

TIME-FRACTIONAL AND MEMORYFUL Δ^{2^k} SIES ON $\mathbb{R}_+ \times \mathbb{R}^d$: HOW FAR CAN WE PUSH WHITE NOISE?

HASSAN ALLOUBA

ABSTRACT. High order and fractional PDEs have become prominent in theory and in modeling many phenomena. Here, we focus on the regularizing effect of a large class of memoryful high-order or time-fractional PDEs—through their fundamental solutions—on stochastic integral equations (SIEs) driven by space–time white noise. Surprisingly, we show that maximum spatial regularity is achieved in the fourth-order-bi-Laplacian case; and any further increase in the *spatial*-Laplacian order is *entirely* translated into additional *temporal* regularization of the SIE. We started this program in [*Discrete Contin. Dyn. Syst. Ser. A* **33** (2013) 413–463, *Stoch. Dyn.* **6** (2006) 521–534], where we introduced two different stochastic versions of the fourth order memoryful PDE associated with the Brownian-time Brownian motion (BTBM): (1) the BTBM SIE and (2) the BTBM SPDE, both driven by space–time white noise. Under wide conditions, we showed the existence of random field locally-Hölder solutions to the BTBM SIE with striking and unprecedented time-space Hölder exponents, in spatial dimensions $d = 1, 2, 3$. In particular, we proved that the spatial regularity of such solutions is nearly locally Lipschitz in $d = 1, 2$. This gave, for the first time, an example of a space–time white noise driven equation whose solutions are smoother than the corresponding Brownian sheet in either time or space.

In this paper, we introduce the $2\beta^{-1}$ -order β -inverse-stable-Lévy-time Brownian motion (β -ISLTBM) SIEs, $\beta \in \{1/2^k; k \in \mathbb{N}\}$, driven by space–time white noise. Based on the dramatic regularizing effect of the BTBM density ($\beta = 1/2$), and since the kernels in these β -ISLTBM SIEs are fundamental solutions to

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higher order Laplacian PDEs; one may suspect that we get even more dramatic spatial regularity than the BTBM SIE case. We show, however, that the BTBM SIE spatial regularity and its random field third spatial dimension limit are maximal among all β -ISLTBM SIEs—no matter how high we take the order $1/\beta$ of the Laplacian. This gives a limit as to how far we can push the SIEs spatial regularity when driven by the rough white noise. Furthermore, we show that increasing the order of the Laplacian β^{-1} beyond the BTBM bi-Laplacian manifests entirely as increased temporal regularity of our random field solutions that asymptotically approaches that of the Brownian sheet as $\beta \searrow 0$. Our solutions are both direct and lattice limit solutions. We treat many stochastic fractional PDEs and their corresponding higher order SPDEs, including BTBM and β -inverse-stable-Lévy-time Brownian motion SPDEs, in separate articles.

1. Introduction, motivation, and statement of results

Lately, many phenomena in mathematical physics, fluids dynamics and turbulence models, mathematical finance, and the modern theory of stochastic processes have been related to and described through deterministic fractional and higher order evolution equations (e.g., see [3], [5]–[12], [14]–[18], [19], [20]–[23], [24], [25]–[28], [30], [31], [36]–[38], and [40]); and it is only natural to investigate these important equations under the influence of a driving random noise.

In the two recent articles [1], [4] we introduced two new stochastic versions of fourth order memory-preserving (which we coin *memoryful*) deterministic PDEs related to Brownian-time processes (BTPs)¹—introduced in [18], [7]—driven by space–time white noise:

- (1) the space–time-white-noise-driven Brownian-time Brownian motion (BTBM) SPDE

$$(1.1) \quad \begin{cases} \partial_t U = \frac{\Delta u_0}{\sqrt{8\pi t}} + \frac{1}{8} \Delta^2 U + a(U) \partial_{t,x}^{1+d} W, & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ U(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

¹ A BTP, in its simplest form, is a process $X^x(|B_t|)$ in which X^x is a Markov process starting at $x \in \mathbb{R}^d$ and B is an independent one dimensional BM starting at 0. A Brownian-time Brownian motion (BTBM) is a BTP in which X^x is also a Brownian motion. BTPs include many new and quite interesting processes (see [18], [7], [24], [37]), which we are currently investigating in several directions (e.g., [16], [12], [17]). With the exception of the Markov snake of Le Gall ([32]), BTPs fall outside the classical theory of Markov, Gaussian, or semimartingale processes. We label BTP PDEs as *memoryful* since the initial data is part of the PDE itself (see (1.4)).

where $\partial_{t,x}^{1+d}W$ is the space–time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$ —and on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ —that corresponds to the Brownian sheet W ; and (2) the stochastic integral equation we called BTBM SIE

$$(1.2) \quad U(t, x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BTBM}^d} u_0(y) dy + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{BTBM}^d} a(U(s, y)) \mathcal{W}(ds \times dy),$$

where $\mathbb{K}_{t;x,y}^{\text{BTBM}^d}$ is the density of a d -dimensional Brownian-time Brownian motion given by:

$$(1.3) \quad \mathbb{K}_{t;x,y}^{\text{BTBM}^d} = 2 \int_0^\infty K_{s;x,y}^{\text{BM}^d} K_{t;0,s}^{\text{BM}} ds$$

with $K_{s;x,y}^{\text{BM}^d} = \frac{e^{-|x-y|^2/2s}}{(2\pi s)^{d/2}}$ and $K_{t;0,s}^{\text{BM}} = \frac{e^{-s^2/2t}}{\sqrt{2\pi t}}$; and where \mathcal{W} is the white noise on $\mathbb{R}_+ \times \mathbb{R}^d$.

Unlike the deterministic case $a \equiv 0$, (1.1) and (1.2) behave differently, and each is quite interesting in its own right. Each of these two equations gives a different stochastic interpretation of the memoryful BTBM PDE in [18], [7]:

$$(1.4) \quad \begin{cases} \partial_t u = \frac{\Delta u_0}{\sqrt{8\pi t}} + \frac{1}{8} \Delta^2 u; & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ u(0, x) = u_0(x); & x \in \mathbb{R}^d, \end{cases}$$

and its equivalent integral form²

$$(1.6) \quad u(t, x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BTBM}^d} u_0(y) dy.$$

As proven in [4], [1], the SIE (1.2)—which we also denote by $e_{\text{BTBM}}^{\text{SIE}}(a, u_0)$ —has real random field solutions in $d = 1, 2, 3$ with striking Hölder regularity in which the time-space Hölder exponents are $(\frac{4-d}{8}^-, (\frac{4-d}{2} \wedge 1)^-)$, as we recall precisely in Section 1.1 below,³ and it is similar in regularity to the following

² For a review of the BTPs higher order and fractional PDEs connections and generalizations, as well as connection to the important Kuramoto–Sivashinsky PDE, we refer the reader to [18], [7], [5], [34], [37], [38] and the references therein. The connection of BTPs to their fourth order PDEs (including (1.4)) was first given in [18]. Also, their connection to time-fractional PDEs was first established implicitly via the half derivative generator in [18]. In [34], [37], [38] the equivalence between a large class of high order and time-fractional PDEs, including (1.4) and

$$(1.5) \quad \begin{cases} \partial_t^{\frac{1}{2}} u = \frac{1}{\sqrt{8}} \Delta u; & t \in (0, \infty), x \in \mathbb{R}^d, \\ u(0, x) = u_0(x); & x \in \mathbb{R}^d, \end{cases}$$

was established explicitly, using the Caputo fractional derivative. For a discussion of interesting aspects of these PDEs, see also the introduction in [1]. In the new multiparameter-time case, the reader is referred to [3], [16].

³ In particular, as was established in [1], the BTBM SIE (1.2) has nearly locally Lipschitz solutions in $d = 1, 2$. This fact provided for the first time a counterexample to the common folklore *non*-wisdom that “a solution to a space–time-white-noise-driven equation cannot have a solution that is more regular, temporally or spatially, than the Brownian-sheet in the

L-Kuramoto–Sivashinsky (L-KS)⁴ SPDE

$$(1.7) \quad \begin{cases} \partial_t U = -\frac{1}{8}\Delta^2 U - \frac{1}{2}\Delta U - \frac{1}{2}U + a(U)\partial_{t,x}^{1+d}W, & (t, x) \in (0, \infty) \times \mathbb{R}^d; \\ U(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

obtained from the linearized KS PDE in [5] by adding a multiplicative space–time white noise term (see [2]). In [14], [15], we treat a large class of higher order and fractional—and rougher—SPDEs, including (1.1) and its equivalent time-fractional SPDE

$$(1.8) \quad \begin{cases} \partial_t^{\frac{1}{2}} U = \frac{1}{\sqrt{8}}\Delta U + a(U)\partial_{t,x}^{1+d}W; & t \in (0, \infty), x \in \mathbb{R}^d, \\ U(0, x) = u_0(x); & x \in \mathbb{R}^d, \end{cases}$$

where $\partial_t^{\frac{1}{2}}$ is a fractional derivative in time (see, e.g., [36]).

In this article, we focus on a large class of fascinating stochastic integral equations driven by space–time white noise and generalizing the BTBM SIE (1.2): the β -inverse-stable-Lévy-time Brownian motion SIEs (β -ISLTBM SIEs), which we discuss in more details in Section 1.2 below. These SIEs are obtained from the BTBM SIE in (1.2) by replacing the BTBM density with the fundamental solution to the $2\beta^{-1} = 2\nu$ order, $\beta^{-1} \in \{2^k; k \in \mathbb{N}\}$, memoryful PDEs

$$(1.9) \quad \begin{cases} \partial_t u_\beta(t, x) = \sum_{\kappa=1}^{\nu-1} \frac{\Delta^\kappa u_0(x)}{2^\kappa t^{1-\kappa/\nu}} E_{\beta, \kappa} + \frac{\Delta^\nu u_\beta(t, x)}{2\nu}, & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ u_\beta(0, x) = u_0(x), & x \in \mathbb{R}^d \end{cases}$$

and their equivalent time-fractional PDEs

$$(1.10) \quad \begin{cases} \partial_t^\beta u_\beta(t, x) = \frac{1}{2}\Delta u_\beta(t, x), & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ u_\beta(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where⁵ $E_{\beta, \kappa} = \frac{\mathbb{E}(\Lambda_\beta(1)^\kappa)}{\kappa!}$, the process Λ_β is the β -inverse-stable-Lévy motion described in Section 1.2 below, and ∂_t^β is the well-known Caputo fractional derivative of order $\beta \in \{1/2^k; k \in \mathbb{N}\}$ in time (see, e.g., [36]).

Based on the dramatic regularizing effect of the BTBM density on the space–time white noise driven BTBM SIE (1.2) as just described above (see also Theorem 1.1 below), and due to the fact that the kernels in the β -ISLTBM SIEs of this article are fundamental solutions to the higher order PDEs (1.9); one may suspect that we get even more dramatic spatial regularity than the BTBM SIE case, possibly obtaining random field solutions in arbitrarily high spatial dimensions as $\beta \searrow 0$ ($\nu \nearrow \infty$) instead of just $d = 3$ as in the BTBM case ($\beta = 1/\nu = 1/2$). We show, however, that the BTBM SIE spatial regularity

underlying white noise”. In a different direction, the explicit kernel approach here and in [1], [2] is useful in extending and generalizing our second order asymptotic analysis in [13] to the higher order case, as we show in [12]. We address this and other aspects separately.

⁴ The L in the name refers to the linearized PDE part. Such L-KS SPDE is treated in [2].

⁵ As usual, \mathbb{E} denotes the expectation operator.

and its random field third spatial dimension limit are maximal among all β -ISLTBM SIEs; no matter how small we take β (how high we take the order β^{-1} of the Laplacian). Further, we show that increasing the order β^{-1} of the *spatial* Laplacian beyond the BTBM order of 2 translates *entirely* into temporal regularization of our β -ISLTBM SIEs.⁶ This surprising result is the regularity content of our two main theorems: Theorem 1.2 and Theorem 1.3 below.

1.1. Recalling the Brownian-time Brownian motion SIE case. Before stating our first main result, it is instructive to recall the BTBM SIE results in [1].⁷ Following [1], we denote by⁸ $H^{\gamma_t^-, \gamma_s^-}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ the space of real-valued locally Hölder functions on $\mathbb{T} \times \mathbb{R}^d$ whose time and space Hölder exponents are in $(0, \gamma_t)$ and $(0, \gamma_s)$, respectively. The first main result in [1] is now restated.

THEOREM 1.1 (Allouba [1]). *Fix $0 < \gamma \leq 1$. Assume the following Lipschitz and growth conditions*

$$\text{(Lip)} \quad \begin{cases} \text{(a) } |a(u) - a(v)| \leq C|u - v|; & u, v \in \mathbb{R}, \\ \text{(b) } a^2(u) \leq C(1 + u^2); & u \in \mathbb{R}, \\ \text{(c) } u_0 \in C_b^{2, \gamma}(\mathbb{R}^d; \mathbb{R}) & \text{and non-random, } \forall d = 1, 2, 3 \end{cases}$$

hold. Then there exists a pathwise-unique strong solution (U, \mathscr{W}) to $e_{\text{BTBM}}^{\text{SIE}}(a, u_0)$ on $\mathbb{R}_+ \times \mathbb{R}^d$, for $d = 1, 2, 3$, which is $L^p(\Omega)$ -bounded on $\mathbb{T} \times \mathbb{R}^d$ for all $p \geq 2$. Furthermore, $U \in H^{\frac{4-d}{8}^-, (\frac{4-d}{2} \wedge 1)^-}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ for every $d = 1, 2, 3$.

Theorem 1.1 states that the stochastic kernel integral equation (1.2) has ultra regular strong⁹ solutions on $\mathbb{R}_+ \times \mathbb{R}^d$, namely $U \in H^{\frac{3}{8}^-, 1^-}(\mathbb{T} \times \mathbb{R}; \mathbb{R})$, $U \in H^{\frac{1}{4}^-, 1^-}(\mathbb{T} \times \mathbb{R}^2; \mathbb{R})$, and $U \in H^{\frac{1}{8}^-, \frac{1}{2}^-}(\mathbb{T} \times \mathbb{R}^3; \mathbb{R})$. That is, in space, the

⁶ I.e., the extra regularizing “energy” of spatial Laplacians of orders higher than that of the bi-Laplacian is converted to extra temporal regularity, when faced with the extremely rough driving space–time white noise.

⁷ Earlier, in [4], the additive noise case $a \equiv 1$ for $e_{\text{BTBM}}^{\text{SIE}}(a, u_0)$ was considered; and the existence of a pathwise unique continuous BTBM SIE solution $U(t, x)$ for $x \in \mathbb{R}^d$ and $d = 1, 2, 3$, such that

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}_{\mathbb{P}} |U(t, x)|^{2p} \leq C \left[1 + t^{\frac{(4-d)p}{4}} \right]; \quad t > 0, d = 1, 2, 3, p \geq 1,$$

was proved.

⁸ Throughout the paper, $\mathbb{T} = [0, T]$ for some fixed but arbitrary $T > 0$. Here and in the sequel $C_b^{p, \gamma}(\mathbb{R}^d; \mathbb{R}) \subset C_b^p(\mathbb{R}^d; \mathbb{R})$ denotes the space of bounded p -times continuously differentiable functions such that all derivatives up to (and including) the p th order are bounded and all p th order derivatives are Hölder continuous, with some Hölder exponent $0 < \gamma \leq 1$. Also, the boundedness conditions on u_0 and its derivatives may easily be relaxed as in [16].

⁹ Here strong is in the stochastic sense of the noise \mathscr{W} and its probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ being fixed a priori. Throughout this article, whenever needed, we will assume that our filtrations satisfy the usual conditions without explicitly stating so.

BTBM paths have a rather remarkable—and initially-surprising—nearly local Lipschitz regularity for $d = 1, 2$; and nearly local Hölder $1/2$ regularity in $d = 3$. This is remarkable because the BTBM kernel is able, in $d = 1, 2$, to spatially regularize such solutions beyond the traditional Hölder- $1/2^-$ spatial regularity of the underlying Brownian sheet corresponding to the driving space–time white noise.¹⁰ This degree of smoothness is unprecedented for space–time white noise driven kernel equations or their corresponding SPDEs; and the BTBM SIE is thus the first such example. In time, our solutions are locally γ -Hölder continuous with dimension-dependent exponent $\gamma \in (0, \frac{4-d}{8})$ for $d = 1, 2, 3$. This is in sharp contrast to traditional second order reaction–diffusion (RD) and other heat-operator-based SPDEs driven by space–time white noise, whose fundamental kernel is the Brownian motion density and whose real-valued mild solutions are confined to the case $d = 1$. In this regard, the dichotomy between the rougher paths of BTBMs as compared to standard Brownian motions on the one hand (quartic vs. quadratic variations) and the stronger regularizing properties of the BTBM density vs. the BM one on the other hand is certainly another interesting point to make.¹¹

1.2. The β -inverse-stable-Lévy-time Brownian motion SIE: The first main theorem. In the first main result of this article, we generalize the first BTBM SIE result in [1] Theorem 1.1 to the interesting case of the inverse-stable-Lévy-time Brownian motion SIE with index $\beta = 1/\nu$, $\nu \in \{2^k; k \in \mathbb{N}\}$ (β -ISLTBM SIE), which we now motivate and introduce.¹² This generalization allows us to better appreciate how hard it is to smooth away space–time white noise.

¹⁰ As noted in [1], it is important to note here that the common “folklore wisdom” of solutions of space–time-white-noise driven equations not being smoother than the associated Brownian sheet—in either space or time—originated from the predominant case of SPDEs, in which either the underlying kernel is that of a Brownian motion or the spatial operator is a Laplacian. The kernel $\mathbb{K}_{t;x,y}^{\text{BTBM}^d}$, however, is much more regularizing to the space–time-white-noise driven $e_{\text{BTBM}}^{\text{SIE}}(a, u_0)$ than the density of BM is to its corresponding equation. This becomes evidently clear in Lemma 2.4, Lemma 2.3, and Lemma 2.2 (compare to the more traditional BM and random walk case in [6]).

¹¹ We observe in passing that—roughly speaking—the paths of $e_{\text{BTBM}}^{\text{SIE}}(a, u_0)$ in $d = 1$ are effectively $3/2$ times as smooth as the RD SPDE paths in $d = 1$, in $d = 2$ the BTBM SIE is as smooth as an RD SPDE in $d = 1$, and in $d = 3$ our BTBM SIE is half as smooth as an RD SPDE in $d = 1$. Also, for $d = 2, 3$, the spatial regularity is roughly four times the temporal one, and in $d = 1$ the spatial regularity is maximized at a near Lipschitz vs. near Hölder $3/8$ in time.

¹² Throughout this article, we assume that $\nu = \beta^{-1} \in \{2^k; k \in \mathbb{N}\}$, where \mathbb{N} is the set of natural numbers. The case $\beta^{-1} = 2$ is the BTBM SIE case, with a minor scaling of the Brownian motion as discussed in [16].

1.2.1. *Recalling β -ISLTBM.* Inverse stable subordinator—which we also call β -inverse-stable-Lévy motion and denote by Λ_β —arise in the work of Meerschaert et al. [35], [33] as scaling limits of continuous time random walks. Let $S(n) = Y_1 + \dots + Y_n$ a sum of independent and identically distributed random variables with $EY_n = 0$ and $EY_n^2 < \infty$. The scaling limit $c^{-1/2}S(\lfloor ct \rfloor) \Rightarrow B(t)$ as $c \rightarrow \infty$ is a Brownian motion B at time t , which is normal with mean zero and variance proportional to t . Consider Y_n to be the random jumps of a particle. If we impose a random waiting time T_n before the n th jump Y_n , then the position of the particle at time $T_n = J_1 + \dots + J_n$ is given by $S(n)$. The number of jumps by time $t > 0$ is $N(t) = \max\{n : T_n \leq t\}$, so the position of the particle at time $t > 0$ is $S(N(t))$, a subordinated process. If $\mathbb{P}(J_n > t) = t^{-\beta}l(t)$ for some $0 < \beta < 1$, where $l(t)$ is slowly varying, then the scaling limit $c^{-1/\beta}T_{\lfloor ct \rfloor} \Rightarrow L_\beta(t)$ is a strictly increasing stable Lévy motion L_β at time t and with index β , sometimes called a stable subordinator. The jump times T_n and the number of jumps $N(t)$ are inverses $\{N(t) \geq x\} = \{T(\lfloor x \rfloor) \leq t\}$ where $\lfloor x \rfloor$ is the smallest integer greater than or equal to x . It follows that the scaling limits are also inverses $c^{-\beta}N(ct) \Rightarrow \Lambda_\beta(t)$ where $\Lambda_\beta(t) = \inf\{x : L(x) > t\}$, so that $\{\Lambda_\beta(t) \leq x\} = \{L_\beta(x) \geq t\}$. We call the process Λ_β a β -inverse-stable-Lévy motion. Since $N(ct) \approx c^\beta \Lambda(t)$, the particle location may, for large c , be approximated by $c^{-\beta/2}S(N(\lfloor ct \rfloor)) \approx (c^\beta)^{-1/2}S(c^\beta \Lambda_\beta(t)) \approx B(\Lambda_\beta(t))$, a Brownian motion subordinated to the inverse or hitting time (or first passage time) process of the stable subordinator L_β . The random variable $L_\beta(t)$ has a smooth density. For properly scaled waiting times, the density of $L_\beta(t)$ has Laplace transform e^{-ts^β} for any $t > 0$, and the random variables $L_\beta(t)$ and $t^{1/\beta}L_\beta(1)$ are identically distributed. Writing $g_\beta(u)$ for the density of $L_\beta(1)$, it follows that $L_\beta(t)$ has density $t^{-1/\beta}g_\beta(t^{-1/\beta}u)$ for any $t > 0$. Using the inverse relation $\mathbb{P}(\Lambda_\beta(t) \leq x) = \mathbb{P}(L_\beta(x) \geq t)$ and taking derivatives, it follows that $\Lambda_\beta(t)$ has density

$$(1.11) \quad K_{t;0,x}^{\Lambda_\beta} = t\beta^{-1}x^{-1-1/\beta}g_\beta(tx^{-1/\beta}).$$

As noted above, we assume throughout this article that

$$\nu = \beta^{-1} \in \{2^k; k \in \mathbb{N}\}.$$

In this case, there is a simple connections between k -iterated Brownian-time Brownian motion and β -ISLTBM. We denote by

$$\mathbb{B}_{\bigcirc_{i=1}^k B_i}^x(t) := B^x(|B_k(\dots B_2(|B_1(t)|)\dots)|)$$

a k -iterated Brownian-time Brownian motion at time t ; where $\{B_i\}_{i=1}^k$ are independent copies of a one dimensional scaled Brownian motion starting at zero, with density $\frac{1}{\sqrt{4\pi t}} \exp(-\frac{z^2}{4t}) = (1/\sqrt{2})K_{t;0,z/\sqrt{2}}^{\text{BM}}$, and independent from the standard d -dimensional Brownian motion B^x , which starts at $x \in \mathbb{R}^d$.

