

## HELICOIDAL MINIMAL SURFACES IN $\mathbf{R}^3$

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ABSTRACT. In 1841, Delaunay (*J. Math. Pures Appl.* **6** (1841) 309–320) showed that the intersection of a constant mean curvature surface of revolution in  $\mathbf{R}^3$  and a plane  $\Pi$  that contains its axis of symmetry  $l$  can be described as the trace of the focus of a conic when this conic rolls without slipping in the plane  $\Pi$  along the line  $l$ . In the same way surfaces of revolution are foliated by circles perpendicular to the axis of symmetry, helicoidal surfaces are foliated by helices, all of them symmetric to a line  $l$ . Roughly speaking, helicoidal surfaces are surfaces invariant under a screw-motion. In this paper, we show that the intersection of a helicoidal minimal surface  $S$  in  $\mathbf{R}^3$  and a plane  $\pi$  perpendicular to line  $l$ —where  $l$  is the axis of symmetry of the screw motion—is characterized by the property that if we roll the curve  $C = S \cap \pi$  on a flat treadmill located on another plane  $\Pi$ , then, the point  $P = \pi \cap l$  describes a hyperbola on the plane  $\Pi$  centered at the fixed point of contact of the treadmill with the curve  $C$ . This way of generating a curve using another curve, similar to the well known “Roulette,” was introduced by the author in (*Pacific J. Math.* **258** (2012) 459–485) and it was called the “TreadmillSled.” We will also prove several properties of the TreadmillSled, in particular we will classify all curves that are the TreadmillSled of another curve.

### 1. Introduction

The surface of revolution generated by the regular curve  $\alpha = (y(s), z(s))$  with  $z(s) \neq 0$  is given by

$$\phi(s, t) = (z(s) \sin(t), y(s), z(s) \cos(t)).$$

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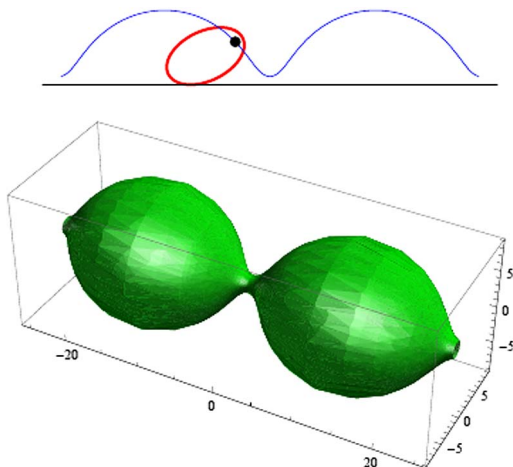


FIGURE 1. An unduloid and the construction of its profile curve.

The curve  $\alpha$  is called the profile curve. In [2], Delaunay proved that a surface of revolution has constant mean curvature, CMC, if and only if, it is a sphere, a cylinder or if its profile curve lies in the trace made by the focus of a conic, when this conic rolls along the  $y$ -axis. When the conic used is a parabola, the surface is minimal and it is called catenoid; if the conic used is a hyperbola, the surface is called a nodoid and if the conic used is an ellipse, the surface is called an unduloid. Since an ellipse has two foci, the trace of each one of them generates an unduloid. It is not difficult to see that these two unduloids are essentially the same, one is a translation of the other. Figure 1 shows how the profile curve of an unduloid is constructed using an ellipse.

When we roll the parabola its focus traces a curve of infinite length. Figure 2 shows a catenoid and its profile curve.

When we roll a set of hyperbolas, their foci trace two curves of finite length, each curve generates a CMC surface. It can be proven that we can translate one of these surfaces to obtain a smooth connected surface with constant mean curvature. If we repeat this connected piece over and over, we obtain a complete CMC surface. In Figure 3 we show, the trace of the foci of the two-branch hyperbola when one of its branches is rolled on a line, the two-piece CMC surface and the connected piece made by gluing the translation of one of the connected components of the initial two-piece surface to the other connected component.

In order to compare the main result in this paper with Delaunay's result, let us think of the operator Roll that takes regular curves in  $\mathbf{R}^2$  into curves in  $\mathbf{R}^2$  given in the following way: For a regular curve  $\alpha : [a, b] \rightarrow \mathbf{R}^2$ , let  $s(t)$

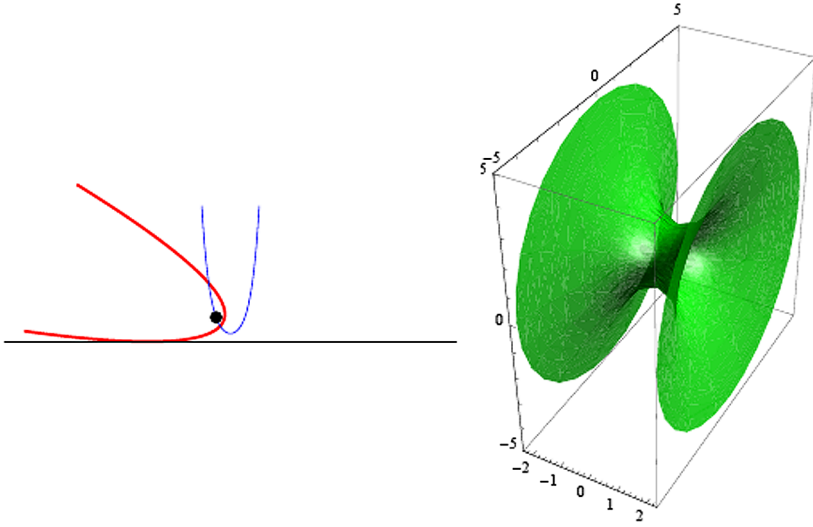


FIGURE 2. A catenoid and the construction of its profile curve.

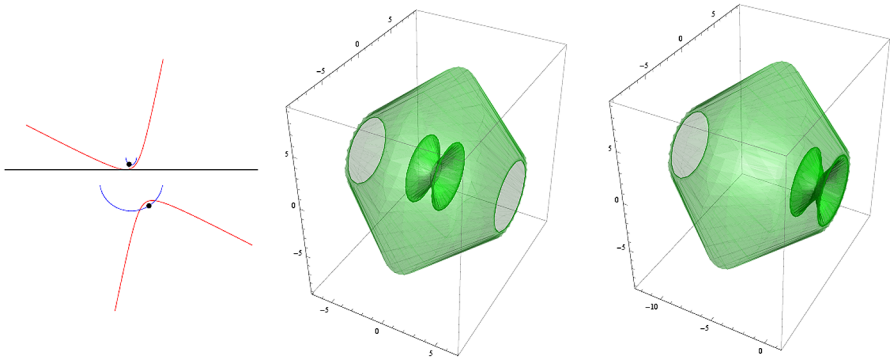


FIGURE 3. A nodoid and the construction of its profile curve.

denote the length of the curve from  $\alpha(a)$  to  $\alpha(t)$  and let us define

$$\text{Roll}(\alpha) = \left\{ T_t \begin{pmatrix} 0 \\ 0 \end{pmatrix} : T_t \text{ is an oriented isometry in } \mathbf{R}^2, T_t(\alpha(t)) = \begin{pmatrix} s(t) \\ 0 \end{pmatrix} \right. \\ \left. \text{and } dT_t \begin{pmatrix} \alpha'(t) \\ |\alpha'(t)| \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Notice that  $\text{Roll}(\alpha)$  is the “Roulette” of the curve  $\alpha$  when the rolling occurs over the  $x$ -axis and the tracing point in the plane that contains  $\alpha$  is the origin.

