

## A GEOMETRIC PROOF OF THE EXISTENCE OF DEFINABLE WHITNEY STRATIFICATIONS

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ABSTRACT. We give a geometric proof of existence of Whitney stratifications of definable sets in o-minimal structures.

### 1. Introduction

It has been known for a long time that semi-varieties (semi-analytic or semi-algebraic for example) can be stratified into smooth manifolds satisfying Whitney conditions (a) and (b). Methods of doing this can be found in Whitney [12], Wall [11], Bochnak, Coste and Roy [1], Lojasiewicz [8], Lojasiewicz, Stasica and Wachta [9], etc. All of the proofs given in the above mentioned literature of the existence of such stratifications use analytical techniques.

Kaloshin [3] has claimed a geometric proof of the existence of stratifications of semivarieties satisfying the Whitney conditions. We show by giving a very simple counterexample that there is a gap in this proof of Kaloshin. In this article, motivated by the idea of Kaloshin, we give a geometric proof of the existence of these stratifications in the more general o-minimal setting. Our method fills the gap in Kaloshin's proof and moreover it works for the case of definable sets in o-minimal structures. Loi [6] also proved this result with a different proof using a wing lemma.

Let us first describe the overview of the idea of Kaloshin.

The following terminology is due to Kaloshin. Let  $V \subset \mathbb{R}^n$  be a closed semivariety and let  $\Sigma$  be a stratification of  $V$ . Given strata  $X$  and  $Y$  of  $\Sigma$  and a point  $y \in \overline{X} \cap Y$ , by a local connected component of  $X$  at  $y$  is meant a connected subset of  $X$  obtained from intersecting  $X$  by a sufficiently small open ball centered at  $y$ . By a result of Lojasiewicz [8], there exist finitely many such connected components for any point  $y \in Y$ .

A local connected component  $X_\alpha$  is said to be an essential component of  $X$  at  $y$  if  $y$  lies in the interior of  $Y \cap \overline{X_\alpha}$  (considered as a subset of  $Y$ ). Now

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$\text{Sing}_a(X, Y)$  is defined as the set of points  $y \in Y$  such that the union of the essential components of  $X$  at  $y$  is not  $(a)$ -regular over  $Y$  at  $y$ . Kaloshin proves that the set  $\text{Sing}_a(X, Y)$  is a semivariety and has dimension less than the dimension of  $Y$ , so showing that Whitney's condition (a) is generic, and the result follows.

We will show pictorially that the set of  $(a)$ -faults (points where the condition (a) fails) of a pair of strata  $(X, Y)$  is in general bigger than  $\text{Sing}_a(X, Y)$ , and that considering only the essential components leaves several  $(a)$ -faults unaccounted for.

Consider the closed subset  $V$  of  $\mathbb{R}^3$  as in Figure 1. It is like Santa's hat except that the conical tip is attached to the round edge of the hat.

Applying the procedure of stratifying  $V$  due to Wall [11],<sup>1</sup> we find that  $\mathbb{R}^3$  will have three strata compatible with  $V$ . The three dimensional stratum will be the complement of  $V$  in  $\mathbb{R}^3$ . The two dimensional stratum will be  $X$  and the one dimensional stratum will be  $Y$ .

Now, take  $y \in Y$  as in the Figure 1 (the tip of the hat) and intersect  $V$  with a small ball around  $y$ . We find that  $X$  has two local connected components at  $y$ , denoted  $X_\alpha$  and  $X_\beta$ . Notice that  $X_\alpha$  is an essential component of  $X$  near  $Y$  while  $X_\beta$  is not. Thus, the set  $\text{Sing}_a(X, Y)$  is empty. Notice also that