By $\mathbb{B}_{\Lambda_{1/2^k}}^x(t) = B^x(\Lambda_{1/2^k}(t))$ we mean a d -dimensional β -ISLTBM—with $\beta = 1/2^k$ —starting at $x \in \mathbb{R}^d$ and evaluated at time t ; in which the outer BM B^x and the inner $\Lambda_{1/2^k}$ are independent.

LEMMA 1.1 (The density of β -ISLTBM). *The probability distributions of $\mathbb{B}_{\bigcirc_{i=1}^k B_i}^x(t)$ and $\mathbb{B}_{\Lambda_{1/2^k}}^x(t)$ are the same for every $k = 1, 2, \dots$ and every $t \geq 0$. In particular, when $\beta = 1/2^k$, $k \in \mathbb{N}$, the Λ_β and the β -ISLTBM transition densities are given by*

$$(1.12) \quad \begin{aligned} K_{t;0,s_1}^{\Lambda_\beta} &= 2^{\frac{k}{2}} \int_{(0,\infty)^{k-1}} K_{t;0,\frac{s_k}{\sqrt{2}}}^{\text{BM}} \prod_{i=0}^{k-2} K_{s_{k-i};0,\frac{s_{k-i-1}}{\sqrt{2}}}^{\text{BM}} ds_2 \cdots ds_k, \\ \mathbb{K}_{t;x}^{\text{BM}^d,\Lambda_\beta} &= 2^{\frac{k}{2}} \int_{(0,\infty)^k} K_{s_1;x}^{\text{BM}^d} K_{t;0,\frac{s_k}{\sqrt{2}}}^{\text{BM}} \prod_{i=0}^{k-2} K_{s_{k-i};0,\frac{s_{k-i-1}}{\sqrt{2}}}^{\text{BM}} ds_1 \cdots ds_k, \end{aligned}$$

respectively.¹³

Proof. Let $\beta = 1/2^k$, $k \in \mathbb{N}$. By Corollary 3.1 in [37], we get that the distributions are the same. Now, equation (0.14) in [16] gives us that

$$(1.13) \quad K_{t;0,x}^{\Lambda_{\frac{1}{2}}} = \frac{2}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) = \frac{2}{\sqrt{2}} K_{t;0,\frac{x}{\sqrt{2}}}^{\text{BM}}.$$

This, together with Lemma 3.1 and Lemma 3.2 in [37] and a simple conditioning argument using the independence of all the Brownian motions, we immediately obtain (1.12) as asserted. \square

We now define our β -ISLTBM SIE as the stochastic integral equation:

$$(1.14) \quad \begin{aligned} U_\beta(t,x) &= \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BM}^d,\Lambda_\beta} u_0(y) dy \\ &\quad + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{BM}^d,\Lambda_\beta} a(U_\beta(s,y)) \mathscr{W}(ds \times dy), \end{aligned}$$

where $\mathbb{K}_{t;x,y}^{\text{BM}^d,\Lambda_\beta}$ is the transition density of a d -dimensional β -ISLTBM, starting from $x \in \mathbb{R}^d$, $\mathbb{B}_{\Lambda_\beta}^x := \{B^x(\Lambda_\beta(t)), t \geq 0\}$ given by:¹⁴

$$(1.15) \quad \mathbb{K}_{t;x,y}^{\text{BM}^d,\Lambda_\beta} = \int_0^\infty K_{s;x,y}^{\text{BM}^d} K_{t;0,s}^{\Lambda_\beta} ds.$$

¹³ We are using the convention $\prod_{i=0}^{-1} c_i = 1$ for any c_i and the convention $\int_{\mathbb{R}_+^0} f(s) ds = f(s)$, for every f . Also, we use the convention that the case $k = 0$ ($\beta = 1$) in the β -ISLTBM is the standard d -dimensional Brownian motion case.

¹⁴ Compare with the expression of $\mathbb{K}_{t;x,y}^{\text{BM}^d,\Lambda_\beta}$ Lemma 1.1 in terms of scaled BM transition densities.

We also denote the β -ISLTBM SIE (1.14) by $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$. Just as in the BTBM SIE case, $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ is one of two *different* stochastic versions¹⁵ of the higher order ($2\nu = 2\beta^{-1}$) memoryful PDEs (1.9) and their equivalent time fractional PDEs (1.10).

Of course, in the deterministic case, both (1.9) and (1.10) are equivalent to their integral form

$$(1.16) \quad u_\beta(t, x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta} dy.$$

1.2.2. *First theorem: $2\beta^{-1}$ order SIEs regularity and third dimension maximality.* Our first main theorem is now stated.

THEOREM 1.2 (Spatio-temporal regularity and third dimension maximality: direct solution). *Fix $\beta = 1/\nu$, $\nu \in \{2^k; k \in \mathbb{N}\}$. Assume the following Lipschitz, growth, and initial smoothness conditions*

$$(Lip) \quad \begin{cases} (a) & |a(u) - a(v)| \leq C|u - v|, & u, v \in \mathbb{R}, \\ (b) & a^2(u) \leq C(1 + u^2), & u \in \mathbb{R}, \\ (c) & u_0 \in C_b^{2\nu-2, \gamma}(\mathbb{R}^d; \mathbb{R}) & \text{and non-random, } \forall d = 1, 2, 3 \end{cases}$$

hold. Then there exists a pathwise-unique strong solution (U_β, \mathscr{W}) to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ on $\mathbb{R}_+ \times \mathbb{R}^d$, for $d = 1, 2, 3$, which is $L^p(\Omega)$ -bounded on $\mathbb{T} \times \mathbb{R}^d$ for all $p \geq 2$. Furthermore, $U_\beta \in H^{(\frac{2\nu-d}{4\nu})^-, (\frac{4-d}{2} \wedge 1)^-}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ for every $d = 1, 2, 3$.

Theorem 1.2 states that, for $\beta = 1/\nu$ and $\nu \in \{2^k; k \in \mathbb{N}\}$, these $2\beta^{-1}$ order β -ISLTBM SIEs have quite interesting locally-Hölder solutions with temporal and spatial Hölder exponents given by $(\frac{2\nu-d}{4\nu})^-$ and $(\frac{4-d}{2} \wedge 1)^-$, respectively, for $d = 1, 2, 3$. Comparing this regularity with the corresponding result for the fourth order BTBM SIE in Theorem 1.1, we see that the spatial regularity—spatial Hölder exponent and the maximum spatial dimension of $\mathfrak{3}$ —is identical. Since, the fundamental density (fundamental solution) estimates leading to the regularity conclusions of Theorem 1.2—Lemma 2.2 to Lemma 2.4—are sharp,¹⁶ this means that there is a limit as to how far we can push against the powerful roughening effect of the driving space-time white noise. Despite the fact that these SIEs are co-driven by fundamental solutions of arbitrarily high order ($2\beta^{-1}$) PDEs involving the spatial β^{-1} -Laplacian operators, we can obtain locally Hölder real random field solutions only up to three spatial

¹⁵ The other stochastic version is the 2ν or the time-fractional β order SPDE obtained from (1.9) or from (1.10) by adding the white noise term as in [14].

¹⁶ We will have more to say about the regularity of these β -ISLTBM SIEs in [17]. We also briefly note that by third dimension maximality, we mean maximality among *integer* dimensions.

TABLE 1. β -ISLTBM SIEs ($\nu = \beta^{-1} \in \{2^k; k \in \mathbb{N}\}$) vs RD SPDEs ($\beta = 1$)

d	Random field solutions		Hölder exponent (time, space)	
	RD SPDE	β -ISLTBM SIE	RD SPDE	β -ISLTBM SIE
1	Yes	Yes	$(\frac{1}{4}^-, \frac{1}{2}^-)$	$((\frac{2\nu-1}{4\nu})^-, 1^-)$
2	No	Yes	N/A	$((\frac{2\nu-2}{4\nu})^-, 1^-)$
3	No	Yes	N/A	$((\frac{2\nu-3}{4\nu})^-, (\frac{1}{2})^-)$

dimensions and with spatial Hölder exponents up to the maximal BTBM bi-Laplacian case ($\beta^{-1} = 2$), for all $\nu = \beta^{-1} \in \{2^k; k \in \mathbb{N}\}$, no matter how large β^{-1} is.

To appreciate the richness of the regularizing effect of these β -ISLTBM SIEs, however, we need to look beyond just the spatial dimensionality and regularity aspects. So, we will now examine the conclusion of Theorem 1.2 regarding the maximum temporal (effective) Hölder exponent,¹⁷ as $\beta \searrow 0$. As observed above, the strong roughening influence of the space–time white noise prevents further spatial smoothing of our β -ISLTBM SIEs beyond the BTBM bi-Laplacian case, no matter how large β^{-1} gets. However, all of the extra smoothing “energy” resulting from increasing the *spatial* Laplacian order β^{-1} cannot simply be “destroyed” by the white noise; and it is converted instead into *temporal* regularization of these β -ISLTBM SIEs (as $\beta \searrow 0$). Theorem 1.2 describes precisely this temporal effect in terms of Hölder exponents. In particular, the maximum effective regularity of the β -ISLTBM SIEs increases asymptotically to the well-known Hölder $(1/2)^-$ regularity of the Brownian sheet; i.e., the maximum effective Hölder exponent $(\frac{2\beta^{-1}-d}{4\beta^{-1}})^- \nearrow \frac{1}{2}^-$ as $\beta \searrow 0$ for every $d = 1, 2, 3$. Table 1 summarizes our regularity findings and compares them to the more standard and classical case of reaction–diffusion SPDEs driven by space–time white noise.

To prepare for the statement of our results under the less-than-Lipschitz conditions in (NLip) (Theorem 1.3 below), we now introduce the spatial lattice version of $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ as well as introduce the new associated process we call β -inverse-stable-Lévy-time random walk and define the lattice limit solutions involved in the statement of Theorem 1.3. The main machinery we use in the proof in this case is our K-martingale approach, which we introduced

¹⁷ The effective Hölder exponent is the minimum of the spatial and temporal Hölder exponents, which of course determine how smooth the random field solutions are as functions of both time and space together.

and used in the BTBM SIE case in [1]. We recall this approach, adapting it to our setting,¹⁸ in Section 4.2.

1.3. The spatial lattice version and the second main result. As in [1], we now spatially discretize $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$. This accomplishes at least two things: (1) it gives a multiscale view of $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ and (2) it allows us to prove our existence and regularity results without the Lipschitz condition on a .

1.3.1. *β -inverse-stable-Lévy-time random walk on the lattice.* In [11], [6], standard continuous-time random walks on a sequence of refining spatial lattices

$$\left\{ \mathbb{X}_n^d := \prod_{i=1}^d \{ \dots, -2\delta_n, -\delta_n, 0, \delta_n, 2\delta_n, \dots \} = \delta_n \mathbb{Z}^d \right\}_{n \geq 1}$$

(with the step size $\delta_n \searrow 0$ as $n \nearrow \infty$) played a crucial role—through their densities—in obtaining our results for second order RD SPDEs. In [1], in the fourth order Brownian-time setting, that role is played by Brownian-time random walks on \mathbb{X}_n^d :

$$(1.17) \quad \mathbb{S}_{B, \delta_n}^x(t) := S_{\delta_n}^x(|B_t|); \quad 0 \leq t < \infty, x \in \mathbb{X}_n^d,$$

where $S_{\delta_n}^x(t)$ is a standard d -dimensional continuous-time symmetric RW starting from $x \in \mathbb{X}_n^d$ and B is an independent one-dimensional BM starting at 0. The subscript δ_n in (1.17) is to remind us that the lattice step size is δ_n in each of the d directions.

In this article, we replace Brownian-time random walk with β -inverse-stable-Lévy-time random walk (β -ISLTRW):

$$(1.18) \quad \mathbb{S}_{\Lambda_\beta, \delta_n}^x(t) := S_{\delta_n}^x(\Lambda_\beta(t)); \quad 0 < \beta < 1, 0 \leq t < \infty, x \in \mathbb{X}_n^d.$$

It is then clear that the transition probability (density) $\mathbb{K}_{t;x,y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta}$ of the β -ISLTRW $\mathbb{S}_{\Lambda_\beta, \delta_n}^x(t)$ on \mathbb{X}_n^d is given by¹⁹

$$(1.19) \quad \mathbb{K}_{t;x,y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta} = 2 \int_0^\infty K_{s;x,y}^{\text{RW}_{\delta_n}^d} K_{t;0,s}^{\Lambda_\beta} ds; \quad 0 < \beta < 1, 0 < t < \infty, x, y \in \mathbb{X}_n^d,$$

where $K_{t;x,y}^{\text{RW}_{\delta_n}^d}$ is the continuous-time random walk transition density starting at $x \in \mathbb{X}_n^d$ and going to $y \in \mathbb{X}_n^d$ in time t , in which the times between transitions are exponentially distributed with mean δ_n^{2d} . That is, $K_{t;x}^{\text{RW}_{\delta_n}^d}$ is the

¹⁸ All we need to adapt it here is to replace the BTRW kernel of [1] with the β -ISLTRW one in (1.19) below.

¹⁹ Throughout this article, $\mathbb{K}_{t,x}^{\text{RW}_{\delta_n}^d, \Lambda_\beta} := \mathbb{K}_{t;x,0}^{\text{RW}_{\delta_n}^d, \Lambda_\beta}$ (with a similar convention for all transition densities).

fundamental solution to the deterministic heat equation on the lattice \mathbb{X}_n^d :

$$(1.20) \quad \frac{du_n^x(t)}{dt} = \frac{1}{2} \Delta_n u_n^x(t); \quad (t, x) \in (0, \infty) \times \mathbb{X}_n^d,$$

where $\mathcal{A}_n := \Delta_n/2$ is the generator of the RW $S_{\delta_n}^x(t)$ on \mathbb{X}_n^d .

By mimicking our proof of Theorem 0.3 in [16], we easily get a 2ν order differential-difference equation connection to β -ISLTRW:

LEMMA 1.2 (β -ISLTRW's DDE). *Fix $\beta = 1/\nu$, $\nu \in \{2^k; k \in \mathbb{N}\}$. Let $u_{\beta,n}^x(t) = \mathbb{E}[u_0(S_{B,\delta_n}^x(t))]$ with u_0 as in (NLip). Then $u_{\beta,n}$ solves the following 2ν order differential-difference equation (DDE) on $\mathbb{R}_+ \times \mathbb{X}_n^d$:*

$$(1.21) \quad \begin{cases} \frac{du_{\beta,n}^x(t)}{dt} = \sum_{\kappa=1}^{\nu-1} \frac{\Delta_n^\kappa u_0(x)}{2^\kappa t^{1-\kappa/\nu}} E_{\beta,\kappa} + \frac{1}{2\nu} \Delta_n^\nu u_{\beta,n}^x(t), \\ (t, x) \in (0, \infty) \times \mathbb{X}_n^d, \\ u_{\beta,n}^x(0) = u_0(x), \quad x \in \mathbb{X}_n^d, \end{cases}$$

where $E_{\beta,\kappa} = \frac{\mathbb{E}_{\mathbb{P}}(\Lambda_\beta(1)^\kappa)}{\kappa!}$. Moreover, $\mathbb{K}_{t,x}^{\text{RW}_{\delta_n}^d, \Lambda_\beta}$ solves (1.21) on $[0, \infty) \times \mathbb{X}_n^d$, with

$$(1.22) \quad u_0(x) = \mathbb{K}_{0;x}^{\text{RW}_{\delta_n}^d, \Lambda_\beta} = K_{0;x}^{\text{RW}_{\delta_n}^d} = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

1.3.2. *Lattice β -ISLTRW SIEs and limits solutions to β -ISLTBM SIEs.* The crucial role of the β -ISLTRW density in our approach to the β -ISLTBM SIEs (1.2) becomes even clearer from the following definition of our approximating spatially-discretized equations:

DEFINITION 1.1 (Lattice β -ISLTRW SIEs). By the β -ISLTRW SIEs associated with the BTBM SIE $e_{\text{BTBM}}^{\text{SIE}}(a, u_0)$, we mean the system

$$\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}_{n=1}^\infty$$

of spatially-discretized stochastic integral equations on $\mathbb{R}_+ \times \mathbb{X}_n^d$ given by

$$(1.23) \quad \tilde{U}_{\beta,n}^x(t) = \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;x,y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta} u_0(y) + \sum_{y \in \mathbb{X}_n^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta} a(\tilde{U}_n^y(s)) \frac{dW_n^y(s)}{\delta_n^{d/2}},$$

where the β -ISLTRW density is given by (1.19). For each $n \in \mathbb{N}$, we think of $\{W_n^x(t); t \geq 0\}$ as a sequence of independent standard Brownian motions indexed by the set \mathbb{X}_n^d (independence within the same lattice). We also assume that if $m \neq n$ and $x \in \mathbb{X}_m^d \cap \mathbb{X}_n^d$ then $W_m^x(t) = W_n^x(t)$, and if $n > m$ and $x \in \mathbb{X}_n^d \setminus \mathbb{X}_m^d$ then $W_m^x(t) = 0$.

NOTATION 1.1. We will denote the deterministic and the random parts of (1.23) by $\tilde{U}_{\beta,n,D}^x(t)$ and $\tilde{U}_{\beta,n,R}^x(t)$ (or $\tilde{U}_{\beta,D}^x(t)$ and $\tilde{U}_{\beta,R}^x(t)$ when we suppress the dependence on n), respectively, whenever convenient.

We define two types of solutions to β -ISLTRW SIEs: direct solutions and limit solutions.

DEFINITION 1.2 (Direct β -ISLTRW SIE solutions). A direct solution to the β -ISLTRW SIE system $\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}_{n=1}^\infty$ on $\mathbb{R}_+ \times \mathbb{X}_n^d$ with respect to the Brownian (in t) system $\{W_n^x(t); t \geq 0\}_{(n,x) \in \mathbb{N} \times \mathbb{X}_n^d}$ on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a sequence of real-valued processes $\{\tilde{U}_n\}_{n=1}^\infty$ with continuous sample paths in t for each fixed $x \in \mathbb{X}_n^d$ and $n \in \mathbb{N}$ such that, for every $(n, x) \in \mathbb{N} \times \mathbb{X}_n^d$, $\tilde{U}_{\beta,n}^x(t)$ is \mathcal{F}_t -adapted, and equation (1.23) holds \mathbb{P} -a.s. A solution is said to be strong if $\{W_n^x(t); t \geq 0\}_{(n,x) \in \mathbb{N} \times \mathbb{X}_n^d}$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ are fixed a priori; and with

$$(1.24) \quad \mathcal{F}_t = \sigma\{\sigma(W_n^x(s); 0 \leq s \leq t, x \in \mathbb{X}_n^d, n \in \mathbb{N}) \cup \mathcal{N}\}; \quad t \in \mathbb{R}_+,$$

where \mathcal{N} is the collection of null sets

$$\{O : \exists G \in \mathcal{G}, O \subseteq G \text{ and } \mathbb{P}(G) = 0\}$$

and where

$$\mathcal{G} = \sigma\left(\bigcup_{t \geq 0} \sigma(W_n^x(s); 0 \leq s \leq t, x \in \mathbb{X}_n^d, n \in \mathbb{N})\right).$$

A solution is termed weak if we are free to choose $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and the Brownian system on it and without requiring \mathcal{F}_t to satisfy (1.24). Replacing \mathbb{R}_+ with $\mathbb{T} := [0, T]$ —for some $T > 0$ in the above, we get the definition of a solution to the β -ISLTRW SIE system $\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}_{n=1}^\infty$ on $\mathbb{T} \times \mathbb{R}^d$.

