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LINDELÖF THEOREMS FOR MONOTONE SOBOLEV FUNCTIONS IN ORLICZ SPACES

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ABSTRACT. Our aim in this paper is to deal with Lindelöf type theorems for monotone Sobolev functions in Orlicz spaces.

1. Introduction and statement of results

Let $\mathbf{B}$ be the unit ball of the n-dimensional Euclidean space $\mathbf{R}^n$. We denote by $\delta_{\mathbf{B}}(x)$ the distance of $x \in \mathbf{B}$ from the boundary $\partial\mathbf{B}$, that is, $\delta_{\mathbf{B}}(x) = 1 - |x|$. We denote by $B(x,r)$ the open ball centered at x with radius r and set $\lambda B(x,r) = B(x,\lambda r)$ for $\lambda > 0$.

A continuous function u on a domain G is called monotone in the sense of Lebesgue (see [6]) if the equalities

$$\max_{\overline{D}} u = \max_{\partial D} u \quad \text{and} \quad \min_{\overline{D}} u = \min_{\partial D} u$$

hold whenever D is a domain with compact closure $\overline{D} \subset G$. If u is a monotone function on G satisfying

$$\int_G \left|\nabla u(z)\right|^p dz < \infty \quad \text{for some } p > n-1,$$

then

$$\left|u(x) - u(y)\right| \le C(n,p) r^{1-n/p} \left(\int_{2B} \left|\nabla u(z)\right|^p dz \right)^{1/p} \tag{1.1}$$

whenever $y \in B = B(x,r)$ with $2B \subset G$, where $C(n,p)$ is a positive constant depending only on n and p (see [10, Chapter 8] and [14, Section 16]). Using this inequality (1.1), Lindelöf theorems for monotone Sobolev functions on

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the half space of $\mathbf{R}^n$ were studied in [2]. For related results, see [1], [3]–[5] and [7]–[12].

In order to give a general result, we consider a nondecreasing positive function φ on the interval $[0,\infty)$ such that φ is of log-type, that is, there exists a positive constant C satisfying

$$\varphi(r^2) \leqq C\varphi(r) \quad \text{for all } r \geqq 0.$$

Set $\Phi_p(r) = r^p\varphi(r)$ for $p > 1$. In this note, we are concerned with boundary limits of monotone Sobolev functions u on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} \Phi_p\big(|\nabla u(z)|\big)\delta_{\mathbf{B}}(z)^\alpha\, dz < \infty. \tag{1.2}$$

Let u be a function on $\mathbf{B}$ and let $\xi \in \partial\mathbf{B}$. For $\gamma \geq 1$ and $c > 0$, set

$$T_\gamma(\xi; c) = \big\{x \in \mathbf{B} : |x-\xi|^\gamma \leq c\delta_{\mathbf{B}}(x)\big\}.$$

We say u has a tangential limit of order γ at ξ if the limit

$$\lim_{T_\gamma(\xi;c)\ni x\to\xi} u(x)$$

exists for every $c > 0$. In particular, a tangential limit of order 1 is called nontangential limit.

Our aim in this paper is to give the following result concerning the Lindelöf type theorem, as an extension of [2], [7] and [12].

THEOREM 1.1. *Let u be a monotone function on $\mathbf{B}$ satisfying (1.2). Suppose $p > n-1$ and $0 \leq n+\alpha-p < 1$. Set*

$$E_1 = \left\{\xi \in \partial\mathbf{B} : \limsup_{r\to 0} r^{p-\alpha-n}\big(\varphi(r^{-1})\big)^{-1} \times \int_{B(\xi,r)\cap\mathbf{B}} \Phi_p\big(|\nabla u(z)|\big)\delta_{\mathbf{B}}(z)^\alpha\, dz > 0\right\}.$$

If $\xi \in \partial\mathbf{B}\setminus E_1$ and there exists a rectifiable curve Γ in $\mathbf{B}$ tending to ξ along which u has a finite limit L, then u has a nontangential limit L at ξ.

