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THE BIERI–NEUMANN–STREBEL INVARIANT OF THE PURE SYMMETRIC AUTOMORPHISMS OF A RIGHT-ANGLED ARTIN GROUP

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ABSTRACT. We compute the BNS-invariant for the pure symmetric automorphism groups of right-angled Artin groups. We use this calculation to show that the pure symmetric automorphism group of a right-angled Artin group is itself not a right-angled Artin group provided that its defining graph contains a separating intersection of links.

1. Introduction

In 1987, the Bieri–Neumann–Strebel (BNS) geometric invariant $\Sigma^1(G)$ was introduced for a discrete group G. The invariant is an open subset of the character sphere $S(G)$ which carries considerable algebraic and geometric information. It determines whether or not a normal subgroup with Abelian quotient is finitely generated; in particular, the commutator subgroup of G is finitely generated if and only if $\Sigma^1(G) = S(G)$. If M is a smooth compact manifold and $G = \pi_1(M)$, then $\Sigma^1(G)$ contains information on the existence of circle fibrations of M. Additionally, if M is a 3-manifold, then $\Sigma^1(G)$ can be described in terms of the Thurston norm. Other aspects of the rich theory of BNS-invariant can be found in [BNS87].

Although $\Sigma^1(G)$ has proven quite difficult to compute in general, it has been computed in the case that G is a right-angled Artin group [MV95], and in the case that G is the pure symmetric automorphism group of a free group [OK00]. In the present article, we generalize the result of [OK00] by computing $\Sigma^1(G)$ when G is the pure symmetric automorphism group of a right-angled Artin group. The outcome of the computation is recorded in Theorem A, to be found in Section 4 below.

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We also provide an application of our computation. It was shown in [CRSV10] that if A is the right-angled Artin group determined by a graph Γ that has no separating intersection of links (no SILS), then the corresponding group of pure symmetric automorphisms $P\Sigma(A)$ is itself a right-angled Artin group. We prove the converse by observing that when Γ has a SIL, the BNS-invariant of $P\Sigma(A)$ does not have a certain distinctive property that the BNS-invariant of a right-angled Artin group must satisfy. Thus we prove the following theorem.

THEOREM B. *The group $P\Sigma(A)$ is isomorphic to a right-angled Artin group if and only if the defining graph Γ contains no SILs.*

Theorem B is indicative of a dichotomy within the family of groups $\{P\Sigma(A)\}$ determined by whether or not Γ has a SIL. Certain algebraic manifestations of this dichotomy were proved in [GPR12]. It would be interesting to understand more geometric manifestations. Since right-angled Artin groups are CAT(0) groups, we are lead to ask the following question:

QUESTION 1.1. If the defining graph Γ contains a SIL, is $P\Sigma(A)$ a CAT(0) group?

This paper is organized as follows: in Section 2 and Section 3, we define the BNS-invariant $\Sigma^1(G)$ and the pure symmetric automorphism group, respectively, and record some useful facts which inform the arguments to follow. We prove Theorem A in Section 4. This proof involves two cases with the first handled in Section 4.1 and the second in Section 4.2. In Section 5, we prove Theorem B.

2. The BNS-invariant

Let G be a finitely generated group. A *character* χ of G is a homomorphism from G to the additive reals. The set of all characters of G, denoted $\mathrm{Hom}(G,\mathbb{R})$, is an n-dimensional real vector space where n is the $\mathbb{Z}$-rank of the Abelianization of G. Two non-zero characters χ_1 and χ_2 are equivalent if there is a real number $r > 0$ such that $\chi_1 = r\chi_2$. The set of equivalence classes $S(G) = \{[\chi] \mid \chi \in \mathrm{Hom}(G,\mathbb{R}) - \{0\}\}$ is called the *character sphere of* G, and this is homeomorphic to an $(n-1)$-dimensional sphere. The BNS invariant $\Sigma^1(G)$, a subset of $S(G)$, may be described in terms of either the geometry of Cayley graphs (see [BNS87]), or G-actions on $\mathbb{R}$-trees (see [Bro87]). For our purposes the latter is more convenient, and we now describe $\Sigma^1(G)$ from that point of view.

Suppose G acts by isometries on an $\mathbb{R}$-tree, T, and let $\ell : G \to \mathbb{R}^+$ be the corresponding length function. For each $g \in G$, let C_g be the characteristic subtree of g. If $\ell(g) = 0$, then g is elliptic, and C_g is its fixed point set; if $\ell(g) \neq 0$, then g is hyperbolic, and C_g is the axis of g. The action is *non-trivial* if at least one element of G is hyperbolic, and *Abelian* if every element of $[G,G]$ is elliptic. A non-trivial Abelian action on an $\mathbb{R}$-tree must fix either one or two ends of the tree, and is considered *exceptional* if it fixes only one end. To each non-trivial Abelian action, and each fixed end e, we associate the character χ such that $|\chi(g)| = \ell(g)$, and $\chi(g)$ is positive if and only if g is a hyperbolic isometry which translates its axis away from the fixed end e. We say g is *χ-elliptic* if $\chi(g)=0$, and *χ-hyperbolic* otherwise.

We are now able to give Brown's formulation of $\Sigma^1(G)$: An equivalence class $[\chi] \in S(G)$ is contained in $\Sigma^1(G)$ unless there exists an $\mathbb{R}$-tree T equipped with an exceptional non-trivial Abelian G-action associated to χ.

To demonstrate that $[\chi] \in \Sigma^1(G)$, it suffices to show that in any $\mathbb{R}$-tree T equipped with a non-trivial Abelian G-action associated to χ, there exists a line X such that $X \subseteq C_g$ for all $g \in G$. For this purpose, the following facts about characteristic subtrees are invaluable (see [OK00]):

Fact A. If $[g,h]=1$ and h is hyperbolic, then $C_h \subseteq C_g$.

Fact B. If $[g,h]=1$, then $C_g \cap C_h \subseteq C_{gh}$.

Essentially, we work with a fixed finite generating set of G, we consider an arbitrary non-trivial Abelian G-action on an arbitrary $\mathbb{R}$-tree T, we let $X \subseteq T$ denote the axis of one χ-hyperbolic generator s, and we use Facts A and B to demonstrate that $X \subseteq C_t$ for every other generator t. For this approach to be successful we typically need a sufficient number of commuting relations in G.

To demonstrate that $[\chi] \in \Sigma^1(G)^c$, it is often convenient to make use of the following well-known facts.

