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# INTERTWINING RELATIONS FOR VOLTERRA OPERATORS ON THE BERGMAN SPACE

CE-ZHONG TONG AND ZE-HUA ZHOU

ABSTRACT. On the Bergman space in the unit disk, we study the intertwining relation for Volterra type operators, whose intertwining operator is a composition operator. We also investigate the "compact" intertwining relations for Volterra type operators. As obvious consequences, the essential commutativity of Volterra type and composition operators are characterized. At the end of the paper, we find a new connection between the Bergman space and little Bloch space through this essential commutativity.

## 1. Introduction

If $X$ and $Y$ are two Banach spaces, the symbol $\mathscr{B}(X,Y)$ denotes the collection of all bounded linear operators from $X$ to $Y$. Let $\mathcal{K}(X,Y)$ be the collection of all compact elements of $\mathscr{B}(X,Y)$, and $\mathscr{Q}(X,Y)$ be the quotient set $\mathscr{B}(X,Y)/\mathcal{K}(X,Y)$.

For linear operators $A \in \mathscr{B}(X,X)$, $B \in \mathscr{B}(Y,Y)$ and $T \in \mathscr{B}(X,Y)$, the phrase "$T$ intertwines $A$ and $B$ in $\mathscr{B}(X,Y)$" means that

$$TA = BT \quad \text{and} \quad T \neq 0. \tag{1.1}$$

Notation $A \propto B$ $(T)$ represents the relation in above equation. In 2008, Bourdon and Shapiro [4] showed that the relation $\propto$ is neither symmetric nor transitive. When the Banach space $X$ becomes a Hilbert space and $\lambda A \propto A$ for some $\lambda \in \mathbb{C}$, we call $\lambda$ an *extended eigenvalue* and the intertwining operator an *extended eigenoperator* of $A$ (see more specific details in [4]).

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Another phrase "$T$ intertwines $A$ and $B$ in $\mathcal{Q}(X,Y)$" means that

$$TA = BT \mod \mathcal{K}(X,Y) \quad \text{with } T \neq 0. \tag{1.2}$$

Notation $A \propto_K B$ $(T)$ represents the relation in equation (1.2). In fact, if $T$ (either in equations (1.1) or (1.2)) is an invertible operator on $X$, then the relation $\propto$ is symmetric, respectively.

Let $\mathbb{D}$ be the unit disk in the complex plane. Denote by $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, and $S(\mathbb{D})$ the collection of all the holomorphic self-maps of $\mathbb{D}$. For $\varphi \in S(\mathbb{D})$, we define the composition operator $C_\varphi$ by $C_\varphi(f) = f \circ \varphi$, where $f \in H(\mathbb{D})$. If $\mathrm{d}A(z)$ represents normalized Lebesgue area measure, it is well known that every composition operator acts boundedly on weighted Bergman spaces

$$\mathcal{A}^p_\alpha := \left\{ f \in H(\mathbb{D}) : \|f\|_p^p = \int_{\mathbb{D}} \left|f(z)\right|^p \mathrm{d}A_\alpha(z) < \infty \right\} \quad (\alpha > -1),$$

where $\mathrm{d}A_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha \, \mathrm{d}A(z)$. For the non-weighted Bergman space, we simplify the notation $\mathcal{A}^p_0$ as $\mathcal{A}^p$.

A function $f \in H(\mathbb{D})$ is called a Bloch function if

$$\|f\|_{\mathcal{B}} := \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right) \left|f'(z)\right| < \infty.$$

The Bloch functions equipped with the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}}$ form a Banach space, which is called the Bloch space and denoted by $\mathcal{B}$. Let $\mathcal{B}_{\mathrm{o}}$ be the subspace of $\mathcal{B}$ consisting of those $f \in \mathcal{B}$ for which

$$\lim_{|z| \to 1} \left(1 - |z|^2\right) \left|f'(z)\right| = 0.$$

And we denote by $H^\infty$ the bounded analytic functions space on $\mathbb{D}$.

If $g \in H(\mathbb{D})$, the *Volterra operator* $J_g$ is defined by

$$J_g f(z) = \int_0^z f(\zeta) g'(\zeta) \, \mathrm{d}\zeta;$$

another integral operator, named *co-Volterra operator*, $I_g$ is defined by

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta) \, \mathrm{d}\zeta,$$

where $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. The operator $J_g$ is actually a generalization of the integral operator (when $g(z) = z$). The operators $J_g$ and $I_g$ are close companions as a consequence of their relations to the multiplication operator $M_g f(z) = g(z) f(z)$. To see this, note that integration by parts gives

$$M_g f = f(0) g(0) + J_g f + I_g f.$$

The discussion of $J_g$ first arose in connection with semigroups of composition operators, and readers may refer to [14] for background. Recently, characterizing the boundedness and compactness of $J_g$ and $I_g$ on certain spaces of analytic functions becomes the most active work. For example, the boundedness of $J_g$ on Hardy spaces, Bergman spaces, BMOA space, Bloch space and $\mathcal{Q}_p$ space are characterized in [1], [2], [14], [15], [17], [18], respectively. In this paper, we use the symbol $V_g$ to represent both $J_g$ and $I_g$.

As a result of Victor Lomonosov's invariant subspace (cf. [10]), the concepts of extended eigenvalue and eigenoperator were introduced by Biswas et al. [3] in 2002. The notion of extended eigenvalue has recently been studied extensively on the space $L^2[0,1]$ (see [3], [9], [12], [13]). Most of them have found some important connections between composition operators (on $L^2[0,1]$) and eigenoperator for Volterra type operators.

Based on those foundations, we investigate the intertwining relation in $\mathscr{B}(\mathcal{A}^p)=\mathscr{B}(\mathcal{A}^p,\mathcal{A}^p)$ for Volterra type operators by intertwining some composition operator, and we give some applications to extended eigenvalues of Volterra type operators. We also find the spectrum of some composition operators. Second, the intertwining relations for Volterra type operators in $\mathscr{Q}(\mathcal{A}^p)=\mathscr{Q}(\mathcal{A}^p,\mathcal{A}^p)$ are studied in Section 4. As consequences, we give necessary and sufficient conditions on the essential commutativity of Volterra type and composition operators. At the end of this paper, we study a new collection of the symbols of Volterra type operators, and the idea together with its concept arise from the essential commutativity of $V_g$ and $C_\varphi$.

Throughout this paper, $C$ is abused to denote any arbitrary positive constant.