REMARK 1.2. In [8, Theorem 2], Manfredi and Villamor treated the case $\varphi \equiv 1$ and a weight is a Muckenhoupt A_q weight, where $1 \leq q < p/(n-1)$. We note that $\delta_{\mathbf{B}}(x)^\alpha$ is in A_q for some $q \in [1, p/(n-1))$ when $-1 < \alpha < (p-n+1)/(n-1)$, but $\delta_{\mathbf{B}}(x)^\alpha$ is not in A_q for all $q \in [1, p/(n-1))$ when $(p-n+1)/(n-1) \leq \alpha < p-n+1$. Hence, our result is a generalization of [8, Theorem 2] in the case when a weight is $\delta_{\mathbf{B}}(x)^\alpha$.

REMARK 1.3. We know that E_1 is of $C_{1,\Phi_p,\alpha}$-capacity zero. For the definition of $(1,\Phi_p,\alpha)$-capacity $C_{1,\Phi_p,\alpha}$ and this fact, we refer to [11, Lemma 7.2 and Corollary 7.2]. See also [10, Section 8].

2. Preliminary lemmas

Throughout this paper, let C denote various constants independent of the variables in question, and $C(\varepsilon)$ a positive constant which depends on ε.

For a proof of Theorem 1.1, we prepare some lemmas. We know the following result from a proof of [13, Theorem 3].

LEMMA 2.1. *Let u be a monotone function on $\mathbf{B}$ satisfying (1.2). Suppose $p > n-1$ and $0 < \varepsilon < 1$. Then*

$$(2.1)\quad \big|u(x) - u(y)\big| \le C\delta_{\mathbf{B}}(x)^{1-n/p}\big(\varphi\big(\delta_{\mathbf{B}}(x)^{-1}\big)\big)^{-1/p}\left(\int_{2B(x)} \Phi_p\big(\big|\nabla u(z)\big|\big)\,dz\right)^{1/p} + C\delta_{\mathbf{B}}(x)^{1-\varepsilon},$$

whenever $x \in \mathbf{B}$ and $y \in B(x)$, where $B(x) = B(x, \delta_{\mathbf{B}}(x)/4)$ and C may depend on ε.

Fix $\xi \in \partial\mathbf{B}$. For $x \in \mathbf{B}$ such that x is close to ξ, set

$$r(x) = |\xi - x| \quad \text{and} \quad y(x) = \big(1 - r(x)\big)\xi.$$

By (2.1), we give the following estimate $|u(x) - u(y(x))|^p$.

LEMMA 2.2. *Let u be a monotone function on $\mathbf{B}$ satisfying (1.2). Suppose $p > n-1$ and $0 < \varepsilon < 1$.*

(1) *If $p < n - \delta$, then for each $x \in T_\gamma(\xi; c)$*

$$\big|u(x) - u\big(y(x)\big)\big|^p \le Cr(x)^{\gamma(p-n+\delta)}\big(\varphi\big(r(x)^{-1}\big)\big)^{-1}\int_{B(\xi,2r(x))\cap\mathbf{B}} \Phi_p\big(\big|\nabla u(z)\big|\big)\delta_{\mathbf{B}}(z)^{-\delta}\,dz + Cr(x)^{p(1-\varepsilon)}.$$

(2) *If $p > n - \delta$, then for each $x \in \mathbf{B}$ with $|x - \xi| < 1/2$*

$$\big|u(x) - u\big(y(x)\big)\big|^p \le Cr(x)^{p-n+\delta}\big(\varphi\big(r(x)^{-1}\big)\big)^{-1}\int_{B(\xi,2r(x))\cap\mathbf{B}} \Phi_p\big(\big|\nabla u(z)\big|\big)\delta_{\mathbf{B}}(z)^{-\delta}\,dz + Cr(x)^{p(1-\varepsilon)},$$

where C may depend on ε.

Proof. We can take a finite chain of balls $B_0, B_1, \dots, B_N$ such that

(i) $B_j = B(x_j)$, $x_j \in \partial B(\xi, r(x)) \cap \mathbf{B}$, $x_0 = x$ and $y(x) \in B_N$;
(ii) $\{\delta_{\mathbf{B}}(x_j)\}$ increase and $\delta_{\mathbf{B}}(x_j) \ge c_1|x - x_j|$ for some constant $c_1 > 0$;
(iii) $B_j \cap B_k \ne \emptyset$ if and only if $|j - k| \le 1$;
(iv) for each $t > 0$, the number of x_j such that $t < \delta_{\mathbf{B}}(x_j) \le 2t$ is less than c_2, where c_2 is a positive constant.