With this operator, Delaunay's theorem implies that if  $\alpha : [0, l] \rightarrow \mathbf{R}^2$  is a piece of a conic with focus at the origin, then  $\text{Roll}(\alpha)$  is the profile curve of a surface of revolution with constant mean curvature. Notice how the origin of the curve  $\alpha$  plays an important role in the definition of  $\text{Roll}(\alpha)$ .

The helicoidal surface generated by the regular curve  $\alpha = (x(s), z(s))$  is given by

$$\phi(s, t) = (x(s) \cos(wt) + z(s) \sin(wt), t, -x(s) \sin(wt) + z(s) \cos(wt)),$$

where  $w > 0$  is fixed. The curve  $\alpha$  is called the profile curve. Before explaining our interpretation for the profile curve of helicoidal minimal surface we need to explain the notion of "TreadmillSled" introduced by the author in [7]. Think of the operator TSS that takes regular curves in  $\mathbf{R}^2$  into curves in  $\mathbf{R}^2$  given in the following way: for a regular curve  $\alpha : [a, b] \rightarrow \mathbf{R}^2$ , let us define

$$\text{TSS}(\alpha) = \left\{ T_t \begin{pmatrix} 0 \\ 0 \end{pmatrix} : T_t \text{ is an oriented isometry in } \mathbf{R}^2, T_t(\alpha(t)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right. \\ \left. \text{and } dT_t \begin{pmatrix} \alpha'(t) \\ |\alpha'(t)| \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

The letters TSS stand for TreadmillSled Set. Notice that the only difference in the geometrical interpretation for  $\text{Roll}(\alpha)$  and  $\text{TSS}(\alpha)$  is that  $\text{Roll}(\alpha)$  is the curve that we obtain by rolling the curve  $\alpha$  *without slipping* along a line, and  $\text{TSS}(\alpha)$  is the curve that we obtain by rolling the curve  $\alpha$  *with full slipping* so that at every time the point of  $\alpha$  that is making contact with the line is not moving forward but staying in the same spot. Therefore, the curve  $\alpha$  will not look like rolling anymore but it will look like moving on a treadmill. Notice how the origin of curve  $\alpha$  plays an important role in the definition of  $\text{TSS}(\alpha)$ . Figure 4 shows the TreadmillSled of the graph of a polynomial of degree 3. In this particular example, the graph of the polynomial does not contain the origin. We can easily see that,  $\alpha$  contains the origin if and only if  $\text{TSS}(\alpha)$  contains the origin. We will use  $x$  and  $z$  for the coordinates of the plane that

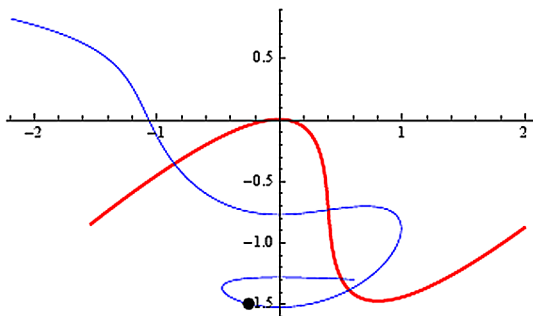


FIGURE 4. The TreadmillSled of the graph of a cubic polynomial.

contains the curve  $\alpha(s)$  and  $u$  and  $v$  for the coordinates of the plane that contains  $\text{TTS}(\alpha)$ .

We are ready to state our main theorem:

**THEOREM 1.1.** *A complete helicoidal surface*

$$\phi(s, t) = (x(s) \cos(wt) + z(s) \sin(wt), t, -x(s) \sin(wt) + z(s) \cos(wt))$$

*is minimal if and only if the TreadmillSled of its profile curve either is the  $u$ -axis and  $\phi$  is a helicoid or it is one of the branches of the hyperbola  $\frac{v^2}{M^2} - w^2u^2 = 1$  for some nonzero  $M$ .*

Besides a detailed proof of the theorem above, in this paper we will show several properties for the TreadmillSled Operator. For a motivation of the study of this operator, we refer to the following papers [3], [5], [6], [7].

As more applications of the TreadmillSled, in [7], the author showed that a helicoidal surface has zero Gauss curvature if and only if the TreadmillSled of its profile curve lies in a vertical semi-line contained in the upper or lower plane (see Figure 5).

Also in the same paper the author showed that a helicoidal surface has constant mean curvature 1 if and only if the TreadmillSled of its profile satisfies the following equation

$$(1.1) \quad u^2 + v^2 - \frac{v}{\sqrt{1 + w^2u^2}} = M \quad \text{for some } M > -\frac{1}{4}.$$

Also, Khuns and Palmer in [6] found a dynamical interpretation for helicoidal surfaces with constant anisotropic mean curvature.

Besides the nice dynamical interpretation of the TreadmillSled, this operator essentially represents a change of coordinates. Actually, one of the main

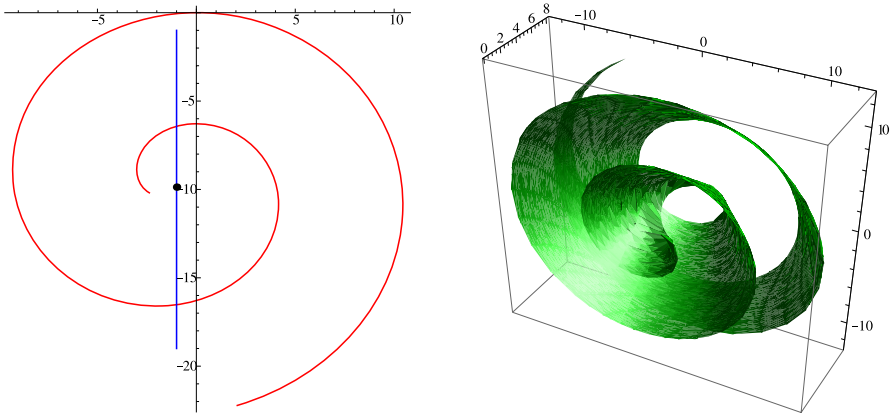


FIGURE 5. The TreadmillSled of the profile curve of a flat helicoidal surface lies in a vertical semiline.

aspects in the papers [6] and [7] is the simplification of an ordinary differential equation, ODE, after changing to “TreadmillSled coordinates,” that is, after considering the ODE for the  $\text{TS}(\alpha)$  instead of the ODE for  $\alpha$ .