## 2. Preliminaries

We consider the weighted Bergman spaces $\mathcal{A}^p_\alpha$ for $p\geq 1$ and $\alpha>-1$ with their norms as defined in the introduction. In the study of the boundedness (and compactness) of integral type operators $J_g$ and $I_g$, we use another equivalent formula of the Bergman norm.

Lemma 2.1 (Theorem 2.17 in [19]). *Let $\alpha>-1$ and $n,p$ be two nonnegative integers. Suppose that $f$ is analytic in $\mathbb{D}$, then $f\in\mathcal{A}^p_\alpha$ if and only if*

$$\left(1-|z|^2\right)^n f^{(n)}(z)\in L^p(\mathbb{D},\mathrm{d}A_\alpha).$$

*Moreover, the norm of $f$ in $\mathcal{A}^p_\alpha$ is equivalent to the following formula*:

$$\left|f(0)\right|+\left|f'(0)\right|+\cdots+\left|f^{(n-1)}(0)\right|+\left\|\left(1-|z|^2\right)^n f^{(n)}(z)\right\|_{L^p(\mathrm{d}A_\alpha)}.$$

The notation $A\lesssim B$ indicates $A\leq C\cdot B$ for positive quantities $A,B$. And if $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$.

Recall that if $\Omega$ is a topological space, a *covering space* of $\Omega$ is a pair $(G;\tau)$ where $G$ is also a connected topological space and $\tau: G\to\Omega$ is a surjective continuous function with the property that for every $\zeta$ in $\Omega$ there is a neighborhood $\Delta$ of $\zeta$ such that each component of $\tau^{-1}(\Delta)$ is mapped by

$\tau$ homeomorphically on to $\Delta$. Such a neighborhood $\Delta$ of $\zeta$ is called a *fundamental neighborhood* of $\zeta$. If $(G;\tau)$ is the covering space for a region $\Omega$ in $\mathbb{C}$ and if $G$ is itself contained in $\mathbb{C}$ and $\tau$ is analytic, we will call $(G;\tau)$ an *analytic covering space*.

LEMMA 2.2 (Theorem 1.3, p. 110 in [6]). *Suppose $(G;\tau)$ is an analytic covering space of $\Omega$, $X$ is a region in the plane, and $f : X \to \Omega$ is an analytic function with $f(x_0) = \alpha_0 = \tau(a_0)$. If $X$ is simply connected, then there is a unique analytic function $T : X \to G$ such that $f = \tau \circ T$ and $T(x_0) = a_0$.*

Boundedness and compactness of $V_g$ on Bergman spaces are characterized in [2], [16] and [11], and we summarize them as below lemma.

LEMMA 2.3. *Let $\alpha, \beta \in (-1,\infty)$, $0 < p \le q < \infty$ and $g \in H(\mathbb{D})$. Then*
(i) *$J_g : \mathcal{A}^p_\alpha \to \mathcal{A}^q_\beta$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \left|g'(z)\right| \left(1-|z|^2\right)^{1-\frac{1+\alpha}{p}+\frac{1+\beta}{q}} < \infty; \tag{2.1}$$

*$J_g : \mathcal{A}^p_\alpha \to \mathcal{A}^q_\beta$ is compact if and only if*

$$\lim_{|z|\to 1} \left|g'(z)\right| \left(1-|z|^2\right)^{1+\frac{1+\beta}{q}-\frac{1+\alpha}{p}} = 0. \tag{2.2}$$

(ii) *$I_g : \mathcal{A}^p_\alpha \to \mathcal{A}^q_\beta$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \left|g(z)\right| \left(1-|z|^2\right)^{\frac{1+\beta}{q}-\frac{1+\alpha}{p}} < \infty; \tag{2.3}$$

*$I_g : \mathcal{A}^p_\alpha \to \mathcal{A}^q_\beta$ is compact if and only if*

$$\lim_{|z|\to 1} \left|g(z)\right| \left(1-|z|^2\right)^{\frac{1+\beta}{q}-\frac{1+\alpha}{p}} = 0. \tag{2.4}$$

We write $V[\varphi; g, h]$ for the operator $C_\varphi V_g - V_h C_\varphi$, which acts on the Bergman space $\mathcal{A}^p_\alpha$. In the following contents, we denote by $\mathcal{K}$ the collection of all compact linear operators on $\mathcal{A}^p_\alpha$. The following lemma is the crucial criterion for compactness. The standard version and its proof can be seen in [7], Proposition 3.11.

LEMMA 2.4. *Suppose that $\varphi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$. Then $V[\varphi; g, h]$ is a compact operator on $\mathcal{A}^p_\alpha$ if and only if for any bounded sequence $\{f_k\}$, $k = 1, 2, \dots$ in $\mathcal{A}^p_\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $V[\varphi; g, h]f_k$ converges to zero in the $\mathcal{A}^p_\alpha$ norm topology as $k$ tends to infinity.*

Next we will state the generalized Carleson measure theorem for $\mathcal{A}^p_\alpha$. It has been obtained by several authors in different forms. We choose to use the version in [20]. We simplify those results in the one dimensional case as follows. For $q > 0$, let $\mu$ be a positive Borel measure on $\mathbb{D}$, and $X$ be a Banach space of analytic functions on $\mathbb{D}$. We say that $\mu$ is an $(X,q,k)$-Carleson measure if there is a constant $C>0$ such that, for any $f\in X$,

$$\int_{\mathbb{D}}\left|f^{(k)}(z)\right|^q\,\mathrm{d}\mu(z)\lesssim\|f\|_X^q.$$

It becomes standard Carleson measure if $k=0$.

For any $\zeta\in\partial\mathbb{D}$ and $r>0$ let

$$Q_r(\zeta)=\bigl\{z\in\mathbb{D}:|\zeta-z|<r\bigr\}.$$

These are Carleson squares in the unit disk. For any $R>0$ and $a\in\mathbb{D}$, we write

$$D(a,R)=\left\{z\in\mathbb{D}:\frac{a-z}{1-\bar{a}z}<R\right\}.$$

When $\alpha>-1$, the condition

$$\mu\bigl(Q_r(\zeta)\bigr)\le Cr^{2+\alpha},\quad r>0,\zeta\in\partial\mathbb{D},$$

is equivalent to the condition

$$\mu\bigl(D(a,R)\bigr)\le C_R\bigl(1-|a|^2\bigr)^{2+\alpha},\quad a\in\mathbb{D}.$$

See Lemma 5.23 and Corollary 5.24 of [19].