See [3, Lemma 2.2]. Pick $z_j \in B_{j-1} \cap B_j$ for $1 \le j \le N$; set $z_0 = x$ and $z_{N+1} = y(x)$. By Lemma 2.1, we see that

$$\begin{aligned}
&\big|u(x) - u\big(y(x)\big)\big| \\
&\quad \le \sum_{j=0}^{N} \big|u(z_{j+1}) - u(z_j)\big| \\
&\quad \le C \sum_{j=0}^{N} \delta_{\mathbf{B}}(x_j)^{1-n/p} \big(\varphi\big(\delta_{\mathbf{B}}(x_j)^{-1}\big)\big)^{-1/p} \\
&\qquad \times \left(\int_{2B_j} \Phi_p\big(|\nabla u(z)|\big)\, dz \right)^{1/p} + C \sum_{j=0}^{N} \delta_{\mathbf{B}}(x_j)^{1-\varepsilon}.
\end{aligned}$$

Taking natural numbers k_0 and k_1 such that $2^{-k_0-1} \le r(x) < 2^{-k_0}$ and $2^{-k_1-1} \le \delta_B(x) < 2^{-k_1}$, we see from (ii) that

$$\begin{aligned}
\sum_{j=0}^{N} \delta_{\mathbf{B}}(x_j)^{1-\varepsilon} &\le \sum_{k=k_0}^{k_1} \left(\sum_{2^{-k-1} \le \delta_B(x_j) < 2^{-k}} \delta_{\mathbf{B}}(x_j)^{1-\varepsilon} \right) \\
&\le c_2 \sum_{k=k_0}^{k_1} 2^{-k(1-\varepsilon)} \le \frac{2^{1-\varepsilon} c_2}{\log 2} \int_{2^{-k_1-1}}^{2^{-k_0}} t^{1-\varepsilon} \frac{dt}{t} \le C \int_{\delta_B(x)/2}^{2r(x)} t^{1-\varepsilon} \frac{dt}{t}.
\end{aligned}$$

Hence, we have by Hölder's inequality

$$\begin{aligned}
&\big|u(x) - u\big(y(x)\big)\big| \\
&\quad \le C \left(\sum_{j=0}^{N} \delta_{\mathbf{B}}(x_j)^{p'\{1-(n-\delta)/p\}} \big(\varphi\big(\delta_{\mathbf{B}}(x_j)^{-1}\big)\big)^{-p'/p} \right)^{1/p'} \\
&\qquad \times \left(\sum_{j=0}^{N} \int_{2B_j} \Phi_p\big(|\nabla u(z)|\big) \delta_{\mathbf{B}}(z)^{-\delta}\, dz \right)^{1/p} + C r(x)^{1-\varepsilon} \\
&\quad \le C \left(I^{p-1} \times \int_{B(\xi, 2r(x)) \cap \mathbf{B}} \Phi_p\big(|\nabla u(z)|\big) \delta_{\mathbf{B}}(z)^{-\delta}\, dz \right)^{1/p} + C r(x)^{1-\varepsilon},
\end{aligned}$$

where $1/p + 1/p' = 1$ and

$$I = \sum_{j=0}^{N} \delta_{\mathbf{B}}(x_j)^{p'\{1-(n-\delta)/p\}} \big(\varphi\big(\delta_{\mathbf{B}}(x_j)^{-1}\big)\big)^{-p'/p}.$$

First, consider the case $p < n - \delta$ and $x \in T_\gamma(\xi; c)$. Then we have