The next type of β -ISLTRW SIE solutions we define is the first step in our K-martingale approach of [1], which we recall in Section 4.2. By first reducing $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$ to the simpler finite dimensional noise setting, it takes full advantage of the notion of β -ISLTRW SIEs limit solutions to β -ISLTBM SIEs.

DEFINITION 1.3 (Limit β -ISLTRW SIE solutions). Let $l \in \mathbb{N}$. By the l -truncated β -ISLTRW SIE on $\mathbb{R}_+ \times \mathbb{X}_n^d$ we mean the β -ISLTRW SIE obtained from (1.23) by restricting the sum in the stochastic term to the finite d -dimensional lattice $\mathbb{X}_{n,l}^d := \mathbb{X}_n^d \cap \{[-l, l]^d; l \in \mathbb{N}\}$ and leaving unchanged the deterministic term $\tilde{U}_{\beta,n,D}^x(t)$:

$$(1.25) \quad \tilde{U}_{\beta,n,l}^x(t) = \begin{cases} \tilde{U}_{\beta,n,D}^x(t) + \sum_{y \in \mathbb{X}_{n,l}^d} \int_0^t \kappa_{\delta_n, s, t}^{x,y} (\tilde{U}_{\beta,n,l}^y(s)) dW_n^y(s); & x \in \mathbb{X}_{n,l}^d, \\ \tilde{U}_{\beta,n,D}^x(t); & x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d, \end{cases}$$

where

$$\kappa_{\beta, \delta_n, s, t}^{x,y} (\tilde{U}_{\beta,n,l}^y(r)) := \frac{\mathbb{K}_{t-s; x, y}^{\text{RW}_{\delta_n^d, \Lambda_\beta}}}{\delta_n^{d/2}} a(\tilde{U}_{\beta,n,l}^y(r)), \quad \forall r, s < t.$$

We denote (1.25) by $e_{\beta\text{-ISLTRW}}^{\text{t-SIE}}(a, u_0, n, l)$. Fix $n \in \mathbb{N}$, a solution to the system of truncated β -ISLTRW SIEs $\{e_{\beta\text{-ISLTRW}}^{\text{t-SIE}}(a, u_0, n, l)\}_{l=1}^\infty$ on $\mathbb{R}_+ \times \mathbb{X}_n^d$ with

respect to the Brownian (in t) system $\{W_n^x(t); t \geq 0\}_{x \in \mathbb{X}_n^d}$ on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a sequence of real-valued processes $\{\tilde{U}_{\beta,n,l}\}_{l \in \mathbb{N}}$ with continuous sample paths in t for each fixed $x \in \mathbb{X}_n^d$ and $l \in \mathbb{N}$, such that, for every $(l, x) \in \mathbb{N} \times \mathbb{X}_n^d$, $\tilde{U}_{\beta,n,l}^x(t)$ is \mathcal{F}_t -adapted, and equation (1.25) holds \mathbb{P} -a.s. We call $\tilde{U}_{\beta,n}$ a limit solution to the β -ISLTRW SIE (1.23) if $\tilde{U}_{\beta,n}$ is a limit of the truncated solutions $\tilde{U}_{\beta,n,l}$ (as $l \rightarrow \infty$). If desired, we may indicate the limit type (a.s., in L^p , weak, \dots , etc.).

REMARK 1.1. In both (1.25) and (1.23), $\tilde{U}_{\beta,n,D}^x(t) = \mathbb{E}[u_0(\mathbb{S}_{B,\delta_n}^x(t))]$. So, by Lemma 1.2, $\tilde{U}_{\beta,n,D}^x(t)$ is differentiable in time t and satisfies (1.21). Also, using linear interpolation, we can extend the definition of an already continuous-in-time process $\tilde{U}_{\beta,n}^x(t)$ on $\mathbb{R}_+ \times \mathbb{X}_n^d$, so as to obtain a continuous process on $\mathbb{R}_+ \times \mathbb{R}^d$, for each $n \in \mathbb{N}$, which we will also denote by $\tilde{U}_{\beta,n}^x(t)$. Henceforth, any such sequence $\{\tilde{U}_{\beta,n}\}$ of interpolated $\tilde{U}_{\beta,n}$'s will be called a continuous or an interpolated solution to the system $\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}_{n=1}^\infty$. Similar comments apply to solutions of the truncated $e_{\beta\text{-ISLTRW}}^{\text{t-SIE}}(a, u_0, n, l)$.

We now define solutions to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ based entirely on their approximating $\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}$, through their limit. Since we defined direct and limit solutions to $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$, for each fixed n , we get two types of β -ISLTRW SIEs limit solutions to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$: direct β -ISLTRW SIEs limit solutions and β -ISLTRW SIE double limit solutions. The ‘‘double’’ in the second type of solutions reminds us that we are taking two limits, one from truncated to non-truncated fixed lattice (as $l \rightarrow \infty$) and the other limit is taken as the lattice mesh size shrinks to zero (as $\delta_n \searrow 0$ or equivalently as $n \nearrow \infty$).

DEFINITION 1.4 (β -ISLTRW SIEs limits solutions to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$). We say that the random field U is a β -ISLTRW SIE limit solution to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ on $\mathbb{R}_+ \times \mathbb{R}^d$ iff there is a solution $\{\tilde{U}_{\beta,n}^x(t)\}_{n \in \mathbb{N}}$ to the lattice SIE system $\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}_{n \in \mathbb{N}}$ on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and with respect to a Brownian system $\{W_n^x(t); t \geq 0\}_{(n,x) \in \mathbb{N} \times \mathbb{X}_n^d}$ such that U is the limit or a modification of the limit of $\{\tilde{U}_{\beta,n}\}_{n \in \mathbb{N}}$ (or a subsequence thereof). A β -ISLTRW SIE limit solution U is called a direct β -ISLTRW SIEs limit solution or a β -ISLTRW SIEs double limit solution according as $\{\tilde{U}_{\beta,n}^x(t)\}_{n \in \mathbb{N}}$ is a sequence of direct or limit solutions to $\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}_{n \in \mathbb{N}}$. The limits may be taken in the a.s., probability, L^p , or weak sense.²⁰ We say that uniqueness in law holds if whenever $U^{(1)}$ and $U^{(2)}$ are β -ISLTRW SIEs limit solutions they have the same law. We say

²⁰ When desired, the types of the solution and the limit are explicitly stated (e.g., we say strong (weak) β -ISLTRW SIEs weak, in probability, $L^p(\Omega)$, or a.s. limit solution to indicate that the solution to the approximating SIEs system is strong (weak) and that the limit of

that pathwise uniqueness holds for β -ISLTRW SIEs limit solutions if whenever $\{\tilde{U}_n^{(1)}\}$ and $\{\tilde{U}_n^{(2)}\}$ are lattice SIEs solutions on the same probability space and with respect to the same Brownian system, their limits $U^{(1)}$ and $U^{(2)}$ are indistinguishable.

1.3.3. *Second main theorem: The lattice-limits solutions case.* We can now state our second main result of the paper. The following theorem gives our lattice-limits solutions result for $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ under the non-Lipschitz conditions (NLip) on a . Our limits solutions result under Lipschitz conditions is stated in Theorem A.1,²¹ which is proved in Appendix A.

THEOREM 1.3 (Spatio-temporal regularity and third dimension maximality: lattice-limits solutions). *Fix $\beta = 1/\nu$, $\nu \in \{2^k; k \in \mathbb{N}\}$. Assume the conditions*

$$(NLip) \quad \begin{cases} (a) \ a(u) \text{ is continuous in } u; & u \in \mathbb{R}, \\ (b) \ a^2(u) \leq C(1 + u^2); & u \in \mathbb{R}, \\ (c) \ u_0 \in C_b^{2\nu-2, \gamma}(\mathbb{R}^d; \mathbb{R}) & \text{and non-random, } \forall d = 1, 2, 3 \end{cases}$$

hold. Then, there exists a β -ISLTRW SIE double weak-limit solution to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$, U , such that $U(t, x)$ is $L^p(\Omega, \mathbb{P})$ -bounded on $\mathbb{T} \times \mathbb{R}^d$ for every $p \geq 2$ and $U_\beta \in H^{(\frac{2\nu-d}{4\nu})^-, (\frac{d-d}{2} \wedge 1)^-}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ for every $d = 1, 2, 3$.

REMARK 1.2. Of course, we can use change of measure—as we did in our earlier work on Allen–Cahn SPDEs and other second order SPDEs (see, e.g., [10], [9], [8] and all our change of measure references in [6] for statements of results and conditions)—to transfer existence, uniqueness, and law equivalence results between the SIE $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ and the β -ISLTBM SIE $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, b, u_0)$ with measurable drift:

$$(1.26) \quad U_\beta(t, x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta} u_0(y) dy + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta} b(U_\beta(s, y)) ds dy \\ + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta} a(U_\beta(s, y)) \mathscr{W}(ds \times dy),$$

under the same conditions on the drift/diffusion ratio. If it is desired to investigate $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, b, u_0)$ on a bounded domain in \mathbb{R}^d with a regular boundary, we simply replace the β -ISLTBM density $\mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta}$ in (1.26) with its boundary-reflected or boundary-absorbed version (the β -ISLTBM density in which the outside d -dimensional BM is either reflected or absorbed at the boundary).

the SIEs is in the weak, in the probability, in the $L^p(\Omega)$, or in the a.s. sense, respectively). Of course, we may also take limits in any other suitable sense.

²¹ The type of limit solutions in the Lipschitz case is direct limit solutions as opposed to the double limit solution in Theorem 1.3.

The proof of Theorem 1.3 under the conditions (NLip) is neither standard nor straightforward—even after obtaining the new non-trivial spatio-temporal regularity estimates (in Lemma 2.3 and Lemma 2.4 below) on the unconventional kernel $\mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta}$. This is because standard techniques, like the classical martingale problem approach, do not apply directly to kernel equations like the β -ISLTBM SIE $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ or its discretized version $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$ under (NLip). This leads us to use our K-martingale approach, introduced in [1].

2. Key estimates

2.1. Density regularity estimates and third dimension maximality.

The first set of estimates²² we need are bounds on the square of the β -inverse-stable-Lévy-time Brownian motion density $\mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta}$ and its associated lattice β -inverse-stable-Lévy-time random walk density $\mathbb{K}_{t,x}^{\text{RW}_{\delta_n}^d, \Lambda_\beta}$ and their temporal and spatial differences. We obtain these estimates for both kernels simultaneously. The method of proof is to reduce, via an asymptotic argument, these estimates for the β -ISLTRW to the corresponding ones for the β -ISLTBM density $\mathbb{K}_{t;x}^{\text{BM}^d, \Lambda_\beta}$ and perform the computations in the continuous setting of the β -ISLTBM. Since all the results in this part hold for all $n \geq N^*$ (equivalently for all $\delta_n \leq \delta_{N^*}$) for some positive integer N^* , we will suppress the dependence on n , except when it is needed or helpful, to simplify the notation. Also, whenever we need these estimates, we assume that $n \geq N^*$ without explicitly stating it every time; and when we do, we let²³

$$(2.1) \quad \mathbb{N}^* := \{n \in \mathbb{N}; n \geq N^*\}.$$

We start by observing that in the classical setting of Brownian motion and its discretized version continuous-time random walk on $\mathbb{X}_n^d = \delta_n \mathbb{Z}^d$, we have the following well-known asymptotic result relating their densities (see, e.g., [39])

$$(2.2) \quad K_{t;[x]_{\delta_n}, [y]_{\delta_n}}^{\text{RW}_{\delta_n}^d} \sim K_{t;x,y}^{\text{BM}^d} \delta_n^d \quad \text{as } n \rightarrow \infty \text{ (as } \delta_n \rightarrow 0); \forall t > 0, x, y \in \mathbb{R}^d,$$

where for each $x \in \mathbb{R}^d$ we use $[x]_{\delta_n}$ to denote the element of \mathbb{X}_n^d obtained by replacing each coordinate x_i with δ_n times the integer part of $\delta_n^{-1}x_i$, and $a_n \sim b_n$ as $n \rightarrow \infty$ means $a_n/b_n \rightarrow 1$ as $n \rightarrow \infty$. Now, for every continuous

²² As is customary, all constants may change their value from one line to the next without changing their notation. Also, to simplify notation, we will often suppress the dependence on β without further notice. We will denote the Euclidean norm on d -dimensional spaces by $|\cdot|$.

²³ We adopt these simplifications with lattice computations throughout the paper.

and bounded $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$, we have

$$(2.3) \quad \begin{aligned} &\lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d \setminus \{x\}} \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta} u_0(y) \delta_n^d \\ &= \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta} u_0(y) dy; \quad t > 0, x \in \mathbb{R}^d, d \geq 1, \end{aligned}$$

and by the dominated convergence theorem we obtain

$$(2.4) \quad \begin{aligned} &\lim_{\delta_n \searrow 0} \left| \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;[x]_{\delta_n}, [y]_{\delta_n}}^{\text{RW}_{\delta_n}^d, \Lambda_\beta} u_0(y) - \sum_{y \in \mathbb{X}_n^d \setminus \{x\}} \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta} u_0(y) \delta_n^d \right| \\ &= \left| \int_0^\infty \left\{ \lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d \setminus \{x\}} [K_{s;[x]_{\delta_n}, [y]_{\delta_n}}^{\text{RW}_{\delta_n}^d} - K_{s;x,y}^{\text{BM}^d}] u_0(y) \right\} K_{t,0,s}^{\Lambda_\beta} ds \right| = 0 \end{aligned}$$

for $t > 0, x \in \mathbb{R}^d$, and $d \geq 1$; since, by (2.2),

$$\lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d} K_{s;[x]_{\delta_n}, [y]_{\delta_n}}^{\text{RW}_{\delta_n}^d} u_0(y) = \lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d} K_{s;x,y}^{\text{BM}^d} u_0(y) \delta_n^d = \int_{\mathbb{R}^d} K_{s;x,y}^{\text{BM}^d} u_0(y) dy$$

for every $(s, x) \in (0, \infty) \times \mathbb{R}^d$. We then straightforwardly get the following result.

LEMMA 2.1. *For every continuous and bounded $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ and for every $d \geq 1$*

$$(2.5) \quad \begin{aligned} &\lim_{\delta_n \searrow 0} \sum_{y \in \mathbb{X}_n^d} \mathbb{K}_{t;[x]_{\delta_n}, [y]_{\delta_n}}^{\text{RW}_{\delta_n}^d, \Lambda_\beta} u_0(y) \\ &= \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta} u_0(y) dy; \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^d, \end{aligned}$$

and the following asymptotic relation holds between the β -ISLTBM and β -ISLTRW densities:

$$(2.6) \quad \begin{aligned} &\mathbb{K}_{t;[x]_{\delta_n}, [y]_{\delta_n}}^{\text{RW}_{\delta_n}^d, \Lambda_\beta} \sim \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta} \delta_n^d \\ &\text{as } n \rightarrow \infty \text{ (as } \delta_n \rightarrow 0); t > 0, x, y \in \mathbb{R}^d, x \neq y. \end{aligned}$$

REMARK 2.1. Equation (2.5) confirms the intuitively clear fact that the kernel form of the β -ISLTRW DDE (1.21) converges pointwise—as $\delta_n \searrow 0$ —to the kernel form of its continuous version, the β -ISLTRW PDE in [37], [16]. We also remind the reader that the right-hand side of (2.5) is in $C^{1,2\beta-1}$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^d$ under the u_0 conditions in (NLip).

Our first regularity lemma for the densities is now stated. It implies, among other things, that there is a considerable smoothing effect of $\mathbb{K}_{s;x}^{\text{BM}^d, \Lambda_\beta}$ as β gets smaller; however it also implies that our SIEs do not possess random field solutions beyond the third spatial dimension, no matter how small β gets.

LEMMA 2.2 (Smoothing and third dimension maximality). *There are constants C and \tilde{C} , depending only on d and $\beta = 1/\nu$, $\nu \in \{2^k; k \in \mathbb{N}\}$, and a $\delta^* > 0$ such that for all $\delta \leq \delta^*$*

$$\int_{\mathbb{R}^d} [\mathbb{K}_{t;x}^{\text{BM}^d, \Lambda_\beta}]^2 dx = Ct^{\frac{-d}{2\nu}} \quad \text{and} \quad \sum_{x \in \mathbb{X}^d} [\mathbb{K}_{t;x}^{\text{RW}_\delta^d, \Lambda_\beta}]^2 \leq \tilde{C}\delta^d t^{\frac{-d}{2\nu}};$$

for all $t > 0, d = 1, 2, 3$. Hence,

$$\int_0^t \int_{\mathbb{R}^d} [\mathbb{K}_{s;x}^{\text{BM}^d, \Lambda_\beta}]^2 dx ds = Ct^{\frac{2\nu-d}{2\nu}} \quad \text{and} \quad \int_0^t \sum_{x \in \mathbb{X}^d} [\mathbb{K}_{s;x}^{\text{RW}_\delta^d, \Lambda_\beta}]^2 ds \leq \tilde{C}\delta^d t^{\frac{2\nu-d}{2\nu}};$$

for all $t > 0, d = 1, 2, 3$. In addition,

$$\int_{\mathbb{R}^d} [\mathbb{K}_{t;x}^{\text{BM}^d, \Lambda_\beta}]^2 dx = \int_0^t \int_{\mathbb{R}^d} [\mathbb{K}_{s;x}^{\text{BM}^d, \Lambda_\beta}]^2 dx ds = \infty,$$

for all $d \geq 4$.

Proof. First, fix an arbitrary $\beta^{-1} = \nu \in \{2^k, k \in \mathbb{N}\}$. Using the definition of $\mathbb{K}_{t;x}^{\text{BM}^d, \Lambda_\beta}$, Lemma 2.1 and Lemma 1.1 here together with Lemma 3.1 and Lemma 3.2 in [37] we obtain²⁴

$$\begin{aligned} (2.7) \quad & \lim_{\delta \searrow 0} \sum_{x \in \mathbb{X}^d} \frac{[\mathbb{K}_{t;x}^{\text{RW}_\delta^d, \Lambda_\beta}]^2}{\delta^d} \\ &= \int_{\mathbb{R}^d} [\mathbb{K}_{t;x}^{\text{BM}^d, \Lambda_\beta}]^2 dx \\ &= \int_0^\infty \int_0^\infty \left[\int_{\mathbb{R}^d} K_{s_1;x}^{\text{BM}^d} K_{u_1;x}^{\text{BM}^d} dx \right] K_{t;0,s_1}^{\Lambda_\beta} K_{t;0,u_1}^{\Lambda_\beta} ds_1 du_1 \\ &= \int_0^\infty \int_0^\infty \left[\frac{1}{[2\pi(s_1 + u_1)]^{d/2}} \right] K_{t;0,s_1}^{\Lambda_\beta} K_{t;0,u_1}^{\Lambda_\beta} ds_1 du_1 \\ &= \left\{ \int_0^\infty \int_0^\infty \left[\frac{2^k}{[2\pi(s_1 + u_1)]^{d/2}} \right] \right. \\ & \quad \times \left(\int_{(0,\infty)^{k-1}} K_{t;0, \frac{s_k}{\sqrt{2}}}^{\text{BM}} \prod_{i=0}^{k-2} K_{s_{k-i};0, \frac{s_{k-i-1}}{\sqrt{2}}}^{\text{BM}} ds_2 \cdots ds_k \right) \\ & \quad \left. \times \left(\int_{(0,\infty)^{k-1}} K_{t;0, \frac{u_k}{\sqrt{2}}}^{\text{BM}} \prod_{i=0}^{k-2} K_{u_{k-i};0, \frac{u_{k-i-1}}{\sqrt{2}}}^{\text{BM}} du_2 \cdots du_k \right) ds_1 du_1 \right\}. \end{aligned}$$

Gathering the two inside integrals and transforming to polar coordinates $(s_i, u_i) \mapsto (\rho_i, \theta_i)$, letting $\underline{\rho} = (\rho_1, \dots, \rho_k)$ and $\underline{\theta} = (\theta_1, \dots, \theta_k)$, and noticing

²⁴ Recall that we are using the convention $\int_{\mathbb{R}_+^0} f(s) ds = f(s)$, for every f .

that all ρ_i for $i = 2, 3, \dots, \rho_k$ cancel when $k \geq 2$; equation (2.7) becomes²⁵

$$(2.8) \quad C \int_{(0, \pi/2)^k} \int_{(0, \infty)^k} \frac{e^{-\frac{\rho_k^2}{4t} \prod_{i=0}^{k-2} e^{-[\frac{\rho_{k-i-1}^2 \cos^2(\theta_{k-i-1})}{4\rho_{k-i} \cos(\theta_{k-i})} + \frac{\rho_{k-i-1}^2 \sin^2(\theta_{k-i-1})}{4\rho_{k-i} \sin(\theta_{k-i})]}}}{\rho_1^{\frac{d}{2}-1} t [\sin(\theta_1) + \cos(\theta_1)]^{\frac{d}{2}} \prod_{i=0}^{k-2} \sqrt{\sin(\theta_{k-i}) \cos(\theta_{k-i})}} d\rho d\theta$$

$$= \begin{cases} Ct^{\frac{-d}{2\nu}}; & d = 1, 2, 3, \\ \infty; & d \geq 4. \end{cases}$$

Then there is a $\delta^* > 0$ such that, whenever $\delta \leq \delta^*$, we obtain

$$\frac{1}{\delta^d} \sum_{x \in \mathbb{X}^d} [\mathbb{K}_{t;x}^{\text{RW}_\delta^d, \Lambda_\beta}]^2 \leq \tilde{C} t^{\frac{-d}{2\nu}}; \quad d = 1, 2, 3,$$

with a finite constant $\tilde{C} > C$. The last assertion of the lemma trivially follows upon integration over the time interval $(0, t]$. \square

The following lemma is key to our Hölder regularity result in time. We give a probabilistically-flavored proof using the notion of 2- β -inverse-stable-Lévy-times random walk and 2- β -inverse-stable-Lévy-times Brownian motion given below.