Lemma 2.1. *Let $\chi \in \mathrm{Hom}(G,\mathbb{R}) - \{0\}$. Suppose there is an epimorphism $\phi : G \to H$ and a character $\psi \in \mathrm{Hom}(H,\mathbb{R})$ such that $\chi = \psi \circ \phi$. If $[\psi] \in \Sigma^1(H)^c$, then $[\chi] \in \Sigma^1(G)^c$.*

Corollary 2.2. *If A and B are non-trivial finitely-generated groups, and $\chi \in \mathrm{Hom}(G,\mathbb{R}) - \{0\}$ factors through an epimorphism $G \to A * B$, then $[\chi] \in \Sigma^1(G)^c$.*

Proof. This follows from Lemma 2.1, and the fact that $\Sigma^1(A*B) = \emptyset$. □

3. Right-angled Artin groups and their pure symmetric automorphisms

Throughout, we fix a simplicial graph Γ, with vertex set V and edge set E. For each vertex $a \in V$, the *link of a* is the set $\mathrm{Lk}(a) = \{b \in V \mid \{a,b\} \in E\}$, and the *star of a* is the set $\mathrm{St}(a) = \mathrm{Lk}(a) \cup \{a\}$. For a set of vertices $W \subseteq V$, we write $\Gamma \setminus W$ for the full subgraph spanned by the vertices in $V \setminus W$.

Let $A = A(\Gamma)$ denote the right-angled Artin group determined by Γ. We shall not distinguish between the vertices of Γ and the generators of A, thus A is the group presented by

$$\big\langle V \mid ab = ba \text{ for all } a,b \in V \text{ such that } \{a,b\} \in E \big\rangle.$$

For each vertex $a \in V \setminus Z$, and each connected component K of $\Gamma \setminus \mathrm{St}(a)$, the map

$$v \mapsto \begin{cases} a^{-1}va & \text{if } v \in K, \\ v & \text{if } v \in V \setminus K, \end{cases}$$

extends to an automorphism $\pi_K^a : A \to A$. We say π_K^a is the *partial conjugation* (*of* A) *with acting letter* a *and domain* K. We write $\mathcal{P}$ for the set comprising the partial conjugations.

The *pure symmetric automorphism group*, $P\Sigma(A)$, comprises those automorphisms $\alpha : A \to A$ which map each vertex to a conjugate of itself. Laurence proved that $P\Sigma(A)$ is generated by $\mathcal{P}$ [Lau95].

We let $Z = \{a \in V \mid \mathrm{St}(a) = V\}$, and we may assume $Z \neq \emptyset$ for the following reason: it follows immediately from Laurence's result, together with the observation that enriching Γ with a new vertex w adjacent to all other vertices does not introduce new partial conjugations, and does not change the domain of any existing partial conjugation. Let $d : V \times V \to \{0,1,2\}$ denote the combinatorial metric on V.

We now record three results, paraphrased from existing literature, which make working with $\mathcal{P}$ tractable. A proof of the first is included because it is so brief; the second follows immediately from the first.

LEMMA 3.1 ([GPR12, Lemma 4.3]). *If* π_K^a, $\pi_L^b \in \mathcal{P}$ *and* $d(a,b) = 2$ *and* $b \notin K$, *then either* $K \cap L = \emptyset$ *or* $K \subseteq L$.

Proof. Assume $\pi_K^a, \pi_L^b \in \mathcal{P}$ and $d(a,b) = 2$ and $b \notin K$. For the sake of contradiction, suppose $\emptyset \neq K \cap L \neq K$. Let $u \in K \cap L$ and $v \in K \setminus L$. Since K is connected, there exists a path α in K from u to v. Since $u \in L$ and $v \notin L$, α passes through a vertex $w \in \mathrm{St}(b)$. Since $d(b,w) \leq 1$ and $w \in K$ and $b \in \Gamma \setminus \mathrm{St}(a)$, $b \in K$—a contradiction. □

LEMMA 3.2 ([GPR12, Corollary 4.4 and Lemma 4.7]). *For each pair of partial conjugations* $(\pi_K^a, \pi_L^b) \in \mathcal{P} \times \mathcal{P}$, *exactly one of the following six cases holds*:

(1) $d(a,b) \leq 1$;
(2) $d(a,b) = 2$, $a \in L$, *and* $b \in K$;
(3) $d(a,b) = 2$, $K \cap L = \emptyset$, *and either* $a \in L$ *or* $b \in K$;
(4) $d(a,b) = 2$, *and either* $\{a\} \cup K \subset L$ *or* $\{b\} \cup L \subset K$;
(5) $d(a,b) = 2$, *and* $(\{a\} \cup K) \cap (\{b\} \cup L) = \emptyset$;
(6) $d(a,b) = 2$, *and* $K = L$.

The relation $[\pi_K^a, \pi_L^b] = 1$ *holds only in the cases* (1), (4) *and* (5).

THEOREM 3.3 ([Toi12, Chapter 3]). *Every relation between partial conjugations is a consequence of the following relations*:

(1) $[\pi_K^a, \pi_L^b] = 1$ *if* (π_K^a, π_L^b) *falls into one of the cases* (1), (4), (5) *of Lemma* 3.2;

(2) $[\pi_K^a \pi_L^a, \pi_L^b] = 1$ *if* $K \neq L$ *and* $b \in K$.

It is convenient to introduce notation for certain products of partial conjugations with the same acting letter. We write $\delta_{K,L}^a$ for the product $\pi_K^a \pi_L^a$, provided $K \neq L$. We write ι^a for the inner automorphism $w \mapsto a^{-1}wa$ for all $w \in A$, and we note ι^a is simply the product of all partial conjugations with acting letter a.

Next, we record some useful facts about the behavior of partial conjugations.

Lemma 3.4. *If* $\pi_K^a, \pi_L^b \in \mathcal{P}$ *are such that* $a \notin L$ *and* $b \in K$ *and* $K \cap L = \emptyset$, *then* $\pi_L^a \in \mathcal{P}$ *and* $[\delta_{K,L}^a, \pi_L^b] = 1$.

Proof. Assume $\pi_K^a, \pi_L^b \in \mathcal{P}$ are such that $a \notin L$ and $b \in K$ and $K \cap L = \emptyset$. Let K' denote the connected component of $\Gamma \setminus \mathrm{St}(a)$ such that $K' \cap L \neq \emptyset$. Since $d(a,b) = 2$ and $a \notin L$ and $b \notin K'$ and $K' \cap L \neq \emptyset$, the pair $(\pi_{K'}^a, \pi_L^b)$ falls into case (6) of Lemma 3.2. Thus, $K' = L$. The relation $[\delta_{K,L}^a, \pi_L^b] = 1$ is (2) in Theorem 3.3. □

Corollary 3.5. *If* $a \in V \setminus Z$ *and* $\pi_L^b \in \mathcal{P}$, *then* $[\iota^a, \pi_L^b] = 1$ *if and only if* $a \notin L$.

4. The BNS-invariant of $P\Sigma(A)$

Throughout this section, we consider an arbitrary non-trivial character $\chi : P\Sigma(A) \to \mathbb{R}$. We write Σ for $\Sigma^1(P\Sigma(A))$, and Σ^c for the complement of Σ in $S(P\Sigma(A))$.