$$\begin{aligned}
I^{p-1} &\le C \left(\int_{\delta_{\mathbf{B}}(x)/2}^{2r(x)} t^{\frac{p-n+\delta}{p-1}} \big(\varphi\big(t^{-1}\big)\big)^{-\frac{1}{p-1}} \frac{dt}{t} \right)^{p-1} \\
&\le C \delta_{\mathbf{B}}(x)^{p-n+\delta} \big(\varphi\big(r(x)^{-1}\big)\big)^{-1}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\big|u(x) - u\big(y(x)\big)\big|^p \\
&\quad\le C\delta_{\mathbf{B}}(x)^{p-n+\delta}\big(\varphi\big(r(x)^{-1}\big)\big)^{-1}\int_{B(\xi,2r(x))\cap\mathbf{B}} \Phi_p\big(\big|\nabla u(z)\big|\big)\delta_{\mathbf{B}}(z)^{-\delta}\,dz \\
&\qquad + Cr(x)^{p(1-\varepsilon)} \\
&\quad\le Cr(x)^{\gamma(p-n+\delta)}\big(\varphi\big(r(x)^{-1}\big)\big)^{-1}\int_{B(\xi,2r(x))\cap\mathbf{B}} \Phi_p\big(\big|\nabla u(z)\big|\big)\delta_{\mathbf{B}}(z)^{-\delta}\,dz \\
&\qquad + Cr(x)^{p(1-\varepsilon)}.
\end{aligned}$$

Next, consider the case $p > n - \delta$. Then we have

$$I^{p-1} \le Cr(x)^{p-n+\delta}\big(\varphi\big(r(x)^{-1}\big)\big)^{-1}.$$

Thus, we can show the second part in the same manner as the first part. □

REMARK 2.3. In Lemma 2.2, we can replace

$$\int_{B(\xi,2r(x))\cap\mathbf{B}} \Phi_p\big(\big|\nabla u(z)\big|\big)\delta_{\mathbf{B}}(z)^{-\delta}\,dz$$

by

$$\int_{B(\xi,2r(x))\cap\mathbf{B}} \Phi_p\big(\big|\nabla u(z)\big|\big)\delta_{\mathbf{B}}(z)^{\alpha}\big|r(x) - |z-\xi|\big|^{-\delta-\alpha}\,dz$$

when $\delta + \alpha > 0$.

REMARK 2.4. The number of balls $B_0, B_1, \dots, B_N$ satisfying (iv) in Lemma 2.2 is less than $c_3 \log(4r(x)/\delta_{\mathbf{B}}(x))$. In fact,

$$\begin{aligned}
N + 1 &\le \sum_{k=k_0}^{k_1} \#\big\{j : 2^{-k-1} \le \delta_{\mathbf{B}}(x_j) < 2^{-k}\big\} \\
&\le \sum_{k=k_0}^{k_1} c_2 = \frac{c_2}{\log 2}\int_{2^{-k_1-1}}^{2^{-k_0}} \frac{dt}{t} \le \frac{c_2}{\log 2}\int_{\delta_{\mathbf{B}}(x)/2}^{2r(x)} \frac{dt}{t} \\
&= \frac{c_2}{\log 2}\log\big(4r(x)/\delta_{\mathbf{B}}(x)\big).
\end{aligned}$$

The following lemma can be proved by (2.1).

LEMMA 2.5. *Let u be a monotone function on $\mathbf{B}$ satisfying (1.2). If $\xi \in \partial\mathbf{B} \setminus E_1$ and there exists a sequence $\{r_j\}$ such that $2^{-j-1} \le r_j < 2^{-j}$ and $u((1-r_j)\xi)$ has a finite limit L, then u has a nontangential limit L at ξ.*

Proof. Fix $\xi \in \partial\mathbf{B} \setminus E_1$. Take $x \in T_1(\xi; c)$ with $2^{-j-1} \le |x-\xi| < 2^{-j}$. Then, as in the proof of Lemma 2.2, we can take a finite chain of balls $B_0, B_1, \dots, B_N$ such that

(i) $B_i = B(x_i)$, $x_i \in T_1(\xi; c) \cap \{y : 2^{-j-1} \le |y-\xi| < 2^{-j}\}$, $x_0 = x$ and $(1-r_j)\xi \in B_N$;
(ii) $B_i \cap B_k \ne \emptyset$ if and only if $|i-k| \le 1$;
(iii) for each $t > 0$, the number of x_i such that $t < \delta_{\mathbf{B}}(x_i) \le 2t$ is less than c', where c' is a positive constant.