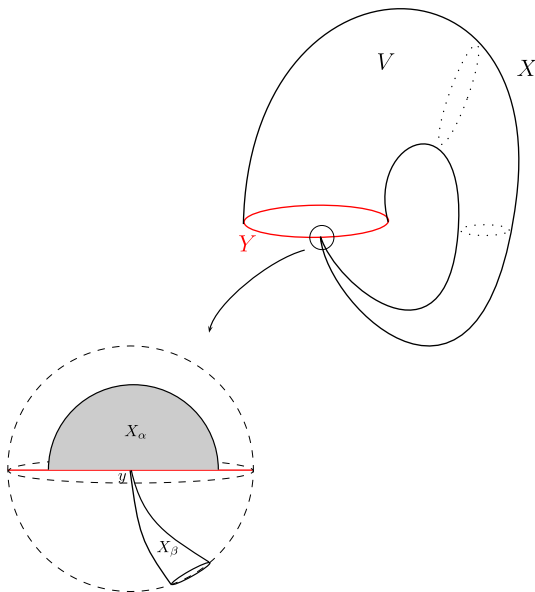


FIGURE 1. A non-essential  $(a)$ -fault.

<sup>1</sup> We must mention here that Wall's method works only for closed semi-varieties.

$X$  is not  $(a)$ -regular over  $Y$  at  $y$ . We call such points  $y$  the non-essential points. Thus the set of  $(a)$ -faults in this stratification of  $V$  is strictly bigger than the set  $\text{Sing}_a(X, Y)$ .

We will now summarize the contents of the article:

In Section 2, we give definitions of  $o$ -minimal structures, definable stratifications, stratifying conditions, Whitney conditions and state the main result (Theorem 2.2). The idea of the proof is to show that Whitney conditions are stratifying conditions (Lemmas 2.3 and 2.4).

In Section 3, we define Kuo functions. These functions give criteria to test Whitney conditions (a) and (b) in a stratification.

In Section 4 after defining essential and non-essential components and points, we show that the set of non-essential points in  $Y$  for a pair  $X, Y$  of definable sets such that  $\overline{X} \cap Y = \emptyset$  has dimension less than the dimension of  $Y$ . This fills the gap in Kaloshin's proof. We then prove that the Whitney conditions (a) and (b) are stratifying conditions. The key to the proof is the existence of a sequence of points in a stratum converging to a point in another stratum in its boundary such that the limit of the sequence of values of the Kuo functions on these points vanish (Lemmas 4.2 and 4.3). This allows us to avoid the use of Rolle's lemma as opposed to the proof of Kaloshin and makes our proof much simpler.

## 2. Preliminaries and statement of results

**2.1.  $o$ -minimal structures.** A *structure* on the ordered field  $(\mathbb{R}, +, \cdot)$  is a family  $\mathcal{D} = (D_n)_{n \in \mathbb{N}}$  satisfying the following properties:

1.  $D_n$  is a boolean algebra of subsets of  $\mathbb{R}^n$ ;
2. If  $A \in D_n$ , then  $\mathbb{R} \times A \in D_{n+1}$  and  $A \times \mathbb{R} \in D_{n+1}$ ;
3.  $D_n$  contains the zero sets of all polynomials in  $n$  variables;
4. If  $A \in D_n$ , then its projection onto the first  $n - 1$  coordinates in  $\mathbb{R}^{n-1}$  is in  $D_{n-1}$ .

Such a  $\mathcal{D}$  is said to be  *$o$ -minimal* if in addition:

5. Any set  $A \in D_1$  is a finite union of open intervals and points.

Elements of  $D_n$  for any  $n$  are called *definable sets* of  $\mathcal{D}$ . A map between two definable sets is said to be a *definable map* if its graph is a definable set.

Let  $\mathcal{D}$  be an  $o$ -minimal structure on  $\mathbb{R}$ . In what follows by definable, we mean in this  $\mathcal{D}$ .

**2.2. Definable stratifications and stratifying conditions.** A definable  $C^p$ -stratification  $\Sigma$  of  $\mathbb{R}^n$  is a partition of  $\mathbb{R}^n$  into finitely many definable  $C^p$  submanifolds<sup>2</sup> of  $\mathbb{R}^n$ , called strata, such that the boundary of every stratum is either empty or a union of some other strata.