Lemma 4.1. *Let* $\pi_K^a, \pi_L^a \in \mathcal{P}$ *with* $K \neq L$. *If* π_K^a, π_L^a *and* $\delta_{K,L}^a$ *are* χ-*hyperbolic, then* $[\chi] \in \Sigma$.

Proof. Suppose π_K^a, π_L^a and $\delta_{K,L}^a$ are χ-hyperbolic. Consider a $P\Sigma(A)$-action on an $\mathbb{R}$-tree T that realizes χ. Let $X = C_{\pi_K^a} = C_{\pi_L^a} = C_{\delta_{K,L}^a}$. Let π_M^c be an arbitrary partial conjugation. If $[\pi_K^a, \pi_M^c] = 1$ or $[\pi_K^a, \pi_M^c] = 1$, then $X \subseteq C_{\pi_M^c}$ by Fact A; thus we may assume $[\pi_K^a, \pi_M^c] \neq 1$ and $[\pi_K^a, \pi_M^c] \neq 1$. It follows that $d(a,c) = 2$. Since $K \cap L = \emptyset$, we may assume without loss of generality that $c \notin K$. Since $d(a,c) = 2$ and $c \notin K$ and $[\pi_K^a, \pi_M^c] \neq 1$, the pair (π_K^a, π_M^c) falls into case (3) or (6) of Lemma 3.2.

First, consider the case that (π_K^a, π_M^c) falls into case (3). Then $a \in M$. By Lemma 3.4, $\pi_K^c \in \mathcal{P}$ and $[\delta_{K,M}^c, \pi_K^a] = 1$. By Fact A, $X \subseteq C_{\delta_{K,M}^c}$. If $c \in L$, then $[\delta_{K,L}^a, \pi_K^c] = 1$ and $X \subseteq C_{\pi_K^c}$ by Fact A. By Fact B, $X \subseteq C_{\pi_M^c}$. If $c \notin L$, then the pair (π_L^a, π_K^c) falls into case (5) of Lemma 3.2 which implies $[\pi_L^a, \pi_K^c] = 1$. By Fact A, $X \subseteq C_{\pi_K^c}$ which implies $X \subseteq C_{\pi_M^c}$ by Fact B.

Now consider the case that (π_K^a, π_M^c) falls into case (6). Then $a \notin M$, $c \notin K$ and $M = K$. Since $M \cap L = K \cap L = \emptyset$ and $a \notin M$ and $[\pi_L^a, \pi_M^c] \neq 1$, the pair (π_L^a, π_M^c) falls into case (3) of Lemma 3.2. Thus, $c \in L$. Since $c \in L$ and $M = K$, $[\delta_{K,L}^a, \pi_M^c] = 1$, and by Fact A, $X \subseteq C_{\pi_M^c}$. □

Corollary 4.2. *If $[\chi] \in \Sigma^c$, then the following properties hold for each vertex $a \in V \setminus Z$:*

(1) *There are at most two χ-hyperbolic partial conjugations with acting letter a.*
(2) *The inner automorphism ι^a is χ-hyperbolic if and only if there is exactly one χ-hyperbolic partial conjugation with acting letter a.*
(3) *If π^a_K and π^a_L are distinct χ-hyperbolic partial conjugations, then $\chi(\pi^a_K) = -\chi(\pi^a_L)$.*

Lemma 4.3. *Let $\pi^a_K, \pi^a_L \in \mathcal{P}$ with $K \neq L$, and let $b \in V$. If π^a_K, π^a_L and ι^b are χ-hyperbolic, then $[\chi] \in \Sigma$.*

Proof. Suppose π^a_K, π^a_L and ι^b are χ-hyperbolic. If $a = b$, then $[\chi] \in \Sigma$ by Corollary 4.2(3). Thus we may assume $b \neq a$. Let T be an $\mathbb{R}$-tree equipped with a $P\Sigma(A)$-action that realizes χ. Let $X = C_{\pi^a_K} = C_{\pi^a_L}$. Since ι^b is χ-hyperbolic, there exists a connected component M of $\Gamma \setminus \mathrm{St}(b)$ such that π^b_M is χ-hyperbolic. If $b \notin K$, then

$$\left[\pi^a_K, \iota^b\right] = \left[\iota^b, \pi^b_M\right] = 1;$$

if $b \in K$, then $b \notin L$ and

$$\left[\pi^a_L, \iota^b\right] = \left[\iota^b, \pi^b_M\right] = 1;$$

in either case, Fact A yields

$$C_{\pi^a_L} = C_{\iota^b} = C_{\pi^b_M} = X.$$

Let π^c_N be an arbitrary partial conjugation. The lemma is proved if we show $X \subseteq C_{\pi^c_N}$, for then the $P\Sigma(A)$-action fixes X setwise and is therefore not exceptional. If π^c_N commutes with any of the automorphisms π^a_K, π^a_L, π^b_M or ι^b, then $X \subseteq C_{\pi^c_N}$ by Fact A. Thus, we may assume π^c_N commutes with none of these automorphisms. It follows that $d(a,c) = d(b,c) = 2$ and $b \in N$. Since $K \cap L = \emptyset$, we may assume without loss of generality that $c \notin L$. We now consider cases based on whether or not N contains a.

First, we consider the case $a \in N$. Since $b \in N$ and $c \notin L$ and $[\pi^c_N, \pi^b_L] \neq 1$, the pair (π^c_N, π^b_L) falls into case (3) of Lemma 3.2; thus $N \cap L = \emptyset$. By Lemma 3.4, π^c_L is a partial conjugation, and $[\pi^a_L, \delta^c_{L,N}] = 1$. By Fact A, $X \subseteq C_{\delta^c_{L,N}}$. Since $b \in N$, $b \notin L$, and by Corollary 3.5, $[\iota^b, \pi^c_L] = 1$. By Fact A, $X \subseteq C_{\pi^c_L}$. By Fact B, $X \subseteq C_{\pi^c_N}$.

Next, we consider the case $a \notin N$. Since $a \notin N$ and $c \notin L$ and $[\pi^a_L, \pi^c_N] \neq 1$, the pair (π^a_L, π^c_N) falls into case (6) of Lemma 3.2; thus $N = L$. Let N' be the component of $\Gamma \setminus \mathrm{St}(c)$ such that $a \in N'$. Therefore, $[\pi^a_L, \delta^c_{N,N'}] = 1$. By Fact A, $X \subseteq C_{\delta^c_{N,N'}}$. Since $b \in N$, $b \notin N'$, and by Corollary 3.5, $[\iota^b, \pi^c_{N'}] = 1$. By Fact A, $X \subseteq C_{\pi^c_{N'}}$. By Fact B, $X \subseteq C_{\pi^c_N}$. ☐

Corollary 4.4. *If $[\chi] \in \Sigma^c$, then exactly one of the following holds:*

(I) *For each vertex* $a \in V \setminus Z$*, there is at most one* χ*-hyperbolic partial conjugation with acting letter* a*.*
(II) *For each vertex* $a \in V \setminus Z$*,* ι^a *is* χ*-elliptic and there are either zero or two* χ*-hyperbolic partial conjugations with acting letter* a*.*

Motivated by the corollary above, we classify characters depending on which case, if any, they fall into.

