}\left(\widetilde{\mathcal{B}}\begin{pmatrix}\phi\\ \psi\end{pmatrix}\bigg|_Y\right)=0 \right\} \\ &= \left\{ \begin{pmatrix}\phi_1+du\wedge\phi_2\\ \psi_1+du\wedge\psi_2\end{pmatrix} \;\middle|\; \phi_1|_Y=0, \right. \\ &\qquad \left. (\partial_u\phi_2)|_Y=0, (\partial_u\psi_1)|_Y=0, \psi_2|_Y=0 \right\}.\end{aligned} \tag{4.3}$$

By (3.13) and the Poincaré duality, we have

$$\begin{aligned}\widetilde{\mathcal{B}}^2_{\widetilde{\mathcal{P}}} &= \begin{pmatrix}\mathcal{B}^2_{M,\mathrm{rel}} & 0\\ 0 & \mathcal{B}^2_{M,\mathrm{abs}}\end{pmatrix}, \\ \log\mathrm{Det}^*\widetilde{\mathcal{B}}^2_{\widetilde{\mathcal{P}}} &= \sum_{q=0}^{m}\big(\log\mathrm{Det}^*\mathcal{B}^2_{M,q,\mathrm{rel}} + \log\mathrm{Det}^*\mathcal{B}^2_{M,q,\mathrm{abs}}\big) \\ &= 2\sum_{q=0}^{m}\log\mathrm{Det}^*\mathcal{B}^2_{M,q,\mathrm{rel}}.\end{aligned} \tag{4.4}$$

We put

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad L=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad S=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{4.5}$$

If we write $\left(\begin{smallmatrix}\phi_1+du\wedge\phi_2\\ \psi_1+du\wedge\psi_2\end{smallmatrix}\right)$ by $\begin{pmatrix}\phi_1\\ \phi_2\\ \psi_1\\ \psi_2\end{pmatrix}$, $\widetilde{\mathcal{B}}$ is written, near the boundary Y, by

$$\begin{aligned}\widetilde{\mathcal{B}} &= \frac{1}{\sqrt{(-1)^m}}\beta\Gamma^Y\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\left\{\partial_u-\left(\nabla^Y+\Gamma^Y\nabla^Y\Gamma^Y\right)\begin{pmatrix}L & 0\\ 0 & L\end{pmatrix}\right\}\\ &=\widetilde{G}(\partial_u+\widetilde{\mathcal{A}}).\end{aligned}\tag{4.6}$$

Comparing (4.6) with (2.1), we have

$$\widetilde{G}=\frac{1}{\sqrt{(-1)^m}}\beta\Gamma^Y\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\qquad \widetilde{\mathcal{A}}=-\left(\nabla^Y+\Gamma^Y\nabla^Y\Gamma^Y\right)\begin{pmatrix}L & 0\\ 0 & L\end{pmatrix},\tag{4.7}$$

which satisfy the relations in (2.2). We denote by $\widetilde{\Pi}_> := \Pi_>\oplus\Pi_>$ the orthogonal projection onto the space spanned by positive eigenforms of $\widetilde{\mathcal{A}}$. We denote the $\pm i$-eigenspace of $\widetilde{G}$ by

$$\left(\Omega^\bullet(M,E\oplus E)|_Y\right)_{\pm i} := \frac{1}{2}(I\mp i\widetilde{G})\left(\Omega^\bullet(M,E)|_Y\oplus\Omega^\bullet(M,E)|_Y\right).\tag{4.8}$$

For instance, if m is odd, simple computation shows that

$$\begin{aligned}\left(\Omega^\bullet(M,E\oplus E)|_Y\right)_{+i} &= \operatorname{span}\begin{pmatrix}\omega_1\\ \omega_2\\ -\beta\Gamma^Y\omega_1\\ -\beta\Gamma^Y\omega_2\end{pmatrix},\\ \left(\Omega^\bullet(M,E\oplus E)|_Y\right)_{-i} &= \operatorname{span}\begin{pmatrix}\omega_1\\ \omega_2\\ \beta\Gamma^Y\omega_1\\ \beta\Gamma^Y\omega_2\end{pmatrix},\end{aligned}\tag{4.9}$$

where $\omega_1,\omega_2\in\Omega^\bullet(Y,E|_Y)$. This fact will be used in (4.23) below. Like (3.14), we write

$$\begin{aligned}\Omega^\bullet(M,E\oplus E)|_Y &= \left(\Omega^\bullet(M,E\oplus E)|_Y\right)^*\oplus\ker\widetilde{\mathcal{A}},\ \left(\Omega^\bullet(M,E\oplus E)|_Y\right)^*\\ &=: (\ker\widetilde{\mathcal{A}})^\perp.\end{aligned}\tag{4.10}$$