For purposes of a better understanding, we will find an explicit parametrization for the TreadmillSled of a curve and we will be referring to this parametrization of the TreadmillSled as just TS; in this way, TS becomes an operator that takes a parametrized regular curve into a parametrized curve. We will show that this operator acts like the derivative operator for functions. For example,

- Given  $\alpha : [a, b] \rightarrow \mathbf{R}^2$ ,  $\text{TS}(\alpha)$  is an expression of  $\alpha$  and  $\alpha'$ .
- If  $\text{TS}(\alpha) = \text{TS}(\beta)$ , then  $\alpha$  and  $\beta$  differ by a “constant.” This time the constant does not represent a translation on the graph like in the case of the derivative operator but it represents an oriented rotation that fixes the origin. More precisely, identifying  $\mathbf{R}^2$  with the complex numbers, if  $\text{TS}(\alpha) = \text{TS}(\beta)$  then  $\beta = e^{ic}\alpha$  for some constant  $c$ .
- When  $\gamma$  is in the image of the operator TS, there is a formula for  $\text{TS}^{-1}(\gamma)$  that depends on  $\gamma$ ,  $\gamma'$  and only one antiderivative. The ambiguity of this antiderivative is responsible for the existence of the whole 1 parametric family of curves with the same TreadmillSled.
- If we change the orientation of  $\alpha$ , that is, if we consider the curve  $\beta(t) = \alpha(-t)$ , then  $\text{TS}(\beta)(t) = -\text{TS}(\alpha)(-t)$ .

If we look at Figure 4, we notice in this example that, the curve that is a TreadmillSled have the property that its velocity vector is horizontal where the curve intercepts the  $v$  axis. This is not a coincidence; actually we will show that a curve  $\gamma(t) = (u(t), v(t))$  is the TreadmillSled of a regular curve  $\alpha$  if and only if

- $v'(t) = -f(t)u(t)$  for some continuous function  $f$  and
- $v(t)f(t) - u'(t)$  is a positive function.

From the first property, we see that if  $u(t_0) = 0$ , then  $v'(t_0) = 0$  and therefore the velocity vector  $\gamma'(t_0)$  is horizontal on points along the  $v$ -axis. The second property is the reason why a whole vertical line cannot be the TreadmillSled of a regular curve. For a vertical line,  $u'(t)$  always vanishes and therefore when the line touches the  $u$ -axis, the function  $v(t)f(t) - u'(t)$  vanishes making the second property fail at this point. Also notice that if the vertical semi-line is contained in the  $v$ -axis, then by the relation  $v'(t) = -f(t)u(t)$ , we must have that  $v'(t)$  vanishes and therefore  $\gamma$  reduces to just a point. It is easy to see that the TreadmillSled of a circle centered at the origin is just a point in the  $v$ -axis. Notice that if the profile curve of a helicoidal surface is a circle centered at the origin then the surface is a cylinder.

## 2. The $\phi$ -TreadmillSled of a curve

Given a curve  $\alpha$ , we have that  $\text{Roll}(\alpha)$  and  $\text{TSS}(\alpha)$  are curves constructed using a motion of the plane containing the curve  $\alpha$  and a pencil placed at the origin of this plane. It is clear that the curve obtained using the curve  $\alpha$  and a tracing point different from the origin can be obtained by using a translation of  $\alpha$  and the origin as tracing point. Therefore, there is not loss of generality by assuming that the point describing  $\text{Roll}(\alpha)$  or  $\text{TSS}(\alpha)$  is the origin. The roulette, which is related with the rolling of a curve  $\alpha$  along another curve  $\beta$ , can be defined as

$$\text{Roulette}(\alpha) = \left\{ T_s \begin{pmatrix} 0 \\ 0 \end{pmatrix} : T_s \text{ is an oriented isometry in } \mathbf{R}^2, \right. \\ \left. T_s(\alpha(s)) = \beta(s) \text{ and } dT_t(\alpha'(s)) = \beta'(s) \right\}.$$

In the previous expression we are assuming that both curves,  $\alpha$  and  $\beta$  are parametrized by arc-length, and again, this is the roulette described when the tracing point is located at the origin.

**2.1. Definition and interpretation.** Let us start this section with a definition that extends the notion of TreadmillSled. This extension is similar to the one that we obtain when we generalize the notion  $\text{Roll}(\alpha)$ , which is the Roulette over a line, to the notion of Roulette over any curve. In this generalization from  $\text{Roll}(\alpha)$  to  $\text{Roulette}(\alpha)$ , we are allowing the point of contact to move along a general curve instead of just a line, and we are also adjusting or moving the plane containing  $\alpha$  to force the velocity vector  $\alpha'(s)$  to be aligned with the velocity vector  $\beta'(s)$  instead of just the vector  $(1,0)$  (the velocity vector of the horizontal line). Since the TreadmillSled of a curve  $\alpha$  is a rolling with full slipping, then, the point of contact is fixed and we cannot generalize this notion by moving the curve  $\alpha$  along a curve  $\beta$ , nevertheless, we can force the velocity vector of  $\alpha$  to be aligned with any direction we wish.

Here is the definition of this generalization of the notion of TreadmillSled.

**DEFINITION 2.1.** Given a regular curve  $\alpha : [a, b] \rightarrow \mathbf{R}^2$  and a function  $\phi : [a, b] \rightarrow \mathbf{R}$ , we define the  $\phi$ -TreadmillSled of  $\alpha$  as the set of points

$$\left\{ T_s \begin{pmatrix} 0 \\ 0 \end{pmatrix} : T_s \text{ is an oriented isometry in } \mathbf{R}^2, T_s(\alpha(s)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right. \\ \left. \text{and } dT_s \left( \frac{\alpha'(s)}{|\alpha'(s)|} \right) = \begin{pmatrix} \cos(\phi(s)) \\ \sin(\phi(s)) \end{pmatrix} \right\}.$$

This set of points will be denoted by  $\phi$ -TS( $\alpha$ ).

**REMARK 2.2.** Notice that the definition of the  $\phi$ -TS( $\alpha$ ) is independent of the parametrization, it only depends on the orientation of the curve. That

is, if  $h : [c, d] \rightarrow [a, b]$  is a function with positive derivative and  $\tilde{\alpha}(t) = \alpha(h(t))$  and  $\tilde{\phi}(t) = \phi(h(t))$ , then  $\tilde{\phi}$ -TS( $\tilde{\alpha}$ ) =  $\phi$ -TS( $\alpha$ ).