Definition 4.5. We say χ is *type* I if for each vertex $a \in V \setminus Z$, there is at most one χ-hyperbolic partial conjugation with acting letter a. We say χ is *type* II if for each vertex $a \in V \setminus Z$, ι^a is χ-elliptic and there are either zero or two χ-hyperbolic partial conjugations with acting letter a.

4.1. Characters of type I.

Definition 4.6 (p-set). A set of partial conjugations $\mathcal{Q} \subseteq \mathcal{P}$ is a *p-set* (or a *partionable* set) if $\mathcal{Q}$ satisfies the following properties:

(1) For each vertex $a \in V \setminus Z$, $\mathcal{Q}$ contains at most one partial conjugation with acting letter a.
(2) The set $\mathcal{Q}$ admits a non-trivial partition $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ with the property that $a \in L$ and $b \in K$ for each pair $(\pi^a_K, \pi^b_L) \in \mathcal{Q}_1 \times \mathcal{Q}_2$.

We say $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ is an *admissible partition* of $\mathcal{Q}$.

Remark 4.7. In the definition above, the first property is implied by the second. In this instance we have preferred transparency to brevity.

Remark 4.8. An arbitrary maximal p-set $\mathcal{Q}$, and an admissible partition $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ may be constructed as follows. Begin with a partial conjugation π^a_K. Let $b_1, \ldots, b_n$ be the vertices of K. For $j = 1, \ldots, n$, let L_j be the connected component of $\Gamma \setminus \mathrm{St}(b_j)$ such that $a \in L_j$. Let $a = a_1, a_2, \ldots, a_m$ be the vertices of $\bigcap_{j=1}^n L_j \neq \emptyset$. For $i = 1, 2, \ldots, m$, let K_i be the connected component of $\Gamma \setminus \mathrm{St}(a_i)$ such that $b_1 \in K_i$. Let

$$\mathcal{Q}_1 = \{\pi^{a_1}_{K_1}, \ldots, \pi^{a_m}_{K_m}\}, \qquad \mathcal{Q}_2 = \{\pi^{b_1}_{L_1}, \ldots, \pi^{b_n}_{L_n}\} \quad \text{and} \quad \mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2.$$

Proposition 4.9. *Suppose* χ *is type* I *and let* $\mathcal{H}$ *denote the set of* χ*-hyperbolic partial conjugations. Then* $[\chi] \in \Sigma^c$ *if and only if* $\mathcal{H}$ *is contained in some p-set* $\mathcal{Q}$*.*

Proof. Suppose χ is type I and $\mathcal{H}$ is contained in a p-set $\mathcal{Q}$. Let $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ be an admissible partition of $\mathcal{Q}$, with

$$\mathcal{Q}_1 = \{\pi^{a_1}_{K_1}, \ldots, \pi^{a_m}_{K_m}\} \quad \text{and} \quad \mathcal{Q}_2 = \{\pi^{b_1}_{L_1}, \ldots, \pi^{b_n}_{L_n}\}.$$

Let G_1 be the free Abelian group with basis $\{u_1, \ldots, u_m\}$, let G_2 be the free Abelian group with basis $\{v_1, \ldots, v_n\}$, and let $G = G_1 * G_2$. Consider a map such that: $\pi^{a_i}_{K_i} \mapsto u_i$ for $i = 1, \ldots, m$; $\pi^{b_j}_{L_j} \mapsto v_j$ for $j = 1, \ldots, n$; and all other partial conjugations are mapped to the identity. It follows from Theorem 3.3

that this map determines an epimorphism $\phi : P\Sigma(A) \to G$. Since χ factors through ϕ, by Corollary 2.2, $[\chi] \in \Sigma^c$.

Now suppose χ is type I and there is no p-set containing $\mathcal{H}$. Let T be an $\mathbb{R}$-tree equipped with a $P\Sigma(A)$-action that realizes χ. Let $\pi_K^a \in \mathcal{H}$, and let $X = C_{\pi_K^a}$. To prove the lemma it suffices to prove that $X \subseteq C_{\pi_M^c}$ for an arbitrary partial conjugation π_M^c, because then we have that the action fixes X setwise and hence is not exceptional. If π_M^c commutes with π_K^a, Fact A gives that $X \subseteq C_{\pi_M^c}$. Thus we may assume that π_M^c does not commute with π_K^a.

Next we show that the elements of $\mathcal{H}$ share the axis X. Let

$$\mathcal{I} = \{\pi_L^b \in \mathcal{H} \mid X \subseteq C_{\pi_L^b}\}.$$

Suppose $\mathcal{H} \neq \mathcal{I}$, and let $\pi_L^b \in \mathcal{H} \setminus I$. Since $X \nsubseteq C_{\pi_L^b}$, we have that $[\pi_L^b, \iota^a] \neq 1$, and $[\pi_K^a, \iota^b] \neq 1$. By Corollary 3.5 we have $a \in L$ and $b \in K$. It follows that $(\mathcal{I}, \mathcal{H} \setminus \mathcal{I})$ is an admissible partition, and $\mathcal{H}$ is a p-set—a contradiction which proves $\mathcal{H} = \mathcal{I}$.

Now consider an arbitrary partial conjugation such that π_M^c does not commute with π_L^b or ι^b whenever $\pi_L^b \in \mathcal{H}$. It follows that $d(b,c) = 2$ for all $\pi_K^b \in \mathcal{H}$. Since π_M^c does not commute with ι^b, $b \in M$ for each $\pi_L^b \in \mathcal{H}$. Since $\mathcal{H} \cup \{\pi_M^c\}$ is not a p-set, $\{\{\pi_M^c\}, \mathcal{H}\}$ is not an admissible partition. Thus there exists $\pi_L^b \in \mathcal{H}$ such that $c \notin L$. Since $d(b,c) = 2$ and $b \in M$ and $c \notin L$ and $[\pi_M^c, \pi_L^b] \neq 1$, Lemma 3.4 gives that π_L^c is a partial conjugation. Since $[\delta_{L,M}^c, \pi_L^b] = 1$, Fact A gives $X \subseteq C_{\delta_{L,M}^c}$. By Corollary 3.5, $[\pi_L^c, \iota^b] = 1$. By Fact A, $X \subseteq C_{\pi_L^c}$. By Fact B, $X \subseteq C_{\pi_M^c}$. □

4.2. Characters of type II.

Definition 4.10 (δ-p-set). A set of partial conjugations $\mathcal{Q} \subseteq \mathcal{P}$ is a *δ-p-set* if $\mathcal{Q}$ satisfies the following properties:

(1) For each vertex $a \in V \setminus Z$, $\mathcal{Q}$ contains either zero or two partial conjugations with acting letter a.
(2) The set $\mathcal{Q}$ admits a non-trivial partition $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ such that $a \in L$ or $b \in K$ or $K = L$ for each pair $(\pi_K^a, \pi_L^b) \in \mathcal{Q}_1 \times \mathcal{Q}_2$.