LEMMA 2.3 (Kernel temporal regularity). *There is a constant C , depending only on d and $\beta = 1/\nu$, $\nu \in \{2^k; k \in \mathbb{N}\}$, and a $\delta^* > 0$ such that for $\delta \leq \delta^*$*

$$(2.9) \quad \begin{cases} \int_0^t \int_{\mathbb{R}^d} [\mathbb{K}_{t-s;x}^{\text{BM}_\delta^d, \Lambda_\beta} - \mathbb{K}_{r-s;x}^{\text{BM}_\delta^d, \Lambda_\beta}]^2 dx ds \leq C(t-r)^{\frac{2\nu-d}{2\nu}}, \\ \int_0^t \sum_{x \in \mathbb{X}^d} [\mathbb{K}_{t-s;x}^{\text{RW}_\delta^d, \Lambda_\beta} - \mathbb{K}_{r-s;x}^{\text{RW}_\delta^d, \Lambda_\beta}]^2 ds \leq C\delta^d(t-r)^{\frac{2\nu-d}{2\nu}}, \end{cases}$$

for $0 < r < t$ and $d = 1, 2, 3$, with the convention that $\mathbb{K}_{t;x}^{\text{RW}_\delta^d, \Lambda_\beta} = 0 = \mathbb{K}_{t;x}^{\text{BTBM}_\delta^d}$ if $t < 0$.

Proof. We will prove that

$$(2.10) \quad \int_0^t \sum_{x \in \mathbb{X}^d} [\mathbb{K}_{s+(t-r);x}^{\text{RW}_\delta^d, \Lambda_\beta} - \mathbb{K}_{s;x}^{\text{RW}_\delta^d, \Lambda_\beta}]^2 ds \leq C\delta^d(t-r)^{\frac{2\nu-d}{2\nu}}; \quad d = 1, 2, 3,$$

for all $\delta \leq \delta^*$, for some $\delta^* > 0$, simultaneously with its corresponding β -inverse-stable-Lévy-time Brownian motion density statement. The first step is to show the identity

$$(2.11) \quad \sum_{x \in \mathbb{X}^d} [\mathbb{K}_{s+(t-r);x}^{\text{RW}_\delta^d, \Lambda_\beta} - \mathbb{K}_{s;x}^{\text{RW}_\delta^d, \Lambda_\beta}]^2$$

$$= \mathbb{K}_{s+(t-r), s+(t-r); 0}^{\text{RW}_{\delta_n}^d, 2\Lambda_\beta} + \mathbb{K}_{s,s; 0}^{\text{RW}_{\delta_n}^d, 2\Lambda_\beta} - 2\mathbb{K}_{s+(t-r), s; 0}^{\text{RW}_{\delta_n}^d, 2\Lambda_\beta},$$

where

$$(2.12) \quad \mathbb{K}_{u,v; 0}^{\text{RW}_{\delta_n}^d, 2\Lambda_\beta} = \int_0^\infty \int_0^\infty K_{r_1+r_2; 0}^{\text{RW}_{\delta_n}^d} K_{u; 0, r_1}^{\Lambda_\beta} K_{v; 0, r_2}^{\Lambda_\beta} dr_1 dr_2$$

²⁵ Equation (2.8) is the reason for the third spatial dimension maximality.

is the density of the $2\text{-}\beta$ -inverse-stable-Lévy-times random walk

$$(2.13) \quad \mathbb{S}_{\Lambda_\beta^{(1)}, \Lambda_\beta^{(2)}, \delta_n^0}(u, v) := S_{\delta_n^0}^0(\Lambda_\beta^{(1)}(u) + \Lambda_\beta^{(2)}(v)); \quad 0 \leq u, v < \infty,$$

in which the d -dimensional random walk $S_{\delta_n^0}^0$ (on \mathbb{X}_n^d) and the two identically-distributed one-dimensional processes Λ_β^1 and Λ_β^2 are all independent. But,

$$(2.14) \quad \begin{aligned} & \sum_{x \in \mathbb{X}^d} \mathbb{K}_{u;x}^{\text{RW}_\delta^d, \Lambda_\beta} \mathbb{K}_{v;x}^{\text{RW}_\delta^d, \Lambda_\beta} \\ &= \int_0^\infty \int_0^\infty \left[\sum_{x \in \mathbb{X}^d} K_{r_1;x}^{\text{RW}_\delta^d} K_{r_2;x}^{\text{RW}_\delta^d} \right] K_{u;0,r_1}^{\Lambda_\beta} K_{v;0,r_2}^{\Lambda_\beta} dr_1 dr_2 \\ &= \int_0^\infty \int_0^\infty K_{r_1+r_2;0}^{\text{RW}_\delta^d} K_{u;0,r_1}^{\Lambda_\beta} K_{v;0,r_2}^{\Lambda_\beta} dr_1 dr_2 = \mathbb{K}_{u,v;0}^{\text{RW}_{\delta_n^0}^d, 2\Lambda_\beta}. \end{aligned}$$

The identity (2.11) immediately follows from (2.14). Similarly, we get the corresponding identity for the β -inverse-stable-Lévy-time Brownian motion setting

$$(2.15) \quad \begin{aligned} & \int_{\mathbb{R}^d} [\mathbb{K}_{s+(t-r);x}^{\text{BM}^d, \Lambda_\beta} - \mathbb{K}_{s;x}^{\text{BM}^d, \Lambda_\beta}]^2 dx \\ &= \mathbb{K}_{s+(t-r), s+(t-r);0}^{\text{BM}^d, 2\Lambda_\beta} + \mathbb{K}_{s,s;0}^{\text{BM}^d, 2\Lambda_\beta} - 2\mathbb{K}_{s+(t-r), s;0}^{\text{BM}^d, 2\Lambda_\beta}, \end{aligned}$$

where

$$(2.16) \quad \mathbb{K}_{u,v;0}^{\text{BM}^d, 2\Lambda_\beta} = \int_0^\infty \int_0^\infty K_{r_1+r_2;0}^{\text{BM}^d} K_{u;0,r_1}^{\Lambda_\beta} K_{v;0,r_2}^{\Lambda_\beta} dr_1 dr_2$$

is the density of the $2\text{-}\beta$ -inverse-stable-Lévy-times Brownian motion

$$(2.17) \quad \mathbb{X}_{\Lambda_\beta^{(1)}, \Lambda_\beta^{(2)}}^0(u, v) := X^0(\Lambda_\beta^{(1)}(u) + \Lambda_\beta^{(2)}(v)); \quad 0 \leq u, v < \infty,$$

in which the d -dimensional BM X^0 and the two identically-distributed one-dimensional processes $\Lambda_\beta^{(1)}$ and $\Lambda_\beta^{(2)}$ are all independent. Using the identities (2.11) and (2.15), along with a similar asymptotic argument to the one we used in the proof of Lemma 2.2 together with the dominated convergence theorem, yield

$$(2.18) \quad \begin{aligned} & \lim_{\delta \searrow 0} \frac{1}{\delta^d} \left[\int_0^t \mathbb{K}_{s+(t-r), s+(t-r);0}^{\text{RW}_{\delta_n^0}^d, 2\Lambda_\beta} ds \right. \\ & \quad \left. + \int_0^t \mathbb{K}_{s,s;0}^{\text{RW}_{\delta_n^0}^d, 2\Lambda_\beta} ds - 2 \int_0^t \mathbb{K}_{s+(t-r), s;0}^{\text{RW}_{\delta_n^0}^d, 2\Lambda_\beta} ds \right] \\ &= \lim_{\delta \searrow 0} \int_0^t \sum_{x \in \mathbb{X}^d} \frac{[\mathbb{K}_{s+(t-r);x}^{\text{RW}_\delta^d, \Lambda_\beta} - \mathbb{K}_{s;x}^{\text{RW}_\delta^d, \Lambda_\beta}]^2}{\delta^d} ds \\ &= \int_0^t \int_{\mathbb{R}^d} [\mathbb{K}_{s+(t-r);x}^{\text{BM}^d, \Lambda_\beta} - \mathbb{K}_{s;x}^{\text{BM}^d, \Lambda_\beta}]^2 dx ds \end{aligned}$$

$$\begin{aligned}
 &= \left[\int_0^t \mathbb{K}_{s+(t-r),s+(t-r);0}^{\text{BM}^d,2\Lambda_\beta} ds + \int_0^t \mathbb{K}_{s,s;0}^{\text{BM}^d,2\Lambda_\beta} ds - 2 \int_0^t \mathbb{K}_{s+(t-r),s;0}^{\text{BM}^d,2\Lambda_\beta} ds \right] \\
 &= \left[\int_0^t \tilde{\mathbb{K}}_{2s+2(t-r)} ds + \int_0^t \tilde{\mathbb{K}}_{2s} ds - 2 \int_0^t \tilde{\mathbb{K}}_{2s+(t-r)} ds \right] \\
 &= \left[\int_0^{\frac{t-r}{2}} \tilde{\mathbb{K}}_{2s} ds - \int_{\frac{t-r}{2}}^{t-r} \tilde{\mathbb{K}}_{2s} ds - \int_t^{t+\frac{t-r}{2}} \tilde{\mathbb{K}}_{2s} ds + \int_{t+\frac{t-r}{2}}^{2t-r} \tilde{\mathbb{K}}_{2s} ds \right]
 \end{aligned}$$

for $d = 1, 2, 3$, where $\tilde{\mathbb{K}}_w$ is defined in terms of $\mathbb{K}_{u,v;0}^{\text{BM}^d,2\Lambda_\beta}$ by the relation

$$\begin{aligned}
 (2.19) \quad &\tilde{\mathbb{K}}_w = \mathbb{K}_{u,v;0}^{\text{BM}^d,2\Lambda_\beta} \iff w = u + v \text{ and } (u, v) \text{ has one of the forms} \\
 &(u, v) = (a, a) \quad \text{or} \quad (u, v) = (a + b, a) \quad \text{or} \\
 &(u, v) = (a, a + b) \quad \text{for some } a, b \geq 0.
 \end{aligned}$$

We observe that

$$\begin{aligned}
 (2.20) \quad &\tilde{\mathbb{K}}_{2u} = \mathbb{K}_{u,u;0}^{\text{BM}^d,2\Lambda_\beta} = \int_0^\infty \int_0^\infty K_{r_1+r_2;0}^{\text{BM}^d} K_{u;0,r_1}^{\Lambda_\beta} K_{u;0,r_2}^{\Lambda_\beta} dr_1 dr_2 \\
 &= \int_0^\infty \int_0^\infty \left[\int_{\mathbb{R}^d} K_{r_1;x}^{\text{BM}^d} K_{r_2;x}^{\text{BM}^d} dx \right] K_{u;0,r_1}^{\Lambda_\beta} K_{u;0,r_2}^{\Lambda_\beta} dr_1 dr_2 \\
 &= \int_{\mathbb{R}^d} [\mathbb{K}_{u;x}^{\text{BM}^d,\Lambda_\beta}]^2 dx = Cu^{\frac{-d}{2\nu}}; \quad d = 1, 2, 3.
 \end{aligned}$$

The last assertion follows from the computation in (2.7) and (2.8). It is clear then that $\tilde{\mathbb{K}}_{2u}$ is decreasing in u , for every $\nu = 1/\beta \in \{2^k; k \in \mathbb{N}\}$. Thus, the sum of the last three terms of the (2.18) is ≤ 0 . This and (2.20) give us (2.10) for all $\delta \leq \delta^*$, for some $\delta^* > 0$ and for some constant $C > 0$, together with its corresponding β -inverse-stable-Lévy-time Brownian motion density statement; and Lemma 2.3 follows at once. \square

The following spatial difference second moment inequality for the β -ISLTRW and β -ISLTBM densities reflects their critical spatial-regularizing effect on our solutions. The following lemma captures the surprising fact that we cannot improve on the spatial regularity of the BTBM SIE by decreasing β below $1/2$. This implies the maximality of the BTBM SIEs spatial regularity among the family of β -ISLTBM SIE family.

LEMMA 2.4 (Kernel spatial regularity). *Let $\beta \in \{1/2^k; k \in \mathbb{N}\}$ and define the intervals*

$$I_d = \begin{cases} (0, 1]; & d = 1, \\ (0, 1); & d = 2, \\ (0, \frac{1}{2}); & d = 3. \end{cases}$$

For any given positive numbers $\{\alpha_d \in I_d\}_{d=1}^3$, there exists a constant C depending only on β, d and $\{\alpha_d\}_{d=1}^3$, and a $\delta^* > 0$ such that for $\delta \leq \delta^*$

$$(2.21) \quad \begin{cases} \int_0^t \int_{\mathbb{R}^d} [\mathbb{K}_{s;x}^{\text{BM}^d, \Lambda_\beta} - \mathbb{K}_{s;x+z}^{\text{BM}^d, \Lambda_\beta}]^2 dx ds \leq C|z|^{2\alpha_d} t^{p(\alpha_d, \beta)}, \\ \int_0^t \sum_{x \in \mathbb{X}^d} [\mathbb{K}_{s;x}^{\text{RW}^d, \Lambda_\beta} - \mathbb{K}_{s;x+z}^{\text{RW}^d, \Lambda_\beta}]^2 ds \leq C\delta^d |z|^{2\alpha_d} t^{p(\alpha_d, \beta)}, \end{cases}$$

for $t > 0$, where $0 < C < \infty$ and $0 \leq p(\alpha_d, \beta) < 1$ for every $\alpha_d \in I_d$ for $d = 1, 2, 3$ and for every $\beta \in \{1/2^k; k \in \mathbb{N}\}$.

REMARK 2.2. For a given $\beta^{-1} \in \{2, 3, 4, \dots\}$, and on any compact time interval $\mathbb{T} = [0, T]$, the inequality (2.21) may—for any given value α_d —be rewritten as

$$(2.22) \quad \begin{cases} \int_0^t \sum_{x \in \mathbb{X}^d} [\mathbb{K}_{s;x}^{\text{RW}^d, \Lambda_\beta} - \mathbb{K}_{s;x+z}^{\text{RW}^d, \Lambda_\beta}]^2 ds \leq \tilde{C}\delta^d |z|^{2\alpha_d}; \\ \int_0^t \int_{\mathbb{R}^d} [\mathbb{K}_{s;x}^{\text{BM}^d, \Lambda_\beta} - \mathbb{K}_{s;x+z}^{\text{BM}^d, \Lambda_\beta}]^2 dx ds \leq \tilde{C}|z|^{2\alpha_d}; \end{cases}$$

where, for each $d = 1, 2, 3$

$$\tilde{C} = C \sup_{\substack{\alpha_d \in I_d, \\ \beta \in \{1/2^k; k \in \mathbb{N}\}}} T^{p(\alpha_d, \beta)} < \infty$$

also depends on T in (2.22).

Proof of Lemma 2.4. Let $\beta = 1/2^k$ for $k \in \mathbb{N}$. Starting with the L^2 estimate involving the spatial difference of the β -ISLTBM density in (2.21), letting $u_1 = r_2$, using the polar transformation $(r_i, u_i) \mapsto (\rho_i, \theta_i)$, letting $\underline{\rho} = (\rho_1, \dots, \rho_k)$ and $\underline{\theta} = (\theta_1, \dots, \theta_k)$, and noticing that all ρ_i for $i = 2, 3, \dots, \rho_k$ cancel when $k \geq 2$, we have

$$(2.23) \quad \begin{aligned} & \int_0^t \int_{\mathbb{R}^d} [\mathbb{K}_{s;x}^{\text{BM}^d, \Lambda_\beta} - \mathbb{K}_{s;x+z}^{\text{BM}^d, \Lambda_\beta}]^2 dx ds \\ &= \int_0^t \left[\int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \prod_{i=1}^2 (K_{r_i;x}^{\text{BM}^d} - K_{r_i;x+z}^{\text{BM}^d}) K_{s;0,r_i}^{\Lambda_\beta} dx dr_1 dr_2 \right] ds \\ &= \int_0^t \int_0^\infty \int_0^\infty (2K_{r_1+u_1;0}^{\text{BM}^d} - 2K_{r_1+u_1;z}^{\text{BM}^d}) K_{s;0,r_1}^{\Lambda_\beta} K_{s;0,u_1}^{\Lambda_\beta} dr_1 du_1 ds \\ &= 2 \int_0^t \int_0^\infty \int_0^\infty \frac{1 - e^{-\frac{|z|^2}{2(r_1+u_1)}}}{[2\pi(r_1 + u_1)]^{d/2}} \\ & \quad \times \left(\int_{(0,\infty)^{k-1}} K_{s;0,\frac{r_k}{\sqrt{2}}}^{\text{BM}} \prod_{i=0}^{k-2} K_{r_{k-i};0,\frac{r_{k-i-1}}{\sqrt{2}}}^{\text{BM}} du_2 \cdots du_k \right) dr_1 du_1 ds \\ & \quad \times \left(\int_{(0,\infty)^{k-1}} K_{s;0,\frac{u_k}{\sqrt{2}}}^{\text{BM}} \prod_{i=0}^{k-2} K_{u_{k-i};0,\frac{u_{k-i-1}}{\sqrt{2}}}^{\text{BM}} du_2 \cdots du_k \right) dr_1 du_1 ds \end{aligned}$$

$$\begin{aligned}
 &\leq C \int_0^t \int_{(0, \frac{\pi}{2})^k} \int_{(0, \infty)^k} \frac{(1 - e^{-\frac{|z|^2}{2\rho_1}}) e^{-\frac{\rho_k^2}{4s}} \prod_{i=0}^{k-2} e^{-[\frac{\rho_{k-i-1}^2}{4\rho_{k-i}}]}}{\rho_1^{\frac{d}{2}-1} s [\sin(\theta) + \cos(\theta)]^{\frac{d}{2}} \prod_{i=0}^{k-2} \sqrt{\sin(\theta_{k-i}) \cos(\theta_{k-i})}} d\underline{\rho} d\underline{\theta} ds \\
 &\leq C \int_0^t \int_{(0, \infty)^k} \frac{(1 - e^{-\frac{|z|^2}{2\rho_1}}) e^{-\frac{\rho_k^2}{4s}} \prod_{i=0}^{k-2} e^{-[\frac{\rho_{k-i-1}^2}{4\rho_{k-i}}]}}{\rho_1^{\frac{d}{2}-1} s} d\underline{\rho} ds \\
 &\leq C \int_0^t \int_{(0, \infty)^k} \frac{|z|^{2\alpha} e^{-\frac{\rho_k^2}{4s}} \prod_{i=0}^{k-2} e^{-[\frac{\rho_{k-i-1}^2}{4\rho_{k-i}}]}}{\rho_1^{\alpha + \frac{d}{2} - 1} s} d\underline{\rho} ds \\
 &\leq \begin{cases} C_1 |z|^{2\alpha} t^{p_1(\alpha, \beta)}; & d = 1, \alpha \in (0, 1], \\ C_2 |z|^{2\alpha} t^{p_2(\alpha, \beta)}; & d = 2, \alpha \in (0, 1), \\ C_3 |z|^{2\alpha} t^{p_3(\alpha, \beta)}; & d = 3, \alpha \in (0, \frac{1}{2}), \end{cases}
 \end{aligned}$$

for some finite constants C_i , $i = 1, 2, 3$, where C_2 and C_3 depend on α ,²⁶ and where we have used the simple facts that $\min_{0 \leq \theta \leq \pi/2} [\sin(\theta) + \cos(\theta)] = 1$ and that $1 - e^{-u} \leq u^\alpha$ for $u \geq 0$ and $0 < \alpha \leq 1$. This proves the L^2 estimate for the β -ISLTBM density in (2.21). Then, an asymptotic argument similar to the one in the proofs of Lemma 2.2 and Lemma 2.3 yields

$$\begin{aligned}
 (2.24) \quad &\lim_{\delta \searrow 0} \int_0^t \sum_{x \in \mathbb{X}^d} \frac{[\mathbb{K}_{s;x}^{\text{RW}^d, \Lambda_\beta} - \mathbb{K}_{s;x+z}^{\text{RW}^d, \Lambda_\beta}]^2}{\delta^d} ds \\
 &= \int_0^t \int_{\mathbb{R}^d} [\mathbb{K}_{s;x}^{\text{BM}^d, \Lambda_\beta} - \mathbb{K}_{s;x+z}^{\text{BM}^d, \Lambda_\beta}]^2 dx ds,
 \end{aligned}$$

together with the desired β -ISLTRW density L^2 estimate in (2.21) for all $\delta \leq \delta^*$, for some $\delta^* > 0$, with possibly different constants. \square

2.2. Spatio-temporal estimates for β -ISLTRW and β -ISLTBM SIEs.

In this subsection, and assuming only the less-than-Lipschitz conditions (NLip) on a —together with a temporary moment condition—we obtain spatial and temporal differences moments estimates that are crucial in obtaining the regularity of the β -ISLTRW SIE $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$ for each fixed $n \in \mathbb{N}^*$ (see (2.1)), the tightness of the β -ISLTRW SIEs sequence

$$\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}_{n \in \mathbb{N}^*},$$

as well as the Hölder regularity for their limiting β -ISLTBM SIE. To make it more convenient for the proof of our first main result in the direct solution case, Theorem 1.2, we include the corresponding spatio-temporal statements

²⁶ See Remark 2.2 in [1] for a detailed discussion in the BTBM case $\beta = 1/2$.

for the β -ISLTBM SIE in the same lemmas, together with those for their lattice cousins.