By Remark 2.4, we note that N is less than C_1, where C_1 is a positive constant depending only on c. Since

$$2^{-j-1} \le |x_i - \xi| \le c\delta_{\mathbf{B}}(x_i) \le c|x_i - \xi| \le c2^{-j},$$

as in the proof of Lemma 2.2, we obtain by (2.1)

$$\begin{aligned}
&\big|u(x) - u\big((1-r_j)\xi\big)\big| \\
&\quad \le C\sum_{i=0}^{N} \delta_{\mathbf{B}}(x_i)^{1-n/p}\big(\varphi\big(\delta_{\mathbf{B}}(x_i)^{-1}\big)\big)^{-1/p} \times \left(\int_{2B(x_i)} \Phi_p\big(|\nabla u(z)|\big)\,dz\right)^{1/p} \\
&\qquad + C\sum_{i=0}^{N} \delta_{\mathbf{B}}(x_i)^{1-\varepsilon} \\
&\quad \le C\sum_{i=0}^{N} \delta_{\mathbf{B}}(x_i)^{1-\varepsilon} \\
&\qquad + C\sum_{i=0}^{N}\Bigg(\delta_{\mathbf{B}}(x_i)^{p-\alpha-n}\big(\varphi\big(\delta_{\mathbf{B}}(x_i)^{-1}\big)\big)^{-1} \\
&\qquad \times \int_{2B(x_i)} \Phi_p\big(|\nabla u(z)|\big)\delta_{\mathbf{B}}(z)^{\alpha}\,dz\Bigg)^{1/p} \\
&\quad \le C2^{-j(1-\varepsilon)} \\
&\qquad + C\left(2^{-j(p-\alpha-n)}\big(\varphi(2^j)\big)^{-1}\int_{B(\xi,2^{-j+1})} \Phi_p\big(|\nabla u(z)|\big)\delta_{\mathbf{B}}(z)^{\alpha}\,dz\right)^{1/p},
\end{aligned}$$

where $0 < \varepsilon < 1$. Since $\xi \in \partial\mathbf{B} \setminus E_1$ and $\lim_{j\to\infty} u((1-r_j)\xi) = L$, u has a nontangential limit L at ξ. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Take a number δ such that $n + \alpha - p < \delta + \alpha < 1$. For $r > 0$ sufficiently small, take $x(r) \in \Gamma \cap \partial B(\xi, r)$ and set $y(x(r)) = (1-r)\xi$. By Lemma 2.2(2) and Remark 2.3, we have

$$\begin{aligned}
&\big|u\big(x(r)\big) - u\big(y\big(x(r)\big)\big)\big|^p \\
&\quad \le Cr^{p-n+\delta}\big(\varphi\big(r^{-1}\big)\big)^{-1} \\
&\qquad \times \int_{B(\xi,2r)\cap\mathbf{B}} \Phi_p\big(|\nabla u(z)|\big)\delta_{\mathbf{B}}(z)^{\alpha}\big|r - |z-\xi|\big|^{-\delta-\alpha}\,dz + Cr^{p(1-\varepsilon)}.
\end{aligned}$$

Moreover, since $0<\delta+\alpha<1$, we see that

$$\int_{2^{-j-1}}^{2^{-j}} \big|r-|z-\xi|\big|^{-\delta-\alpha}\,dr \le C2^{-j(1-\delta-\alpha)}.$$

Hence, it follows that

$$\begin{aligned}
&\inf_{2^{-j-1}\le r<2^{-j}} \big|u\big(x(r)\big)-u\big(y\big(x(r)\big)\big)\big|^p \\
&\le C\int_{2^{-j-1}}^{2^{-j}} \bigg(r^{p-n+\delta}\big(\varphi\big(r^{-1}\big)\big)^{-1} \\
&\quad \times \int_{B(\xi,2r)\cap\mathbf{B}} \Phi_p\big(|\nabla u(z)|\big)\delta_{\mathbf{B}}(z)^\alpha\big|r-|z-\xi|\big|^{-\delta-\alpha}\,dz\bigg)\frac{dr}{r} \\
&\quad + C\big(2^{-j}\big)^{p(1-\varepsilon)} \\
&\le C2^{-j\{p-n+\delta-1\}}\big(\varphi\big(2^j\big)\big)^{-1} \\
&\quad \times \int_{B(\xi,2^{-j+1})\cap\mathbf{B}} \Phi_p\big(|\nabla u(z)|\big)\delta_{\mathbf{B}}(z)^\alpha\bigg(\int_{2^{-j-1}}^{2^{-j}}\big|r-|z-\xi|\big|^{-\delta-\alpha}\,dr\bigg)\,dz \\
&\quad + C\big(2^{-j}\big)^{p(1-\varepsilon)} \\
&\le C2^{-j\{p-\alpha-n\}}\big(\varphi\big(2^j\big)\big)^{-1} \\
&\quad \times \int_{B(\xi,2^{-j+1})\cap\mathbf{B}} \Phi_p\big(|\nabla u(z)|\big)\delta_{\mathbf{B}}(z)^\alpha\,dz + C\big(2^{-j}\big)^{p(1-\varepsilon)}.
\end{aligned}$$