We say $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ is an *admissible δ-partition* of $\mathcal{Q}$.

Remark 4.11. It follows from the definitions that if $\pi_{K_1}^a, \pi_{K_{-1}}^a \in \mathcal{Q}$ and $K_1 \neq K_{-1}$, then either $\pi_{K_1}^a, \pi_{K_{-1}}^a \in \mathcal{Q}_1$ or $\pi_{K_1}^a, \pi_{K_{-1}}^a \in \mathcal{Q}_2$. Further, for each quadruple

$$(\pi_{K_1}^a, \pi_{K_{-1}}^a, \pi_{L_1}^b, \pi_{L_{-1}}^b) \in \mathcal{Q}_1 \times \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_2,$$

$a \in L_i$ and $b \in K_j$ and $K_{-i} = L_{-j}$ for some $i, j \in \{-1, 1\}$.

Lemma 4.12. *Let $\pi_{K_1}^a, \pi_{K_2}^a, \pi_{L_1}^b, \pi_{L_2}^b \in \mathcal{P}$ be distinct partial conjugations. Then $[\pi_{K_i}^a, \pi_{L_j}^b] \neq 1$ for all $i, j \in \{1, 2\}$ if and only if $\{\pi_{K_1}^a, \pi_{K_2}^a, \pi_{L_1}^b, \pi_{L_2}^b\}$ is a δ-p-set.*

Proof. Assume $[\pi^a_{K_i}, \pi^b_{L_j}] \neq 1$ for all $i,j \in \{1,2\}$. Without loss of generality, assume $a \notin L_2$ and $b \notin K_2$. Since $a \notin L_2$ and $b \notin K_2$ and $[\pi^a_{K_2}, \pi^b_{L_2}] \neq 1$, the pair $(\pi^a_{K_2}, \pi^b_{L_2})$ falls into case (6) of Lemma 3.2; thus $K_2 = L_2$. Since $b \notin K_2$ and $K_2 \cap L_1 = L_2 \cap L_1 = \emptyset$ and $[\pi^a_{K_2}, \pi^b_{L_1}] \neq 1$, the pair $(\pi^a_{K_2}, \pi^b_{L_1})$ falls into case (3) of Lemma 3.2; thus $a \in L_1$. Since $a \notin L_2$ and $K_1 \cap L_2 = K_1 \cap K_2 = \emptyset$ and $[\pi^a_{K_1}, \pi^b_{L_2}] \neq 1$, the pair $(\pi^a_{K_1}, \pi^b_{L_2})$ falls into case (3) of Lemma 3.2; thus $b \in K_1$. Thus $\{\{\pi^a_{K_1}, \pi^a_{K_2}\}, \{\pi^b_{L_1}, \pi^b_{L_2}\}\}$ is an admissible δ-partition of $\{\pi^a_{K_1}, \pi^a_{K_2}, \pi^b_{L_1}, \pi^b_{L_2}\}$. The converse follows immediately from the definitions and Lemma 3.2. □

LEMMA 4.13. *Let $\pi^a_{K_1}$, $\pi^a_{K_2}$, π^c_M be distinct partial conjugations, and let T be a $\mathbb{R}$-tree equipped with a $P\Sigma(A)$-action that realizes χ. If $\pi^a_{K_1}$ and $\pi^a_{K_2}$ are χ-hyperbolic, $c \notin K_1$ and $C_{\pi^a_{K_1}} \nsubseteq C_{\pi^c_M}$, then $c \in K_2$ and $\pi^c_{K_1} \in \mathcal{P}$.*

Proof. Suppose $\pi^a_{K_1}$ and $\pi^a_{K_2}$ are χ-hyperbolic and $c \notin K_1$. Let T be an $\mathbb{R}$-tree equipped with a $P\Sigma(A)$-action that realizes χ, and suppose $C_{\pi^a_{K_1}} \nsubseteq C_{\pi^c_M}$. It follows that $d(a,c) = 2$.

Since $c \notin K_1$ and $[\pi^a_{K_1}, \pi^c_M] \neq 1$, the pair $(\pi^a_{K_1}, \pi^c_M)$ falls into either case (3) or case (6) of Lemma 3.2. If $(\pi^a_{K_1}, \pi^c_M)$ falls into case (3), $a \in M$. By Lemma 3.4, $\pi^c_{K_1} \in \mathcal{P}$. Since $[\delta^c_{K_1,M}, \pi^a_{K_1}] = 1$, but Fact B cannot be used, we must have that $[\pi^c_{K_1}, \pi^a_{K_2}] \neq 1$; thus $(\pi^c_{K_1}, \pi^a_{K_2})$ falls into case (3) of Lemma 3.2, and $c \in K_2$. If $(\pi^a_{K_1}, \pi^c_M)$ falls into case (6), we have $a \notin M$ and $M = K_1$. But then since $a \notin M$ and $M \cap K_2 = \emptyset$ and $[\pi^a_{K_2}, \pi^c_M] \neq 1$, the pair $(\pi^a_{K_2}, \pi^c_M)$ falls into case (3) of Lemma 3.2. Thus $c \in K_2$. □

PROPOSITION 4.14. *Suppose χ is type* II *and let $\mathcal{H}$ denote the set of χ-hyperbolic partial conjugations. Then $[\chi] \in \Sigma^c$ if and only if $\mathcal{H}$ is contained in some δ-p-set $\mathcal{Q}$.*

Proof. Suppose $\mathcal{H}$ is contained in some δ-p-set $\mathcal{Q}$. Let $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ be an admissible partition of $\mathcal{Q}$ with

$$\mathcal{Q}_1 = \left\{\pi^{a_1}_{K_1}, \pi^{a_1}_{L_1}, \ldots, \pi^{a_m}_{K_m}, \pi^{a_m}_{L_m}\right\} \quad \text{and} \quad \mathcal{Q}_2 = \left\{\pi^{b_1}_{M_1}, \pi^{b_1}_{N_1}, \ldots, \pi^{b_n}_{M_n}, \pi^{b_n}_{N_n}\right\}.$$