Fix $n \in \mathbb{N}^*$, and assume $\tilde{U}_{\beta,n}$ solves $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$ in (1.23) and U_β solves the β -ISLTBM SIE $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ in (1.14). Suppressing the dependence on n , let $\tilde{M}_{\beta,q}(t) = \sup_x \mathbb{E}|\tilde{U}_\beta^x(t)|^{2q}$, and $M_{\beta,q}(t) = \sup_x \mathbb{E}|U_\beta(t, x)|^{2q}$ for $q \geq 1$ and $\beta \in \{1/2^k, k \in \mathbb{N}\}$. Writing \tilde{U}_β and U_β in terms of their deterministic and random parts $\tilde{U}_\beta^x(t) = \tilde{U}_{\beta,D}^x(t) + \tilde{U}_{\beta,R}^x(t)$ and $U_\beta(t, x) = U_{\beta,D}(t, x) + U_{\beta,R}(t, x)$, we observe that $\tilde{U}_{\beta,D}^x(t)$ is smooth in time by Lemma 1.2 and $U_{\beta,D}(t, x)$ is smooth in time and space as it is a solution to PDEs of order $2\beta^{-1}$ as in [16], [37]. The next two lemmas give us estimates on the random part.

LEMMA 2.5 (Spatial differences). *Assume that (NLip) holds and that $M_{\beta,q}(t)$ and $\tilde{M}_{\beta,q}(t)$ are bounded on any time interval²⁷ $\mathbb{T} = [0, T]$. There exists a constant C depending only on $q \geq 1$, $\max_x |u_0(x)|$, $\beta = 1/\nu$, $\nu \in \{2^k; k \in \mathbb{N}\}$, the spatial dimension $d = 1, 2, 3$, α_d , and T such that*

$$(2.25) \quad \begin{cases} \mathbb{E}|\tilde{U}_{\beta,R}^x(t) - \tilde{U}_{\beta,R}^y(t)|^{2q} \leq C|x - y|^{2q\alpha_d}, \\ \mathbb{E}|U_{\beta,R}(t, x) - U_{\beta,R}(t, y)|^{2q} \leq C|x - y|^{2q\alpha_d}, \end{cases}$$

for all $x, y \in \mathbb{X}^d$, $t \in \mathbb{T}$, and $d = 1, 2, 3$; where $\{\alpha_d\}_{d=1}^3$ are as in Lemma 2.4. That is, in $d = 1$, we may take $\alpha_1 = 1$; in $d = 2$ we may take any fixed $\alpha_2 \in (0, 1)$; and in $d = 3$, α_3 may be taken to be any fixed value in $(0, \frac{1}{2})$.

Proof. We prove the lattice SIE statement in (2.25) for \tilde{U}_β ; the proof of the statement for U_β follows the exact same steps, with obvious modifications and will not be repeated. Using Burkholder inequality, we have for any $(t, x, y) \in \mathbb{T} \times \mathbb{X}^{2d}$

$$(2.26) \quad \begin{aligned} & \mathbb{E}|\tilde{U}_{\beta,R}^x(t) - \tilde{U}_{\beta,R}^y(t)|^{2q} \\ & \leq C\mathbb{E}\left|\sum_{z \in \mathbb{X}^d} \int_0^t [\mathbb{K}_{t-s;x,z}^{\text{RW}_\delta^d, \Lambda_\beta} - \mathbb{K}_{t-s;y,z}^{\text{RW}_\delta^d, \Lambda_\beta}]^2 a^2(\tilde{U}_\beta^z(s)) \frac{ds}{\delta^d}\right|^q. \end{aligned}$$

For any fixed but arbitrary point $(t, x, y) \in \mathbb{T} \times \mathbb{X}^{2d}$ let $\mu_t^{x,y}$ be the measure defined on $[0, t] \times \mathbb{X}^d$ by

$$d\mu_t^{x,y}(s, z) = [\mathbb{K}_{t-s;x,z}^{\text{RW}_\delta^d, \Lambda_\beta} - \mathbb{K}_{t-s;y,z}^{\text{RW}_\delta^d, \Lambda_\beta}]^2 \frac{ds}{\delta^d},$$

and let $|\mu_t^{x,y}| = \mu_t^{x,y}([0, t] \times \mathbb{X}^d)$. We see from (2.27), Jensen’s inequality applied to the probability measure $\mu_t^{x,y}/|\mu_t^{x,y}|$, the growth condition on a , the

²⁷ This is the aforementioned temporary moment condition. It is assumed here (in Lemma 2.5 and Lemma 2.6 below) only to simplify the presentation and to get to the proof of Theorem 1.1 as quickly as possible in Section 3. In Section 4.1, this moment condition is shown to automatically hold under (NLip).

definition of $\tilde{M}_{\beta,q}(t)$, and elementary inequalities, that we have

$$\begin{aligned}
 (2.27) \quad & \mathbb{E}|\tilde{U}_{\beta,R}^x(t) - \tilde{U}_{\beta,R}^y(t)|^{2q} \\
 & \leq C\mathbb{E}\left[\int_{[0,t] \times \mathbb{X}^d} |a(\tilde{U}_{\beta}^z(s))|^{2q} \frac{d\mu_t^{x,y}(s,z)}{|\mu_t^{x,y}|}\right] |\mu_t^{x,y}|^q \\
 & \leq C\left[\int_{[0,t] \times \mathbb{X}^d} (1 + \tilde{M}_{\beta,q}(s)) \frac{d\mu_t^{x,y}(s,z)}{|\mu_t^{x,y}|}\right] |\mu_t^{x,y}|^q.
 \end{aligned}$$

Now, using the boundedness assumption on $\tilde{M}_{\beta,q}$ on \mathbb{T} for $d = 1, 2, 3$, we get

$$\begin{aligned}
 \mathbb{E}|\tilde{U}_{\beta,R}^x(t) - \tilde{U}_{\beta,R}^y(t)|^{2q} & \leq C|\mu_t^{x,y}|^q \leq [C_d t^{p_d(\alpha_d)}]^q |x - y|^{2q\alpha_d} \\
 & \leq \tilde{C}_d |x - y|^{2q\alpha_d}; \quad \alpha_d \in I_d,
 \end{aligned}$$

where the last inequality follows from Lemma 2.4 and (2.22) in Remark 2.2, and where the constant $\tilde{C} < \infty$ is as in Remark 2.2. □

LEMMA 2.6 (Temporal differences). *Assume that (NLip) holds and that $M_{\beta,q}(t)$ and $\tilde{M}_{\beta,q}(t)$ are bounded on any time interval $\mathbb{T} = [0, T]$. There exists a constant C depending only on $q \geq 1$, $\max_x |u_0(x)|$, $\beta = 1/\nu$, $\nu \in \{2^k; k \in \mathbb{N}\}$, the spatial dimension $d = 1, 2, 3$, and T such that*

$$(2.28) \quad \begin{cases} \mathbb{E}|U_{\beta,R}(t,x) - U_{\beta,R}(r,x)|^{2q} \leq C|t - r|^{\frac{(2\nu-d)q}{2\nu}}; & x \in \mathbb{R}^d, t, r \in \mathbb{T}, \\ \mathbb{E}|\tilde{U}_{\beta,R}^x(t) - \tilde{U}_{\beta,R}^x(r)|^{2q} \leq C|t - r|^{\frac{(2\nu-d)q}{2\nu}}; & x \in \mathbb{X}^d, t, r \in \mathbb{T}, \end{cases}$$

for $d = 1, 2, 3$.

Proof. We prove the lattice SIE statement in (2.28) for \tilde{U}_{β} ; the proof of the statement for U_{β} follows the exact same steps, with obvious modifications. Assume without loss of generality that $r < t$. Using Burkholder inequality, and using the change of variable $\rho = t - s$, we have for $(r, t, x) \in \mathbb{T}^2 \times \mathbb{X}^d$

$$\begin{aligned}
 (2.29) \quad & \mathbb{E}|\tilde{U}_{\beta,R}^x(t) - \tilde{U}_{\beta,R}^x(r)|^{2q} \\
 & \leq C\mathbb{E}\left|\sum_{z \in \mathbb{X}^d} \int_0^r [\mathbb{K}_{t-s;x,z}^{\text{RW}_{\delta}^d, \Lambda_{\beta}} - \mathbb{K}_{r-s;x,z}^{\text{RW}_{\delta}^d, \Lambda_{\beta}}]^2 a^2(\tilde{U}^z(s)) \frac{ds}{\delta^d}\right|^q \\
 & \quad + C\mathbb{E}\left|\sum_{z \in \mathbb{X}^d} \int_0^{t-r} [\mathbb{K}_{\rho;x,z}^{\text{RW}_{\delta}^d, \Lambda_{\beta}}]^2 a^2(\tilde{U}^z(t - \rho)) \frac{d\rho}{\delta^d}\right|^q.
 \end{aligned}$$

For a fixed point (r, t, x) and a fixed β , let $\mu_{\beta,t,r}^x$ be the measure defined on $[0, r] \times \mathbb{X}^d$ by

$$d\mu_{\beta,t,r}^x(s, z) = [\mathbb{K}_{t-s;x,z}^{\text{RW}_{\delta}^d, \Lambda_{\beta}} - \mathbb{K}_{r-s;x,z}^{\text{RW}_{\delta}^d, \Lambda_{\beta}}]^2 \frac{ds}{\delta^d}$$

and let $|\mu_{\beta,t,r}^x| = \mu_{\beta,t,r}^x([0, r] \times \mathbb{X}^d)$. Also, for a fixed $x \in \mathbb{X}^d$ and β , let κ^x be the measure defined on $[0, t - r] \times \mathbb{X}^d$ by

$$d\kappa_{\beta}^x(\rho) = [\mathbb{K}_{\rho;x,z}^{\text{RW}_{\delta}^d, \Lambda_{\beta}}]^2 \frac{d\rho}{\delta^d}$$

and let $|\kappa_{\beta}^x| = \kappa_{\beta}^x([0, t - r] \times \mathbb{X}^d)$. Then, arguing as in Lemma 2.5 above we get that

$$\mathbb{E}|\tilde{U}_{\beta,R}^x(t) - \tilde{U}_{\beta,R}^x(r)|^{2q} \leq C(|\mu_{\beta,t,r}^x|^q + |\kappa_{\beta}^x|^q) \leq C(t - r)^{\frac{(2\nu - d)q}{2\nu}},$$

for $d = 1, 2, 3$, where the last inequality follows from Lemma 2.2 and Lemma 2.3, completing the proof. \square

3. Proof of the first main theorem

Here, we prove Theorem 1.1. We start first by recalling a useful elementary Gronwall-type lemma whose proof can be found in Walsh [41].

LEMMA 3.1. *Let $\{g_n(t)\}_{n=0}^{\infty}$ be a sequence of positive functions such that g_0 is bounded on $\mathbb{T} = [0, T]$ and*

$$g_n(t) \leq C \int_0^t g_{n-1}(s)(t - s)^{\alpha} ds, \quad n = 1, 2, \dots$$

for some constants $C > 0$ and $\alpha > -1$. Then, there exists a (possibly different) constant $C > 0$ and an integer $k > 1$ such that for each $n \geq 1$ and $t \in \mathbb{T}$

$$g_{n+mk}(t) \leq C^m \int_0^t g_n(s) \frac{t - s}{(m - 1)!} ds; \quad m = 1, 2, \dots$$

We are now ready for our proof.

Proof of Theorem 1.2. For the existence proof, we construct a solution iteratively. So, given a space–time white noise \mathscr{W} , on some $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, define

$$(3.1) \quad \begin{cases} U_{\beta}^{(0)}(t, x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_{\beta}} u_0(y) dy, \\ U_{\beta}^{(n+1)}(t, x) = U_{\beta}^{(0)}(t, x) + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_{\beta}} a(U_{\beta}^{(n)}(s, y)) \mathscr{W}(ds \times dy). \end{cases}$$

We will show that, for any $p \geq 2$ and all $d = 1, 2, 3$, the sequence

$$\{U_{\beta}^{(n)}(t, x)\}_{n \geq 1}$$

converges in $L^p(\Omega)$ to a solution. Let

$$\begin{aligned} D_{\beta,n,p}(t, x) &:= \mathbb{E}|U_{\beta}^{(n+1)}(t, x) - U_{\beta}^{(n)}(t, x)|^p, \\ D_{\beta,n,p}^*(t) &:= \sup_{x \in \mathbb{R}^d} D_{\beta,n,p}(t, x). \end{aligned}$$

Starting with the case $p > 2$, we bound $D_{\beta,n,p}$ using Burkholder inequality, the Lipschitz condition (a) in (Lip), and then Hölder inequality with $0 \leq \varepsilon \leq 1$

and $q = p/(p - 2)$ to get

$$\begin{aligned} & D_{\beta,n,p}(t, x) \\ &= \mathbb{E} \left| \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta} [a(U_\beta^{(n)}(s, y)) - a(U_\beta^{(n-1)}(s, y))] \mathcal{W}(ds \times dy) \right|^p \\ &\leq C \mathbb{E} \left| \int_{\mathbb{R}^d} \int_0^t (\mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta})^2 [U_\beta^{(n)}(s, y) - U_\beta^{(n-1)}(s, y)]^2 ds dy \right|^{p/2} \\ &\leq C \left(\int_{\mathbb{R}^d} \int_0^t [\mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta}]^{2\varepsilon q} ds dy \right)^{p/2q} \\ &\quad \times \int_{\mathbb{R}^d} \int_0^t (\mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta})^{(1-\varepsilon)p} D_{\beta,n-1,p}(s, y) ds dy. \end{aligned}$$

Take $\varepsilon = (p - 2)/p$ in the above ($2\varepsilon q = (1 - \varepsilon)p = 2$), take the supremum over the space variables, and use Lemma 2.2 to see that, for $d = 1, 2, 3$ the above reduces to

$$(3.2) \quad D_{\beta,n,p}^*(t) \leq C \left(t^{\frac{2\nu-d}{2\nu}} \right)^{\frac{p-2}{2}} \int_0^t D_{\beta,n-1,p}^*(s) [t - s]^{\frac{-d}{2\nu}} ds.$$

The case $p = 2$ is simpler. We apply Burkholder’s inequality to $D_{n,2}$ and then take the space supremum to get

$$(3.3) \quad D_{\beta,n,2}^*(t) \leq C \int_0^t D_{\beta,n-1,2}^*(s) [t - s]^{\frac{-d}{2\nu}} ds,$$

i.e., on any time interval $\mathbb{T} = [0, T]$, the integral multiplier on the right-hand side of (3.2) is bounded; and if $D_{\beta,n-1,p}^*$ is bounded on \mathbb{T} then so is $D_{\beta,n,p}^*$, for every $p \geq 2$. Now,

$$D_{\beta,0,p}^*(t) \leq C \sup_{x \in \mathbb{R}^d} \mathbb{E} \left| \int_{\mathbb{R}^d} \int_0^t [\mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta}]^2 a^2(U_\beta^{(0)}(s, y)) ds dy \right|^{\frac{p}{2}}.$$

Since u_0 is bounded and deterministic, then so are $U^{(0)}$ and $a(U^{(0)})$. The latter assertion follows from the growth condition on a in (Lip). Thus, by Lemma 2.2 $D_{\beta,0,p}^*$ is bounded on \mathbb{T} for $d = 1, 2, 3$ and so are all the $D_{\beta,n,p}^*$. Lemma 3.1 now implies that for each $d = 1, 2, 3$, the series $\sum_{m=0}^\infty [D_{\beta,n+mk,p}^*(t)]^{1/p}$ converges uniformly on compacts for each n , which in turn implies that $\sum_{n=0}^\infty [D_{\beta,n,p}^*(t)]^{1/p}$ converges uniformly on compacts. Thus $U_\beta^{(n)}$ converges in $L^p(\Omega)$ for $p \geq 2$, uniformly on $\mathbb{T} \times \mathbb{R}^d$ for $d = 1, 2, 3$. Let $U_\beta(t, x) := \lim_{n \rightarrow \infty} U_\beta^{(n)}(t, x)$. It is easy to see that U_β satisfies (1.14), and hence solves the β -ISLTBM SIE $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$. This follows from (3.1) since the Lipschitz condition in (Lip) gives

$$\mathbb{E} |a(U_\beta(t, x)) - a(U_\beta^{(n)}(t, x))|^2 \leq C \mathbb{E} |U_\beta(t, x) - U_\beta^{(n)}(t, x)|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly on $\mathbb{T} \times \mathbb{R}^d$. Therefore, the stochastic integral term in (3.1) converges to the same term with $U_\beta^{(n)}$ replaced with the limiting U_β —that is, it converges

to the corresponding term in $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ —as $n \rightarrow \infty$, for

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta} (a(U_\beta(s, y)) - a(U_\beta^{(n)}(s, y))) \mathscr{W}(ds \times dy) \right]^2 \\ & \leq C \int_{\mathbb{R}^d} \int_0^t [\mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta}]^2 \mathbb{E}[U_\beta(s, y) - U_\beta^{(n)}(s, y)]^2 ds dy \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. It follows that U_β satisfies the β -ISLTBM SIE $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$. Also, the solution is strong since the $U_\beta^{(n)}$ are constructed for a given white noise \mathscr{W} , and the limit U_β satisfies (1.2) with respect to that same \mathscr{W} . Clearly U_β is $L^p(\Omega)$ bounded on $\mathbb{T} \times \mathbb{R}^d$, $d = 1, 2, 3$, for any $p \geq 2$ and for any $T > 0$.

To show uniqueness, fix an arbitrary $\beta^{-1} \in \{2^k; k \in \mathbb{N}\}$ —and suppress the dependence of solutions on β —and let $d = 1, 2, 3$, let $T > 0$ be fixed but arbitrary, and let U_1 and U_2 be two solutions to our β -ISLTBM SIE (1.14) that are $L^2(\Omega)$ -bounded on $\mathbb{T} \times \mathbb{R}^d$. Fix an arbitrary $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Let $D(t, x) = U_2(t, x) - U_1(t, x)$, $L_2(t, x) = \mathbb{E}D^2(t, x)$, and $L_2^*(t) = \sup_{x \in \mathbb{R}^d} L_2(t, x)$ (which is bounded on \mathbb{T} by hypothesis). Then, using (1.14), the Lipschitz condition in (Lip), and taking the supremum over the space variable and using Lemma 2.2 we have

$$\begin{aligned} (3.4) \quad L_2(t, x) &= \int_{\mathbb{R}^d} \int_0^t \mathbb{E}[a(U_2(s, y)) - a(U_1(s, y))]^2 [\mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta}]^2 ds dy \\ &\leq C \int_{\mathbb{R}^d} \int_0^t L_2(s, y) [\mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta}]^2 ds dy \\ &\leq C \int_0^t L_2^*(s) \int_{\mathbb{R}^d} [\mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta}]^2 dy ds \leq C \int_0^t \frac{L_2^*(s)}{(t-s)^{\frac{d}{2\nu}}} ds. \end{aligned}$$

Iterating and interchanging the order of integration we get

$$\begin{aligned} (3.5) \quad L_2(t, x) &\leq C \left\{ \int_0^t L_2^*(r) \left(\int_r^t \frac{ds}{(t-s)^{\frac{d}{2\nu}} (s-r)^{\frac{d}{2\nu}}} \right) dr \right\} \\ &\leq C \left(\int_0^t L_2^*(s) ds \right) \end{aligned}$$

for any $d = 1, 2, 3$. Hence,

$$(3.6) \quad L_2^*(t) \leq C \left(\int_0^t L_2^*(s) ds \right)$$

for every $t \geq 0$. An easy application of Gronwall’s lemma gives that $L_2^* \equiv 0$. So for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $d = 1, 2, 3$ we have $U_1(t, x) = U_2(t, x)$ with probability one. The indistinguishability of U_1 from U_2 , and hence pathwise uniqueness, follows immediately from their Hölder regularity, which we now turn to.