It is not difficult to see that the  $\phi$ -TreadmillSled of  $\alpha$  can be viewed as the curve generated by doing the following steps:

- Imagine that the curve  $\alpha$  is in a plane which can freely move. Moreover, let us assume that there is a hole in the origin of this plane and also let us assume that we have placed a pencil in this hole.
- Imagine that another plane, this one fixed, contains a treadmill based at the origin with a device that allows the treadmill to incline at any angle.
- The curve  $\alpha$  in the moving plane will generate another curve in the fixed plane, the  $\phi$ -TreadmillSled of  $\alpha$ .
- The  $\phi$ -TreadmillSled of  $\alpha$  is the curve drawn on the fixed plane by the pencil located at the origin of the moving plane, when the curve  $\alpha$  passes on the treadmill with the property that, anytime the point  $\alpha(s)$  is on the treadmill, the treadmill is aligned in the direction  $(\cos(\phi(s)), \sin(\phi(s)))$ .

**2.1.1. Generalization of the Roulette.** We can extend the  $\phi$ -TreadmillSled and the Roulette in the following way.

**DEFINITION 2.3.** Given two curves  $\alpha : [0, l_1] \rightarrow \mathbf{R}^2$  and  $\beta : [0, l_2] \rightarrow \mathbf{R}^2$  parametrized by arc-length, and two functions  $\rho : [0, l_1] \rightarrow [0, l_2]$  and  $\phi : [0, l_1] \rightarrow \mathbf{R}$ , we can define the Generalized Roulette of  $\alpha$  over  $\beta$  with functions  $\rho$  and  $\phi$  as

$$\text{GR}(\alpha) = \left\{ T_s \begin{pmatrix} 0 \\ 0 \end{pmatrix} : T_s \text{ is an oriented isometry in } \mathbf{R}^2, T_s(\alpha(s)) = \beta(\rho(s)) \right. \\ \left. \text{and } dT_s(\alpha'(s)) = \begin{pmatrix} \cos(\phi(s)) \\ \sin(\phi(s)) \end{pmatrix} \right\}.$$

**REMARK 2.4.** With this definition we have that if  $\rho(s) = s$  and  $\phi(s)$  satisfies the equation  $\beta'(s) = (\cos \phi(s), \sin \phi(s))$ , then  $\text{GR}(\alpha)$  agrees with the Roulette of  $\alpha$ . If  $\rho(s) = 0$  for all  $s$  and  $\beta$  is any curve that satisfies  $\beta(0) = (0, 0)$ , then  $\text{GR}(\alpha)$  agrees with the  $\phi$ -TreadmillSled of  $\alpha$ .

**2.2. Parametrization of the  $\phi$ -TreadmillSled.** The following proposition will provide a formula to find the  $\phi$ -TreadmillSled of a curve  $\alpha$ .

**PROPOSITION 2.5.** Let  $\alpha : [a, b] \rightarrow \mathbf{R}^2$  be a regular curve in  $\mathbf{R}^2$ , if  $\alpha(s) = (x(s), y(s))^T = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$ , then,

$$(2.1) \quad \beta(s) = A(\theta(s))\alpha(s) = -A(-\phi(s))A(\rho(s))\alpha(s)$$

is a parametrization of the  $\phi$ -TreadmillSled of  $\alpha$ . Here

$$A(\tau) = \begin{pmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{pmatrix}$$



and

$$\theta(s) = \rho(s) - \phi(s) + \pi \quad \text{and} \quad \begin{pmatrix} \cos(\rho(s)) \\ \sin(\rho(s)) \end{pmatrix} = \frac{1}{|\alpha'(s)|} \alpha'(s).$$

*Proof.* We will use the parameter  $s$  to describe points in the set  $\phi\text{-TS}(\alpha)$ . For a fixed  $s \in [a, b]$ , let us find an oriented isometry of  $\mathbf{R}^2$  such that  $T_s(\alpha(s)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $dT_s\left(\frac{\alpha'(s)}{|\alpha'(s)|}\right) = \begin{pmatrix} \cos(\phi(s)) \\ \sin(\phi(s)) \end{pmatrix}$ . We know that

$$T_s \begin{pmatrix} u \\ v \end{pmatrix} = A(\tilde{\theta}(s)) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} c_1(s) \\ c_2(s) \end{pmatrix}.$$

Notice that once we find  $\tilde{\theta}(s)$ ,  $c_1(s)$  and  $c_2(s)$ , using the Definition 2.1, we get that  $\beta(s) = (c_1(s), c_2(s))^T$  is a point in  $\phi\text{-TS}(\alpha)$ ; and therefore, when we vary  $s$  in the interval  $[a, b]$ , we obtain that  $\beta(s) = (c_1(s), c_2(s))$  is a parametrization of  $\phi\text{-TS}(\alpha)$ .

Since

$$dT_s \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = A(\tilde{\theta}(s)) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{and} \quad dT_s \left( \frac{\alpha'(s)}{|\alpha'(s)|} \right) = \begin{pmatrix} \cos(\phi(s)) \\ \sin(\phi(s)) \end{pmatrix}$$

we have that

$$A(\tilde{\theta}(s)) \begin{pmatrix} \cos(\rho(s)) \\ \sin(\rho(s)) \end{pmatrix} = \begin{pmatrix} \cos(\phi(s)) \\ \sin(\phi(s)) \end{pmatrix}$$

and therefore,

$$A(\tilde{\theta}(s))A(-\rho(s)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A(-\phi(s)) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since  $A(\tau_1 + \tau_2) = A(\tau_1)A(\tau_2)$ , the last equation implies that  $A(\tilde{\theta}(s) - \rho(s) + \phi(s)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , which implies that  $\tilde{\theta}(s) = \rho(s) - \phi(s)$ .

Now, using the equation  $T_s(\alpha(s)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  we get that

$$\begin{pmatrix} c_1(s) \\ c_2(s) \end{pmatrix} = -A(\tilde{\theta}(s))\alpha(s) = A(\theta(s))\alpha(s).$$

Since  $\beta(s) = (c_1(s), c_2(s))^T$ , then the proposition follows.  $\square$

**2.3. TreadmillSled as a parametrized curve.** The definition of TreadmillSled of a curve given in the Introduction corresponds with the  $\phi$ -TreadmillSled when  $\phi$  is the zero function. Sometimes we will view  $\phi\text{-TS}(\alpha)$  not as a set but as the parametrized curve described in (2.1). In particular, we have the following way to define the TreadmillSled not as a set but as a parametrized curve.

