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TEST EXPONENTS FOR MODULES WITH FINITE PHANTOM PROJECTIVE DIMENSION

MELVIN HOCHSTER AND YONGWEI YAO

ABSTRACT. Let $(R,\mathfrak{m})$ be an equidimensional excellent local ring of prime characteristic $p>0$. We give an alternate proof of the existence of a uniform test exponent for any given $c\in R^\circ$ and all ideals generated by (full or partial) systems of parameters. This follows from a more general result about the existence of a test exponent for any given Artinian R-module. If we further assume R is Cohen–Macaulay, then there exists a test exponent for any given $c\in R^\circ$ and all perfect modules with finite projective dimension.

0. Introduction

Throughout this paper, R is a Noetherian ring of prime characteristic $p>0$. By $(R,\mathfrak{m},k)$, we indicate that R is a local ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m}=k$.

Also, we always use $q=p^e, Q=p^E, q_0=p^{e_0}, q'=p^{e'}, q''=p^{e''}$, etcetera, to denote varying powers of p with $e,E,e_0,e',e''\in\mathbb{N}$.

Let M be an R-module. Then for any $e\geq 0$, we can derive a left R-module structure on the set M by $r\cdot m:=r^{p^e}m$ for any $r\in R$ and $m\in M$. For technical reasons, we keep the original right R-module structure on M by default. We denote the derived R-R-bimodule by ${}^eM$. Thus, in ${}^eM$, we have $r\cdot m=m\cdot r^{p^e}$, which is equal to $r^q m$ in the original M. If R is reduced, then ${}^eR$, as a left R-module, is isomorphic to $R^{1/q}:=\{r^{1/p}\mid r\in R\}$. We use $\lambda^l(-),\lambda^r(-)$ to denote the left and right lengths of a bimodule. It is easy to

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see that $\lambda^l({}^eM) = q^{\alpha(R)}\lambda^r({}^eM) = q^{\alpha(R)}\lambda(M)$ for any finite length module M over $(R, \mathfrak{m}, k)$, in which $\alpha(R) = \log_p[k : k^p]$.

We say that R is *F-finite* if ${}^1R$ (or, equivalently, ${}^eR$ for all e) is finitely generated as an left R-module.

For any R-module M and e, we can always form a new R-module $F^e(M)$ by scalar extension via $F^e : R \to R$ by $r \mapsto r^q$. In other words, $F^e(M)$ has the R-module structure that is determined by the right R-module structure of $M \otimes_R {}^eR$; and it is this R-module structure of $F^e(M)$ that we mean unless otherwise specified. If $h \in \mathrm{Hom}_R(M, N)$, then we correspondingly have $F^e(h) : \mathrm{Hom}_R(F^e(M), F^e(N))$. Sometimes, especially when both M and N are free, we may write $F^e(h)$ as $h^{[q]}$.

A very important concept in studying rings of characteristic p is tight closure. Tight closure was first studied and developed by Hochster and Huneke in the 1980s.

Definition 0.1 (Hochster–Huneke [HH1]). Let R be a Noetherian ring of prime characteristic p and $N \subseteq M$ be R-modules. The *tight closure* of N in M, denoted by N_M^*, is defined as follows: An element $x \in M$ is said to be in N_M^* if there exists an element $c \in R^\circ$ such that $x \otimes c \in N_M^{[q]} \subseteq M \otimes_R {}^eR$ for all $e \gg 0$, where R° is the complement of the union of all minimal primes of the ring R and $N_M^{[q]}$ denotes the (right) R-submodule of $F_R^e(M) = M \otimes_R {}^eR$ generated by $\{x \otimes 1 \in M \otimes_R {}^eR \mid x \in N\}$. The element $x \otimes 1 \in M \otimes_R {}^eR$ is denoted by $x_M^{p^e} = x_M^q$. (By our convention on $F_R^e(M)$, we have $cx_M^q = x \otimes c \in N_M^{[q]}$.)

Definition 0.2 ([HH2]). Let R be a Noetherian ring of prime characteristic p, $q_0 = p^{e_0}$ and let $N \subseteq M$ be R-modules. We say $c \in R^\circ$ is a q_0-*weak test element* for $N \subseteq M$ if $c(N_M^*)_M^{[q]} \subseteq N_M^{[q]}$ for all $q \geq q_0$. In case $N = 0$, we may simply call it a *test element* for M. By a q_0-weak test element, we simply mean a q_0-weak test element for all R-modules. If a q_0-weak test element c remains a q_0-weak test element under every localization, then we call c a *locally stable* q_0-weak test. Finally, in case $q_0 = 1$, we simply call c a *test element* or *locally stable* test element.

