Illinois Journal of Mathematics
Volume 57, Number 4, Winter 2013, Pages 1019–1024
S 0019-2082

ON THE KÄHLER STRUCTURES OVER QUOT SCHEMES

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Abstract. Let $S^n(X)$ be the n-fold symmetric product of a compact connected Riemann surface X of genus g and gonality d. We prove that $S^n(X)$ admits a Kähler structure such that all the holomorphic bisectional curvatures are nonpositive if and only if $n < d$. Let $\mathcal{Q}_X(r,n)$ be the Quot scheme parametrizing the torsion quotients of $\mathcal{O}_X^{\oplus r}$ of degree n. If $g \geq 2$ and $n \leq 2g-2$, we prove that $\mathcal{Q}_X(r,n)$ does not admit a Kähler structure such that all the holomorphic bisectional curvatures are nonnegative.

1. Introduction

Let X be a compact connected Riemann surface of genus g and gonality d. For a positive integer n, let $S^n(X)$ denote the n-fold symmetric product of X. More generally, $\mathcal{Q}_X(r,n)$ will denote that Quot scheme that parametrizes all the torsion quotients of $\mathcal{O}_X^{\oplus r}$ of degree n. So $S^n(X) = \mathcal{Q}_X(1,n)$. This $\mathcal{Q}_X(r,n)$ is a complex projective manifold.

We prove the following (see Theorem 3.1):

The symmetric product $S^n(X)$ admits a Kähler structure satisfying the condition that all the holomorphic bisectional curvatures are nonpositive if and only if $n < d$.

The "only if" part was proved in [Bi1].

The main theorem of [BR] says the following (see [BR, Theorem 1.1]): If $g \geq 2$ and $n \leq 2(g-1)$, then $S^n(X)$ does not admit any Kähler metric for which all the holomorphic bisectional curvatures are nonnegative. A simpler proof of this result was given in [Bi2]. Here, we prove the following generalization of it (see Proposition 4.1):

Received April 9, 2013; received in final form November 17, 2013.
2010 *Mathematics Subject Classification.* 14C20, 32Q05, 32Q10.

Assume that $g \geq 2$ and $n \leq 2(g-1)$. Then $\mathcal{Q}_X(r,n)$ does not admit any Kähler structure such that all the holomorphic bisectional curvatures are non-negative.

If $r > 1$, the method in [Bi2] give a much weaker version of Proposition 4.1.

2. Preliminaries

Let X be a compact connected Riemann surface of genus g. For any positive integer n, consider the Cartesian product X^n. Denote by P_n the group of permutations of $\{1,\ldots,n\}$. The group P_n has a natural action on X^n. The quotient X^n/P_n will be denoted by $S^n(X)$; it is called the *n-fold symmetric product* of X.

Let $\mathcal{O}_X$ denote the sheaf of germs of holomorphic functions on X. For a positive integer r, consider the sheaf $\mathcal{O}_X^{\oplus r}$ of germs of holomorphic sections of the trivial holomorphic vector bundle on X of rank r. For any positive integer n, let

$$\mathcal{Q} := \mathcal{Q}_X(r,n)$$

be the Quot scheme parametrizing all the torsion quotients of degree n of the $\mathcal{O}_X$-module $\mathcal{O}_X^{\oplus r}$. Equivalently, points of $\mathcal{Q}$ parametrize coherent analytic subsheaves of $\mathcal{O}_X^{\oplus r}$ of rank r and degree $-n$. This $\mathcal{Q}$ is an irreducible smooth complex projective variety of dimension rn [Be, p. 1, Theorem 2].

Note that $\mathcal{Q}_X(1,n)$ is identified with the symmetric product $S^n(X)$ by sending a quotient of $\mathcal{O}_X$ to the scheme–theoretic support of it. If we consider $\mathcal{Q}_X(1,n)$ as the parameter space for the coherent analytic subsheaves of $\mathcal{O}_X$ of rank 1 and degree $-n$, then the above identification of $\mathcal{Q}_X(1,n)$ with $S^n(X)$ sends a subsheaf $\psi : L \hookrightarrow \mathcal{O}_X$ to the divisor of ψ.

The *gonality* of X is the smallest integer d such that there is a nonconstant holomorphic map $X \longrightarrow \mathbb{CP}^1$ of degree d (see [Ei, p. 171]). Therefore, the gonality of X is one if and only if $g = 0$. If $g \in \{1,2\}$, then the gonality of X is two. More generally, the gonality of X is two if and only if X is hyperelliptic of positive genus.

3. Nonpositive holomorphic bisectional curvatures

THEOREM 3.1. *Let d denote the gonality of X. The symmetric product $S^n(X)$ admits a Kähler structure satisfying the condition that all the holomorphic bisectional curvatures are nonpositive if and only if $n < d$.*

Proof. If $n \geq d$, then we know that $S^n(X)$ does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonpositive [Bi1, p. 1491, Proposition 3.2]. We recall that this follows from the fact that there is a nonconstant holomorphic embedding of $\mathbb{CP}^1$ in $S^n(X)$ if $n \geq d$.