For any given $\beta^{-1} = \nu \in \{2^k; k \in \mathbb{N}\}$, we have just shown that, under the Lipschitz conditions (Lip), our β -ISLTBM SIE in (1.14) has an $L^p(\Omega)$ -

bounded solution $U_\beta(t, x)$ on $\mathbb{T} \times \mathbb{R}^d$ for any $T > 0$ and any $p \geq 2$. Equivalently, $M_{\beta,q}(t) = \sup_x \mathbb{E}|U_\beta(t, x)|^{2q}$, $q \geq 1$, is bounded on any time interval \mathbb{T} . Recalling that the deterministic part²⁸ of U_β is a $C^{1,2\nu}(\mathbb{R}_+, \mathbb{R}^d)$ function, we can then use Lemma 2.5 and Lemma 2.6 above, on the random part of U_β for $d = 1, 2, 3$ to straightforwardly get the desired local Hölder regularity for the direct solution of $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$, U_β , as follows: we let $q_n = n + d$ for $n \in \{0, 1, \dots\}$ and let $n = m + d$ for $m = \{0, 1, \dots\}$, we then have from Lemma 2.5 and Lemma 2.6 that

$$(3.7) \quad \begin{cases} \mathbb{E}|U_\beta(t, x) - U_\beta(t, y)|^{2n+2d} \leq C_d|x - y|^{(2n+2d)\alpha_d}, \\ \mathbb{E}|U_\beta(t, x) - U_\beta(r, x)|^{2m+4d} \leq C|t - r|^{\frac{(2\nu-d)(m+2d)}{2\nu}} \end{cases}$$

for $d = 1, 2, 3$. Thus as in Theorem 2.8, p. 53, and Problem 2.9, p. 55, in [29], we get that the spatial Hölder exponent is $\gamma_s \in (0, \frac{2(n+d)\alpha_d-d}{2n+2d})$ and the temporal exponent is $\gamma_t \in (0, \frac{m(1-d/2\nu)+d(1-d/\nu)}{2m+4d}) \forall m, n$. Taking the limits as $m, n \rightarrow \infty$, we get $\gamma_t \in (0, \frac{2\nu-d}{4\nu})$ and $\gamma_s \in (0, \alpha_d)$, for $d = 1, 2, 3$. The proof is complete. \square

4. Proof of the second main theorem

4.1. Regularity and tightness without the Lipschitz condition. As we mentioned in Section 2.2, the finiteness assumption of $M_{\beta,q}(t)$ and $\tilde{M}_{\beta,q}(t)$ on \mathbb{T} in Lemma 2.5 and Lemma 2.6 is for convenience only. We now proceed to show how to remove that assumption by showing it automatically holds under the weaker conditions (NLip). It is easily seen that if a is bounded then, for all spatial dimensions $d = 1, 2, 3$, $\tilde{M}_{\beta,q}$ is bounded on any compact time interval $\mathbb{T} = [0, T]$ (see Remark 4.1 below). The following proposition gives an exponential upper bound on the growth of $\tilde{M}_{\beta,q}$ in time in all $d = 1, 2, 3$ under the conditions in (NLip). The same result holds for $M_{\beta,q}$ with only notational and obvious changes to the following proofs.

PROPOSITION 4.1 (Exponential bound for $\tilde{M}_{\beta,q}$). *Assume that $\tilde{U}_\beta^x(t)$ is a solution of the β -ISLTRW SIE $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$ on $\mathbb{T} \times \mathbb{X}^d$, and assume that the conditions in (NLip) are satisfied. There exists a constant C depending only on $q, \max_x |u_0(x)|$, the dimension d, β , and T such that*

$$\tilde{M}_{\beta,q}(t) \leq C \left(1 + \int_0^t \tilde{M}_{\beta,q}(s) ds \right); \quad 0 \leq t \leq T,$$

for every $q \geq 1, \beta \in \{\frac{1}{2k}; k \in \mathbb{N}\}$ and $d = 1, 2, 3$. Hence, $\tilde{M}_{\beta,q}(t) \leq C \exp\{Ct\}$ for $0 \leq t \leq T, q \geq 1, \beta \in \{\frac{1}{2k}; k \in \mathbb{N}\}$, and $d = 1, 2, 3$. In particular, $\tilde{M}_{\beta,q}$ is bounded on \mathbb{T} for all $q \geq 1, \beta \in \{\frac{1}{2k}; k \in \mathbb{N}\}$, and $d = 1, 2, 3$.

²⁸ Of course, the deterministic part of $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ is, as discussed before, the integral $\int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta} u_0(y) dy$; and the random part is $\int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{BM}^d, \Lambda_\beta} a(U_\beta(s, y)) \mathscr{W}(ds \times dy)$.

The proof of Proposition 4.1 proceeds via the following lemma and its corollary.

LEMMA 4.1. *Under the same assumptions as in Proposition 4.1 there exists a constant C depending only on $q, \max_x |u_0(x)|$, the dimension d, β , and T such that*

$$\tilde{M}_{\beta,q}(t) \leq \begin{cases} C(1 + \int_0^t \frac{\tilde{M}_{\beta,q}(s)}{(t-s)^{2\nu}} ds); & 0 < t \leq T, \\ C; & t = 0, \end{cases}$$

for every $q \geq 1, \beta \in \{\frac{1}{2^k}; k \in \mathbb{N}\}$, and $d = 1, 2, 3$.

Proof. Fix $q \geq 1$, let $\tilde{U}_{\beta,D}^x(t) \triangleq \sum_{y \in \mathbb{X}^d} \mathbb{K}_{t;x,y}^{\text{RW}_\delta^d, \Lambda_\beta} u_0(y)$ (the deterministic part of \tilde{U}_β). Then, for any $(t, x) \in \mathbb{T} \times \mathbb{X}^d$, we apply Burkholder inequality to the random term $\tilde{U}_{\beta,R}^x(t)$ to get

$$(4.1) \quad \mathbb{E}|\tilde{U}_\beta^x(t)|^{2q} = \mathbb{E} \left| \sum_{y \in \mathbb{X}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{\text{RW}_\delta^d, \Lambda_\beta} \frac{a(\tilde{U}_\beta^y(s))}{\delta^{d/2}} dW^y(s) + \tilde{U}_{\beta,D}^x(t) \right|^{2q} \\ \leq C \left(\mathbb{E} \left| \sum_{y \in \mathbb{X}^d} \int_0^t (\mathbb{K}_{t-s;x,y}^{\text{RW}_\delta^d, \Lambda_\beta})^2 \frac{a^2(\tilde{U}_\beta^y(s))}{\delta^d} ds \right|^q + |\tilde{U}_{\beta,D}^x(t)|^{2q} \right).$$

Now, for a fixed point $(t, x) \in \mathbb{T} \times \mathbb{X}^d$ let μ_t^x be the measure on $[0, t] \times \mathbb{X}^d$ defined by $d\mu_t^x(s, y) = [(\mathbb{K}_{t-s;x,y}^{\text{RW}_\delta^d, \Lambda_\beta})^2 / \delta^d] ds$, and let $|\mu_t^x| = \mu_t^x([0, t] \times \mathbb{X}^d)$. Then, we can rewrite (4.1) as

$$(4.2) \quad \mathbb{E}|\tilde{U}_\beta^x(t)|^{2q} \leq C \left(\mathbb{E} \left| \int_{[0,t] \times \mathbb{X}^d} a^2(\tilde{U}_\beta^y(s)) \frac{d\mu_t^x(s, y)}{|\mu_t^x|} \right|^q + |\mu_t^x|^q + |\tilde{U}_{\beta,D}^x(t)|^{2q} \right).$$

Observing that $\mu_t^x / |\mu_t^x|$ is a probability measure, we apply Jensen’s inequality, the growth condition on a in (NLip), and other elementary inequalities to (4.2) to obtain

$$\mathbb{E}|\tilde{U}_\beta^x(t)|^{2q} \\ \leq C \left(\mathbb{E} \left[\int_{[0,t] \times \mathbb{X}^d} |a(\tilde{U}_\beta^y(s))|^{2q} \frac{d\mu_t^x(s, y)}{|\mu_t^x|} \right] |\mu_t^x|^q + |\tilde{U}_{\beta,D}^x(t)|^{2q} \right) \\ \leq C \left[\int_{[0,t] \times \mathbb{X}^d} (1 + \mathbb{E}|\tilde{U}_\beta^y(s)|^{2q}) d\mu_t^x(s, y) \right] |\mu_t^x|^{q-1} + C|\tilde{U}_{\beta,D}^x(t)|^{2q} \\ = C \left(\left[\sum_{y \in \mathbb{X}^d} \int_0^t \frac{(\mathbb{K}_{t-s;x,y}^{\text{RW}_\delta^d, \Lambda_\beta})^2}{\delta^d} (1 + \mathbb{E}|\tilde{U}_\beta^y(s)|^{2q}) ds \right] |\mu_t^x|^{q-1} + |\tilde{U}_{\beta,D}^x(t)|^{2q} \right).$$

Using Lemma 2.2, we see that $|\mu_t^x|$ is uniformly bounded for $t \leq T$ and $d = 1, 2, 3$. So, using the boundedness of u_0 , and hence of $\tilde{U}_{\beta,D}^x(t)$ by the simple

fact that $\sum_{y \in \mathbb{X}^d} \mathbb{K}_{t;x,y}^{\text{RW}_\delta^d, \Lambda_\beta} = 1$, Lemma 2.2 and the definition of $\tilde{M}_{\beta,q}(s)$, we get

$$\begin{aligned} \mathbb{E}|\tilde{U}_\beta^x(t)|^{2q} &\leq C \left(1 + \sum_{y \in \mathbb{X}^d} \int_0^t \frac{(\mathbb{K}_{t-s;x,y}^{\text{RW}_\delta^d, \Lambda_\beta})^2}{\delta^d} \tilde{M}_{\beta,q}(s) ds \right) \\ &\leq_{R_1} C \left(1 + \int_0^t \frac{\tilde{M}_{\beta,q}(s)}{(t-s)^{\frac{d}{2\nu}}} ds \right). \end{aligned}$$

Here, R_1 holds for $d = 1, 2, 3$. This implies that

$$\tilde{M}_{\beta,q}(t) \leq C \left(1 + \int_0^t \frac{\tilde{M}_{\beta,q}(s)}{(t-s)^{\frac{d}{2\nu}}} ds \right).$$

Of course, $\tilde{M}_{\beta,q}(0) = \sup_x |u_0(x)|^{2q} \leq C$, by the boundedness and non-randomness assumptions on $u_0(x)$ in (NLip). The proof is complete. \square

REMARK 4.1. It is clear that for a bounded a , $\tilde{M}_{\beta,q}$ is locally bounded in time. This follows immediately from Lemma 2.2 along with (4.2) above.

COROLLARY 4.1. *Under the same assumptions as those in Proposition 4.1 there exists a constant C depending only on $q, \max_x |u_0(x)|$, the dimension d, β , and T such that*

$$\tilde{M}_{\beta,q}(t) \leq C \left(1 + \int_0^t \tilde{M}_{\beta,q}(s) ds \right), \quad 0 \leq t \leq T, q \geq 1, \beta \in \left\{ \frac{1}{2^k}; k \in \mathbb{N} \right\},$$

for $d = 1, 2, 3$; and hence

$$\tilde{M}_{\beta,q}(t) \leq C \exp \{ Ct \}; \quad \forall 0 \leq t \leq T, q \geq 1, \beta \in \left\{ \frac{1}{2^k}; k \in \mathbb{N} \right\}, \text{ and } d = 1, 2, 3.$$

Proof. Iterating the bound in Lemma 4.1 once, and changing the order of integration, we obtain

$$\begin{aligned} (4.3) \quad \tilde{M}_{\beta,q}(t) &\leq C \left\{ 1 + C \left[\int_0^t \frac{ds}{(t-s)^{\frac{d}{2\nu}}} \right. \right. \\ &\quad \left. \left. + \int_0^t \tilde{M}_{\beta,q}(r) \left(\int_r^t \frac{ds}{(t-s)^{\frac{d}{2\nu}}(s-r)^{\frac{d}{2\nu}}} \right) dr \right] \right\} \\ &\leq C \left(1 + \int_0^t \tilde{M}_{\beta,q}(s) ds \right) \end{aligned}$$

for $d = 1, 2, 3$. The proof of the last statement is a straightforward application of Gronwall’s lemma to (4.3). This finishes the proof of Corollary 4.1 and thus of Proposition 4.1. \square

The regularity, tightness, and weak limit conclusions for the β -ISLTRW SIEs now follow.

LEMMA 4.2 (Regularity and tightness). *Assume that the conditions (NLip) hold, and that $\{\tilde{U}_{\beta,n}^x(t)\}_{n \in \mathbb{N}^*}$ is a sequence of spatially-linearly-interpolated solutions to the β -ISLTRW SIEs $\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}_{n \in \mathbb{N}^*}$ in (1.23). Then*

- (a) *For every n , $\tilde{U}_{\beta,n}^x(t)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^d$. Moreover, with probability one, the continuous map $(t, x) \mapsto \tilde{U}_{\beta,n}^x(t)$ is locally γ_t -Hölder continuous in time with $\gamma_t \in (0, \frac{2\nu-d}{4\nu})$ for $d = 1, 2, 3$.*
- (b) *There is a β -ISLTRW SIE weak limit solution to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$, call it U_β , such that $U_\beta(t, x)$ is $L^p(\Omega, \mathbb{P})$ -bounded on $\mathbb{T} \times \mathbb{R}^d$ for every $p \geq 2$ and $U_\beta \in H^{(\frac{2\nu-d}{4\nu})^-, \alpha_d}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ for every $d = 1, 2, 3$ and $\alpha_d \in I_d$, where α_d and I_d are as in Lemma 2.4.*

REMARK 4.2. Of course in part (a) above, even without linear interpolation in space, $\tilde{U}_\beta^x(t)$ is locally Hölder continuous in time with Hölder exponent $\gamma_t \in (0, \frac{2\nu-d}{4\nu})$ for $d = 1, 2, 3$.

Proof of Lemma 4.2. For each n , let $\tilde{U}_{\beta,n}^x(t) = \tilde{U}_{\beta,n,D}^x(t) + \tilde{U}_{\beta,n,R}^x(t)$ be the decomposition of $\tilde{U}_{\beta,n}^x(t)$ in (1.23) into its deterministic and random parts, respectively.

- (a) By Lemma 1.2, $\tilde{U}_{\beta,n,D}^x(t)$ is clearly smooth in time; so it is enough to consider the random term $\tilde{U}_{\beta,n,R}^x(t)$. We let $q_m = m + 2$ for $m \in \{0, 1, \dots\}$, we then have from Lemma 2.6 that

$$(4.4) \quad \mathbb{E}|\tilde{U}_{\beta,R}^x(t) - \tilde{U}_{\beta,R}^x(r)|^{4+2m} \leq C|t - r|^{\frac{(2\nu-d)(m+2)}{2\nu}},$$

for $d = 1, 2, 3$. Thus as in Theorem 2.8, p. 53, in [29], we get that $\gamma_t \in (0, \frac{m(1-d/2\nu)+2-d-d/\nu}{2m+4})$ for every m . Taking the limit as $m \rightarrow \infty$, we get $\gamma_t \in (0, \frac{2\nu-d}{4\nu})$ for $d = 1, 2, 3$.

- (b) By Lemma 2.1 it follows that $\tilde{U}_{\beta,n,D}^x(t)$ converges pointwise to the deterministic part of $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ in (1.2); that is,

$$(4.5) \quad \lim_{n \rightarrow \infty} \tilde{U}_{\beta,n,D}^x(t) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{BM}^d, \Lambda_\beta} u_0(y) dy.$$

We also conclude from Lemma 2.5 and Lemma 2.6 that the sequence $\{\tilde{U}_{\beta,n,R}^x(t)\}_{n \in \mathbb{N}^*}$ is tight on $C(\mathbb{T} \times \mathbb{R}^d)$ for $d = 1, 2, 3$. Thus there exists a weakly convergent subsequence $\{\tilde{U}_{\beta,n_k}\}_{k \in \mathbb{N}}$ and hence a β -ISLTRW SIE weak limit solution U to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$. Then, following Skorokhod, we construct processes²⁹ $Y_{\beta,k} \stackrel{\mathcal{L}}{=} \tilde{U}_{\beta,n_k}$ on some filtered probability space $(\Omega^S, \mathcal{F}^S, \{\mathcal{F}_t^S\}, \mathbb{P}^S)$ such that with probability 1, as $k \rightarrow \infty$, $Y_{\beta,k}(t, x)$ converges to a random field $Y_\beta(t, x)$ uniformly on compact subsets of $\mathbb{T} \times \mathbb{R}^d$ for $d = 1, 2, 3$. Now, for the β -ISLTRW SIEs limit regularity

²⁹ As usual, $\stackrel{\mathcal{L}}{=}$ denotes equal in law or distribution.

assertions, clearly the deterministic term on the right-hand side of (4.5) is $C^{1,2\nu}$ and bounded as in [16], so we use Proposition 4.1, Lemma 2.5, and Lemma 2.6 to obtain the regularity results for the random part. We provide the steps here for completeness. First, $Y_{\beta,k} \stackrel{\mathcal{L}}{=} \tilde{U}_{\beta,n_k}$ and so Proposition 4.1 gives us, for each $p \geq 2$:

$$(4.6) \quad \mathbb{E}|Y_{\beta,k}(t,x)|^p = \mathbb{E}|\tilde{U}_{\beta,n_k}^x(t)|^p \leq C < \infty; \quad \forall (t,x,k) \in \mathbb{T} \times \mathbb{R}^d \times \mathbb{N}, d = 1, 2, 3,$$

for some constant C that is independent of k, t, x but that depends on the dimension d . It follows that, for each $(t,x) \in \mathbb{T} \times \mathbb{R}^d$ the sequence $\{|Y_k(t,x)|^p\}_k$ is uniformly integrable for each $p \geq 2$ and each $d = 1, 2, 3$. Thus,

$$(4.7) \quad \mathbb{E}|U_\beta(t,x)|^p = \mathbb{E}|Y_\beta(t,x)|^p = \lim_{k \rightarrow \infty} \mathbb{E}|Y_{\beta,k}(t,x)|^p \leq C < \infty; \quad \forall (t,x) \in \mathbb{T} \times \mathbb{R}^d,$$

for all $d = 1, 2, 3$ and $p \geq 2$. Equation (4.7) establishes the L^p boundedness assertion. In addition, for $q \geq 1$ and $d = 1, 2, 3$ we have by Proposition 4.1

$$(4.8) \quad \begin{aligned} &\mathbb{E}|Y_{\beta,k}(t,x) - Y_{\beta,k}(t,y)|^{2q} + \mathbb{E}|Y_{\beta,k}(t,x) - Y_{\beta,k}(r,x)|^{2q} \\ &\leq C [\mathbb{E}|Y_{\beta,k}(t,x)|^{2q} + \mathbb{E}|Y_{\beta,k}(t,y)|^{2q} + \mathbb{E}|Y_{\beta,k}(r,x)|^{2q}] \\ &\leq C; \quad \forall (k,r,t,x,y) \in \mathbb{N} \times \mathbb{T}^2 \times \mathbb{R}^2. \end{aligned}$$

So, for each $(r,t,x,y) \in \mathbb{T}^2 \times \mathbb{R}^2$, the sequences

$$\{|Y_{\beta,k}(t,x) - Y_{\beta,k}(t,y)|^{2q}\}_k$$

and

$$\{|Y_{\beta,k}(t,x) - Y_{\beta,k}(r,x)|^{2q}\}_k$$

are uniformly integrable, for each $q \geq 1$. Therefore, using Lemma 2.5 and Lemma 2.6, we obtain

$$(4.9) \quad \left\{ \begin{aligned} &\mathbb{E}|U_\beta(t,x) - U_\beta(t,y)|^{2q} \\ &= \mathbb{E}|Y_\beta(t,x) - Y_\beta(t,y)|^{2q} \\ &= \lim_{k \rightarrow \infty} \mathbb{E}|Y_{\beta,k}(t,x) - Y_{\beta,k}(t,y)|^{2q} \leq C_d |x - y|^{2q\alpha_d}; \quad \alpha_d \in I_d, \\ &\mathbb{E}|U_\beta(t,x) - U_\beta(r,x)|^{2q} \\ &= \mathbb{E}|Y_\beta(t,x) - Y_\beta(r,x)|^{2q} \\ &= \lim_{k \rightarrow \infty} \mathbb{E}|Y_{\beta,k}(t,x) - Y_{\beta,k}(r,x)|^{2q} \leq C |t - r|^{\frac{(2\nu-d)q}{2\nu}}, \end{aligned} \right.$$

for $d = 1, 2, \dots, 3$. The local Hölder regularity is then obtained using exactly the same steps as in (3.7) and the following conclusions.