We define unitary maps $U_{\widetilde{\mathcal{P}}}$, $U_{\widetilde{\Pi}_>} : (\Omega^\bullet(M,E\oplus E)|_Y)^*\to(\Omega^\bullet(M,E\oplus E)|_Y)^*$ by [(3.15)]

$$\begin{aligned}U_{\widetilde{\mathcal{P}}} &= \sqrt{(-1)^{m+1}}\beta\Gamma^Y\begin{pmatrix}0 & -S\\ S & 0\end{pmatrix},\\ U_{\widetilde{\Pi}_>} &= \left(\mathcal{B}_Y^2\right)^{-\frac{1}{2}}\left(\nabla^Y+\Gamma^Y\nabla^Y\Gamma^Y\right)\begin{pmatrix}-L & 0\\ 0 & -L\end{pmatrix}.\end{aligned}\tag{4.11}$$

Here the domain of $U_{\widetilde{\mathcal{P}}}$ can be naturally extended to $\Omega^\bullet(M,E\oplus E)|_Y$. The following lemma is an analogue of Lemma 3.2, whose proof is straightforward.

Lemma 4.1. (1) *$U_{\widetilde{\mathcal{P}}}$ and $U_{\widetilde{\Pi}_>}$ map $(\Omega^\bullet(M, E\oplus E)|_Y)^*_{\pm i}$ onto $(\Omega^\bullet(M, E\oplus E)|_Y)^*_{\mp i}$.*

(2) *$U_{\widetilde{\mathcal{P}}}U_{\widetilde{\mathcal{P}}} = -\mathrm{Id}$, $U_{\widetilde{\Pi}_>}U_{\widetilde{\Pi}_>} = \mathrm{Id}$ and $U_{\widetilde{\Pi}_>}U_{\widetilde{\mathcal{P}}} = U_{\widetilde{\mathcal{P}}}U_{\widetilde{\Pi}_>}$. Hence, $U^*_{\widetilde{\mathcal{P}}} = -U_{\widetilde{\mathcal{P}}}$ and $U^*_{\widetilde{\Pi}_>} = U_{\widetilde{\Pi}_>}$.*

(3) *$U^*_{\widetilde{\Pi}_>}U_{\widetilde{\mathcal{P}}} + U^*_{\widetilde{\mathcal{P}}}U_{\widetilde{\Pi}_>} = 0$.*

(4) *$\mathrm{Im}\,\widetilde{\mathcal{P}} = \{\omega + U_{\widetilde{\mathcal{P}}}\omega \mid \omega \in (\Omega^\bullet(M, E\oplus E)|_Y)_{+i}\}$, $\mathrm{Im}\,\widetilde{\Pi}_> = \{\omega + U_{\widetilde{\Pi}_>}\omega \mid \omega \in (\Omega^\bullet(M, E\oplus E)|_Y)^*_{+i}\}$.*

Remark. It is not difficult to see that there is no unitary map from $(\Omega^\bullet(M, E)|_Y)_{+i}$ to $(\Omega^\bullet(M, E)|_Y)_{-i}$ whose graph is $\mathrm{Im}\,\mathcal{P}_{\mathrm{rel}}$ or $\mathrm{Im}\,\mathcal{P}_{\mathrm{abs}}$. Hence, we cannot apply Theorem 2.5 to this case. This is the reason why we consider the double of de Rham complexes as above.