THEOREM 2.5 (Theorem 50 in [20]). *Suppose* $0<p\le q<\infty$, $\alpha$ *is real, and* $\mu$ *is a positive Borel measure on* $\mathbb{D}$. *Then for any nonnegative integer* $k$ *with* $\alpha+kp>-1$ *the following conditions are equivalent.*

(a) $\mu$ *is an* $(\mathcal{A}^p_\alpha,q,k)$*-Carleson measure, that is*

$$\int_{\mathbb{D}}\left|f^{(k)}(w)\right|^q\,\mathrm{d}\mu(w)\lesssim\|f\|^q_{\mathcal{A}^p_\alpha}$$

*for all* $f\in\mathcal{A}^p_\alpha$.

(b) *For each (or some)* $s>0$ *there is a constant* $C>0$ *such that*

$$\int_{\mathbb{D}}\frac{(1-|z|^2)^s}{|1-\bar{w}z|^{s+(2+\alpha+kp)q/p}}\,\mathrm{d}\mu(w)\le C$$

*for all* $z\in\mathbb{D}$.

(c) $\mu(Q_r(\zeta))\lesssim r^{(2+\alpha+kp)q/p}$ *for all* $r>0$ *and* $\zeta\in\partial\mathbb{D}$.

(d) *For each (or some)* $R>0$, $\mu(D(a,R))\lesssim(1-|a|^2)^{(2+\alpha+kp)q/p}$ *for all* $a\in\mathbb{D}$.

Then its little oh version can be formulated as below.

THEOREM 2.6 (pp. 55–56 in [20]). *Let* $p,q,\alpha,\mu$ *and* $k$ *be defined as in Theorem 2.5. Then the following four conditions are equivalent.*

(a) *If* $\{f_j\}$ *is a bounded sequence in* $\mathcal{A}^p_\alpha$ *and* $f_j(z)\to 0$ *for every* $z\in\mathbb{D}$, *then*

$$\lim_{j\to\infty}\int_{\mathbb{D}}\left|f_j^{(k)}(z)\right|^q\,\mathrm{d}\mu(z)=0.$$

(b) *For every (or some) $s>0$, we have*

$$\lim_{|z|\to 1^-}\int_{\mathbb{D}}\frac{(1-|z|^2)^s\,\mathrm{d}\mu(w)}{|1-\bar{w}z|^{s+(2+\alpha+kp)q/p}}=0.$$

(c) *The limits*

$$\lim_{r\to 0^+}\frac{\mu(Q_r(\zeta))}{r^{(2+\alpha+kp)q/p}}=0$$

*holds uniformly for $\zeta\in\partial\mathbb{D}$.*

(d) *For every (or some) $R>0$, we have*

$$\lim_{|a|\to 1^-}\frac{\mu(D(a,R))}{(1-|a|^2)^{(2+\alpha+kp)q/p}}=0.$$

## 3. Intertwining relations for integral-type operators in $\mathscr{B}(\mathcal{A}^p)$

PROPOSITION 3.1. *Let $\varphi$ and $\psi$ be analytic self maps of $\mathbb{D}$, and $g,h\in H(\mathbb{D})$. As four operators $C_\varphi, C_\psi, J_g, J_h$ acting on $\mathcal{A}^p$, we have $C_\varphi J_g=J_hC_\psi$ if and only if*

(a) *either $g$ is a constant, or $\varphi(0)=0$; and*
(b) *$\varphi=\psi$; and*
(c) *$h=g\circ\varphi+C$, where $C$ is an arbitrary constant.*

*Proof.* $C_\varphi J_g=J_hC_\psi$ if and only if $\|C_\varphi J_g-J_hC_\psi\|=0$, that is

$$\sup_f\frac{\|(C_\varphi J_g-J_hC_\psi)f\|_p^p}{\|f\|_p}=0.$$

By Lemma 2.1, we have

$$\begin{aligned}&\big\|(C_\varphi J_g-J_hC_\psi)f\big\|_p^p\\ &\quad\approx\left|\int_0^{\varphi(0)}g'(\zeta)f(\zeta)\,\mathrm{d}\zeta\right|^p\\ &\qquad+\int_{\mathbb{D}}\big|f\big(\varphi(z)\big)g'\big(\varphi(z)\big)\varphi'(z)-f\big(\psi(z)\big)h'(z)\big|^p\big(1-|z|^2\big)^p\,\mathrm{d}A(z).\end{aligned}$$

Hence, for each $f\in\mathcal{A}^p$,

(1) $\int_0^{\varphi(0)}g'(\zeta)f(\zeta)\,\mathrm{d}\zeta=0$; and
(2) $|f(\varphi(z))g'(\varphi(z))\varphi'(z)-f(\psi(z))h'(z)|(1-|z|^2)=0$ a.e. on $\mathbb{D}$.

Item (1) holds trivially when $\varphi(0)=0$. Assuming that $\varphi(0)\neq 0$ and $g'(z)=\sum_{j=0}^\infty a_jz^j$, we put $f(z)=z^j$ $(j=0,1,2,\dots)$ in item (1), then the following equations can be obtained:

$$
\begin{aligned}
a_0 + \frac{1}{2}a_1\varphi(0) + \frac{1}{3}a_2\varphi(0)^2 + \cdots = 0;\\
\frac{1}{2}a_0 + \frac{1}{3}a_1\varphi(0) + \frac{1}{4}a_2\varphi(0)^2 + \cdots = 0;\\
\frac{1}{3}a_0 + \frac{1}{4}a_1\varphi(0) + \frac{1}{5}a_2\varphi(0)^2 + \cdots = 0;\\
\cdots
\end{aligned}
$$

We write those equations in matrix form:

$$
A\big(a_0, a_1\varphi(0), a_2\varphi(0)^2, \ldots\big)^T = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1\varphi(0) \\ a_2\varphi(0)^2 \\ \cdots \end{pmatrix} = 0.
$$

According to discussion from p. 306 to p. 307 in [5], matrix $A$ is formally one-to-one. Hence, $a_j = 0$ for all $j = 0, 1, 2, \ldots$, that is $g \equiv C$.