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<sup>2</sup> A definable  $C^p$  submanifold of  $\mathbb{R}^n$  meaning a definable subset and also a  $C^p$  submanifold of  $\mathbb{R}^n$ .

Let  $\mathcal{A} = \{A_1, \dots, A_k\}$  be a family of definable subsets of  $\mathbb{R}^n$ . A stratification  $\Sigma$  of  $\mathbb{R}^n$  is said to be compatible with  $\mathcal{A}$  if each  $A_i$  is the union of some strata of  $\Sigma$ . In the rest of the paper, by definable we mean of class  $C^p$ .

Let  $(X, Y)$  be a pair of definable submanifolds of  $\mathbb{R}^n$  such that  $Y \subset \overline{X} \setminus X$ . Let  $\gamma$  be a condition on the pair  $(X, Y)$  at points in  $Y$ . A point  $y \in Y$  is said to be a  $(\gamma)$ -fault if the condition  $\gamma$  fails to be satisfied for the pair  $(X, Y)$  at  $y$ . We denote by  $\mathcal{F}_\gamma(X, Y)$  the set of all  $(\gamma)$ -faults for the pair  $(X, Y)$ . If  $\mathcal{F}_\gamma(X, Y)$  is empty then we say that the pair  $(X, Y)$  is  $(\gamma)$ -regular. Moreover, a stratification is said to be  $(\gamma)$ -regular if every pair of its strata is  $(\gamma)$ -regular.

A condition  $(\gamma)$  is said to be a *stratifying condition* if for any pair  $(X, Y)$  as above the set  $\mathcal{F}_\gamma(X, Y)$  is definable and  $\dim \mathcal{F}_\gamma(X, Y) < \dim Y$ . Using cell decomposition theorem [10] and arguments as in the proof of Proposition 2 in [9], we have the following result (see also [7]).

**THEOREM 2.1.** *Let  $\mathcal{A} = \{A_1, \dots, A_k\}$  be a family of definable subsets of  $\mathbb{R}^n$ . If  $(\gamma)$  is a stratifying condition then there exists a  $(\gamma)$ -regular definable stratification of  $\mathbb{R}^n$  compatible with  $\mathcal{A}$ .*

**2.3. Whitney conditions.** Let  $X$  be a definable submanifold of  $\mathbb{R}^n$  and  $y \in \overline{X}$ . A sequence of points  $\{x_n\}$  in  $X$  converging to  $y$  is said to be a *good sequence* if the corresponding sequences  $\{T_{x_n} X\}$  of tangent spaces in the Grassmannian converges. The limit  $\lim_{n \rightarrow \infty} T_{x_n} X$  will be called the *Grassmannian limit* of the sequence  $\{x_n\}$ . Since the Grassmannian is a compact metric space, for every sequence in  $X$  there exists a subsequence which is a good sequence.

Let  $(X, Y)$  be a pair of definable submanifolds of  $\mathbb{R}^n$  such that  $Y \subset \overline{X} \setminus X$ . Consider the following conditions on  $(X, Y)$  at a point  $y \in Y$ .

- (a) The Grassmannian limit of every good sequence  $\{x_n\}$  in  $X$  converging to  $y$  contains the tangent space  $T_y Y$ .
- (b) For every sequence  $\{y_n\}$  in  $Y$  converging to  $y$ , the Grassmannian limit of every good sequence  $\{x_n\}$  in  $X$  converging to  $y$  contains  $v := \lim_{n \rightarrow \infty} \frac{x_n - y_n}{\|x_n - y_n\|}$  if  $v$  exists.

The reader must have realized that the conditions (a) and (b) are the usual Whitney conditions (a) and (b) written differently.