It is straightforward that

$$\widetilde{\mathcal{P}}\widetilde{\mathcal{A}}\widetilde{\mathcal{P}} = (I - \widetilde{\mathcal{P}})\widetilde{\mathcal{A}}(I - \widetilde{\mathcal{P}}) = 0, \tag{4.12}$$

and by Theorem 2.1 in [14] (cf. (3.19)) we have

$$\widetilde{Q}(t) = \begin{pmatrix} Q(t) & 0 \\ 0 & Q(t) \end{pmatrix} = \begin{pmatrix} \sqrt{\mathcal{A}^2 + t} & 0 \\ 0 & \sqrt{\mathcal{A}^2 + t} \end{pmatrix} + \text{a smoothing operator}, \tag{4.13}$$

which shows that $\widetilde{\mathcal{P}}$ and $I - \widetilde{\mathcal{P}}$ satisfy the item (4) in the Condition A. This fact and the assertion (3) in Lemma 4.1 show that $\widetilde{\mathcal{P}}$ and $I - \widetilde{\mathcal{P}}$ satisfy the Condition A, as in the previous section,

We next consider a partitioned manifold $\widehat{M} = M_1 \cup_Y M_2$ as before. We assume the same assumptions as in Theorem 2.5. Let $\widetilde{K}_i : (\Omega^\bullet(M_i, E\oplus E)|_Y)_{+i} \to (\Omega^\bullet(M_i, E\oplus E)|_Y)_{-i}$ be a unitary operator such that $\mathrm{graph}(\widetilde{K}_i) = \mathrm{Im}\,\widetilde{\mathcal{C}}_i$, the Cauchy data space with respect to $\widetilde{\mathcal{B}}_{M_i}$. We denote by $\widetilde{Q}_i$ the Neumann jump operator for $\widetilde{\mathcal{B}}^2_{M_i}$ on M_i and by $\widetilde{\mathcal{A}}_i$ the tangential Dirac operator of $\widetilde{\mathcal{B}}_{M_i}$. We now apply Theorem 2.5 with $\mathcal{P}_1 = \widetilde{\mathcal{P}}$ and $\mathcal{P}_2 = I - \widetilde{\mathcal{P}}$. Since $U_{I-\widetilde{\mathcal{P}}} = -U_{\widetilde{\mathcal{P}}}$, we have the following equality.

$$\begin{aligned} &\log \mathrm{Det}^* \widetilde{\mathcal{B}}^2_{\widehat{M}} - \log \mathrm{Det}^* \widetilde{\mathcal{B}}^2_{M_1,\widetilde{\mathcal{P}}} - \log \mathrm{Det}^* \widetilde{\mathcal{B}}_{M_2,I-\widetilde{\mathcal{P}}} \\ &\quad = -\log 2 \cdot \big(\zeta_{\widetilde{\mathcal{A}}^2}(0) + \widetilde{l}\big) \\ &\qquad + 2\log\det \widetilde{A}_0 - 2(\log\det \widetilde{V}_{M_1,\widetilde{\mathcal{P}}} + \log\det \widetilde{V}_{M_2,I-\widetilde{\mathcal{P}}}) \\ &\qquad + 2\log\left|\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\big(I - \widetilde{K}_1^{-1}\widetilde{K}_2\big)\right)\right| \\ &\qquad - \left\{\log\det{}^*_{\mathrm{Fr}}\left(I + \frac{1}{2}\big(U^{-1}_{\widetilde{\mathcal{P}}}\widetilde{K}_1 + \widetilde{K}_1^{-1}U_{\widetilde{\mathcal{P}}}\big)\right)\right. \end{aligned} \tag{4.14}$$

$$+ \log \det\nolimits^*_{\mathrm{Fr}} \left(I - \frac{1}{2}\big(U_{\widetilde{\mathcal{P}}}^{-1} \widetilde{K}_2 + \widetilde{K}_2^{-1} U_{\widetilde{\mathcal{P}}}\big) \right) \Bigg\}$$
$$+ \log 2 \cdot \big(\zeta_{((I-\widetilde{\mathcal{P}})(\widetilde{Q}_1 - \widetilde{\mathcal{A}}_1)(I-\widetilde{\mathcal{P}}))}(0) + \zeta_{(\widetilde{\mathcal{P}}(\widetilde{Q}_2 - \widetilde{\mathcal{A}}_2)\widetilde{\mathcal{P}})}(0)\big).$$

With the same notations in (3.21), we note that

$$\log \det \widetilde{A}_0 = 2 \sum_{q=0}^{m} \log \det A_{0,q},$$

$$\begin{aligned} \log \det \widetilde{V}_{M_i, \widetilde{\mathcal{P}}} &= \log \det \widetilde{V}_{M_i, I-\widetilde{\mathcal{P}}} \\ &= \sum_{q=0}^{m} (\log \det V_{M_i,q,\mathrm{rel}} + \log \det V_{M_i,q,\mathrm{abs}}) \\ &= 2 \sum_{q=0}^{m} \log \det V_{M_i,q,\mathrm{rel}} = 2 \sum_{q=0}^{m} \log \det V_{M_i,q,\mathrm{abs}}. \end{aligned} \tag{4.15}$$