By item (2),

$$
\big|f\big(\varphi(z)\big)g'\big(\varphi(z)\big)\varphi'(z) - f\big(\psi(z)\big)h'(z)\big| = 0 \quad \text{a.e. on } \delta\mathbb{D}
$$

for some $0 < \delta < 1$ and each $f$ in $\mathcal{A}^p$. That implies

$$
f\big(\varphi(z)\big)g'\big(\varphi(z)\big)\varphi'(z) = f\big(\psi(z)\big)h'(z)
$$

for each $f \in \mathcal{A}^p$ from Uniqueness theorem. By choosing $f(z) = z$ and $f(z) = z^2$, we obtain (c) and (b).

Conversely, if (a), (b) and (c) hold, the equality $C_\varphi J_g = J_h C_\psi$ follows by a direct computation. □

Similarly, we have the following proposition.

PROPOSITION 3.2. *Let $\varphi$ and $\psi$ be analytic self maps of $\mathbb{D}$, and $g, h \in H(\mathbb{D})$. As four operators $C_\varphi, C_\psi, I_g, I_h$ acting on $\mathcal{A}^p$, we have $C_\varphi I_g = I_h C_\psi$ if and only if*

(a) *either $g \equiv 0$ or $\varphi(0) = 0$; and*
(b) *$\varphi = \psi$; and*
(c) *$h = g \circ \varphi$.*

Having Propositions 3.1 and 3.2, we obtain next theorem in summary.

THEOREM 3.3. *Let $\varphi$ be an analytic self map of $\mathbb{D}$, and $g, h \in H(\mathbb{D})$.*

(a) *When $g, h \in \mathcal{B}$, then $J_g \propto J_h(C_\varphi)$ if and only if*

$$
\textit{either } g \textit{ is a constant, or } \varphi(0) = 0, \textit{ and } h = g \circ \varphi + C;
$$

(b) *when $g, h \in H^\infty(\mathbb{D})$, $I_g \propto I_h(C_\varphi)$ if and only if*

$$
\textit{either } g \equiv 0 \textit{ or } \varphi(0) = 0, \textit{ and } h = g \circ \varphi.
$$

Let $f, g$ be two holomorphic functions on $\mathbb{D}$. If there is an analytic self map of $\mathbb{D}$, namely $\varphi$, such that $f = g \circ \varphi$, then it is clear that $f(\mathbb{D}) \subset g(\mathbb{D})$. Conversely, we can get a simple conclusion from Lemma 2.2 for a covering map $g$ as follows.

PROPOSITION 3.4. *Let nonconstant $g, h \in H(\mathbb{D})$. Suppose $g$ is a covering map.*

(a) *For some constant $C$, if $(h + C)(\mathbb{D}) \subset g(\mathbb{D})$ and $h(0) + C = g(0)$, then there exists a $\varphi \in S(\mathbb{D})$ such that $J_g \propto J_h$ $(C_\varphi)$;*
(b) *if $h(\mathbb{D}) \subset g(\mathbb{D})$ and $h(0) = g(0)$, then there exists a $\varphi \in S(\mathbb{D})$ such that $I_g \propto I_h$ $(C_\varphi)$.*

*Proof.* For (a), $g$ is a covering map, and the pair $(\mathbb{D}; g)$ is an analytic covering space of $g(\mathbb{D})$. Since

$$h + C : \mathbb{D} \to (h + C)(\mathbb{D}) \subset g(\mathbb{D})$$

is a surjective map with $(h + C)(0) = g(0)$, Lemma 2.2 shows that there is a $\varphi \in S(\mathbb{D})$ such that $h + C = g \circ \varphi$ and $\varphi(0) = 0$. Then item (a) follows immediately from Theorem 3.3.

The proof of (b) is similar. □

Next, we look into some simple applications of intertwining relations for two Volterra operators with a composition operator as their intertwining operator.

When $p = 2$, the classic Bergman space $\mathcal{A}^2$ is a Hilbert space. Its evaluation function at $w$ is denoted by

$$K_w(z) = \frac{1}{(1 - \bar{w}z)^2},$$

and $\langle f, K_w \rangle = f(w)$ for every $f \in \mathcal{A}^2$ where $\langle \cdot, \cdot \rangle$ stands for the inner product of $\mathcal{A}^2$. The evaluation of the derivative of functions in $\mathcal{A}^2$ at $w$ is denoted by $K_w^{(1)}$, and $\langle f, K_w^{(1)} \rangle = f'(w)$. (Interested reader can see more details on those kernel functions of $\mathcal{A}^2$ from p. 16 to p. 20 in [7].)

First, we use intertwining relations to observe some simple properties of extended eigenvalues and extended eigenoperators of $V_g$:

Let $\lambda \in \mathbb{C}$ and $g \in \mathcal{B}$.

(1) 1 is always an extended eigenvalues of $V_g$;
(2) If $|\lambda| < 1$, then $\lambda$ cannot be an extended eigenvalue of $J_g$ with some composition operator as its extended eigenoperator;
(3) If $g \in H^\infty$ and $|\lambda| < 1$, then $\lambda$ can not be an extended eigenvalue of $I_g$ with some composition operator as its extended eigenoperator;
(4) If $g \in H^\infty$ with non symmetric image then $\lambda \neq 1$ cannot be an extended eigenvalue of $I_g$ with some composition operator as its extended eigenoperator.

Another application of the intertwining relation for Volterra operators occurs in the study of the eigenvalues of some composition operators on a subspace of the Bergman space $\mathcal{A}^2$. In Proposition 3.1, the constant term in the equivalent norm leads to $\varphi(0)=0$, which seems too strict. And if we consider a subspace of $\mathcal{A}^2$ denoted by $\mathcal{A}^2/\mathbb{C}$ (whose elements are in $\mathcal{A}^2$ without distinguishing a constant), the item $\|(1-|z|^2)^n f^{(n)}(z)\|_{L^2(\mathrm{d}A)}$ for $f\in\mathcal{A}^2/\mathbb{C}$ becomes a norm for that subspace. On this subspace, the characterizations in Proposition 3.1 should be reduced to item (b) and (c). Also according to Littlewood subordination principle, every composition operator is bounded on $\mathcal{A}^2/\mathbb{C}$. To study the eigenvalues for some of those composition operators, we may look into the adjoint of Volterra operator $V_g$ first.