**DEFINITION 2.6.** Let  $\alpha : [a, b] \rightarrow \mathbf{R}^2 = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$  be a regular curve. We define the TreadmillSled of  $\alpha$  as the parametric curve  $\text{TS}(\alpha) : [a, b] \rightarrow \mathbf{R}^2$  given by

$$\text{TS}(\alpha)(s) = \frac{1}{\sqrt{x'(s)^2 + y'(s)^2}} \begin{pmatrix} -x'(s)x(s) - y'(s)y(s) \\ x(s)y'(s) - y(s)x'(s) \end{pmatrix}.$$

The following remark gives us some insight about the nature of the operator  $\phi$ -TreadmillSled defined in the set of regular curves. As we already noticed, the  $\phi$ -TreadmillSled of a curve is independent of the parametrization as long as the orientation is preserved. Therefore, there is not loss of generality if we assume that the curves in the domain of the operator  $\phi$ -TreadmillSled are parametrized by arc-length.

REMARK 2.7. Let us define  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . If  $\alpha$  is an arc-length parametrized curve and  $\begin{pmatrix} u(s) \\ v(s) \end{pmatrix} = \text{TS}(\alpha)$ , then,

$$u = -\langle \alpha, \alpha' \rangle \quad \text{and} \quad v = \langle \alpha', J\alpha \rangle$$

where  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product.

With this definition of  $J$  we have that the curvature of  $\alpha$  is  $k(s) = \langle \alpha''(s), J(\alpha'(s)) \rangle$ . Notice that if  $\alpha'(s) = \begin{pmatrix} \cos(\rho(s)) \\ \sin(\rho(s)) \end{pmatrix}$ , then  $k(s) = \rho'(s)$ . Therefore, if we know the curvature  $k(s)$  of a curve parametrized by arc-length and a given angle for the velocity vector, let's say  $\alpha'(a) = \begin{pmatrix} \cos(\rho_0) \\ \sin(\rho_0) \end{pmatrix}$ , then

$$\text{TS}(\alpha) = -A(\rho(s))\alpha(s) \quad \text{where} \quad \rho(s) = \int_a^s k(u) du + \rho_0.$$

**2.4. The  $\phi$ -TreadmillSled and complex numbers.** The parametrization given in Proposition 2.5 can be viewed in the following way.

COROLLARY 2.8. *If we identify each point  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2$  with the complex number  $x_1 + ix_2$ , then*

$$\phi\text{-TS}(\alpha) = e^{i\phi} \text{TS}(\alpha).$$

*For any curve  $\alpha(s) = x_1(s) + ix_2(s)$ . Moreover, if the function  $\phi$  is fixed, then, the  $\phi$ -TreadmillSled of two curves is the same, if and only if the TreadmillSled of the curves is the same.*

One of the reasons we introduce this extension to the notion of TreadmillSled is because it provides an interpretation for the curve  $h(t)\alpha(t)$  when  $h: [a, b] \rightarrow \mathbb{C}$ ,  $\alpha: [a, b] \rightarrow \mathbb{C}$  are curves in the complex plane with  $|h(t)| = 1$ . The following corollary provides a way to program the inclination on a treadmill (find the function  $\phi$ ) if we want to get the curve  $e^{ig(t)}\alpha(t)$  as the  $\phi$ -TreadmillSled of  $\alpha$ .

COROLLARY 2.9. *If  $\alpha: [a, b] \rightarrow \mathbb{C} \cong \mathbf{R}^2$  is a regular curve with curvature function  $\kappa$  and  $g: [a, b] \rightarrow \mathbf{R}$  is a function, then*

$$e^{ig(t)}\alpha(t) = \phi\text{-TS}(\alpha),$$

where  $\phi(t) = \int_a^t \kappa(\tau) |\alpha'(\tau)| d\tau + \rho_0 + g(t) + \pi$  and  $\alpha'(a) = \begin{pmatrix} \cos(\rho_0) \\ \sin(\rho_0) \end{pmatrix}$ .

**2.5. The inverse image of the TreadmillSled operator.** In this section we will point out that even though the TreadmillSled operator is not one to one, given a curve  $\beta$  in the range of the operator TreadmillSled, there is only a one-parametric family of curves whose image is the curve  $\beta$ , moreover, all these curves in the inverse image of the curve  $\beta$  differ only by an oriented rotation about the origin. More precisely, we have the following proposition.

**PROPOSITION 2.10.** *Let  $\alpha_1 : [a, b] \rightarrow \mathbf{R}^2$  and  $\alpha_2 : [a, b] \rightarrow \mathbf{R}^2$  be two curves parametrized by arc-length.  $\text{TS}(\alpha_1) = \text{TS}(\alpha_2)$  if and only if  $\alpha_2(s) = A(\tau)\alpha_1(s)$  for some constant  $\tau$ .*

*Proof.* Let  $\rho_1(s)$  and  $\rho_2(s)$  be functions such that  $\begin{pmatrix} \cos(\rho_i(s)) \\ \sin(\rho_i(s)) \end{pmatrix} = \alpha'_i(s)$ . If  $\alpha_2(s) = A(\tau)\alpha_1(s)$ , then

$$\alpha'_2(s) = A(\tau) \begin{pmatrix} \cos(\rho_1(s)) \\ \sin(\rho_1(s)) \end{pmatrix} = \begin{pmatrix} \cos(\rho_1(s) - \tau) \\ \sin(\rho_1(s) - \tau) \end{pmatrix}.$$

Therefore, we may assume that  $\rho_2(s) = \rho_1(s) - \tau$ . Using Proposition 2.5, we obtain that

$$\text{TS}(\alpha_2) = A(\rho_2 + \pi)\alpha_2 = A(\rho_1 - \tau + \pi)A(\tau)\alpha_1 = A(\rho_1 + \pi)\alpha_1 = \text{TS}(\alpha_1).$$

Therefore, we have proven that if  $\alpha_2 = A(\tau)\alpha_1$ , then  $\text{TS}(\alpha_1) = \text{TS}(\alpha_2)$ . Now let us assume that  $\text{TS}(\alpha_1) = \text{TS}(\alpha_2)$ . Let us fix an  $s_0 \in [a, b]$  such that  $|\alpha_1(s_0)| \neq 0$ . Since  $\text{TS}(\alpha_1)(s_0) = \text{TS}(\alpha_2)(s_0)$  then  $|\alpha_1(s_0)| = |\alpha_2(s_0)|$ . Let  $\tau$  be a real number such that  $A(\tau)\alpha_1(s_0) = \alpha_2(s_0)$  and let us consider  $\alpha_3(s) = A(\tau)\alpha_1(s)$ . We have,  $\text{TS}(\alpha_3) = \text{TS}(\alpha_2)$ , and moreover, we have that  $\alpha_3(s_0) = \alpha_2(s_0)$ . Using Definition 2.6 we get that if  $\alpha(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$  is a curve parametrized by arc-length and  $\text{TS}(\alpha)(s) = \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}$ , then

$$\begin{aligned} u(s) &= -x'(s)x(s) - y'(s)y(s), \\ v(s) &= x(s)y'(s) - y(s)x'(s). \end{aligned}$$