The proof is complete. □

4.2. Recalling the K-martingale approach. For the article to be self-contained, we now recall and briefly discuss the K-martingale approach from [1]—adapting it to this paper’s setting.³⁰ This approach is tailor-made for kernel SIEs like $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ and other mild formulations for many SPDEs on the lattice. The first step is to truncate to a finite lattice model as in (1.25). Of course, even after we truncate the lattice, a remaining hurdle to applying a martingale problem approach is that the finite sum of stochastic integrals in (1.25) is not a local martingale. So, we introduce a key ingredient in this K-martingale method: the auxiliary problem associated with the truncated β -ISLTRW SIE in (1.25), which we now give. Fix $(l, n) \in \mathbb{N}^2$ and $\tau \in \mathbb{R}_+$. We define the τ -auxiliary β -ISLTRW SIE associated with (1.25) on $[0, \tau] \times \mathbb{X}_n^d$ by

$$\begin{aligned}
 \text{(Aux)} \quad X_{\beta, n, l}^{\tau, x}(t) &= \begin{cases} \tilde{U}_{\beta, n, D}^x(t) + \sum_{y \in \mathbb{X}_{n, l}^d} \int_0^t \kappa_{\beta, \delta_n, s, \tau}^{x, y}(X_{\beta, n, l}^{\tau, y}(s)) dW_n^y(s); & x \in \mathbb{X}_{n, l}^d, \\ \tilde{U}_{\beta, n, D}^x(t); & x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n, l}^d, \end{cases}
 \end{aligned}$$

where the independent BMs sequence $\{W_n^y\}_{y \in \mathbb{X}_{n, l}^d}$ in (Aux) is the same for all $\tau > 0$, as well as $x \in \mathbb{X}_{n, l}^d$. We denote (Aux) by $e_{\beta\text{-ISLTRW}}^{\text{aux-SIE}}(a, u_0, n, l, \tau)$. We say that the pair of families $(\{X_{\beta, n, l}^\tau\}_{\tau \geq 0}, \{W_n^y\}_{y \in \mathbb{X}_{n, l}^d})$ is a solution to $\{e_{\beta\text{-ISLTRW}}^{\text{aux-SIE}}(a, u_0, n, l, \tau)\}_{\tau \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ if there is one family of independent BMs $\{W_n^y(t); 0 \leq t < \infty\}_{y \in \mathbb{X}_{n, l}^d}$ —up to indistinguishability—on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ such that, for every fixed $\tau \in \mathbb{R}_+$

- (a) the process $\{X_{\beta, n, l}^{\tau, x}(t), \mathcal{F}_t; 0 \leq t \leq \tau, x \in \mathbb{X}_n^d\}$ has continuous sample paths in t for each fixed $x \in \mathbb{X}_n^d$ and $X_{\beta, n, l}^{\tau, x}(t) \in \mathcal{F}_t$ for all $x \in \mathbb{X}_n^d$ for every $0 \leq t \leq \tau$; and
- (b) equation (Aux) holds on $[0, \tau] \times \mathbb{X}_n^d$, \mathbb{P} -almost surely.

Naturally, implicit in our definition above the assumption that, for each fixed $\tau \in \mathbb{R}_+$, we have

$$\mathbb{P} \left[\int_0^\tau (\kappa_{\beta, \delta_n, s, \tau}^{x, y}(X_{\beta, n, l}^{\tau, y}(s)))^2 ds < \infty \right] = 1; \quad \forall x, y \in \mathbb{X}_{n, l}^d, 0 \leq t \leq \tau.$$

For simplicity, we will sometimes say that the random field

$$X_{\beta, n, l}^\tau = \{X_{\beta, n, l}^{\tau, x}(t), \mathcal{F}_t; 0 \leq t \leq \tau, x \in \mathbb{X}_n^d\}$$

is a solution to (Aux) to mean the above. Clearly, if $X_{\beta, n, l}^{\tau, x}(t)$ satisfies (Aux) then $\tilde{U}_{\beta, n, l}^x(\tau) := X_{\beta, n, l}^{\tau, x}(\tau)$ satisfies (1.25) at $t = \tau$ for all $x \in \mathbb{X}_n^d$. Also, for

³⁰ All we need to adapt it here is a notational change, replacing the BTRW transition density in [1] with the β -ISLTRW one.

each n and each $d = 1, 2, 3$

$$|\kappa_{\beta, \delta_n, s, \tau}^{x, y}(X_{\beta, n, l}^{\tau, y}(s))| = \left| \frac{\mathbb{K}_{\tau-s; x, y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta}}{\delta_n^{d/2}} a(X_{\beta, n, l}^{\tau, y}(s)) \right| \leq \frac{|a(X_{\beta, n, l}^{\tau, y}(s))|}{\delta_n^{d/2}}.$$

In addition, for each fixed $\tau \in \mathbb{R}_+$ and each fixed $x, y \in \mathbb{X}_{n, l}^d$ we have for a solution $X_{\beta, n, l}^\tau$ to (Aux) that

$$\kappa_{\beta, \delta_n, s, \tau}^{x, y}(X_{\beta, n, l}^{\tau, y}(s)) \in \mathcal{F}_s; \quad \forall s \leq \tau,$$

since, of course the deterministic $\mathbb{K}_{\tau-s; x, y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta} / \delta_n^{d/2} \in \mathcal{F}_s$ and $a(X_{\beta, n, l}^{\tau, y}(s)) \in \mathcal{F}_s$. Thus, if $X_{\beta, n, l}^\tau$ solves (Aux); then, for each fixed $\tau > 0$ and $x, y \in \mathbb{X}_{n, l}^d$, each stochastic integral in (Aux)

$$I_{\beta, n, l}^{\tau, x, y} = \left\{ I_{\beta, n, l}^{\tau, x, y}(t) := \int_0^t \kappa_{\beta, \delta_n, s, \tau}^{x, y}(X_{\beta, n, l}^{\tau, y}(s)) dW_n^y(s), \mathcal{F}_t; 0 \leq t \leq \tau \right\}$$

is a continuous local martingale in t on $[0, \tau]$. This is clear since by a standard localization argument we may assume the boundedness of a ($|a(u)| \leq C$); in this case we have for each fixed $x, y \in \mathbb{X}_{n, l}^d$ and $\tau \in \mathbb{R}_+$ that

$$\mathbb{E}[I_{\beta, n, l}^{\tau, x, y}(t) | \mathcal{F}_r] = \int_0^r \kappa_{\beta, \delta_n, s, \tau}^{x, y}(X_{\beta, n, l}^{\tau, y}(s)) dW_n^y(s) = I_{\beta, n, l}^{\tau, x, y}(r), \quad r \leq t \leq \tau.$$

So, the finite sum over $\mathbb{X}_{n, l}^d$ in (Aux) is also a continuous local martingale in t on $[0, \tau]$. That is, for each $\tau > 0$ and $x \in \mathbb{X}_{n, l}^d$, $M_{\beta, n, l}^{\tau, x} \in \mathcal{M}_2^{c, \text{loc}}$, where

$$M_{\beta, n, l}^{\tau, x} = \left\{ M_{\beta, n, l}^{\tau, x}(t) := \sum_{y \in \mathbb{X}_{n, l}^d} \int_0^t \kappa_{\beta, \delta_n, s, \tau}^{x, y}(X_{\beta, n, l}^{\tau, y}(s)) dW_n^y(s), \mathcal{F}_t; 0 \leq t \leq \tau \right\}$$

with quadratic variation

$$(4.10) \quad \langle M_{\beta, n, l}^{\tau, x}(\cdot) \rangle_t = \sum_{y \in \mathbb{X}_{n, l}^d} \int_0^t [\kappa_{\beta, \delta_n, s, \tau}^{x, y}(X_{\beta, n, l}^{\tau, y}(s))]^2 ds,$$

where we have used the independence of the BMs $\{W_n^y\}_{y \in \mathbb{X}_{n, l}^d}$ within the lattice $\mathbb{X}_{n, l}^d$. For each $\tau > 0$, we call $M_{\beta, n, l}^{x, \tau}$ a kernel local martingale (or K-local martingale).

There is another complicating factor in formulating our K-martingale problem approach that is not present in the standard SDEs setting. To easily extract solutions to the truncated β -ISLTRW SIEs in (1.25) from the family of auxiliary problems $\{e_{\beta\text{-ISLTRW}}^{\text{aux-SIE}}(a, u_0, n, l, \tau)\}_{\tau > 0}$ in (Aux), we want the independent BMs sequence $\{W_n^y\}_{y \in \mathbb{X}_{n, l}^d}$ to not depend on the choices of τ and x . That is, we want all the K-local martingales in (Aux) to be stochastic integrals with respect to the same sequence $\{W_n^y\}_{y \in \mathbb{X}_{n, l}^d}$, regardless of τ and x .

With this in mind, we now formulate the K-martingale problem associated with the auxiliary β -ISLTRW SIEs in (Aux). Let

$$(4.11) \quad C_{n,l} := \{u : \mathbb{R}_+ \times (\mathbb{X}_{n,l}^d)^2 \rightarrow \mathbb{R}^2; t \mapsto u^{x_1, x_2}(t) \text{ is continuous } \forall x_1, x_2\}.$$

For $u \in C_{n,l}$, let $u^{x_1, x_2}(t) = (u_1^{x_1}(t), u_2^{x_2}(t))$ with $u^x(t) = u^{x,x}(t)$; and for any $\tau_1, \tau_2 > 0$ and any $x_1, x_2, y \in \mathbb{X}_{n,l}^d$, let

$$(4.12) \quad \Upsilon_{\delta_n, t, \tau_{i,j}}^{x_i, j, y}(u^y(t)) := \frac{\mathbb{K}_{\tau_i - t; x_i, y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta}}{\delta_n^{d/2}} a(u_i^y(t)) \frac{\mathbb{K}_{\tau_j - t; x_j, y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta}}{\delta_n^{d/2}} a(u_j^y(t)); \quad 1 \leq i, j \leq 2$$

(we are allowing the cases $\tau_1 = \tau_2$ and/or $x_1 = x_2$) where for typesetting convenience we denoted the points (τ_i, τ_j) and (x_i, x_j) by $\tau_{i,j}$ and $x_{i,j}$, respectively. We denote by ∂_i and ∂_{ij}^2 the first order partial derivative with respect to the i th argument and the second order partials with respect to the i and j arguments, respectively. Let $C^2 = C^2(\mathbb{R}^2, \mathbb{R})$ be the class of twice continuously differentiable real-valued functions on \mathbb{R}^2 and let

$$(4.13) \quad C_b^2 = \{f \in C^2; f \text{ and its derivatives up to second order are bounded}\}.$$

Now, for $\tau_1, \tau_2 > 0$, for $f \in C_b^2$, and for $(t, x_1, x_2, u) \in [0, \tau_1 \wedge \tau_2] \times (\mathbb{X}_{n,l}^d)^2 \times C_{n,l}$ let

$$(4.14) \quad (\mathcal{A}_\Upsilon^{\tau_1, 2} f)(t, x_1, x_2, u) := \sum_{1 \leq i \leq 2} \partial_i f(u^{x_1, x_2}(t)) \frac{\partial}{\partial t} \tilde{U}_{\beta, n, D}^{x_i}(t) + \frac{1}{2} \sum_{1 \leq i, j \leq 2} \partial_{ij}^2 f(u^{x_i, x_j}(t)) \sum_{y \in \mathbb{X}_{n,l}^d} \Upsilon_{\delta_n, t, \tau_{i,j}}^{x_i, j, y}(u^y(t)).$$

Let $X_{\beta, n, l}^\tau = \{X_{\beta, n, l}^{\tau, x}(t); 0 \leq t \leq \tau, x \in \mathbb{X}_n^d\}$ be a continuous in t adapted real-valued process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. For every $\tau_1, \tau_2 > 0$ define the two-dimensional stochastic process $Z_{\beta, n, l}^{\tau_1, 2}$:

$$(4.15) \quad \{Z_{\beta, n, l}^{x_1, 2, \tau_1, 2}(t) = (X_{\beta, n, l}^{\tau_1, x_1}(t), X_{\beta, n, l}^{\tau_2, x_2}(t)); (t, x_1, x_2) \in [0, \tau_1 \wedge \tau_2] \times (\mathbb{X}_{n,l}^d)^2\}$$

with $Z_{\beta, n, l}^{y, \tau_1, 2}(t) = (X_{\beta, n, l}^{\tau_1, y}(t), X_{\beta, n, l}^{\tau_2, y}(t))$ and let $U_0^{x_1, x_2} = (u_0(x_1), u_0(x_2))$. We say that the family $\{X_{\beta, n, l}^\tau\}_{\tau \geq 0}$ satisfies the K-martingale problem associated with the auxiliary β -ISLTRW SIEs in (Aux) on $\mathbb{R}_+ \times \mathbb{X}_n^d$ if for every $f \in C_b^2$, $0 < \tau_1, \tau_2 < \infty$, $\tau = \tau_1 \wedge \tau_2$, $t \in [0, \tau]$, $x_1, x_2 \in \mathbb{X}_{n,l}^d$, and $x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d$ we have

$$(KM) \quad \begin{cases} f(Z_{\beta, n, l}^{x_1, 2, \tau_1, 2}(t)) - f(U_0^{x_1, x_2}) \\ \quad - \int_0^t (\mathcal{A}_\Upsilon^{\tau_1, 2} f)(s, x_1, x_2, Z_{\beta, n, l}^{\tau_1, 2}) ds \in \mathcal{M}_2^{c, \text{loc}}; \\ X_{\beta, n, l}^{\tau, x}(t) = \tilde{U}_{\beta, n, D}^x(t). \end{cases}$$

We are now ready to state the equivalence of the K-martingale problem in (KM) to the auxiliary SIEs in (Aux) and its implication for the β -ISLTRW SIE in (1.25). This result is of independent interest and is stated as the following theorem.³¹

THEOREM 4.1. *The existence of a solution pair $(\{X_{\beta,n,l}^\tau\}_{\tau \geq 0}, \{W_n^y\}_{y \in \mathbb{X}_{n,l}^d})$ to $\{e_{\beta\text{-ISLTRW}}^{\text{aux-SIE}}(a, u_0, n, l, \tau)\}_{\tau \geq 0}$ in (Aux) on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is equivalent to the existence of a processes family $\{X_{\beta,n,l}^\tau\}_{\tau \geq 0}$ satisfying (KM). Furthermore, if there is $\{X_{\beta,n,l}^\tau\}_{\tau \geq 0}$ satisfying (KM) then there is a solution to (1.25) on $\mathbb{R}_+ \times \mathbb{X}_n^d$.*

The proof follows the exact same steps as the proof of Theorem 1.3 in [1] and will not be repeated.

4.3. Completing the proof of the second main result. We now complete the proof of Theorem 1.3. In Section 2.2 and Section 4.1, we assumed the existence of a β -ISLTRW SIE solution and we obtained regularity and tightness for the sequence of lattice SIEs $\{e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)\}_{n \in \mathbb{N}^*}$. This, in turn, implied the existence and regularity for a β -ISLTRW SIE limit solution to our $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ in (1.14). To complete the existence of the desired double limit solution³² for $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ it suffices then to prove the existence of a solution to $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$ for each fixed $n \in \mathbb{N}^*$, under the condition (NLip), that is uniformly $L^p(\Omega, \mathbb{P})$ bounded on $[0, T] \times \mathbb{X}^d$ for every $T > 0$ and every $p \geq 2$. We establish this existence via the K-martingale approach just recalled and adapted from [1], using Theorem 4.1.

First, the following proposition summarizes the results in this case for the β -ISLTRW SIEs spatial lattice scale.³³

PROPOSITION 4.2 (Existence for β -ISLTRW SIEs with non-Lipschitz a). *Assume the conditions (NLip) hold. Then,*

- (a) *For every $(n, l) \in \mathbb{N}^* \times \mathbb{N}$, every $\beta = 1/\nu \in \{1/2^k, k \in \mathbb{N}\}$, and for every $p \geq 2$, there exists an L^p -bounded solution $\tilde{U}_{\beta,n,l}^x(t)$ to the truncated*

³¹ This is because it is easily adaptable to many mild formulations of SPDEs, of different orders, not just for the β -ISLTBM SIEs. Since we do not prove uniqueness under less than Lipschitz conditions for our β -ISLTBM SIEs, we have not explicitly mentioned the uniqueness implications of our K-martingale approach. More on that in future articles.

³² The type of our lattice limit solution to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ in (1.14) depends on the conditions: under the Lipschitz conditions (Lip) we get a direct solution to the lattice SIE $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$ for every n and a direct β -ISLTRW SIE limit solution to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$ (see Theorem A.1); whereas under the non-Lipschitz conditions in (NLip) we obtain a limit β -ISLTRW SIE solution, thanks to our K-martingale approach, and a β -ISLTRW SIEs double limit solution to $e_{\beta\text{-ISLTBM}}^{\text{SIE}}(a, u_0)$.

³³ We remind the reader that we will, without further notice, suppress the dependence on β whenever it is more convenient notationally to do so.

- β -ISLTRW SIE (1.25) on $\mathbb{T} \times \mathbb{X}_n^d$. Moreover, if we linearly interpolate $\tilde{U}_{\beta,n,l}^x(t)$ in space; then, with probability one, the continuous map $(t, x) \mapsto \tilde{U}_{\beta,n,l}^x(t)$ is locally γ_t -Hölder continuous in time with $\gamma_t \in (0, \frac{2\nu-d}{4\nu})$ for $\nu = \beta^{-1} \in \{2^k; k \in \mathbb{N}\}$ and $d = 1, 2, 3$.
- (b) For any fixed $n \in \mathbb{N}^*$, the sequence $\{\tilde{U}_{\beta,n,l}^x(t)\}_{l \in \mathbb{N}}$ of linearly interpolated solutions in (a) has a subsequential weak limit $\tilde{U}_{\beta,n}$ in $C(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$. We thus have a limit solution $\tilde{U}_{\beta,n}$ to $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$, and $\tilde{U}_{\beta,n}$ is locally γ_t -Hölder continuous in time with $\gamma_t \in (0, \frac{2\nu-d}{4\nu})$ for $\nu = \beta^{-1} \in \{2^k; k \in \mathbb{N}\}$ and $d = 1, 2, 3$.

Proof. First, recall that the deterministic term $\tilde{U}_{\beta,D}^x(t)$ in (1.25) is completely determined by u_0 . Moreover, under the conditions in (NLip) on u_0 , $\tilde{U}_{\beta,D}^x(t)$ is clearly bounded and it is smooth in time as in Remark 1.1. Fix an arbitrary $T > 0$, and let $\mathbb{T} = [0, T]$. We now prove the existence of a family of adapted processes $\{\tilde{X}_{\beta,n,l}^\tau\}_{\tau \in \mathbb{T}}$ satisfying our K-martingale problem (KM), which by Theorem 4.1 implies the existence of a solution to the l -truncated β -ISLTRW SIE (1.25) on $\mathbb{T} \times \mathbb{X}_n^d$. On a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ we prepare a family of r -independent BMs $\{W_n^y(t)\}_{y \in \mathbb{X}_{n,l}^d}$. For each $\tau \in \mathbb{T}$ and each $i = 1, 2, \dots$ define a continuous process $X_{\beta,n,l,i}^\tau$ on $[0, \tau] \times \mathbb{X}_n^d$ inductively for $k/2^i \leq t \leq ((k+1)/2^i) \wedge \tau$ ($k = 0, 1, 2, \dots$) as follows: $X_{\beta,n,l,i}^{x,\tau}(0) = u_0(x)$ ($x \in \mathbb{X}_n^d$) and if $X_{\beta,n,l,i}^{x,\tau}(t)$ is defined for $t \leq k/2^i$, then we define $X_{\beta,n,l,i}^{x,\tau}(t)$ for $k/2^i \leq t \leq ((k+1)/2^i) \wedge \tau$, by

$$(4.16) \quad X_{\beta,n,l,i}^{x,\tau}(t) = \begin{cases} X_{\beta,n,l,i}^{x,\tau}(k/2^i) + \sum_{y \in \mathbb{X}_{n,l}^d} \kappa_{\delta_n, \frac{k}{2^i}, \tau}^{x,y} (X_{\beta,n,l,i}^{y,\tau}(k/2^i)) (\Delta_{t, \frac{k}{2^i}} W_n^y) \\ \quad + [\tilde{U}_{\beta,n,D}^x(t) - \tilde{U}_{n,D}^x(k/2^i)]; & x \in \mathbb{X}_{n,l}^d, \\ \tilde{U}_{\beta,n,D}^x(t); & x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d, \end{cases}$$

where $\Delta_{t, \frac{k}{2^i}} W_n^y = W_n^y(t) - W_n^y(k/2^i)$. Clearly, $X_{\beta,n,l,i}^\tau$ is the solution to the equation

$$(4.17) \quad X_{\beta}^{x,\tau}(t) = \begin{cases} \sum_{y \in \mathbb{X}_{n,l}^d} \int_0^t \kappa_{\delta_n, \phi_i(s), \tau}^{x,y} (X^{y,\tau}(\phi_i(s))) dW_n^y(s) + \tilde{U}_{\beta,n,D}^x(t); & x \in \mathbb{X}_{n,l}^d, \\ \tilde{U}_{\beta,n,D}^x(t); & x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d \end{cases}$$

with $X_{\beta}^{x,\tau}(0) = u_0(x)$, where $\phi_i(t) = k/2^i$ for $k/2^i \leq t < (k+1)/2^i \wedge \tau$ ($k = 0, 1, 2, \dots$).