Lemma 2.5 in [15] shows that

$$\begin{aligned} \dim \ker\big(\widetilde{\mathcal{P}}(\widetilde{Q}_i - \widetilde{\mathcal{A}}_i)\widetilde{\mathcal{P}}\big) &= \dim \ker\big((I-\widetilde{\mathcal{P}})(\widetilde{Q}_i - \widetilde{\mathcal{A}}_i)(I-\widetilde{\mathcal{P}})\big) \\ &= \sum_{q=0}^{m} \big(\beta_q(M_i) + \beta_q(M_i, Y)\big). \end{aligned} \tag{4.16}$$

Since $\mathrm{Im}\, \widetilde{\mathcal{P}} = \mathrm{Im}(I - \widetilde{\mathcal{P}}) = \bigoplus_{q=0}^{m-1} (\Omega^q(Y, E|_Y) \oplus \Omega^q(Y, E|_Y))$, the equalities (4.12) and (4.13) lead to

$$\begin{aligned} &\zeta_{(\widetilde{\mathcal{P}}\widetilde{Q}_i\widetilde{\mathcal{P}})}(0) + \dim \ker(\widetilde{\mathcal{P}}\widetilde{Q}_i\widetilde{\mathcal{P}}) \\ &\quad = \zeta_{((I-\widetilde{\mathcal{P}})\widetilde{Q}_i(I-\widetilde{\mathcal{P}}))}(0) + \dim \ker\big((I-\widetilde{\mathcal{P}})\widetilde{Q}_i(I-\widetilde{\mathcal{P}})\big) \\ &\quad = 2\Big(\zeta_{\sqrt{\mathcal{B}_Y^2}}(0) + \dim \ker \sqrt{\mathcal{B}_Y^2}\Big) = 2 \sum_{q=0}^{m-1} \big(\zeta_{\mathcal{B}^2_{Y,q}}(0) + \dim \ker \mathcal{B}^2_{Y,q}\big). \end{aligned} \tag{4.17}$$

Hence, by (4.16) and (4.17) we have

$$\begin{aligned} &\zeta_{(\widetilde{\mathcal{P}}(\widetilde{Q}_1 - \widetilde{\mathcal{A}}_1)\widetilde{\mathcal{P}}))}(0) + \zeta_{((I-\widetilde{\mathcal{P}})(\widetilde{Q}_2 - \widetilde{\mathcal{A}}_2)(I-\widetilde{\mathcal{P}}))}(0) \\ &\quad = 4 \sum_{q=0}^{m-1} \big(\zeta_{\mathcal{B}^2_{Y,q}}(0) + \dim \ker \mathcal{B}^2_{Y,q}\big) - 2 \sum_{q=0}^{m} \big(\beta_q(M_1) + \beta_q(M_2)\big). \end{aligned} \tag{4.18}$$

Similarly, by (4.7) we have

$$\begin{aligned} &\widetilde{\mathcal{A}}^2 = \mathcal{B}_Y^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \text{and} \\ &\zeta_{\widetilde{\mathcal{A}}^2}(0) + \widetilde{l} = 4 \sum_{q=0}^{m-1} \big(\zeta_{\mathcal{B}^2_{Y,q}}(0) + \dim \ker \mathcal{B}^2_{Y,q}\big). \end{aligned} \tag{4.19}$$

Hence, (4.14) can be rewritten as follows.