So assume that $n < d$.

Let

$$\varphi : S^n(X) \longrightarrow \mathrm{Pic}^n(X) \tag{3.1}$$

be the natural holomorphic map that sends any $\{x_1, \ldots, x_n\} \in S^n(X)$ to the holomorphic line bundle $\mathcal{O}_X(\sum_{i=1}^n x_i)$. We will show that φ is an immersion.

Take any point $\underline{x} = \{x_1, \ldots, x_n\} \in S^n(X)$. The divisor $\sum_{i=1}^n x_i$ will be denoted by D. Let

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow Q'(\underline{x}) := \mathcal{O}_X/\mathcal{O}_X(-D) \longrightarrow 0 \tag{3.2}$$

be the short exact sequence corresponding to the point $\underline{x}$. tensoring it with $\mathcal{O}_X(-D)^* = \mathcal{O}_X(D)$ we get the short exact sequence

$$\begin{aligned} 0 \longrightarrow End\big(\mathcal{O}_X(-D)\big) = \mathcal{O}_X \longrightarrow Hom\big(\mathcal{O}_X(-D), \mathcal{O}_X\big) = \mathcal{O}_X(D) \\ \longrightarrow Q(\underline{x}) := Hom\big(\mathcal{O}_X(-D), Q'(\underline{x})\big) \longrightarrow 0. \end{aligned}$$

Let

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{O}_X) \xrightarrow{\alpha} H^0\big(X, \mathcal{O}_X(D)\big) \\ \xrightarrow{\beta} H^0\big(X, Q(\underline{x})\big) \xrightarrow{\gamma} H^1(X, \mathcal{O}_X) \end{aligned} \tag{3.3}$$

be the long exact sequence of cohomologies associated to this short exact sequence of sheaves.

The holomorphic tangent space to $S^n(X)$ at $\underline{x}$ is

$$T_{\underline{x}} S^n(X) = H^0\big(X, Q(\underline{x})\big)$$

and the tangent bundle of $\mathrm{Pic}^n(X)$ is the trivial vector bundle with fiber $H^1(X, \mathcal{O}_X)$. The differential at $\underline{x}$ of the map φ in (3.1)

$$(d\varphi)(\underline{x}) : T_{\underline{x}} S^n(X) = H^0\big(X, Q(\underline{x})\big) \longrightarrow T_{\varphi(\underline{x})} \mathrm{Pic}^n(X) = H^1(X, \mathcal{O}_X)$$

satisfies the identity

$$(d\varphi)(\underline{x}) = \gamma, \tag{3.4}$$

where γ is the homomorphism in (3.3).

Now, $H^0(X, \mathcal{O}_X) = \mathbb{C}$. Since $n < d$, it can be shown that

$$H^0\big(X, \mathcal{O}_X(D)\big) = \mathbb{C}.$$

Indeed, $\dim H^0(X, \mathcal{O}_X(D)) \geq 1$ because D is effective. If

$$\dim H^0\big(X, \mathcal{O}_X(D)\big) \geq 2,$$

then considering the partial linear system given by two linearly independent sections of $\mathcal{O}_X(D)$ we get a holomorphic map from X to $\mathbb{CP}^1$ whose degree coincides with the degree of D. This contradicts the fact that the gonality of X is strictly bigger than n. Therefore, $H^0(X, \mathcal{O}_X(D)) = \mathbb{C}$.

Since $H^0(X, \mathcal{O}_X(D)) = \mathbb{C}$, the homomorphism α in (3.3) is an isomorphism. Hence β in the exact sequence (3.3) is the zero homomorphism and γ in (3.3) is injective.

Since γ in (3.3) is injective, from (3.4) we conclude that φ is an immersion.

The compact complex torus $\mathrm{Pic}^n(X)$ admits a flat Kähler metric ω. The pullback $\varphi^*\omega$ is a Kähler metric on $S^n(X)$ because φ is an immersion. Since ω is flat, all the holomorphic bisectional curvatures of $\varphi^*\omega$ are nonpositive. □

Lemma 3.2. *Take $r \geq 2$ and take any positive integer n. The Quot scheme $\mathcal{Q}_X(r,n)$ does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonpositive.*

Proof. Let

$$f : \mathcal{Q}_X(r,n) \longrightarrow S^n(X) \tag{3.5}$$

be the holomorphic map that sends any quotient Q of $\mathcal{O}_X^{\oplus r}$ to the quotient of $\mathcal{O}_X$ corresponding to the rth exterior product of the kernel for Q. Take any $\underline{x} = \{x_1, \ldots, x_n\} \in S^n(X)$ such that all x_i are distinct points. Then $f^{-1}(\underline{x})$ is isomorphic to $(\mathbb{CP}^{r-1})^n$. In particular, there are embeddings of $\mathbb{CP}^1$ in $\mathcal{Q}_X(r,n)$. This immediately implies that $\mathcal{Q}_X(r,n)$ does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonpositive. □

4. Nonnegative holomorphic bisectional curvatures

In this section we assume that $g \geq 2$.