Proposition 3.5. *Let $K_w$ and $K_w^{(1)}$ be defined as above, $V_g^*$ be the adjoint of $V_g$.*

(1) *If $g\in\mathcal{B}$, then $J_g^*$ is bounded on $\mathcal{A}^2$, and*

$$J_g^* K_w = \overline{\int_0^w g'(\zeta)\overline{K_\zeta(z)}\,\mathrm{d}\zeta} \quad (\forall z\in\mathbb{D}).$$

(2) *If $g\in H^\infty$, then $I_g^*$ is bounded on $\mathcal{A}^2$, and*

$$I_g^* K_w = \overline{\int_0^w g(\zeta)\overline{K_\zeta^{(1)}(z)}\,\mathrm{d}\zeta} \quad (\forall z\in\mathbb{D}).$$

*In particular, $V_g^* 1 = V_g^* K_0 = 0$.*

*Proof.* The boundedness of $J_g^*$ and $I_g^*$ can be obtained by Lemma 2.3. For each $f\in\mathcal{A}^2$ and $z\in\mathbb{D}$,

$$\begin{aligned}
\big\langle f(z), J_g^* K_w(z)\big\rangle &= \langle J_g f, K_w\rangle \\
&= \left\langle \int_0^z f(\zeta)g'(\zeta)\,\mathrm{d}\zeta, K_w(z)\right\rangle \\
&= \int_0^w f(\zeta)g'(\zeta)\,\mathrm{d}\zeta \\
&= \int_0^w \big\langle f(z), K_\zeta(z)\big\rangle g'(\zeta)\,\mathrm{d}\zeta \\
&= \int_0^w g'(\zeta)\int_{\mathbb{D}} f(z)\overline{K_\zeta(z)}\,\mathrm{d}A(z)\,\mathrm{d}\zeta \\
&= \int_{\mathbb{D}} f(z)\int_0^w g'(\zeta)\overline{K_\zeta(z)}\,\mathrm{d}\zeta\,\mathrm{d}A(z) \\
&= \left\langle f(z), \overline{\int_0^w g'(\zeta)\overline{K_\zeta(z)}\,\mathrm{d}\zeta}\right\rangle,
\end{aligned}$$

where Fubini's theorem is applied in the next-to-last equation.

Similarly, we can compute $I_g^* K_w$ as the following:

$$\begin{aligned}\langle f(z), I_g^* K_w(z)\rangle &= \langle I_g f(z), K_w(z)\rangle = I_g f(w) \\ &= \int_0^w g(\zeta)\langle f(z), K_\zeta^{(1)}(z)\rangle \,\mathrm{d}\zeta \\ &= \left\langle f(z), \overline{\int_0^w g(\zeta)\overline{K_\zeta^{(1)}(z)}\,\mathrm{d}\zeta}\right\rangle.\end{aligned}$$

Since $f \in \mathcal{A}^2$ is arbitrary, we have obtained the desired conclusions. □

THEOREM 3.6. *Assume that $\varphi \in S(\mathbb{D})$ has an nonzero fixed point $\omega \in \mathbb{D}$. If there exist a non constant $g \in \mathcal{B}$ and a nonzero complex number $\lambda$ such that at least one of the following two holds*:

(a) $K_\omega \notin \ker(J_g^*)$ *and*

$$\lambda g + (1-\lambda) g(\omega) = g \circ \varphi; \tag{3.1}$$

(b) $K_\omega^{(1)} \notin \ker(I_g^*)$ *and equation* (3.1) *with* $g(0) = 0$.

*Then $1/\bar{\lambda}$ is an eigenvalue of $C_\varphi^*$, which acts as the adjoint operator of $C_\varphi$ on $\mathcal{A}^2/\mathbb{C}$.*

From the theorem above, we firstly note that $|\lambda| \le 1$ since the range of $\lambda g + c$ should be in the range of $g$. Second, the term "$(1-\lambda)g(\omega)$" in (3.1) is just for the purpose of making $\lambda$ independent with respect to the value of $g(\omega)$. For example, $\lambda$ dose not need to be 1 when $g(\omega) = 0$, and we note that, in that case, all the extended eigenvalues of $C_\varphi$ with some $J_g$'s as their extended eigenoperators are the eigenvalues of $C_\varphi$.

*Proof of Theorem* 3.6. If condition (a) holds, we have $J_g \propto J_{\lambda g}$ $(C_\varphi)$ from equation (3.1). Hence

$$\begin{aligned}&(C_\varphi J_g)^* = (J_{\lambda g} C_\varphi)^* \\ &\quad\Rightarrow\quad J_g^* C_\varphi^* = C_\varphi^* J_{\lambda g}^* \\ &\quad\Rightarrow\quad J_g^* C_\varphi^* K_\omega(z) = C_\varphi^* J_{\lambda g}^* K_\omega(z) \\ &\quad\Rightarrow\quad J_g^* K_{\varphi(\omega)}(z) = C_\varphi^* \overline{\int_0^\omega \lambda g'(\zeta)\overline{K_\zeta(z)}\,\mathrm{d}\zeta} \\ &\quad\Rightarrow\quad \bar{\lambda} C_\varphi^* \overline{\int_0^\omega g'(\zeta)\overline{K_\zeta(z)}\,\mathrm{d}\zeta} = J_g^* K_\omega = \overline{\int_0^\omega g'(\zeta)\overline{K_\zeta(z)}\,\mathrm{d}\zeta}.\end{aligned}$$

Since $K_\omega \notin \operatorname{ran}(J_g)^\perp = \ker(J_g^*)$, we have $J_g^* K_\omega \neq 0$, and subsequently $1/\bar{\lambda}$ is an eigenvalue of $C_\varphi^*$ with $J_g^* K_\omega$ as its eigenvector.

Similarly, if condition (b) holds, we obtain $I_g \propto I_{\lambda g}$ $(C_\varphi)$ this time. Then the same computations give

$$\bar{\lambda} C_\varphi^* \overline{\int_0^w g(\zeta)\overline{K_\zeta^{(1)}(z)}\,\mathrm{d}\zeta} = \overline{\int_0^w g(\zeta)\overline{K_\zeta^{(1)}(z)}\,\mathrm{d}\zeta}.$$

By applying $K_\omega^{(1)} \notin \operatorname{ran}(I_g)^\perp$ then, the proof is complete. □

REMARK 3.7. On the Bergman space $\mathcal{A}^2$, the former theorem dose not work, since each analytic self map of the unit disk shall have only one inner fixed point (Schwarz–Pick theorem, see Theorem 2.39 in [7]), and $V_g^* K_0 = 0$ by Proposition 3.5.