$$\begin{aligned}
\sum_{q=0}^{m}\big(\log \operatorname{Det}^* \mathcal{B}^2_{\widehat{M},q} - \log \operatorname{Det}^* \mathcal{B}^2_{M_1,q,\mathrm{rel}} - \log \operatorname{Det}^* \mathcal{B}^2_{M_2,q,\mathrm{abs}}\big) \\
= -\log 2 \sum_{q=0}^{m}\big(\beta_q(M_1) + \beta_q(M_2)\big) \\
+ 2\sum_{q=0}^{m} \log\det A_{0,q} - 2\sum_{q=0}^{m}(\log\det V_{M_{1,q,\mathrm{rel}}} \\
+ \log\det V_{M_{2,q,\mathrm{abs}}}) + \log\left|\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\big(I - \widetilde{K}_1^{-1}\widetilde{K}_2\big)\right)\right| \\
- \frac{1}{2}\left\{\log\det{}^*_{\mathrm{Fr}}\left(I + \frac{1}{2}\big(U_{\widetilde{\mathcal{P}}}^{-1}\widetilde{K}_1 + \widetilde{K}_1^{-1}U_{\widetilde{\mathcal{P}}}\big)\right)\right. \\
\left. + \log\det{}^*_{\mathrm{Fr}}\left(I - \frac{1}{2}\big(U_{\widetilde{\mathcal{P}}}^{-1}\widetilde{K}_2 + \widetilde{K}_2^{-1}U_{\widetilde{\mathcal{P}}}\big)\right)\right\}.
\end{aligned} \tag{4.20}$$

Finally, we analyze the last three terms in (4.20). We discuss only the case when the dimension of $\widehat{M}$ is odd. The same method can be used for an even dimensional case. From now on, we assume that $\widehat{M}$ is odd dimensional. From (3.5) and (3.16), we have

$$\big(\Omega^{\bullet}(M,E)|_Y\big)_{\pm i} = \left\{\frac{1}{2}\big(I \mp \beta\Gamma^Y\big)\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \,\middle|\, \omega_1,\omega_2 \in \Omega^{\bullet}(Y,E|_Y)\right\}. \tag{4.21}$$

We recall the Calderón projector $\mathcal{C}_1 : \Omega^{\bullet}(M,E)|_Y \to \Omega^{\bullet}(M,E)|_Y$ for $\mathcal{B}_{M_1}$ and the corresponding unitary operator $K_1 : (\Omega^{\bullet}(M,E)|_Y)_{+i} \to (\Omega^{\bullet}(M,E)|_Y)_{-i}$ so that $\mathrm{graph}(K_1) = \operatorname{Im}\mathcal{C}_1$. Let $\widetilde{\mathcal{C}}_1 : \Omega^{\bullet}(M,E\oplus E)|_Y \to \Omega^{\bullet}(M,E\oplus E)|_Y$ be the Calderón projector for $\widetilde{\mathcal{B}}_{M_1}$. From the definition of $\widetilde{\mathcal{B}}$ [(4.1)], we have

$$\operatorname{Im}\widetilde{\mathcal{C}}_1 = \operatorname{Im}\mathcal{C}_1 \oplus \operatorname{Im}\mathcal{C}_1. \tag{4.22}$$

Now consider the unitary operator $\widetilde{K}_1 : (\Omega^{\bullet}(M,E\oplus E)|_Y)_{+i} \to (\Omega^{\bullet}(M,E\oplus E)|_Y)_{-i}$ for $\widetilde{\mathcal{C}}_1$. Then, (4.22) implies that for $x \in (\Omega^{\bullet}(M,E\oplus E)|_Y)_{+i}$, $x + \widetilde{K}_1 x$ is expressed by $\binom{y+K_1 y}{z+K_1 z}$ for some $y,z \in (\Omega^{\bullet}(M,E)|_Y)_{+i}$. Hence, using (4.9), $\widetilde{K}_1$ is described explicitly as follows.

$$\begin{aligned}
\widetilde{K}_1\begin{pmatrix}\omega_1\\ \omega_2\\ -\beta\Gamma^Y\omega_1\\ -\beta\Gamma^Y\omega_2\end{pmatrix} &= \widetilde{K}_1\begin{pmatrix}\frac{I-\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} + \frac{I+\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2}\\ \frac{I-\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} - \frac{I+\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2}\end{pmatrix} \\
&= \begin{pmatrix}K_1\frac{I-\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} + K_1^{-1}\frac{I+\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2}\\ K_1\frac{I-\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} - K_1^{-1}\frac{I+\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2}\end{pmatrix}.
\end{aligned} \tag{4.23}$$