**THEOREM 2.2.** *Let  $\mathcal{A} = \{A_1, \dots, A_k\}$  be a family of definable subsets of  $\mathbb{R}^n$ . Then there exists an (a)-regular (resp. (b)-regular) definable stratification of  $\mathbb{R}^n$  compatible with  $\mathcal{A}$ .*

By Theorem 2.1, to prove Theorem 2.2, it suffices to show that conditions (a) and (b) are stratifying conditions. For any definable submanifolds  $X, Y \subset \mathbb{R}^n$  such that  $Y \subset \overline{X} \setminus X$ , it is easy to see that the set of (a)-faults  $\mathcal{F}_a(X, Y)$  (resp. (b)-faults  $\mathcal{F}_b(X, Y)$ ) is definable once we write it using quantifiers, see for example [7]. Thus, we need to prove the following lemmas:

LEMMA 2.3. For any definable submanifolds  $X, Y \subset \mathbb{R}^n$  such that  $Y \subset \overline{X} \setminus X$ , we have  $\dim \mathcal{F}_a(X, Y) < \dim Y$ .

LEMMA 2.4. For any definable submanifolds  $X, Y \subset \mathbb{R}^n$  such that  $Y \subset \overline{X} \setminus X$ , we have  $\dim \mathcal{F}_b(X, Y) < \dim Y$ .

We will prove Lemmas 2.3 and 2.4 in Section 4.

### 3. Kuo functions

Let  $X, Y$  be definable submanifolds of  $\mathbb{R}^n$  such that  $Y \subset \overline{X} \setminus X$ . Suppose that  $\dim Y = k$ . Since (a) (resp. (b)) regularity is a local property we can assume that locally  $Y$  is a  $k$ -plane with a basis of unit vectors  $\{e_1, \dots, e_k\}$ .

Given a linear subspace  $L$  of  $\mathbb{R}^n$  we denote by  $\pi_L : \mathbb{R}^n \rightarrow L$  the canonical orthogonal projection of  $\mathbb{R}^n$  onto  $L$ . Let  $x \in X$  and consider  $T_x X$  as a linear subspace of  $\mathbb{R}^n$ . Using the idea of Kuo [4] (see also [3]) we define functions, which we call Kuo functions, that give criteria to test (a) and (b)-regularity.

Let  $p_a : X \rightarrow \mathbb{R}$  be the function defined by

$$p_a(x) := \sum_{i=1}^k \left\| \pi_{N_x X}(e_i) \right\|^2,$$

where  $N_x X$  is the orthogonal complement of  $T_x X$ .

Let  $p_{b'} : X \rightarrow \mathbb{R}$  be the function defined by

$$p_{b'}(x) := \left\| \pi_{N_x X}(p(x)) \right\|^2,$$

where  $p(x) := \frac{x - \pi_Y(x)}{\|x - \pi_Y(x)\|}$ .

Let  $p_b : X \rightarrow \mathbb{R}$  be the function defined by

$$p_b(x) := p_a(x) + p_{b'}(x).$$

Kuo [4] (see also [3]) proved that a pair  $(X, Y)$  satisfies the condition (a) (resp. (b)) at  $y \in Y$  if and only if for every good sequence  $\{x_n\}$  in  $X$  converging to  $y$ ,  $\lim_{n \rightarrow \infty} p_a(x_n) = 0$  (resp.  $\lim_{n \rightarrow \infty} p_b(x_n) = 0$ ).

### 4. Existence of Whitney stratifications for definable sets

Let  $P$  and  $Q$  be linear subspaces of  $\mathbb{R}^n$ . The angle between  $P$  and  $Q$  is defined by

$$\delta(P, Q) := \sup_{\lambda \in P, \|\lambda\|=1} \left\{ \left\| \lambda - \pi_Q(\lambda) \right\| \right\}.$$

The function  $\delta$  takes values in  $[0, 1]$ . In general  $\delta$  is not symmetric, for instance, if  $P \subset Q$  and  $P \neq Q$  then  $\delta(P, Q) = 0$  while  $\delta(Q, P) = 1$ . The following properties are easy to verify.