Now, for $q \geq 1$, let $M_{\beta,q,l,i}^\tau(t) = \sup_{x \in \mathbb{X}_n^d} \mathbb{E}|X_{\beta,n,l,i}^{x,\tau}(t)|^{2q}$. By the boundedness of $\tilde{U}_{\beta,n,D}^x(t)$ over the whole infinite lattice \mathbb{X}_n^d , we have

$$(4.18) \quad M_{\beta,q,l,i}^\tau(t) \leq C + \sup_{x \in \mathbb{X}_{n,l}^d} \mathbb{E}|X_{\beta,n,l,i}^{x,\tau}(t)|^{2q}.$$

Then, replacing \mathbb{X}_n^d by $\mathbb{X}_{n,l}^d$ and following the same steps as in the proof of Proposition 4.1, we get that

$$(4.19) \quad \sup_{\tau \in \mathbb{T}} \sup_{t \in [0,\tau]} M_{\beta,q,l,i}^\tau(t) \leq C, \quad d = 1, 2, 3,$$

where, here and in the remainder of the proof, the constant C depends only on $q, \beta, \max_x |u_0(x)|$, the spatial dimension $d = 1, 2, 3$, and T but may change its value from one line to the next. Remembering that $\delta_n \searrow 0$ as $n \nearrow \infty$ and $n \in \mathbb{N}^*$, the independence in l is trivially seen since Lemma 2.2 implies

$$\sum_{y \in \mathbb{X}_{n,l}^d} [\mathbb{K}_{t;x,y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta}]^2 \leq \sum_{y \in \mathbb{X}_n^d} [\mathbb{K}_{t;x,y}^{\text{RW}_{\delta_n}^d, \Lambda_\beta}]^2 \leq \frac{C}{t^{d/2\nu}}; \quad \forall d = 1, 2, 3, l \in \mathbb{N}.$$

Similarly, letting $X_{n,l,i,R}^{x,\tau}$ denote the random part of $X_{\beta,n,l,i}^{x,\tau}$ on the truncated lattice $\mathbb{X}_{n,l}^d$, using (4.19), and repeating the arguments in Lemma 2.5 and Lemma 2.6—replacing \mathbb{X}_n^d by $\mathbb{X}_{n,l}^d$ and noting that Lemma 2.3 and Lemma 2.4 hold on $\mathbb{X}_{n,l}^d$ —we obtain

$$(4.20) \quad \begin{aligned} & \mathbb{E}|X_{\beta,n,l,i,R}^{x,\tau_1}(t) - X_{\beta,n,l,i,R}^{y,\tau_1}(t)|^{2q} + \mathbb{E}|X_{\beta,n,l,i,R}^{x,\tau_2}(t) - X_{\beta,n,l,i,R}^{y,\tau_2}(t)|^{2q} \\ & \leq C_d |x - y|^{2q\alpha_d}; \quad \alpha_d \in I_d, \\ & \mathbb{E}|X_{\beta,n,l,i,R}^{x,\tau_1}(t) - X_{\beta,n,l,i,R}^{x,\tau_1}(r)|^{2q} + \mathbb{E}|X_{\beta,n,l,i,R}^{x,\tau_2}(t) - X_{\beta,n,l,i,R}^{x,\tau_2}(r)|^{2q} \\ & \leq C |t - r|^{\frac{(2\nu-d)q}{2\nu}}, \end{aligned}$$

for all $x, y \in \mathbb{X}_{n,l}^d$, $r, t \in [0, \tau_1 \wedge \tau_2]$, $\tau_1, \tau_2 \in \mathbb{T}$, and $d = 1, 2, 3$. It follows that, for every point $\tau_{1,2} = (\tau_1, \tau_2) \in \mathbb{T}^2$, there is a subsequence $\{(\tilde{X}_{\beta,n,l,i_m}^{\tau_1}, \tilde{X}_{\beta,n,l,i_m}^{\tau_2})\}_{m=1}^\infty$ on a probability space $(\tilde{\Omega}_{\tau_{1,2}}, \tilde{\mathcal{F}}_{\tau_{1,2}}, \tilde{\mathbb{P}}_{\tau_{1,2}})$ such that

$$(\tilde{X}_{\beta,n,l,i_m}^{\tau_1}, \tilde{X}_{\beta,n,l,i_m}^{\tau_2}) \stackrel{\mathcal{L}}{\rightrightarrows} (X_{\beta,n,l,i_m}^{\tau_1}, X_{\beta,n,l,i_m}^{\tau_2})$$

and

$$(\tilde{X}_{\beta,n,l,i_m}^{x,\tau_1}(t), \tilde{X}_{\beta,n,l,i_m}^{x,\tau_2}(t)) \longrightarrow (\tilde{X}_{\beta,n,l}^{x,\tau_1}(t), \tilde{X}_{\beta,n,l}^{x,\tau_2}(t))$$

uniformly on compact subsets of $[0, \tau_1 \wedge \tau_2] \times \mathbb{X}_n^d$, as $m \rightarrow \infty$ a.s. Let $\mathbb{T}_\mathbb{Q} = \mathbb{T} \cap \mathbb{Q}$, where \mathbb{Q} is the set of rationals, and define the product probability space

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) := \left(\bigotimes_{\tau_{1,2} \in \mathbb{T}_\mathbb{Q}^2} \tilde{\Omega}_{\tau_{1,2}}, \bigotimes_{\tau_{1,2} \in \mathbb{T}_\mathbb{Q}^2} \tilde{\mathcal{F}}_{\tau_{1,2}}, \bigotimes_{\tau_{1,2} \in \mathbb{T}_\mathbb{Q}^2} \tilde{\mathbb{P}}_{\tau_{1,2}} \right).$$

If $s < t$, then for every $f \in C_b^2(\mathbb{R}^2; \mathbb{R})$, $\tau_1, \tau_2 \in \mathbb{T}_{\mathbb{Q}} \setminus \{0\}$, $t \in [0, \tau_1 \wedge \tau_2]$, $x_1, x_2 \in \mathbb{X}_{n,l}^d$, and for every bounded continuous $F: C(\mathbb{R}_+; \mathbb{R}^2) \rightarrow \mathbb{R}$ that is $\mathcal{B}_s(C(\mathbb{R}_+; \mathbb{R}^2)) := \sigma(z(r); 0 \leq r \leq s)$ -measurable function, we have

$$\begin{aligned}
 (4.21) \quad & \mathbb{E}_{\tilde{\mathbb{P}}} \left[\left\{ f(\tilde{Z}_{\beta,n,l}^{x_1,2,\tau_1,2}(t)) - f(\tilde{Z}_{\beta,n,l}^{x_1,2,\tau_1,2}(s)) \right. \right. \\
 & \quad \left. \left. - \int_s^t (\mathcal{A}_{\Upsilon}^{\tau_1,2} f)(r, x_1, x_2, \tilde{Z}_{\beta,n,l}^{\tau_1,2}) dr \right\} F(\tilde{Z}_{\beta,n,l}^{x_1,2,\tau_1,2}(\cdot)) \right] \\
 &= \lim_{m \rightarrow \infty} \mathbb{E}_{\tilde{\mathbb{P}}} \left[\left\{ f(\tilde{Z}_{\beta,n,l,i_m}^{x_1,2,\tau_1,2}(t)) - f(\tilde{Z}_{\beta,n,l,i_m}^{x_1,2,\tau_1,2}(s)) \right. \right. \\
 & \quad \left. \left. - \int_s^t (\mathcal{A}_{\Upsilon,i_m}^{\tau_1,2} f)(r, x_1, x_2, \tilde{Z}_{\beta,n,l,i_m}^{\tau_1,2}) dr \right\} F(\tilde{Z}_{\beta,n,l,i_m}^{x_1,2,\tau_1,2}(\cdot)) \right] \\
 &= 0,
 \end{aligned}$$

where, by a standard localization argument, we have assumed that a is also bounded; and where $\tilde{Z}_{\beta,n,l}^{\tau_1,2}$ and $\tilde{Z}_{\beta,n,l,i_m}^{\tau_1,2}$ are obtained from the definition of $Z_{\beta,n,l}^{\tau_1,2}$ in (4.15) by replacing $X_{\beta,n,l}^{\tau_j}$ by $\tilde{X}_{\beta,n,l}^{\tau_j}$ and $\tilde{X}_{\beta,n,l,i_m}^{\tau_j}$, $j = 1, 2$, respectively. The operator $\mathcal{A}_{\Upsilon,i_m}^{\tau_1,2}$ is obtained from $\mathcal{A}_{\Upsilon}^{\tau_1,2}$ by replacing $\Upsilon_{\delta_n,t,\tau_i,j}^{x_i,j,y}(u^y(t))$ in (4.14) by $\Upsilon_{\delta_n,\phi_{i_m}(t),\tau_{i,j}}^{x_i,j,y}(u^y(\phi_{i_m}(t)))$. Also, obviously, for any $\tau \in \mathbb{T}_{\mathbb{Q}}$ and $t \in [0, \tau]$

$$(4.22) \quad \tilde{X}_{\beta,n,l}^{x,\tau}(t) = \lim_{m \rightarrow \infty} \tilde{X}_{\beta,n,l,i_m}^{x,\tau}(t) = \tilde{U}_{\beta,n,D}^x(t); \quad x \in \mathbb{X}_n^d \setminus \mathbb{X}_{n,l}^d, \text{ a.s. } \tilde{\mathbb{P}}.$$

It follows from (4.21) and (4.22) that $\{\tilde{X}_{\beta,n,l}^{\tau}\}_{\tau \in \mathbb{T}_{\mathbb{Q}}}$ satisfies the K-martingale problem (KM) with respect to the filtration $\{\tilde{\mathcal{F}}_t\}$, with

$$\tilde{\mathcal{F}}_t = \bigcap_{\varepsilon > 0} \sigma\{\tilde{X}_{\beta,n,l}^{x,\tau}(u); u \leq (t + \varepsilon) \wedge \tau, \tau \in \mathbb{T}_{\mathbb{Q}} \cap (t, T]\}.$$

Thus, by Theorem 4.1, with $\tau \in \mathbb{R}_+$ replaced by $\tau \in \mathbb{T}_{\mathbb{Q}}$, there is a solution $\tilde{U}_{\beta,n,l}^x(t)$ to the l -truncated β -ISLTRW SIE (1.25) on $\mathbb{T}_{\mathbb{Q}} \times \mathbb{X}_n^d$. Use continuous extension in time of $\tilde{U}_{\beta,n,l}^x(t)$ to extend its definition to $\mathbb{T} \times \mathbb{X}_n^d$, and denote the extension also by $\tilde{U}_{\beta,n,l}^x(t)$. Clearly $\tilde{U}_{\beta,n,l}^x(t)$ solves the l -truncated β -ISLTRW SIE (1.25) on $\mathbb{T} \times \mathbb{X}_n^d$.

Now, for $q \geq 1$, let $M_{\beta,q,l}(t) = \sup_{x \in \mathbb{X}_n^d} \mathbb{E}|\tilde{U}_{\beta,n,l}^x(t)|^{2q}$. As above, the boundedness of $\tilde{U}_{\beta,n,D}^x(t)$, implies

$$(4.23) \quad M_{\beta,q,l}(t) \leq C + \sup_{x \in \mathbb{X}_{n,l}^d} \mathbb{E}|\tilde{U}_{\beta,n,l}^x(t)|^{2q}.$$

Then, replacing \mathbb{X}_n^d by $\mathbb{X}_{n,l}^d$ and following the same steps as in the proof of Proposition 4.1, we get that

$$(4.24) \quad M_{\beta,q,l}(t) \leq C, \quad \forall t \in \mathbb{T}, \beta \in \{1/2^k; k \in \mathbb{N}\} \text{ and } d = 1, 2, 3.$$

Similarly, letting $\tilde{U}_{\beta,n,l,R}^x(t)$ denote the random part of $\tilde{U}_{\beta,n,l}^x(t)$ on the truncated lattice $\mathbb{X}_{n,l}^d$, using (4.24), and repeating the arguments in Lemma 2.5 and Lemma 2.6—replacing \mathbb{X}_n^d by $\mathbb{X}_{n,l}^d$ and noting that the inequalities in Lemma 2.3 and Lemma 2.4 trivially hold if we replace \mathbb{X}_n^d by $\mathbb{X}_{n,l}^d$ —we obtain

$$(4.25) \quad \begin{aligned} \mathbb{E}|\tilde{U}_{\beta,n,l,R}^x(t) - \tilde{U}_{\beta,n,l,R}^y(t)|^{2q} &\leq C_d|x - y|^{2q\alpha_d}; \quad \alpha_d \in I_d, \\ \mathbb{E}|\tilde{U}_{\beta,n,l,R}^x(t) - \tilde{U}_{\beta,n,l,R}^x(r)|^{2q} &\leq C|t - r|^{\frac{(2\nu-d)q}{2\nu}}, \end{aligned}$$

for all $x, y \in \mathbb{X}_{n,l}^d$, $r, t \in \mathbb{T}$, and $d = 1, 2, 3$. By Remark 1.1, $\tilde{U}_{\beta,n,D}^x(t)$ is differentiable in t . So, linearly interpolating $\tilde{U}_{\beta,n,l}^x(t)$ in space and using (4.25) and arguing as in the proof of part (a) of Lemma 4.2, we get that the continuous map $(t, x) \mapsto \tilde{U}_{\beta,n,l}^x(t)$ is locally γ_t -Hölder continuous in time with $\gamma_t \in (0, \frac{2\nu-d}{4\nu})$ for $\nu = \beta^{-1} \in \{2^k; k \in \mathbb{N}\}$ and $d = 1, 2, 3$.

(b) Clearly, $\tilde{U}_{\beta,n,D}^x(t)$ in (1.25) is the same for every l , so it is enough to show convergence of the random part $\tilde{U}_{\beta,n,l,R}^x(t)$. Using (4.25) we get tightness for $\{\tilde{U}_{\beta,n,l,R}^x(t)\}_l$ and consequently a subsequential weak limit $\tilde{U}_{\beta,n}$, which is our limit solution for $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$. For the regularity assertion, $\tilde{U}_{\beta,n,D}^x(t)$ is smooth and bounded as noted above. So, using (4.24) and (4.25), and imitating the argument in the proof of part (b) of Lemma 4.2 (remembering that here we are taking the limit as $l \rightarrow \infty$); we get the desired L^p boundedness for $\tilde{U}_{\beta,n}$ as in Proposition 4.1 and the spatial and temporal moments bounds in Lemma 2.5 and Lemma 2.6

$$(4.26) \quad \begin{cases} \mathbb{E}|\tilde{U}_{\beta,n}^x(t)|^{2q} \leq C, \\ \mathbb{E}|\tilde{U}_{\beta,n,R}^x(t) - \tilde{U}_{n,R}^y(t)|^{2q} \leq C_d|x - y|^{2q\alpha_d}; \quad \alpha_d \in I_d, \\ \mathbb{E}|\tilde{U}_{\beta,n,R}^x(t) - \tilde{U}_{n,R}^x(r)|^{2q} \leq C|t - r|^{\frac{(2\nu-d)q}{2\nu}}, \end{cases}$$

for $(t, x, n) \in \mathbb{T} \times \mathbb{X}_n^d \times \mathbb{N}^*$ and for $\nu = \beta^{-1} \in \{2^k; k \in \mathbb{N}\}$, $d = 1, 2, 3$, and $q \geq 1$ and the desired Hölder regularity follows. The proof is complete. \square

We now get Theorem 1.3 for $e_{\beta\text{TBM}}^{\text{SIE}}(a, u_0)$ as the following corollary.

COROLLARY 4.2. *Theorem 1.3 holds.*

Proof. The desired conclusion follows upon using the argument in the proof of part (b) of Lemma 4.2 along with Definition 1.4 and the L^p -boundedness and the spatial and temporal moments bounds for $\{\tilde{U}_{\beta,n}\}_n$ that we got in (4.26) above. \square

Appendix A. Limit solutions in the Lipschitz case

We now state prove our lattice-limit solution existence, uniqueness, and regularity for our BTBM SIE on $\mathbb{R}_+ \times \mathbb{R}^d$ under Lipschitz conditions.

THEOREM A.1 (Lattice-limits solutions: the Lipschitz case). *Under the Lipschitz conditions there exists a unique-in-law direct β -ISLTRW SIE weak-limit solution to $e_{\text{BTBM}}^{\text{SIE}}(a, u_0)$, U , such that $U(t, x)$ is $L^p(\Omega, \mathbb{P})$ -bounded on $\mathbb{T} \times \mathbb{R}^d$ for every $p \geq 2$ and $U \in H^{(\frac{2\nu-d}{4\nu})^-, (\frac{4-d}{2} \wedge 1)^-}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ for every $d = 1, 2, 3$.*

Theorem A.1 follows as a corollary to the results of Section 2.2 combined with the following proposition.

PROPOSITION A.1. *Under the Lipschitz conditions (Lip) there exists a unique direct solution to $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$, $\tilde{U}_{\beta, n}$, on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ that is $L^p(\Omega, \mathbb{P})$ -bounded on $[0, T] \times \mathbb{X}_n^d$ for every $T > 0$, $p \geq 2$, $n \in \mathbb{N}^*$, and $d = 1, 2, 3$.*

The proof of Proposition A.1 follows the same steps as the non-discretization Picard-type direct proof of the corresponding part in the continuous case in Section 3, with obvious changes, and we leave the details to the interested reader.

COROLLARY A.1. *Theorem A.1 holds.*

Proof. The conclusion follows from Proposition A.1, Lemma 2.5, Lemma 2.6, and Lemma 4.2(b). \square

REMARK A.1. With extra work, it is possible to prove the existence of a strong limit solution under Lipschitz conditions. We plan to address that in a future article.

Appendix B. Glossary of frequently used acronyms and notations

B.1. Acronyms.

- BM: Brownian motion.
- BTBM: Brownian-time Brownian motion.
- BTBM SIE: Brownian-time Brownian motion stochastic integral equation.
- BTP: Brownian-time processes.
- BTP SIE: Brownian-time process stochastic integral equation.
- BTC: Brownian-time chain.
- BTRW: Brownian-time random walk.
- β -ISLTRW DDE: Brownian-time random walk differential difference equation.
- β -ISLTRW SIE: Brownian-time random walk stochastic integral equation.
- DDE: Differential difference equation.
- KS: Kuramoto–Sivashinsky.
- RW: Random walk.
- SIE: Stochastic integral equation.

B.2. Notations.

- \mathbb{N} : The usual set of natural numbers $\{1, 2, 3, \dots\}$.
- $K_{t;x,y}^{\text{RW}_{\delta_n}^d}$: The d -dimensional continuous-time random walk transition density, starting at $x \in \mathbb{X}_n^d$ and going to $y \in \mathbb{X}_n^d$ in time t .
- $K_{s;x,y}^{\text{BM}^d}$: The density of a d -dimensional BM.
- $K_{t;0,s}^{\text{BM}}$: The density of a 1-dimensional BM, starting at 0.
- $\mathbb{K}_{t;x,y}^{\text{BTBM}^d}$: The kernel or density of a d -dimensional Brownian-time Brownian motion.
- $\mathbb{K}_{t;x,y}^{\text{RW}_{\delta_n}^d, \Lambda^\beta}$: The kernel or density of a d -dimensional Brownian-time random walk on a spatial lattice with step size δ_n in each of the d -dimensions.
- $e_{\text{BTBM}}^{\text{SIE}}(a, u_0)$: The BTBM SIE with diffusion coefficient a and initial function u_0 .
- $e_{\beta\text{-ISLTRW}}^{\text{SIE}}(a, u_0, n)$: The β -ISLTRW SIE on the lattice $\mathbb{X}_n^d = \delta_n \mathbb{Z}^d$ with diffusion coefficient a and initial function u_0 .

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HASSAN ALLOUBA, DEPARTMENT OF MATHEMATICAL SCIENCES, KENT STATE UNIVERSITY, KENT, OH 44242, USA

E-mail address: allouba@math.kent.edu