Since $K_1 \frac{I-\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} \in (\Omega^\bullet(M,E)|_Y)_{-i}$ and $K_1^{-1}\frac{I+\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} \in (\Omega^\bullet(M, E)|_Y)_{+i}$, by (4.21) we have

$$\beta\Gamma^Y K_1 \frac{I-\beta\Gamma^Y}{2}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = K_1 \frac{I-\beta\Gamma^Y}{2}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix},$$
$$\beta\Gamma^Y K_1^{-1} \frac{I+\beta\Gamma^Y}{2}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = -K_1^{-1} \frac{I+\beta\Gamma^Y}{2}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix},$$

which leads to

$$U_{\widetilde{\mathcal{P}}}^{-1}\widetilde{K}_1\begin{pmatrix}\omega_1\\ \omega_2\\ -\beta\Gamma^Y\omega_1\\ -\beta\Gamma^Y\omega_2\end{pmatrix} = \begin{pmatrix} SK_1\frac{I-\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} + SK_1^{-1}\frac{I+\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} \\ -SK_1\frac{I-\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} + SK_1^{-1}\frac{I+\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2}\end{pmatrix} = \begin{pmatrix} SK_1\frac{I-\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} + SK_1^{-1}\frac{I+\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} \\ -\beta\Gamma^Y\{SK_1\frac{I-\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2} + SK_1^{-1}\frac{I+\beta\Gamma^Y}{2}\binom{\omega_1}{\omega_2}\}\end{pmatrix}. \tag{4.24}$$

We define an isomorphism

$$\Psi : \Omega^\bullet(M,E)|_Y \to \big(\Omega^\bullet(M,E\oplus E)|_Y\big)_{+i} \quad \text{by } \Psi\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\omega_1\\ \omega_2\\ -\beta\Gamma^Y\omega_1\\ -\beta\Gamma^Y\omega_2\end{pmatrix}, \tag{4.25}$$

which leads to

$$\Psi^{-1}U_{\widetilde{\mathcal{P}}}^{-1}\widetilde{K}_1\Psi\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = SK_1\frac{I-\beta\Gamma^Y}{2}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} + SK_1^{-1}\frac{I+\beta\Gamma^Y}{2}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}. \tag{4.26}$$

By the same way, we have

$$\Psi^{-1}\widetilde{K}_1^{-1}U_{\widetilde{\mathcal{P}}}\Psi\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = K_1S\frac{I-\beta\Gamma^Y}{2}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} + K_1^{-1}S\frac{I+\beta\Gamma^Y}{2}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}. \tag{4.27}$$

Since $\Omega^\bullet(M,E)|_Y = (\Omega^\bullet(Y,E)|_Y)_{+i} \oplus (\Omega^\bullet(Y,E)|_Y)_{-i}$, we use this decomposition to write

$$\Psi^{-1}\big(U_{\widetilde{\mathcal{P}}}^{-1}\widetilde{K}_1 + \widetilde{K}_1^{-1}U_{\widetilde{\mathcal{P}}}\big)\Psi = \begin{pmatrix} 0 & SK_1^{-1}+K_1^{-1}S \\ SK_1+K_1S & 0\end{pmatrix} = \begin{pmatrix} 0 & (SK_1+K_1S)^* \\ SK_1+K_1S & 0\end{pmatrix}. \tag{4.28}$$

Hence, we have

$$\begin{aligned}\text{(4.29)}\qquad &\det\nolimits^*_{\mathrm{Fr}}\Big(I+\frac{1}{2}\big(U_{\widetilde{\mathcal{P}}}^{-1}\widetilde{K}_1+\widetilde{K}_1^{-1}U_{\widetilde{\mathcal{P}}}\big)\Big)\\ &=\det\nolimits^*_{\mathrm{Fr}}\begin{pmatrix} I & \frac{1}{2}(SK_1+K_1S)^* \\ \frac{1}{2}(SK_1+K_1S) & I\end{pmatrix}\\ &=\det\nolimits^*_{\mathrm{Fr}}\Big(I-\frac{1}{4}(SK_1+K_1S)^*(SK_1+K_1S)\Big).\end{aligned}$$

To analyze (4.29), we note that

$$\text{(4.30)}\qquad \big(\Omega^\bullet(Y,E)|_Y\big)_{\pm i}=\begin{pmatrix}\Omega^\bullet(Y,E|_Y)_\pm\\ \oplus\\ \Omega^\bullet(Y,E|_Y)_\pm\end{pmatrix},$$

$$\text{where } \Omega^\bullet(Y,E|_Y)_\pm := \frac{I\mp\beta\Gamma^Y}{2}\Omega^\bullet(Y,E|_Y).$$