1. If  $\dim P = \dim Q$  then  $\delta(P, Q) = \delta(Q, P)$ .
2. If  $P \subset Q$  then  $\delta(P, Q) = 0$ .
3. If  $\dim T \leq \dim P \leq \dim Q$  then  $\delta(T, Q) \leq \delta(T, P) + \delta(P, Q)$ .

For a real number  $\varepsilon > 0$ , a definable submanifold  $X$  is said to be  $\varepsilon$ -flat if for every  $x, x'$  in  $X$ ,  $\delta(T_x X, T_{x'} X) < \varepsilon$ . If  $\dim X = 0$  then we assume that  $X$  is  $\varepsilon$ -flat for every  $\varepsilon > 0$ .

LEMMA 4.1. *Let  $X \subset \mathbb{R}^n$  be a definable set of dimension  $k < n$  and let  $\varepsilon > 0$  be a real number. Then there is a definable stratification of  $X$  such that every stratum is  $\varepsilon$ -flat.*

*Proof.* This is proved for subanalytic sets in Proposition 5 in Kurdyka [5], but the idea also works for definable sets. □

LEMMA 4.2. *Let  $X, Y$  be definable submanifolds of  $\mathbb{R}^n$  such that  $Y \subset \overline{X} \setminus X$  and let  $y$  be a point in  $Y$ . Then there exists a good sequence  $\{x_n\}$  in  $X$  converging to  $y$  such that  $p_a(x_n)$  converges to 0.*

*Proof.* Suppose on the contrary that there is an  $\varepsilon > 0$  such that for every good sequence  $\{x_n\}$  in  $X$  converging to  $y$ , the limit of the sequence  $p_a(x_n)$  is greater than  $\varepsilon$ . In other words, we can choose  $\varepsilon$  sufficient small such that for any given good sequence  $\{x_n\}$  with the Grassmannian limit  $\tau$ , we have  $\delta(T_y Y, \tau) > \varepsilon$ .

Take a stratification of  $\mathbb{R}^n$  compatible with  $X$  such that its strata are  $\frac{\varepsilon}{4}$ -flat (this is possible by Lemma 4.1). We can write  $X = \bigsqcup_{i=1}^m X_i$  where the  $X_i$ 's are the strata. Set  $Y' := \bigcup_{i=1}^m \text{Int}_Y(\overline{X_i} \cap Y)$ . Notice that  $Y'$  is open and dense in  $Y$ . The proof now breaks into the two following cases.

*Case 1:  $y \in Y'$ .*

There is an  $X_i$ ,  $1 \leq i \leq m$ , such that  $y \in \text{Int}_Y(\overline{X_i} \cap Y)$ . Fix a good sequence  $\{x_n\}$  in  $X_i$  and denote by  $\tau$  its Grassmannian limit.

Since  $\delta(T_y Y, \tau) > \varepsilon$ , we can choose a line  $l \subset T_y Y$  satisfying  $\delta(l, \tau) > \frac{\varepsilon}{2}$ . We define the  $\frac{\varepsilon}{4}$ -cone around  $l$  centered at  $y$  as follows:

$$C_y := \left\{ x \in \mathbb{R}^n : \delta(\mu(x - y), l) < \frac{\varepsilon}{4} \right\},$$

where  $\mu(x - y)$  denotes the line spanned by the unit vector  $\frac{x - y}{\|x - y\|}$ .