Since $\xi\notin E_1$, we see that

$$\lim_{j\to\infty}\inf_{2^{-j-1}\le r<2^{-j}} \big|u\big(x(r)\big)-u\big(y\big(x(r)\big)\big)\big|^p = 0.$$

Hence, we find a sequence $\{r_j\}$ such that $2^{-j-1}\le r_j<2^{-j}$ and

$$\lim_{j\to\infty}\big|u\big(x(r_j)\big)-u\big(y\big(x(r_j)\big)\big)\big|^p = 0.$$

Since u has a finite limit L at ξ along Γ, we have

$$\lim_{j\to\infty} u\big(y(r_j)\big) = \lim_{j\to\infty} u\big(x(r_j)\big) = L.$$

Thus u has a nontangential limit L at ξ by Lemma 2.5. □

REMARK 3.1. Let u be a monotone function on $\mathbf{B}$ satisfying (1.2) and let $\gamma\ge 1$. Suppose $p>n-1$ and $n+\alpha-p\ge 0$ and set

$$\begin{aligned}
E_\gamma = \bigg\{\xi\in\partial\mathbf{B}:\ &\limsup_{r\to 0} r^{\gamma(p-\alpha-n)}\big(\varphi\big(r^{-1}\big)\big)^{-1} \\
&\times\int_{B(\xi,r)\cap\mathbf{B}} \Phi_p\big(|\nabla u(z)|\big)\delta_{\mathbf{B}}(z)^\alpha\,dz > 0\bigg\}.
\end{aligned}$$

If $\xi \in \partial\mathbf{B} \setminus E_\gamma$ and u has a radial limit at ξ, then u has a tangential limit of order γ at ξ. See [12, Theorem 4] and [10, Section 8].

In fact, since $\xi \notin E_\gamma$, we have by Lemma 2.2(1) with $\delta = -\alpha$

$$\lim_{T_\gamma(\xi;c)\ni x\to\xi} \left|u(x) - u\big(y(x)\big)\right|^p = 0,$$

so that

$$\lim_{T_\gamma(\xi;c)\ni x\to\xi} \left|u(x) - u\big(y(x)\big)\right| = 0.$$

Since the radial limit $\lim_{x\to\xi} u(y(x))$ exists by our assumption, the limit $\lim_{T_\gamma(\xi;c)\ni x\to\xi} u(x)$ exists.

Remark 3.2. Let H_h denote the Hausdorff measure with the measure function h. We know that $H_h(E_\gamma) = 0$, where $h(r) = r^{\gamma(n+\alpha-p)}\varphi(r^{-1})$. For this fact, we refer to [11, Lemma 7.2].

Remark 3.3. Let u be a monotone function on $\mathbf{B}$ satisfying (1.2). Then u has a nontangential limit at $\xi \in \partial\mathbf{B}$ except in a set of $C_{1,\Phi_p,\alpha}$-capacity zero. For the case $\varphi \equiv 1$, see [8, Theorem 5.2].

In fact, to show this, we define

$$\tilde{E} = \left\{\xi \in \partial\mathbf{B} : \int_{\mathbf{B}} |\xi - y|^{1-n} \left|\nabla u(y)\right| dy = \infty\right\}$$

and set $F = \tilde{E} \cup E_1$, where E_1 is as in Theorem 1.1. Note that $\tilde{E}$ is of $C_{1,\Phi_p,\alpha}$-capacity zero by the definition of $C_{1,\Phi_p,\alpha}$-capacity, and F is of $C_{1,\Phi_p,\alpha}$-capacity zero by Remark 1.3. If $\xi \notin \tilde{E}$, then u has a finite radial limit L. In view of Theorem 1.1, we see that if $\xi \in \partial\mathbf{B} \setminus F$, then u has a nontangential limit L at ξ.

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