For values of  $s$  such that  $x(s)^2 + y(s)^2 > 0$ , we get that

$$\begin{aligned} x'(s) &= -\frac{1}{x(s)^2 + y(s)^2} (x(s)u(s) + y(s)v(s)), \\ y'(s) &= \frac{1}{x(s)^2 + y(s)^2} (x(s)v(s) - y(s)u(s)). \end{aligned}$$

By the Existence and Uniqueness theorem of ordinary differential equations we get that the conditions  $\alpha_3(s_0) = \alpha_2(s_0)$  and  $\text{TS}(\alpha_3) = \text{TS}(\alpha_2)$  imply that  $\alpha_2(s) = \alpha_3(s)$  for all  $s$  near  $s_0$ . Since both curves are regular, by a continuity argument we conclude that the real number  $\tau$  is independent of  $s_0$  and therefore  $\alpha_2(s) = \alpha_3(s)$  for all  $s$ . We then get  $\alpha_2 = A(\tau)\alpha_1$  for some  $\tau$ . This finishes the proof of the proposition.  $\square$

**2.6. The range of the TreadmillSled operator and a formula for the inverse of the TreadmillSled.** In this section, we will characterize the range of the operator TreadmillSled. Moreover, we will provide two easy-to-check properties such that, any curve that satisfies them, must be the TreadmillSled of another curve. Moreover, for any curve  $\gamma$  that satisfy these two easy-to-check properties we will find an explicit formula for a curve  $\alpha$  whose TreadmillSled is the curve  $\gamma$ . Under the assumption that a curve  $\gamma$  is in the range of the operator TS, the formula for the inverse of the TreadmillSled provided below was found in [6].

**PROPOSITION 2.11.** *Let  $\gamma(s) = \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}$  be a regular curve.  $\gamma$  is the TreadmillSled of a regular curve  $\alpha$  if and only if  $v'(s) = -f(s)u(s)$  for some continuous function  $f$  and  $vf - u' > 0$ . More precisely, if  $f, v$  and  $u$  satisfy the two previous conditions, and  $F(s)$  is an antiderivative of  $f(s)$ , then,*

$$\text{TS}(\alpha) = \gamma \quad \text{where } \alpha(t) = -A(-F(t))\gamma(t).$$

*Proof.* Let us assume that  $\gamma(s)$  is the TreadmillSled of a curve  $\alpha$ . Let us first consider the case when  $\alpha$  is parametrized by arc-Length. If we denote by  $k_\alpha$  the curvature of  $\alpha$ , then, using Remark 2.7 we obtain,

$$u = -\langle \alpha, \alpha' \rangle \quad \text{and} \quad v = \langle \alpha', J\alpha \rangle.$$

Therefore,

$$v' = \langle \alpha'', J\alpha \rangle + \langle \alpha', J\alpha' \rangle = k_\alpha \langle J\alpha', J\alpha \rangle = k_\alpha \langle \alpha', \alpha \rangle = -k_\alpha u$$

and

$$u' = -1 - \langle \alpha, \alpha'' \rangle = -1 - k_\alpha \langle \alpha, J\alpha' \rangle = -1 + k_\alpha \langle J\alpha, \alpha' \rangle = -1 + k_\alpha v.$$

Taking  $f = k_\alpha$ , we conclude that  $v' = -fu$  and  $fv - u' = 1$ . If we now consider a regular curve  $\tilde{\alpha}$ , then we have that  $\tilde{\alpha}(t) = \alpha(h(t))$  where  $\alpha$  is parametrized by arc-length and  $h(t)$  is a function with  $h'(t) > 0$ . Therefore, by either Remark 2.2 or by Definition 2.6, we get that if  $\tilde{\gamma} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$  is the TreadmillSled of  $\tilde{\alpha}$ , then  $\tilde{\gamma}(t) = \gamma(h(t))$  where  $\gamma = \begin{pmatrix} u \\ v \end{pmatrix}$  is the TreadmillSled of  $\alpha$ . Since  $\alpha$  is parametrized by arc-length, then  $v'(h(t)) = -f(h(t))u(h(t))$  for some continuous function  $f$  and  $fv - u' = 1$ . A direct verification shows that  $\tilde{f}(t) = h'(t)f(h(t))$  is a continuous function that satisfies  $\tilde{v}'(t) = -\tilde{f}(t)\tilde{u}(t)$ , and

$$\tilde{v}(t)\tilde{f}(t) - \tilde{u}'(t) = h'(t)v(h(t))f(h(t)) - h'(t)u'(h(t)) = h'(t) > 0.$$

This inequality finishes the proof of one of the implications of the proposition. Let us assume now that the functions  $u, v$  are given and that  $v' = -fu$  for some continuous function  $f$ , and that  $fv - u' > 0$ . We need to prove that

if  $F' = f$  then

$$\alpha(t) = -A(-F(t))\gamma(t)$$

satisfies that  $\text{TS}(\alpha) = \gamma$ . Using the fact that

$$\frac{dA(\tau)}{d\tau} = A\left(\tau + \frac{\pi}{2}\right) = A(\tau)A\left(\frac{\pi}{2}\right) = -A(\tau)J$$

we get that

$$\alpha' = -fA(-F)J\gamma - A(-F)\gamma' = -A(-F)(fJ\gamma + \gamma').$$

Therefore,

$$\langle \alpha', \alpha' \rangle = \langle fJ\gamma + \gamma', fJ\gamma + \gamma' \rangle = f^2\langle \gamma, \gamma \rangle + 2f\langle J\gamma, \gamma' \rangle + \langle \gamma', \gamma' \rangle.$$

Before we continue with the proof, we point out that the set  $\{s : u(s) = 0\}$  cannot contain an open set because of the regularity of the curve  $\gamma$ . Indeed, if  $(a, b)$  is an open interval contained in  $\{s : u(s) = 0\}$  and  $s_0 \in (a, b)$ , then clearly  $u'(s_0) = 0$  and, using the equation  $v'(s_0) = f(s_0)u(s_0)$  we conclude that  $v'(s_0)$  is also zero, which is impossible because we are assuming that  $\gamma'(s)$  does not vanish for any  $s$ . As a consequence of this observation, we have that if two continuous functions agree in the complement of the set  $\{s : u(s) = 0\}$ , then, they agree in the whole domain of the function  $\gamma$ .