Since  $y \in \text{Int}_Y(\overline{X_i} \cap Y)$ , the intersection  $X_i(y) := X_i \cap C_y$  is a non-empty definable set and  $y \in \overline{X_i(y)}$ . The curve selection lemma (see van den Dries [10]) says that there is a  $C^1$  curve  $\gamma : (0, 1) \rightarrow X_i(y)$  such that  $\lim_{t \rightarrow 0} \gamma(t) = y$ . Choose a good sequence  $\{x'_n\}$  along the curve  $\gamma$  converging to  $y$  and denote by  $\tau'$  its Grassmannian limit. Put  $l' := \lim_{n \rightarrow \infty} T_{x'_n} \gamma$ , then  $l' \subset \tau'$  and

$$\delta(l, \tau') \leq \delta(l, l') \leq \frac{\varepsilon}{4}.$$

Since  $X_i$  is  $\frac{\varepsilon}{4}$ -flat,  $\delta(\tau, \tau') < \frac{\varepsilon}{4}$ . Thus,

$$\delta(l, \tau) \leq \delta(l, \tau') + \delta(\tau', \tau) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

a contradiction.

Case 2:  $y \notin Y'$ .

Because  $Y'$  is dense in  $Y$  we can find a sequence  $\{y_n\}$  in  $Y'$  tending to  $y$ . By case 1, for each  $y_n$  there is a good sequence  $\{x_{n,m}\}$  in  $X$  converging to  $y_n$  such that  $p_a(x_{n,m})$  converges to 0. It is possible to choose a good sequence  $\{x'_n\}$  in  $X$  converging to  $y$  such that  $x'_n \in \{x_{n,m}\}$  and  $p_a(x'_n) < \varepsilon$ . The limit of the sequence  $p_a(x'_n)$  is clearly less than  $\varepsilon$ . This provides a contradiction.  $\square$

To prove Lemma 2.3, we need the following definitions. For  $y \in Y$ , denote by  $B_r(y)$  the open ball in  $\mathbb{R}^n$  of radius  $r$  centered at  $y$ . By Hardt's theorem about topological triviality for definable sets (Theorem 5.19, p. 60 in [2]), the topological type of the intersection  $B_r(y) \cap X$  is stable, that is, there is an  $r > 0$  sufficiently small such that for every  $0 < r' < r$  the sets  $B_r(y) \cap X$  and  $B_{r'}(y) \cap X$  are topologically equivalent. Denote by  $N_y$  the number of connected components of the intersection  $B_r(y) \cap X$ . This number is uniformly bounded on  $Y$ . More precisely, there exists an integer  $\kappa$  such that  $N_y \leq \kappa$  for all  $y \in Y$ . A connected component  $X_i(y)$  ( $i = 1, \dots, N_y$ ) of the intersection  $B_r(y) \cap X$  is said to be *essential* if  $y$  is in the interior of  $\overline{X_i(y)} \cap Y$  in  $Y$ , denoted by  $\text{Int}_Y(\overline{X_i(y)} \cap Y)$  ( $i = 1, \dots, N_y$ ). We say that  $y$  is an *essential point* if  $X_i(y)$  is essential for all  $i$ .

Observe that every point in  $\bigcap_i^{N_y} \text{Int}_Y(\overline{X_i(y)} \cap Y)$  is essential. Set  $T_j(X, Y) := \{y \in Y : N_y = j\}$ . Then the set of essential points can be written as follows

$$\Omega(X, Y) := \bigcup_{j=1}^{\kappa} \left\{ y \in T_j(X, Y) : y \in \bigcap_{i \leq j} \text{Int}_Y(\overline{X_i(y)} \cap Y) \right\}.$$

This implies that  $\Omega(X, Y)$  is an open definable set in  $Y$ . In addition, we can cover  $Y$  by countably many balls  $B_{r_\alpha}(y_\alpha)$  where  $y_\alpha \in Y \cap (\{0\}^{n-k} \times \mathbb{Q}^k)$  and  $r_\alpha \in \mathbb{Q}$  such that the intersection  $B_{r_\alpha}(y_\alpha) \cap X$  is stable. It is clear that the set of non-essential points has dimension less than the dimension of  $Y$  since it is contained in the countable union of boundaries of  $\overline{X_i(y_\alpha)} \cap Y$  in  $Y$  for all  $y_\alpha$  and all  $i = 1, \dots, N_{y_\alpha}$ . The set  $\Omega(X, Y)$  thus is a definable set open and dense in  $Y$ .