According to this decomposition, we may write $K_1:\begin{pmatrix}\Omega^\bullet(Y,E|_Y)_+\\ \oplus\\ \Omega^\bullet(Y,E|_Y)_+\end{pmatrix}\to\begin{pmatrix}\Omega^\bullet(Y,E|_Y)_-\\ \oplus\\ \Omega^\bullet(Y,E|_Y)_-\end{pmatrix}$ by

$$\text{(4.31)}\qquad K_1=\begin{pmatrix}A_1 & B_1\\ C_1 & D_1\end{pmatrix},$$

$$\text{where } A_1,B_1,C_1,D_1:\Omega^\bullet(Y,E|_Y)_+\to\Omega^\bullet(Y,E|_Y)_-.$$

We note that $\Gamma=i\beta\Gamma^Y\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ preserves the decomposition $(\Omega^\bullet(Y,E)|_Y)_{\pm i}$ and commutes with $\mathcal{B}_{M_1}$, which implies that K_1 commutes with Γ. Since $\Omega^\bullet(Y,E|_Y)_\pm$ are (∓1)-eigenspaces of $\beta\Gamma^Y$, we have $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)K_1=K_1\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, which shows that

$$\text{(4.32)}\qquad B_1=C_1,\qquad A_1=-D_1.$$

Hence, we have

$$\text{(4.33)}\qquad SK_1+K_1S=\begin{pmatrix}2A_1 & 0\\ 0 & 2A_1\end{pmatrix}.$$

Since $K_1-U_{\Pi_>}$ is a trace class operator ((2.10)) and $U_{\Pi_>}=(\mathcal{B}_Y^2)^{-1}(\nabla^Y+\Gamma^Y\nabla^Y\Gamma^Y)\left(\begin{smallmatrix}0&-1\\-1&0\end{smallmatrix}\right)$ ((3.15)), A_1 is a trace class operator. Hence, we have

$$\begin{aligned}\text{(4.34)}\qquad &\det\nolimits^*_{\mathrm{Fr}}\Big(I+\frac{1}{2}\big(U_{\widetilde{\mathcal{P}}}^{-1}\widetilde{K}_1+\widetilde{K}_1^{-1}U_{\widetilde{\mathcal{P}}}\big)\Big)\\ &=\det\nolimits^*_{\mathrm{Fr}}\Big(I-\frac{1}{4}(SK_1+K_1S)^*(SK_1+K_1S)\Big)\\ &=\det\nolimits^*_{\mathrm{Fr}}\begin{pmatrix}I-A_1^*A_1 & 0\\ 0 & I-A_1^*A_1\end{pmatrix}=\big(\det\nolimits^*_{\mathrm{Fr}}\big(I-A_1^*A_1\big)\big)^2.\end{aligned}$$

Putting $K_2 = \left(\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right)$, the same method shows that

$$\det{}^*_{\mathrm{Fr}}\left(I - \frac{1}{2}\big(U_{\widetilde{\mathcal{P}}}^{-1}\widetilde{K}_2 + \widetilde{K}_2^{-1}U_{\widetilde{\mathcal{P}}}\big)\right) = \big(\det{}^*_{\mathrm{Fr}}\big(I - A_2^*A_2\big)\big)^2. \tag{4.35}$$

In view of (4.20), we note that

$$\begin{aligned} \widetilde{K}_1^{-1}\widetilde{K}_2 \begin{pmatrix} \omega_1 \\ \omega_2 \\ -\beta\Gamma^Y\omega_1 \\ -\beta\Gamma^Y\omega_2 \end{pmatrix} &= \begin{pmatrix} K_1^{-1}K_2\frac{I-\beta\Gamma^Y}{2}\left(\begin{smallmatrix}\omega_1\\ \omega_2\end{smallmatrix}\right) + K_1K_2^{-1}\frac{I+\beta\Gamma^Y}{2}\left(\begin{smallmatrix}\omega_1\\ \omega_2\end{smallmatrix}\right) \\ K_1^{-1}K_2\frac{I-\beta\Gamma^Y}{2}\left(\begin{smallmatrix}\omega_1\\ \omega_2\end{smallmatrix}\right) - K_1K_2^{-1}\frac{I+\beta\Gamma^Y}{2}\left(\begin{smallmatrix}\omega_1\\ \omega_2\end{smallmatrix}\right) \end{pmatrix} \\ &= \begin{pmatrix} K_1^{-1}K_2\frac{I-\beta\Gamma^Y}{2}\left(\begin{smallmatrix}\omega_1\\ \omega_2\end{smallmatrix}\right) + K_1K_2^{-1}\frac{I+\beta\Gamma^Y}{2}\left(\begin{smallmatrix}\omega_1\\ \omega_2\end{smallmatrix}\right) \\ -\beta\Gamma^Y\{K_1^{-1}K_2\frac{I-\beta\Gamma^Y}{2}\left(\begin{smallmatrix}\omega_1\\ \omega_2\end{smallmatrix}\right) + K_1K_2^{-1}\frac{I+\beta\Gamma^Y}{2}\left(\begin{smallmatrix}\omega_1\\ \omega_2\end{smallmatrix}\right)\} \end{pmatrix}, \end{aligned} \tag{4.36}$$