Let G_1 be the free Abelian group with basis $\{u_1, \ldots, u_m\}$, G_2 be the free Abelian group with basis $\{v_1, \ldots, v_n\}$, and $G = G_1 * G_2$. Define $\phi : P\Sigma(A) \to G$ by $\pi^{a_i}_{K_i} \mapsto u_i$ and $\pi^{a_i}_{L_i} \mapsto u_i^{-1}$ for $i = 1, \ldots, m$, $\pi^{b_j}_{M_j} \mapsto v_j$ and $\pi^{b_j}_{N_j} \mapsto v_j^{-1}$ for $j = 1, \ldots, n$, and all other generators map to the identity. For $\pi^{a_i}_{K_i} \in \mathcal{Q}_1$ and $\pi^{b_j}_{M_j} \in \mathcal{Q}_2$, we have either $a_i \in M_j$ or $K_i = M_j$, and in either case, $[\pi^{a_i}_{K_i}, \pi^{b_j}_{M_j}] \neq 1$. Thus, ϕ is a well-defined epimorphism. Since χ factors through this map, by Corollary 2.2, we have $[\chi] \in \Sigma^c$.

Suppose $\mathcal{H}$ is not contained in some δ-p-set $\mathcal{Q}$. Let T be an $\mathbb{R}$-tree equipped with an $P\Sigma(A)$-action that realizes χ. Since χ is type II, we have $\pi_{a,K}, \pi_{a,L} \in \mathcal{H}$ for some vertex $a \in V \setminus Z$. Let $X = C_{\pi^a_K} = C_{\pi^a_L}$.

First, we will show $X = C_{\pi^b_M}$ for each $\pi^b_M \in \mathcal{H}$. Define $\mathcal{I} = \{\pi^b_M \in \mathcal{H} \mid X = C_{\pi^b_M}\}$. Assume $\mathcal{H} \neq \mathcal{I}$, and let $\pi^b_M \in \mathcal{H} \setminus \mathcal{I}$. Since $\pi^b_M \in \mathcal{H}$, there exists $\pi^b_N \in \mathcal{H}$ where $M \neq N$, and clearly $\pi^b_N \in \mathcal{H} \setminus \mathcal{I}$. Let $\pi^c_Q \in \mathcal{I}$. Again, there must be $\pi^c_R \in \mathcal{I}$ such that $Q \neq R$. By Lemma 4.12, $(\mathcal{I}, \mathcal{H} \setminus \mathcal{I})$ is an admissible δ-partition which is a contradiction, so $\mathcal{H} = \mathcal{I}$.

Now let π^b_M be an arbitrary element of $\mathcal{P}$, and let

$$\mathcal{H} = \{\pi^{a_1}_{K_1}, \pi^{a_1}_{L_1}, \ldots, \pi^{a_m}_{K_m}, \pi^{a_m}_{L_m}\}.$$

By Lemma 4.13, either $X \subseteq C_{\pi^b_M}$ or without loss of generality, $b \in K_i$ and $\pi^b_{L_i} \in \mathcal{P}$ for each $i = 1, \ldots, m$. Assume the latter is true, so either $a_i \notin M$ for some $i \in \{1, \ldots, m\}$ or $a_i \in M$ for each $i \in \{1, \ldots, m\}$. If $a_i \notin M$, then π^b_M commutes with $\pi^{a_i}_{L_i}$ which implies by Fact A that $X \subseteq C_{\pi^b_M}$. Suppose for each $i = 1, \ldots, m$, $a_i \in M$. If $L_i \cap L_j = \emptyset$ for some $i \neq j$, then $[\pi^b_{L_i}, \pi^{a_j}_{L_j}] = 1$ which implies $X \subseteq C_{\pi^b_{L_i}}$. Since $a \in M$ and $b \notin L_i$ and $L_i \cap M = \emptyset$, we have $[\delta^b_{L_i,M}, \pi^{a_i}_{L_i}] = 1$. By Fact A, $X \subseteq C_{\delta^b_{L_i,M}}$, and by Fact B, $X \subseteq C_{\pi^b_M}$. Suppose $L_i \cap L_j \neq \emptyset$ for each pair (i,j). Then $L_i = L_j$ for each pair (i,j) since these are connected components of $\Gamma \setminus \mathrm{St}(b)$. Denote by L this connected component. Then $(\{\pi^b_M, \pi^b_L\}, \mathcal{H})$ is an admissible partition of the δ-p-set $\mathcal{H} \cup \{\pi^b_M, \pi^b_L\}$ which is a contradiction. Therefore, $X \subseteq C_{\pi^b_M}$, and $[\chi] \in \Sigma$. ☐

Proposition 4.9 and Proposition 4.14 prove our first main theorem.

Theorem A. *Let $\chi : P\Sigma(A) \to \mathbb{R}$ be a character, and let $\mathcal{H}$ denote the set of χ-hyperbolic partial conjugations. Then $[\chi] \in \Sigma^c$ if and only if $\mathcal{H}$ is contained in a set of partial conjugations $\mathcal{Q}$ such that either:*

(1) *The set $\mathcal{Q}$ admits a partition $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ with the property that $a \in L$ and $b \in K$ for each pair $(\pi^a_K, \pi^b_L) \in \mathcal{Q}_1 \times \mathcal{Q}_2$; or*
(2) *For each vertex $a \in V \setminus Z$, ι^a is χ-elliptic, and $\mathcal{Q}$ contains either zero or two partial conjugations with acting letter a; and $\mathcal{Q}$ admits a partition $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ with the property that $a \in L$ or $b \in K$ or $K = L$ for each pair $(\pi^a_K, \pi^b_L) \in \mathcal{Q}_1 \times \mathcal{Q}_2$.*

Example 4.15. Let $A = \langle a, b, c, d, e \mid [a,b], [b,c], [c,d], [c,e]\rangle$. The pure symmetric automorphism group $P\Sigma(A)$ is generated by the set

$$\{\pi^a_{\{c,d,e\}}, \pi^b_{\{d\}}, \pi^b_{\{e\}}, \pi^c_{\{a\}}, \pi^d_{\{a,b\}}, \pi^d_{\{e\}}, \pi^e_{\{a,b\}}, \pi^e_{\{d\}}\},$$

so $S(P\Sigma(A))$ is a 7-dimensional sphere. The maximal p-sets are:

(1) $\mathcal{Q}_1 = \{\pi^a_{\{c,d,e\}}, \pi^c_{\{a\}}, \pi^d_{\{a,b\}}, \pi^e_{\{a,b\}}\}$ with admissible partition $\{\pi^a_{\{c,d,e\}}\}$ and $\{\pi^c_{\{a\}}, \pi^d_{\{a,b\}}, \pi^e_{\{a,b\}}\}$,
(2) $\mathcal{Q}_2 = \{\pi^a_{\{c,d,e\}}, \pi^b_{\{d\}}, \pi^d_{\{a,b\}}\}$ with admissible partition $\{\pi^a_{\{c,d,e\}}, \pi^b_{\{d\}}\}$ and $\{\pi^d_{\{a,b\}}\}$,

(3) $\mathcal{Q}_3 = \{\pi^a_{\{c,d,e\}}, \pi^b_{\{e\}}, \pi^e_{\{a,b\}}\}$ with admissible partition $\{\pi^a_{\{c,d,e\}}, \pi^b_{\{e\}}\}$ and $\{\pi^e_{\{a,b\}}\}$, and
(4) $\mathcal{Q}_4 = \{\pi^d_{\{e\}}, \pi^e_{\{d\}}\}$.