*Proof of Lemma 2.3.* Since the set of essential points in  $Y$  is definable, dense and open in  $Y$ , we can assume without loss of generality that every point in  $Y$  is essential.

Take a point  $y$  in  $\mathcal{F}_a(X, Y)$ . By Lemma 4.2, there is an essential component  $X_i(y)$  with two sequences of points  $\{x'_n\}$  and  $\{x''_n\}$  converging to  $y$  such that  $p_a(x'_n) \rightarrow \varepsilon'$  and  $p_a(x''_n) \rightarrow \varepsilon''$  for some non-negative numbers  $\varepsilon' < \varepsilon''$ . Notice that the function  $p_a(x)$  takes values in  $[0, k]$  where  $k$  is the dimension of  $Y$ . By Sard's lemma there exists a regular value  $\varepsilon \in (\varepsilon', \varepsilon'')$  of the function  $p_a$ , so the set  $X^\varepsilon := (p_a)^{-1}(\varepsilon)$  is a definable submanifold of  $X$  of codimension 1 in  $X$ . Since  $X_i(y)$  is locally connected at  $y$ ,  $x'_n$  and  $x''_n$  can be connected by

a curve  $\gamma_n$ . Choosing points  $x_n \in \gamma_n$  such that  $p_a(x_n) = \varepsilon$ , we get a sequence  $\{x_n\} \subset X^\varepsilon$  converging to  $y$ , and hence  $y \in \overline{X^\varepsilon} \setminus X^\varepsilon$ .

Now choose countably many regular values  $\{\varepsilon_\nu\}_{\nu \in \mathbb{Z}}$  of the function  $p_a$  whose union is dense in  $[0, k]$  and define  $X^{\varepsilon_\nu} := (p_a)^{-1}(\varepsilon_\nu)$ . Then the union  $\bigcup_{\nu \in \mathbb{Z}} \overline{X^{\varepsilon_\nu}} \setminus X^{\varepsilon_\nu}$  contains all (a)-faults of the pair  $(X, Y)$ .

Put  $I := \{\nu \in \mathbb{Z} : \dim \overline{X^{\varepsilon_\nu}} \cap Y = \dim Y\}$ . For  $\nu \in I$ , denote by  $Y^{\varepsilon_\nu} = \text{Int}_Y(\overline{X^{\varepsilon_\nu}} \cap Y)$ , then  $(X^{\varepsilon_\nu}, Y^{\varepsilon_\nu})$  is again a pair of definable submanifolds with  $Y^{\varepsilon_\nu} \subset \overline{X^{\varepsilon_\nu}} \setminus X^{\varepsilon_\nu}$ . Let  $p_a^{\varepsilon_\nu}$  be the Kuo function on  $(X^{\varepsilon_\nu}, Y^{\varepsilon_\nu})$  constructed as in Section 3. Observe that  $p_a^{\varepsilon_\nu}(x) \geq p_a(x)$  for every  $x \in X^{\varepsilon_\nu}$ . This shows

$$\mathcal{F}_a(X, Y) \subset \bigcup_{\nu \in I} \mathcal{F}_a(X^{\varepsilon_\nu}, Y^{\varepsilon_\nu}) \cup Z,$$

where  $Z := \bigcup_{\nu \in \mathbb{Z} \setminus I} (\overline{X^{\varepsilon_\nu}} \cap Y) \cup \bigcup_{\nu \in I} ((\overline{X^{\varepsilon_\nu}} \cap Y) \setminus Y^{\varepsilon_\nu})$  a subset of positive codimension in  $Y$ .