## 4. Intertwining relations for integral-type operators in $\mathscr{Q}(\mathcal{A}^p)$

In this section, we apply the equivalent norm (in Lemma 2.1) without concerning the constant item, since the constant has no effect on the boundedness and compactness of Volterra operators. Our results of "compact" intertwining relations for $J_g$ can be seen as simple corollaries of the main theorem in [8]. We modify the notation in [8] slightly, and define a integral operator

$$\mathrm{I}_{\varphi,\alpha,\beta}^{p,q}(u)(a) = \int_{\mathbb{D}} \left( \frac{1-|a|^2}{|1-\bar{a}\varphi(w)|^2} \right)^{(2+\alpha)q/p} \big|u(w)\big|^q \,\mathrm{d}A_\beta(w).$$

THEOREM 4.1 (Theorem 1 in [8]). *Let $1 < p \le q < \infty$, $\alpha, \beta > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the weighted composition operator $uC_\varphi$ is bounded from $\mathcal{A}_\alpha^p$ into $\mathcal{A}_\beta^q$ if and only if*

$$\sup_{a\in\mathbb{D}} \mathrm{I}_{\varphi,\alpha,\beta}^{p,q}(u)(a) < \infty. \tag{4.1}$$

THEOREM 4.2 (Theorem 2 in [8]). *Let $1 < p \le q < \infty$, $\alpha, \beta > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Assume that $uC_\varphi$ is bounded from $\mathcal{A}_\alpha^p$ into $\mathcal{A}_\beta^q$. Then*

$$\|uC_\varphi\|_e^q \approx \limsup_{|a|\to 1} \mathrm{I}_{\varphi,\alpha,\beta}^{p,q}(u)(a).$$

Now we are ready to discuss the "compact" intertwining relations for Volterra type operators.

COROLLARY 4.3. *Let $0 < p \le q < \infty$, and $\alpha, \beta > -1$. Assume that $\varphi \in S(\mathbb{D})$, both $g$ and $h$ are in $H(\mathbb{D})$. Then:*

(i) *$J[\varphi; g, h]$ is a bounded operator from $\mathcal{A}_\alpha^p$ into $\mathcal{A}_\beta^q$ if and only if*

$$\sup_{a\in\mathbb{D}} \mathrm{I}_{\varphi,\alpha,q+\beta}^{p,q}\big((g\circ\varphi - h)'\big)(a) < \infty.$$

(ii) *$J[\varphi; g, h]$ is a compact operator on $\mathcal{A}_\alpha^p$ into $\mathcal{A}_\beta^q$ if and only if $J[\varphi; g, h]$ is bounded and*

$$\lim_{|a|\to 1} \mathrm{I}_{\varphi,\alpha,q+\beta}^{p,q}\big((g\circ\varphi - h)'\big)(a) = 0.$$

*Proof.* We note that

$$
\begin{aligned}
(C_\varphi J_g - J_h C_\varphi) f(z) &= C_\varphi\left(\int_0^z f(w) g'(w)\,\mathrm{d}w\right) - J_h (f\circ\varphi)(z)\\
&= \int_0^{\varphi(z)} f(w) g'(w)\,\mathrm{d}w - \int_0^z f\big(\varphi(w)\big) h'(w)\,\mathrm{d}w,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\big\|J[\varphi; g,h] f\big\|^q_{A^q_\beta}\\
&\quad\approx \int_{\mathbb{D}} \big|f\big(\varphi(z)\big) g'\big(\varphi(z)\big)\varphi'(z) - f\big(\varphi(z)\big) h'(z)\big|^q \big(1-|z|^2\big)^q\,\mathrm{d}A_\beta(z)\\
&\quad\approx \int_{\mathbb{D}} \big|(g\circ\varphi - h)'(z)\big|^q \big|f\big(\varphi(z)\big)\big|^q\,\mathrm{d}A_{q+\beta}(z)\\
&\quad= \big\|(g\circ\varphi - h)' C_\varphi(f)\big\|^q_{A^q_{q+\beta}}.
\end{aligned}
$$

Hence, by Theorems 4.1 and 4.2, we have (i) and (ii). □

THEOREM 4.4. *Let $\varphi \in S(\mathbb{D})$, and $g, h \in \mathcal{B}$. Assume that both Volterra operator $J_g, J_h$ and composition operator $C_\varphi$ are bounded linear operators on the Bergman space $\mathcal{A}^p$. Then $J_g \propto_K J_h$ $(C_\varphi)$ if and only if*

$$
\lim_{|a|\to 1} \mathrm{I}^{p,p}_{\varphi,0,p}\big((g\circ\varphi - h)'\big)(a) = 0.
$$

*Proof.* See the definition of "intertwining relation" and Corollary 4.3. □

Next, we are going to discuss the intertwining relations for $I_g$ and $I_h$ with some composition operator $C_\varphi$ intertwining. For $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$, a *weighted differential composition operator* is defined by

$$
uC'_\varphi f := u\cdot (f\circ\varphi)',
$$

where $f$ is in $H(\mathbb{D})$. We first give a lemma on the boundedness and compactness of this operator from $\mathcal{A}^p_\alpha$ to $\mathcal{A}^q_\beta$. We define another integral operator by

$$
\mathrm{II}^{p,q}_{\varphi,\alpha,\beta}(u)(a) := \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(w)|^2}\right)^{(2+\alpha+p)q/p} \big|u(z)\varphi'(z)\big|^q\,\mathrm{d}A_\beta(z).
$$

LEMMA 4.5. *Let $u$ be in $H(\mathbb{D})$ and $\varphi$ in $S(\mathbb{D})$. For $0<p\le q<\infty$ and $\alpha,\beta > -1$, the weighted differential operators $uC'_\varphi$ is defined as above from $\mathcal{A}^p_\alpha$ to $\mathcal{A}^q_\beta$. Then:*

(i) *$uC'_\varphi$ is bounded if and only if*

$$
\sup_{a\in\mathbb{D}} \mathrm{II}^{p,q}_{\varphi,\alpha,\beta}(u)(a) < \infty;
$$

(ii) *$uC'_\varphi$ is compact if and only if $uC'_\varphi$ is bounded and*

$$\lim_{|a|\to 1} \mathrm{II}^{p,q}_{\varphi,\alpha,\beta}(u)(a) = 0.$$

*Proof.* For (i): by definition, $uC'_\varphi$ is bounded from $\mathcal{A}^p_\alpha$ into $\mathcal{A}^q_\beta$ if and only if for any $f \in \mathcal{A}^p_\beta$,

$$\big\|\big(uC'_\varphi\big)f\big\|^q_{\mathcal{A}^q_\beta} \lesssim \|f\|^q_{\mathcal{A}^p_\alpha}. \tag{4.2}$$