For every point where the function  $u$  does not vanish, we have that  $f = -\frac{v'}{u}$ . Moreover, we get that

$$\begin{aligned} \langle \alpha', \alpha' \rangle &= \frac{(v')^2}{u^2}(u^2 + v^2) - 2\frac{v'}{u}(uv' - vu') + (u')^2 + (v')^2 \\ &= \left(u' + \frac{vv'}{u}\right)^2 = (fv - u')^2. \end{aligned}$$

Since we have that  $fv - u' > 0$ , then we conclude that  $|\alpha'| = fv - u'$  anytime  $u$  does not vanish.

Since we have that  $\{s : u(s) = 0\}$  does not contains an open interval, then, by the continuity of the functions  $|\langle \alpha', \alpha' \rangle|$  and  $fv - u'$  we conclude that  $|\langle \alpha', \alpha' \rangle| = fv - u' > 0$  everywhere, and therefore  $\alpha$  is a regular curve. Moreover, we have

$$\begin{aligned} \text{TS}(\alpha) &= \frac{1}{|\alpha'|} \begin{pmatrix} -\langle \alpha', \alpha \rangle \\ \langle \alpha', J\alpha \rangle \end{pmatrix} \\ &= \frac{1}{|\alpha'|} \begin{pmatrix} -\langle \gamma, fJ\gamma + \gamma' \rangle \\ \langle fJ\gamma + \gamma', J\gamma \rangle \end{pmatrix} \\ &= \frac{1}{|\alpha'|} \begin{pmatrix} -\langle \gamma, \gamma' \rangle \\ f\langle \gamma, \gamma \rangle + \langle \gamma', J\gamma \rangle \end{pmatrix}. \end{aligned}$$

Since,

$$-\langle \gamma, \gamma' \rangle = -vv' - uu' = vfu - uu' = u(vf - u') = u|\alpha'|$$

and,

$$\begin{aligned} f\langle\gamma, \gamma\rangle + \langle\gamma', J\gamma\rangle &= -\frac{v'}{u}(u^2 + v^2) + uv' - vu' = -\frac{v^2v'}{u} - vu' \\ &= v(fv - u') = v|\alpha'| \end{aligned}$$

we conclude that  $\text{TS}(\alpha) = \gamma$ . This completes the proof of the proposition.  $\square$

REMARK 2.12. The regularity condition for the curve  $\gamma$  in the previous proposition can be replaced by the weaker condition that  $\gamma'$  does not vanish on an open set, that is, it can be replaced by the condition that the curve  $\gamma$  is not constant on an open interval.

### 3. A dynamical interpretation for helicoidal minimal surfaces

Helicoidal minimal hypersurfaces have been understood for a long time. For a detailed study, we refer to the last section of the last chapter of the book of Differential Geometry by Graustein [4]. We have that all the isometric surfaces (except for the catenoid) from the well-known family of surfaces that starts with a helicoid and ends with a catenoid are helicoidal minimal surfaces. See Figure 6. Actually, every helicoidal minimal surface belongs to one of these families.

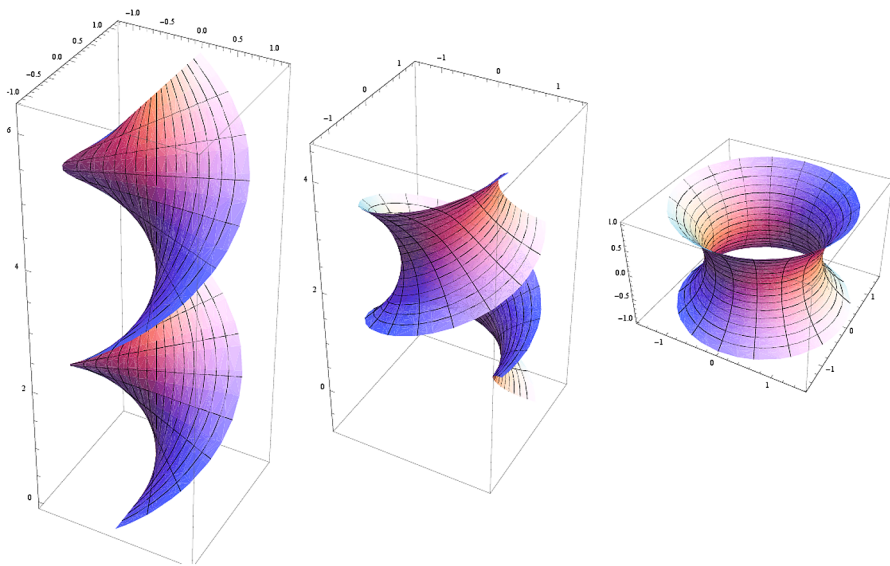


FIGURE 6. A helicoid (left), a helicoidal minimal surface (center) and a catenoid (right) are part of a family of isometric minimal surfaces.

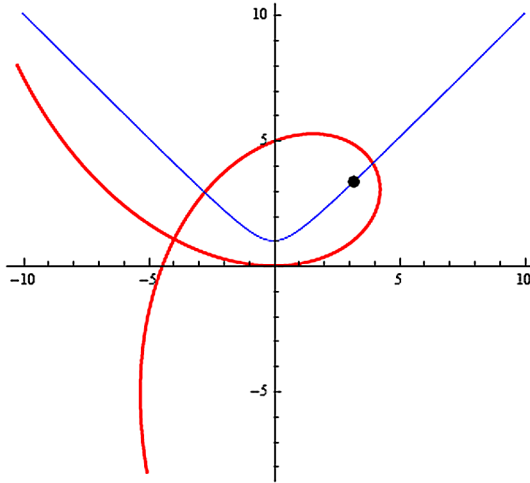


FIGURE 7. The TreadmillSled of the profile curve of a helicoidal minimal surface is either the  $u$ -axis or a hyperbola. In the notation of Theorem 3.1, in this figure, we have  $w = 1$  and  $M = 1$ .

A similar result for helicoidal CMC surfaces was proven in [1] by Do Carmo and Dajczer. They proved that every helicoidal surface belongs to a family of isometric surfaces that continuously move from an unduloid to a nodoid. In this section, we provide a dynamical interpretation for the profile curve of a helicoidal minimal surface (see Figure 7). Let us state and prove the main theorem in this section.

**THEOREM 3.1.** *A complete helicoidal surface*

$$\phi(s, t) = (x(s) \cos(wt) + z(s) \sin(wt), t, -x(s) \sin(wt) + z(s) \cos(wt))$$

is minimal if and only if the TreadmillSled of its profile either is the  $u$ -axis and  $\phi$  is a helicoid or it is one of the branches of the hyperbola  $\frac{v^2}{M^2} - w^2 u^2 = 1$  for some nonzero  $M$ .