Because a countable union of subsets of positive codimensions in  $Y$  is a subset of positive codimension in  $Y$ , it remains to show that  $\dim \mathcal{F}_a(X^{\varepsilon_\nu}, Y^{\varepsilon_\nu}) < \dim Y^{\varepsilon_\nu}$  for  $\nu \in I$ . This follows from the inductive application of the above arguments for  $(X^{\varepsilon_\nu}, Y^{\varepsilon_\nu})$ . The induction stops when  $\dim X^{\varepsilon_\nu} \leq \dim Y$ .  $\square$

In order to prove Lemma 2.4, we will use the following Lemma 4.3 which plays the same role for (b)-regularity as Lemma 4.2 does for (a)-regularity.

LEMMA 4.3. *Let  $X, Y$  be definable submanifolds of  $\mathbb{R}^n$  such that  $Y \subset \overline{X} \setminus X$  and let  $y$  be a point in  $Y$ . There is a good sequence  $\{x_n\}$  in  $X$  converging to  $y$  such that  $p_b(x_n)$  converges to 0.*

*Proof.* Suppose that there is an  $\varepsilon > 0$  such that  $p_b(x_n) > \varepsilon$  for every good sequence  $\{x_n\}$  in  $X$  converging to  $y$ . We will show a contradiction by giving a sequence  $\{x_n\}$  in  $X$  converging to  $y$  such that  $p_b(x_n)$  converges to a value less than  $\varepsilon$ .

For  $y \in Y$ , we define  $Z(y) := X \cap (Y^\perp + \{y\})$  and

$$\omega(y) := \inf \{d(x, y) : x \in Z(y)\},$$

where  $d(x, y)$  is the usual distance from  $x$  to  $y$ . Put  $\omega(y) = 1$  if  $Z(y) = \emptyset$ . Clearly  $\omega(y)$  is a definable function on  $Y$ .

We claim that the set  $\Delta := \{y \in Y : y \in \overline{Z(y)}\}$  is open and dense in  $Y$ . In other words, its complement  $\Delta^c := \{y \in Y : \omega(y) > 0\}$  is of dimension less than the dimension of  $Y$ . Thus, suppose to the contrary that  $\dim \Delta^c = \dim Y$ . By the cell decomposition theorem and local compactness of  $Y$ , there is an open set  $U$  in  $Y$  and a constant  $c > 0$  such that  $\omega(y) > c$  for every  $y \in U$ . This means  $U \not\subset \overline{X} \setminus X$ , a contradiction.

Denote by  $Y'$  the set of points in  $\Delta$  which are not (a)-faults. Take  $y \in Y'$ . The curve selection lemma says that there exists a  $C^1$  definable curve  $\gamma : (0, 1) \rightarrow Z(y)$  such that  $\lim_{t \rightarrow 0} \gamma(t) = y$ . Choose  $\{z_m\}$  a good sequence in the curve  $\gamma$  converging to  $y$  and denote by  $\tau$  its Grassmannian limit.



From the construction we have  $\pi_Y(z_m) = y$ , hence  $p(z_m) = \frac{z_m - y}{\|z_m - y\|}$ . Obviously  $\lim_{m \rightarrow \infty} p(z_m) \in \lim_{m \rightarrow \infty} T_{z_m} \gamma \subset \tau$ . This implies that  $\lim_{m \rightarrow \infty} p_{b'}(z_m) = 0$ . Moreover, since  $y$  is not an  $(a)$ -fault,  $\lim_{m \rightarrow \infty} p_b(z_m) = 0$ .

Since  $Y'$  is dense in  $Y$ , for  $y \in Y$  there is a sequence  $\{y_n\} \subset Y'$  converging to  $y$ . Let  $\{\gamma_n\}$  be the corresponding sequence of curves as above. Choose a sequence  $x_n$  converging to  $y$  with  $x_n \in \gamma_n$  and  $p_b(x_n) < \varepsilon$ , then  $\lim_{n \rightarrow \infty} p_b(x_n)$  is obviously less than  $\varepsilon$ . This gives a contradiction.  $\square$

Lemma 4.3 together with the arguments of the proof of Lemma 2.3 provide a proof for Lemma 2.4.

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