Letting $w = \varphi(z)$, we get

$$\int_{\mathbb{D}} \big|f(w)\big|^q \,\mathrm{d}\mu_{u,\varphi}(w) \lesssim \|f\|^q_{\mathcal{A}^p_\alpha},$$

where $\mu_{u,\varphi} = \nu_{u,\varphi} \circ \varphi^{-1}$ and $\mathrm{d}\nu_{u,\varphi}(z) = |u(z)\varphi'(z)|^q \,\mathrm{d}A_\beta(z)$. And equation (4.2) implies that $\mathrm{d}\mu_{u,\varphi}$ is a $(\mathcal{A}^p_\alpha, q, 1)$-Carleson measure. By Theorem 2.5, this is equivalent to

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{(1-|a|^2)^s}{|1-\bar{a}w|^{s+(2+\alpha+kp)q/p}}\right) \mathrm{d}\mu_{u,\varphi}(w) < \infty,$$

for every $s > 0$. Putting $s = (2+\alpha+kp)q/p$ and changing the variable back to $z$ we get (i).

For (ii) similarly as the proof of (i), the equivalence of (a) and (b) in Theorem 2.6 is suitable to obtain the compactness of $uC'_\varphi$ from $\mathcal{A}^p_\alpha$ into $\mathcal{A}^q_\beta$. □

By direct calculation and Lemma 4.5, the following corollary can be obtained similarly as Corollary 4.3, thus we omit the proof.

Corollary 4.6. *Let $1 < p \le q < \infty$, $\alpha, \beta > -1$. Assume that $\varphi \in S(\mathbb{D})$, both $g$ and $h$ are in $H(\mathbb{D})$. Then:*

(i) *$I[\varphi; g, h]$ is a bounded operator from $\mathcal{A}^p_\alpha$ into $\mathcal{A}^q_\beta$ if and only if*

$$\sup_{a\in\mathbb{D}} \mathrm{II}^{p,q}_{\varphi,\alpha,q+\beta}(g\circ\varphi - h)(a) < \infty.$$

(ii) *$I[\varphi; g, h]$ is a compact operator from $\mathcal{A}^p_\alpha$ into $\mathcal{A}^q_\beta$ if and only if $I[\varphi; g, h]$ is bounded and*

$$\lim_{|a|\to 1} \mathrm{II}^{p,q}_{\varphi,\alpha,q+\beta}(g\circ\varphi - h)(a) = 0.$$

Let $q = p > 0$ and $\alpha = \beta = 0$, we obtain the following theorem.

Theorem 4.7. *Let $\varphi \in S(\mathbb{D})$ and $g, h \in H^\infty$. Assume that $I_g, I_h$ and $C_\varphi$ are bounded linear operators on the Bergman space $\mathcal{A}^p$. Then $I_g \propto_K I_h$ $(C_\varphi)$ if and only if*

$$\lim_{|a|\to 1} \mathrm{II}^{p,p}_{\varphi,0,p}(g\circ\varphi - h)(a) = 0.$$

## 5. Essential commutativity for Volterra and composition operators

If $V_g \propto_K V_g\ (C_\varphi)$, we say $V_g$ and $C_\varphi$ are *essential commutative*. In this section we will discuss the essential commutativity for $V_g$ and $C_\varphi$. In fact it is an obvious consequence of Theorem 4.4 and 4.7. That is,

(A) If $g \in \mathcal{B}$, $J_g$ and $C_\varphi$ are essential commutative if and only if

$$\lim_{|a|\to 1} \mathrm{I}^{p,p}_{\varphi,0,p}\big((g\circ\varphi - g)'\big)(a) = 0;$$

(B) If $g \in H^\infty$, $I_g$ and $C_\varphi$ are essential commutative if and only if

$$\lim_{|a|\to 1} \mathrm{II}^{p,p}_{\varphi,0,p}(g\circ\varphi - g)(a) = 0.$$

Since every analytic self map of unit disk induces a composition operator on $\mathcal{A}^p$, it is natural to ask whether there is a non-constant holomorphic function $g$ on $\mathbb{D}$ such that $V_g$ is bounded and essentially commutes with every $C_\varphi$. Furthermore, can we characterize the set of all such $g$? By the way, we call this collection of $g$ the *universal set* of $V_g$, and we denote it by $\Omega_{\mathrm{co}}(V_g)$ where "co" stands for the intertwining operator—"composition operator." The existence is quite simple, just note that each compact $V_g$ on $\mathcal{A}^p$ can be made to commute essentially with every composition operator $C_\varphi$ on $\mathcal{A}^p$.

According to item (A) in this section, we can draw the following conclusion.

THEOREM 5.1. *Let $\Omega_{\mathrm{co}}(J_g)$ be the universal set of $J_g$, then*

$$\Omega_{\mathrm{co}}(J_g) = \mathcal{B}_{\mathrm{o}}.$$

To prove this, we need the following approximation. See Theorem 1 and 10 in [8].

LEMMA 5.2. *Let $u \in H(\mathbb{D})$, $1 < p \le q < \infty$, and $\alpha, \beta > -1$. Let the multiplication operator $M_u$ be bounded from $\mathcal{A}^p_\alpha$ into $A^q_\beta$. Then*

$$\begin{aligned}\|M_u\|_e &\approx \limsup_{|a|\to 1} \mathrm{I}_{\mathrm{id},\alpha,\beta}(u)(a)\\ &\approx \limsup_{|a|\to 1} \big|u(a)\big|\big(1-|a|^2\big)^{(\beta+2)/q-(\alpha+2)/p}.\end{aligned}$$

*Consequently, $M_u$ is compact from $\mathcal{A}^p_\alpha$ into $\mathcal{A}^q_\beta$ if and only if*

$$\lim_{|a|\to 1} \big|u(a)\big|\big(1-|a|^2\big)^{(\beta+2)/q-(\alpha+2)/p} = 0.$$

*Proof of Theorem 5.1.* First, we prove $\mathcal{B}_{\mathrm{o}} \subset \Omega_{\mathrm{co}}(J_g)$. If $g \in \mathcal{B}_{\mathrm{o}}$, then $J_g$ is an compact linear operator on $\mathcal{A}^p$ from Lemma 2.3. Since every composition operator $C_\varphi$ is bounded on $\mathcal{A}^p$, the operator $J[\varphi; g, g]$ is certainly compact on $\mathcal{A}^p$. Hence, $g \in \Omega_{\mathrm{co}}(J_g)$.