The only maximal δ-p-set is $\{\pi^b_{\{d\}}, \pi^b_{\{e\}}, \pi^d_{\{a,b\}}, \pi^d_{\{e\}}, \pi^e_{\{a,b\}}, \pi^e_{\{d\}}\}$ with admissible partition $\{\pi^b_{\{d\}}, \pi^b_{\{e\}}\}$ and $\{\pi^d_{\{a,b\}}, \pi^d_{\{e\}}, \pi^e_{\{a,b\}}, \pi^e_{\{d\}}\}$. Therefore, Σ^c consists of the characters $[\chi]$ such that:

(1) χ sends all generators to zero except maybe those generators in $\mathcal{Q}_i$ for some $1 \leq i \leq 4$, or
(2) $\chi(\pi^b_{\{d\}}) = -(\pi^b_{\{e\}}), \chi(\pi^d_{\{a,b\}}) = -\chi(\pi^d_{\{e\}}), \chi(\pi^e_{\{a,b\}}) = -\chi(\pi^e_{\{d\}})$, and χ sends all other generators to zero.

5. Right-angled Artin groups with separating intersecting links

A graph Γ has a separating intersection of links (SIL) if there exists a pair a, b of distinct non-adjacent vertices such that $\Gamma \setminus (\mathrm{Lk}(a) \cap \mathrm{Lk}(b))$ has a connected component M containing neither a nor b. The following proposition was proven in [CRSV10], and we state the result in terms of our particular circumstance.

Proposition 5.1 ([CRSV10, Theorem 3.6]). *If the defining graph Γ contains no SILs, then $P\Sigma(A)$ is isomorphic to a right-angled Artin group.*

In this section we prove the converse to Proposition 5.1, which completes the proof of Theorem B. We continue to use the notation described above.

Given a non-trivial character $\psi : A \to \mathbb{R}$, we write Γ_ψ for the full subgraph of Γ spanned by the set of ψ-hyperbolic vertices. The subgraph Γ_ψ is called *dominating* if every vertex in Γ is either in, or adjacent to a vertex in, Γ_ψ. It was shown in [MV95] that:

Theorem 5.2 ([MV95, Theorem 4.1]). *Suppose $[\psi] \in S(A)$. Then $[\psi] \in \Sigma^1(A)$ if and only if Γ_ψ is connected and dominating.*

For each set of vertices $U \subseteq V$, we write $S(U)$ for the sub-sphere

$$\big\{[\psi] \in S(A) \mid \psi(v) = 0 \text{ for all } v \in V \setminus U\big\}.$$

We note that $S(U)$ is a sub-sphere of dimension $|U| - 1$ (we consider $S(\emptyset)$ to be a sub-sphere of dimension -1). We say $S(U)$ is a *missing sub-sphere* if $S(U) \subseteq \Sigma(A)^c$, and we note this holds exactly when the full subgraph spanned by U is disconnected or non-dominating. If U spans a subgraph of Γ which is non-dominating, then every subset of U spans a subset of Γ which is non-dominating; if U spans a subgraph of Γ which is disconnected, then every subset of U spans a subset of Γ which is disconnected or non-dominating. It follows that if $S(U)$ and $S(W)$ are missing sub-spheres, then $S(U \cap W)$ is a missing sub-sphere. It also follows that $\Sigma^1(A)$ is constructed from $S(A)$

by removing the maximal missing sub-spheres. Viewing the construction of $\Sigma^1(A)$ in this distinctive way, we observe the following:

LEMMA 5.3. *If A is a right-angled Artin group, and $S_1, \ldots, S_p \subseteq S(A)$ are the maximal missing sub-spheres, then*

$$\begin{aligned}
&\operatorname{rk}\bigl(A/[A,A]\bigr) - \operatorname{rk}\bigl(Z(A)\bigr) \\
&\quad = 1 + \sum_i \dim(S_i) - \sum_{i<j} \dim(S_i \cap S_j) \\
&\qquad + \sum_{i<j<k} \dim(S_i \cap S_j \cap S_k) - \cdots + (-1)^{n-1} \dim(S_1 \cap \cdots \cap S_p).
\end{aligned}$$

Proof. Since $\operatorname{rk}(A/[A,A]) = |V|$, and $\operatorname{rk}(Z(A)) = |Z|$, the lemma is proved if we show that the right-hand side of the equation sums to $|V \setminus Z|$. It follows from Theorem 5.2 that, for each i, $S_i = S(U_i)$ for some maximal set of vertices U_i which spans a disconnected or non-dominating subgraph of Γ. For each vertex $v \in V \setminus Z$, the singleton set $\{v\}$ spans a non-dominating subgraph of Γ, and hence v is contained in at least one set U_i. Any set of vertices containing an element of Z spans a connected and dominating subgraph of Γ. Thus we have $V \setminus Z = U_1 \cup U_2 \cup \cdots \cup U_p$. Now the Principle of Inclusion–Exclusion, together with the identity $\sum_{i=1}^{p} (-1)^{i-1} \binom{p}{i} = 1$, gives:

$$\begin{aligned}
&|U_1 \cup U_2 \cup \cdots \cup U_p| \\
&\quad = \sum_i |U_i| - \sum_{i<j} |U_i \cap U_j| \\
&\qquad + \sum_{i<j<k} |U_i \cap U_j \cap U_k| - \cdots + (-1)^{n-1} |U_1 \cap \cdots \cap U_p| \\
&\quad = \sum_i \bigl(\dim(S_i) + 1\bigr) - \sum_{i<j} \bigl(\dim(S_i \cap S_j) + 1\bigr) \\
&\qquad + \sum_{i<j<k} \bigl(\dim(S_i \cap S_j \cap S_k) + 1\bigr) - \cdots + (-1)^{n-1} \bigl(\dim(S_1 \cap \cdots \cap S_p) + 1\bigr) \\
&\quad = 1 + \sum_i \dim(S_i) - \sum_{i<j} \dim(S_i \cap S_j) \\
&\qquad + \sum_{i<j<k} \dim(S_i \cap S_j \cap S_k) - \cdots + (-1)^{n-1} \dim(S_1 \cap \cdots \cap S_p).
\end{aligned}$$

□

Next, we characterize the maximal missing sub-spheres in $S(A)$ by a property which makes no reference to the canonical generating set of A, thereby allowing us to identify the only candidates for maximal missing sub-spheres in $S(G)$ when we do not yet know whether or not G is a right-angled Artin group.