which shows that

$$\begin{aligned} \Psi^{-1}\widetilde{K}_1^{-1}\widetilde{K}_2\Psi &= K_1^{-1}K_2\frac{I-\beta\Gamma^Y}{2} + K_1K_2^{-1}\frac{I+\beta\Gamma^Y}{2} \\ &= \begin{pmatrix} K_1^{-1}K_2 & 0 \\ 0 & K_1K_2^{-1} \end{pmatrix}. \end{aligned} \tag{4.37}$$

Hence, we have

$$\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\big(I - \widetilde{K}_1^{-1}\widetilde{K}_2\big)\right) = \left|\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\big(I - K_1^{-1}K_2\big)\right)\right|^2. \tag{4.38}$$

The same computation for an even dimensional case leads to the same result. Summarizing the above argument, we have the following result, which is the main result of this section.

THEOREM 4.2. *Let $(\widehat{M}, \widehat{g})$ be a closed Riemannian manifold and Y be a hypersurface of $\widehat{M}$ with $\widehat{M} = M_1 \cup_Y M_2$. We assume that $\widehat{g}$ is a product metric near Y. We denote the odd signature operator on $\widehat{M}$ by $\mathcal{B}_{\widehat{M}}$ and its restriction to M_1 and M_2 by $\mathcal{B}_{M_1}$ and $\mathcal{B}_{M_2}$. Then:*

$$\begin{aligned} &\sum_{q=0}^{m}\big(\log \operatorname{Det}^* \mathcal{B}^2_{\widehat{M},q} - \log \operatorname{Det}^* \mathcal{B}^2_{M_1,q,\mathrm{rel}} - \log \operatorname{Det}^* \mathcal{B}^2_{M_2,q,\mathrm{abs}}\big) \\ &\quad = -\log 2 \sum_{q=0}^{m}\big(\beta_q(M_1) + \beta_q(M_2)\big) \\ &\qquad + 2\sum_{q=0}^{m}\log\det A_{0,q} - 2\sum_{q=0}^{m}(\log\det V_{M_{1,q,\mathrm{rel}}} + \log\det V_{M_{2,q,\mathrm{abs}}}) \end{aligned}$$

$$+ 2\log\left|\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\left(I - K_1^{-1}K_2\right)\right)\right|$$
$$- \left\{\log\det{}^*_{\mathrm{Fr}}\left(I - A_1^*A_1\right) + \log\det{}^*_{\mathrm{Fr}}\left(I - A_2^*A_2\right)\right\},$$

where $A_1, A_2 : \Omega^\bullet(Y, E|_Y)_+ \to \Omega^\bullet(Y, E|_Y)_-$ *are first components of* K_1 *and* K_2*, respectively.*

Remark. (1) If all cohomologies vanish, that is, $H^*(M;E) = H^*(M_i;E) = H^*(M_i, Y;E) = 0$, then the first three terms in Theorem 3.3 and Theorem 4.2 do not appear. (2) So far we do not know how to describe $(\log\mathrm{Det}^*\mathcal{B}^2_{\widehat{M},q} - \log\mathrm{Det}^*\mathcal{B}^2_{M_1,q,\mathcal{P}_-,\mathcal{L}_0} - \log\mathrm{Det}^*\mathcal{B}^2_{M_2,q,\mathcal{P}_+,\mathcal{L}_1})$ and $(\log\mathrm{Det}^*\mathcal{B}^2_{\widehat{M},q} - \log\mathrm{Det}^*\mathcal{B}^2_{M_1,q,\mathrm{rel}} - \log\mathrm{Det}^*\mathcal{B}^2_{M_2,q,\mathrm{abs}})$ for each single q.

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