To prove $\Omega_{\mathrm{co}}(J_g) \subset \mathcal{B}_{\mathrm{o}}$, suppose $g \in \Omega_{\mathrm{co}}(J_g)$, then $g \in \mathcal{B}$ (by Lemma 2.3) and $J[\varphi; g, g]$ is compact for each $\varphi \in S(\mathbb{D})$. Putting $\varphi(z) = e^{\mathbf{i}\theta} z$ where $\forall\theta \in [0, 2\pi]$, by Corollary 4.3 we have

$$\begin{aligned}
(5.1)\qquad &\lim_{|a|\to 1} \int_{\mathbb{D}} \left( \frac{1-|a|^2}{|1-\bar{a}e^{\mathbf{i}\theta}z|^2} \right)^2 \left| e^{\mathbf{i}\theta} g'\left(e^{\mathbf{i}\theta}z\right) - g'(z) \right|^p \mathrm{d}A_p(z) \\
&= \lim_{|a|\to 1} \int_{\mathbb{D}} \left( \frac{1-|a|^2}{|1-\bar{a}z|^2} \right)^2 \left| e^{\mathbf{i}\theta} g'\left(e^{\mathbf{i}\theta}z\right) - g'(z) \right|^p \mathrm{d}A_p(z) \\
&\approx \mathrm{I}_{\mathrm{id},0,p}\big((g\circ\varphi - g)'\big) \\
&\approx \lim_{|a|\to 1} \left(1-|a|^2\right) \left| e^{\mathbf{i}\theta} g'\left(e^{\mathbf{i}\theta}a\right) - g'(a) \right|^{\frac{p+2}{2} - \frac{p}{2}},
\end{aligned}$$

where the last line follows from the approximation in Lemma 5.2.

It is necessary to estimate the upper bound of last formula in (5.1).

$$\begin{aligned}
&\lim_{|z|\to 1} \left(1-|z|^2\right) \left| e^{\mathbf{i}\theta} g'\left(e^{\mathbf{i}\theta}z\right) - g'(z) \right| \\
&\le \lim_{|z|\to 1} \left(1-|z|^2\right) \left| e^{\mathbf{i}\theta} g'\left(e^{\mathbf{i}\theta}z\right) \right| + \lim_{|z|\to 1} \left(1-|z|^2\right) \left| g'(z) \right| \\
&= \lim_{|e^{\mathbf{i}\theta}z|\to 1} \left(1-\left|e^{\mathbf{i}\theta}z\right|^2\right) \left| g'\left(e^{\mathbf{i}\theta}z\right) \right| + \lim_{|z|\to 1} \left(1-|z|^2\right) \left| g'(z) \right| \\
&\le 2\|g\|_{\mathcal{B}} < \infty.
\end{aligned}$$

Thus the formula in (5.1) is bounded independent of $\theta$.

We write $g(z) = \sum_{n=0}^{\infty} a_n z^n$, and integrate the left-hand side of (5.1) with respect to $\theta$ from 0 to $2\pi$

$$\begin{aligned}
0 &= \int_0^{2\pi} \lim_{|z|\to 1} \left(1-|z|^2\right) \left| e^{\mathbf{i}\theta} g'\left(e^{\mathbf{i}\theta}z\right) - g'(z) \right| \mathrm{d}\theta \\
&= \lim_{|z|\to 1} \int_0^{2\pi} \left(1-|z|^2\right) \left| e^{\mathbf{i}\theta} g'\left(e^{\mathbf{i}\theta}z\right) - g'(z) \right| \mathrm{d}\theta \\
&= \lim_{|z|\to 1} \int_0^{2\pi} \left(1-|z|^2\right) \left| \sum_{n=1}^{\infty} n a_n z^{n-1} \left(e^{\mathbf{i}n\theta} - 1\right) \right| \mathrm{d}\theta \\
&\ge \lim_{|z|\to 1} \left(1-|z|^2\right) \left| \sum_{n=1}^{\infty} n a_n z^{n-1} \int_0^{2\pi} \left(e^{\mathbf{i}n\theta} - 1\right) \mathrm{d}\theta \right| \\
&= 2\pi \lim_{|z|\to 1} \left(1-|z|^2\right) \left| g'(z) \right|,
\end{aligned}$$

where Dominant Convergent theorem is applied in second line. Thus $g \in \mathcal{B}_{\mathrm{o}}$. $\square$

THEOREM 5.3. *Let $\Omega_{\mathrm{co}}(I_g)$ be the universal set of $I_g$, then*

$$\Omega_{\mathrm{co}}(I_g) = \mathbb{C}.$$

*Proof.* It is obvious that $\mathbb{C} \subset \Omega_{\rm co}(I_g)$, and we just prove the contra-containment. If $g \in \Omega_{\rm co}(I_g)$, we consider the compactness of each $I[e^{\mathbf{i}\theta}z; g, g]$ on $\mathcal{A}^p$ where $\forall\theta \in [0, 2\pi]$. Following item (B), we have

$$\begin{aligned}
0 &= \lim_{|a|\to 1} \mathrm{II}^{p,p}_{e^{\mathbf{i}\theta}z;0,p}(g\circ\varphi - g)(a)\\
&= \lim_{|a|\to 1} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}e^{\mathbf{i}\theta}z|^2}\right)^{2+p} \big|g\big(e^{\mathbf{i}\theta}z\big) - g(z)\big|^p \,\mathrm{d}A_p(z)\\
&= \lim_{|a|\to 1} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{2+p} \big|g\big(e^{\mathbf{i}\theta}z\big) - g(z)\big|^p \,\mathrm{d}A_p(z)\\
&\approx \lim_{|a|\to 1} \big(1-|a|^2\big)^0 \big|g\big(e^{\mathbf{i}\theta}a\big) - g(a)\big|,
\end{aligned}$$

where the last approximation follows from the method of Theorem 10 in [8]. Uniqueness theorem and $0 \approx \lim_{|a|\to 1} |g(e^{\mathbf{i}\theta}a) - g(a)|$ give that $g \equiv$ *constant*. This completes the proof. □

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Ce-Zhong Tong, Department of Mathematics, Hebei University of technology, Tianjin 300401, P.R. China

*E-mail address*: cezhongtong@hotmail.com

Ze-Hua Zhou, Department of Mathematics, Tianjin University, Tianjin 300072, P.R. China

*E-mail address*: zehuazhoumath@aliyun.com; zhzhou@tju.edu.cn