*Proof.* Let us assume that the profile curve  $\alpha(s) = (x(s), z(s))$  is parametrized by arc-length. If  $\text{TS}(\alpha)(s) = (\xi_1(s), \xi_2(s))$  then by Definition 2.6 we have

$$\xi_1(s) = -x'(s)x(s) - z'(s)z(s) \quad \text{and} \quad \xi_2(s) = x(s)z'(s) - z(s)x'(s).$$

Since we are assuming that  $\alpha$  is parametrized by arc-length, there exists a function  $\theta$  such that  $\alpha'(s) = (\cos(\theta(s)), \sin(\theta(s)))$ . From the previous equation, we get that

$$\theta'(s) = x'(s)z''(s) - z'(s)x''(s).$$

With this definition of  $\theta(s)$  and the definition of the function  $\xi_1(s)$  and  $\xi_2(s)$  given above, we obtain that

$$\begin{aligned}x(s) &= -\xi_1(s) \cos(\theta(s)) + \xi_2(s) \sin(\theta(s)) \quad \text{and} \\z(s) &= -\xi_1(s) \sin(\theta(s)) - \xi_2(s) \cos(\theta(s)).\end{aligned}$$

A direct computation shows that

$$\nu = \left( \frac{\sin(wt - \theta)}{\sqrt{1 + w^2\xi_1^2}}, -\frac{w\xi_1}{\sqrt{1 + w^2\xi_1^2}}, \frac{\cos(wt - \theta)}{\sqrt{1 + w^2\xi_1^2}} \right)$$

is a Gauss map of the immersion  $\phi$  and, with respect to this Gauss map, the first and second fundamental forms are given by

$$\begin{aligned}E &= 1, & F &= -w\xi_2, & G &= 1 + w^2(\xi_1^2 + \xi_2^2) \quad \text{and} \\e &= \frac{\theta'}{\sqrt{1 + w^2\xi_1^2}}, & f &= \frac{-w}{\sqrt{1 + w^2\xi_1^2}}, & g &= \frac{w^2\xi_2}{\sqrt{1 + w^2\xi_1^2}}.\end{aligned}$$

Using the values above we get that the mean curvature  $H$  of the  $\phi$  is given by

$$H = \frac{-w^2\xi_2 + \theta'(1 + w^2(\xi_1^2 + \xi_2^2))}{2(1 + w^2\xi_1^2)^{\frac{3}{2}}}.$$

Therefore, the equation  $H = 0$ , that is, the minimality of the immersion  $\phi$ , implies

$$\theta' = \frac{w^2\xi_2}{1 + w^2(\xi_1^2 + \xi_2^2)}.$$

From the definition of  $\xi_1$  and  $\xi_2$  we get that

$$\begin{aligned}\xi_1' &= -x''x - (x')^2 - z''z - (z')^2 \\&= \theta'x \sin(\theta) - \theta'z \cos(\theta) - 1 \\&= \theta'\xi_2 - 1.\end{aligned}$$

Likewise we obtain that  $\xi_2' = -\theta'\xi_1$ . Therefore if  $\phi$  is minimal, replacing the expression for  $\theta'$  above, we get that

$$\begin{aligned}\xi_1' &= \frac{w^2\xi_2^2}{1 + w^2(\xi_1^2 + \xi_2^2)} - 1, \\ \xi_2' &= -\frac{w^2\xi_1\xi_2}{1 + w^2(\xi_1^2 + \xi_2^2)}.\end{aligned}$$

A direct verification shows that if  $(\xi_1(s), \xi_2(s))$  satisfies the differential equation above then,

$$\frac{\xi_2(s)}{\sqrt{1 + w^2\xi_1(s)^2}} = M \quad \text{for some constant } M.$$

If  $M = 0$ , then  $\xi_2(s) = 0$ . That is, in this case the TreadmillSled of  $\alpha$  is the  $u$ -axis. If  $M$  is not zero, we get by squaring the centered equation



above, that the TreadmillSled lies in one of the branches of the hyperbola  $\frac{v^2}{M^2} - w^2u^2 = 1$ . Therefore, one implication of the theorem follows. We will use the inverse formula for the TreadmillSled, see Proposition 2.11, to prove the other implication. Let us assume that the TreadmillSled of the profile curve  $\alpha$  is the horizontal line  $\gamma(s) = (\gamma_1, \gamma_2) = (-s, 0)$ . In this case the function  $f$  from Proposition 2.11 is given by  $f = -\frac{\gamma'_2(s)}{\gamma'_1(s)} = 0$ , since  $f$  is continuous we have that the first of the two easy-to-check properties holds. Since  $\gamma_2f - \gamma'_1 = 1 > 0$ , then the second of the two easy-to-check properties holds too. In this case  $F(s) = c$  where  $c$  is any constant. Therefore, it follows that an inverse of the horizontal line is given by

$$\alpha(s) = - \begin{pmatrix} \cos(-c) & \sin(-c) \\ -\sin(-c) & \cos(-c) \end{pmatrix} \begin{pmatrix} -s \\ 0 \end{pmatrix} = \begin{pmatrix} s \cos(c) \\ s \sin(c) \end{pmatrix}.$$

That is,  $\alpha$  is a line through the origin and therefore the surface  $\phi$  is a helicoid. Now, let us assume that the TreadmillSled of the profile curve satisfies the equation  $\frac{v^2}{M^2} - w^2x^2 = 1$ . We can assume that this TreadmillSled is parametrized as  $\gamma(s) = (\gamma_1, \gamma_2) = (-\frac{1}{w} \sinh(s), M \cosh(s))$ . In this case, the function  $f$  from Proposition 2.11 is given by  $f = -\frac{\gamma'_2(s)}{\gamma'_1(s)} = Mw$ , since  $f$  is continuous we have that the first of the two easy-to-check properties holds. Since  $\gamma_2f - \gamma'_1 = \frac{1+M^2w^2}{w} \cosh(s)$ , then the second of the two easy-to-check properties holds too. In this case, we can take the function  $F(s) = Mws$ . Therefore, using Proposition 2.11, we get that an inverse of the branch of the hyperbola is

$$\begin{aligned} \alpha(s) &= - \begin{pmatrix} \cos(-Mws) & \sin(-Mws) \\ -\sin(-Mws) & \cos(-Mws) \end{pmatrix} \begin{pmatrix} -\frac{1}{w} \sinh(s) \\ M \cosh(s) \end{pmatrix} \\ &= \begin{pmatrix} M \cosh(s) \sin(Mws) + \frac{1}{w} \sinh(s) \cos(Mws) \\ -M \cosh(s) \cos(Mws) + \frac{1}{w} \sinh(s) \sin(Mws) \end{pmatrix}. \end{aligned}$$

A direct verification shows that if we define the functions  $x(s)$  and  $z(s)$  by the equation  $\alpha(s) = (x(s), z(s))$ , then,  $\phi(s, t) = (x(s) \cos(wt) + z(s) \sin(wt), t, -x(s) \sin(wt) + z(s) \cos(wt))$  is minimal. This finishes the proof of the theorem. □

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