A normal subgroup K in a finitely-generated group G is a *complement kernel* if $K = \ker(\psi)$ for some $[\psi] \in \Sigma(G)^c$. For such K, the set

$$\{[\psi] \in \Sigma^1(G)^c \mid K \subseteq \ker(\psi)\}$$

is the *complement subspace determined by* K.

Lemma 5.4. *For each subset $S \subseteq S(A)$, S is a maximal missing sub-sphere if and only if S is the complement subspace determined by some minimal complement kernel K.*

Proof. Suppose $S = S(U)$ is a maximal missing sub-sphere in $S(A)$, with $U = \{u_1, \dots, u_p\}$. Let $\psi_U : A \to \mathbb{R}$ denote the character such that

$$\psi_U(v) = 0 \quad \text{for } v \in V \setminus U \quad \text{and} \quad \psi_U(u_i) = \pi^i \quad \text{for } i = 1, \dots, p.$$

Since π is transcendental, $K_U = \ker(\psi_U)$ consists of those elements $a \in A$ with zero exponent sums in each of the vertices $u_1, \dots, u_p$. It follows that $[\psi_U] \in S(U)$, and $K_U \subseteq \ker(\psi)$ for every $[\psi] \in S(U)$. Thus $S(U)$ is the complement subspace determined by K_U. The maximality of U, together with Theorem 5.2, implies that K_U is minimal amongst the kernels of characters in $\Sigma^1(A)^c$. It also follows from Theorem 5.2 that every minimal complement kernel arises in this way. □

We now have an approach for showing that a finitely-generated torsion-free group G is not a right-angled Artin group: we identify the minimal complement kernels $K_1, \dots, K_p$ in G; use these to identify the corresponding complement subspaces $S_1, \dots, S_p$ in $S(G)$; then show that Lemma 5.3 fails. We carry out this plan for $P\Sigma(A)$ when Γ contains a SIL.

Lemma 5.5. *If S is the complement subspace corresponding to a minimal complement kernel K in $P\Sigma(A)$, then either:*

$$S = \{[\chi] \in S(P\Sigma(A)) \mid \chi(\pi_K^a) = 0 \text{ for all } \pi_K^a \in \mathcal{P} \setminus \mathcal{Q}\}$$

for some maximal p-set $\mathcal{Q}$, in which case $\dim(S) = |\mathcal{Q}| - 1$; *or*

$$S = \{[\chi] \in S(A) \mid \chi(\pi_K^a) = 0 \text{ for all } \pi_K^a \in \mathcal{P} \setminus \mathcal{Q}, \text{ and } \chi(\iota^v) = 0 \text{ for all } v \in V\}$$

for some maximal δ-p-set $\mathcal{Q}$, in which case $\dim(S) = |\mathcal{Q}|/2 - 1$.

Proof. Suppose S is the complement subspace corresponding to a minimal complement kernel K in $P\Sigma(A)$, and let $\chi : P\Sigma(A) \to \mathbb{R}$ be a character with kernel K. By Corollary 4.4, χ is type I or type II.

Consider first the case that χ is type I. By Proposition 4.9, the χ-hyperbolic vertices comprise a p-set $\mathcal{Q}$. The minimality of K implies that $\mathcal{Q}$ is not contained in a larger p-set. That S is as described follows immediately.

Now consider the case that χ is type II. By Proposition 4.14, the χ-hyperbolic vertices comprise a δ-p-set $\mathcal{Q}$. The minimality of K implies that $\mathcal{Q}$ is not contained in a larger δ-p-set. That S is as described follows immediately. □

Lemma 5.6. *If $\mathcal{Q}_1, \dots, \mathcal{Q}_p$ are the maximal p-sets in $P\Sigma(A)$, and $S_1, \dots, S_p$ the corresponding complement subspaces, then*

$$\begin{aligned} &\operatorname{rk}\big(P\Sigma(A)/\big[P\Sigma(A), P\Sigma(A)\big]\big) \\ &\quad = 1 + \sum_i \dim(S_i) - \sum_{i<j} \dim(S_i \cap S_j) \\ &\qquad + \sum_{i<j<k} \dim(S_i \cap S_j \cap S_k) - \dots + (-1)^{n-1} \dim(S_1 \cap \dots \cap S_p). \end{aligned}$$

Proof. It follows from Theorem 3.3 that $\operatorname{rk}(P\Sigma(A)/[P\Sigma(A), P\Sigma(A)]) = |\mathcal{P}|$. Suppose $\pi_K^a \in \mathcal{P}$. Let b be a vertex in K, and let L be the connected component of $\Gamma \setminus \operatorname{St}(b)$ such that $a \in L$. Then $\{\pi_K^a, \pi_L^b\}$ is a p-set. Thus every partial conjugation is contained in at least one p-set. Now, as in the proof of Lemma 5.3, the lemma follows from the Principle of Inclusion–Exclusion and the identity $\sum_{i=1}^{p} (-1)^{i-1} \binom{p}{i} = 1$. □

Corollary 5.7. *If $P\Sigma(A)$ is isomorphic to a right-angled Artin group, then $\Sigma^1(P\Sigma(A))^c$ contains no characters of type* II.

Proof. Assume the notation of Lemma 5.6. Suppose $\Sigma^1(P\Sigma(A))^c$ contains a character of type II. Then there exists a maximal δ-p-set $\mathcal{Q}$, and corresponding complement subspace S. By Lemma 5.5, since $|\mathcal{Q}| \geq 4$, $\dim(S) \geq 1$. Since no character is both type I and type II, $S \cap S_i = \emptyset$ for each i. It follows from Lemma 5.6 that the equation in Theorem 5.3 fails because the right-hand side exceeds the left-hand side. □

Proposition 5.8. *If Γ contains a SIL, then $P\Sigma(A)$ is not isomorphic to a right-angled Artin group.*

Proof. Suppose Γ contains a SIL. Let a, b and M be as in the definition of a SIL, let K be the connected component of $\Gamma \setminus \operatorname{St}(a)$ that contains b, and let L be the connected component of $\Gamma \setminus \operatorname{St}(b)$ that contains a. The set $\{\pi_K^a, \pi_M^a, \pi_L^b, \pi_M^b\}$ is a δ-p-set. In particular, $\Sigma^1(P\Sigma(A))$ contains at least one character of type II. By Corollary 5.7, $P\Sigma(A)$ is not isomorphic to a right-angled Artin group. □

Proposition 5.8 and [CRSV10, Theorem 3.6] prove Theorem B.

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