Corollary 4.1. *If $n \leq 2(g-1)$, then $\mathcal{Q}_X(r,n)$ does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonnegative.*

Proof. Assume that $\mathcal{Q}_X(r,n)$ has a Kähler structure ω such that all the holomorphic bisectional curvatures for ω are nonnegative. Consequently, tangent bundle $T\mathcal{Q}_X(r,n)$ is nef. See [DPS, p. 305, Definition 1.9] for the definition of a nef vector bundle; nef line bundles are introduced in [DPS, p. 299, Definition 1.2]. Since $\mathcal{Q}_X(r,n)$ is a complex projective manifold, nef bundles on $\mathcal{Q}_X(r,n)$ in the sense of [DPS] coincide with the nef bundles on $\mathcal{Q}_X(r,n)$ in the algebraic geometric sense (see lines 13–14 (from top) in [DPS, p. 296]).

Let

$$\delta := \varphi \circ f : \mathcal{Q}_X(r,n) \longrightarrow \mathrm{Pic}^n(X) \tag{4.1}$$

be the composition, where φ and f are constructed in (3.1) and (3.5) respectively. The homomorphism

$$H^1\big(\mathrm{Pic}^n(X), \mathbb{Q}\big) \longrightarrow H^1\big(S^n(X), \mathbb{Q}\big), \qquad c \longmapsto \varphi^* c$$

is an isomorphism [Ma, p. 325, (6.3)]. Also, the homomorphism

$$H^1\big(S^n(X), \mathbb{Q}\big) \longrightarrow H^1\big(\mathcal{Q}_X(r,n), \mathbb{Q}\big), \qquad c \longmapsto f^* c$$

is an isomorphism [BGL, p. 647, Proposition 4.2] (see also the last line of [BGL, p. 647]). Combining these we conclude that the homomorphism

$$H^1\big(\mathrm{Pic}^n(X), \mathbb{Q}\big) \longrightarrow H^1\big(\mathcal{Q}_X(r,n), \mathbb{Q}\big), \qquad c \longmapsto \delta^* c$$

is an isomorphism, where δ is constructed in (4.1).

Since the fibers of δ are connected, this implies that δ is the Albanese morphism for $\mathcal{Q}_X(r,n)$. Since the tangent bundle of $\mathcal{Q}_X(r,n)$ is nef, the Albanese map δ is a holomorphic surjective submersion onto $\mathrm{Pic}^n(X)$ [CP], [DPS, p. 321, Proposition 3.9].

Since δ is surjective, the map φ in (3.1) is surjective. Therefore,

$$g = \dim \mathrm{Pic}^n(X) \leq \dim S^n(X) = n.$$

The map φ is a submersion because δ is a submersion and f is surjective.

We will show that φ is not a submersion if $n \leq 2(g-1)$.

Take any $n \leq 2(g-1)$. Let D' be the divisor of a holomorphic 1-form on X. We note that the degree of D' is $2(g-1)$. Take an effective divisor D on X of degree n such that $D' - D$ is effective. Writing $D' = x_1 + \cdots + x_{2g-2}$, we may take $D = x_1 + \cdots + x_n$. Substitute this D in (3.2) and consider the corresponding long exact sequence of cohomologies

$$(4.2)\quad H^0\big(X, Q(\underline{x})\big) \xrightarrow{\gamma} H^1(X, \mathcal{O}_X) \xrightarrow{\gamma'} H^1\big(X, \mathcal{O}_X(D)\big) \longrightarrow H^1\big(X, Q(\underline{x})\big)$$

as in (3.3). We note that $H^1(X, Q(\underline{x})) = 0$ because $Q(\underline{x})$ is a torsion sheaf on X. From Serre duality,

$$H^1\big(X, \mathcal{O}_X(D)\big) = H^0\big(X, \mathcal{O}_X\big(D' - D\big)\big)^*.$$

Now $H^0(X, \mathcal{O}_X(D' - D)) \neq 0$ because $D' - D$ is effective. Combining these, we conclude that γ' in (4.2) is nonzero. Hence γ in (4.2) is not surjective. Therefore, from (3.4) we conclude that the differential $d\varphi$ of φ is not surjective at the point $D \in S^n(X)$. In particular, φ is not a submersion if $n \leq 2(g-1)$. This completes the proof. □

Acknowledgment. We thank the referee for helpful comments. The first-named author acknowledges the support of J. C. Bose Fellowship.

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