Definition 0.3 ([HH4]). Let R be a Noetherian ring of prime characteristic p, $c \in R$, and $N \subseteq M$ (finitely generated) R-modules. We say that $Q = p^E$ is a *test exponent* for c and $N \subseteq M$ (over R) if, for any $x \in M$, the occurrence of $cx^q \in N_M^{[q]}$ for one single $q \geq Q$ implies $x \in N_M^*$. In case $N = 0$, we may simply call it a *test exponent* for c and M.

Remark 0.4. (1) It is easy to check the following statements: To say $c \in R^\circ$ is a test element for $N \subseteq M$ is the same as to say c is a test element for $(0 \subseteq)$ M/N. Similarly, to say $Q = p^E$ is a test exponent for c and $N \subseteq M$ is the same as to say Q is a test exponent for c and $(0 \subseteq)$ M/N

(2) However, by "a (q_0-weak) test element for an ideal I", we usually mean "a (q_0-weak) test element for $I \subseteq R$" rather than "a (q_0-weak) test element for $0 \subseteq I$". Similarly, when we say "a test exponent for c and an ideal I", we usually mean "a test exponent for c and $I \subseteq R$" rather than "a test exponent for c and $0 \subseteq I$".

Under mild conditions, test elements exist.

THEOREM 0.5. *Let R be F-finite or essentially of finite type over an excellent local ring $(A, \mathfrak{n})$ of characteristic p. Say $\sqrt{0}^{[q_0]} = 0$, where $\sqrt{0}$ is the nilradical of R.*

(1) *One may choose $c \in R^\circ$ such that $(R_{\mathrm{red}})_c$ is regular. Then c has a power c^k that is a completely stable q_0-weak test element for all finitely generated R-modules.*
(2) *In fact, there is a power c^k that is a completely stable q_0-weak test element for all (not necessarily finitely generated) R-modules.*

Proof. (1) See [HH2, Theorem (6.1)].

(2) It suffices to prove the case where R (and hence A) is reduced. Under the assumption that R is F-finite, this was proved under the hypothesis that R_c is weakly F-regular and Gorenstein in the thesis of Haggai Elitzur [El]. From this, we can see the remaining case as follows. Since A is excellent, $R \otimes_A \widehat{A}$ is reduced; and it is faithfully flat over R. First, we replace A by its completion and R by $R \otimes_A \widehat{A}$. Henceforth, assume that A is complete. It remains true that R_c is regular. In particular, this means that R_c is weakly F-regular and Gorenstein. We next make use of the Γ construction from [HH2, §6]. Choose a coefficient field K for A and a p-base Λ for K. For each cofinite subset Γ of Λ the ring A has a faithfully flat purely inseparable extension A^Γ, and for all sufficiently small cofinite sets $\Gamma \subseteq \Lambda$, $R^\Gamma = A^\Gamma \otimes_A R$ is reduced by [HH2, Lemma (6.13)], and R^Γ_c is weakly F-regular and Gorenstein by [HH2, Lemma (6.19)]. The ring R^Γ is F-finite and $(R^\Gamma)_c$ is weakly F-regular and Gorenstein. Therefore, c has the required property for R^Γ, and since this ring is faithfully flat over R, for R as well. $\square$

If there exists a test exponent for a locally stable test element $c \in R^\circ$ and (finitely generated) R-modules $N \subseteq M$, then the tight closure of N in M commutes with localization. This result is implicit in [McD] and is explicitly stated in [HH4, Proposition 2.3]. Moreover, Hochster and Huneke showed in [HH4] that the converse is true as below.

THEOREM 0.6 ([HH4]). *Let R be a Noetherian ring of prime characteristic p with a given locally stable test element c, and $N \subseteq M$ finitely generated R-modules. Assume that the tight closure of N in M commutes with localization. Then there exists a test exponent for c and $N \subseteq M$.*

Given $\underline{x} = x_1, \ldots, x_h$ in a local ring $(R, \mathfrak{m})$, we say that $\underline{x}$ is a (full) system of parameters of R if $h = \dim(R)$ and $\sqrt{(\underline{x})} = \mathfrak{m}$; we say $\underline{x}$ is a partial system of parameters of R if $\underline{x}$ can be expanded to a system of parameters of R.

In [HH4], Hochster and Huneke asked, among other questions, whether there exists a uniform test exponent for a given test element and all ideals generated by systems of parameters. This question has been recently answered positively by R. Y. Sharp.

Theorem 0.7 (Sharp [Sh, Theorem 3.2]). *Let $(R, \mathfrak{m})$ be an equidimensional excellent local ring of prime characteristic p and $c \in R^\circ$. Then there exists a test exponent for c and all ideals generated by (partial or full) systems of parameters of R.*

In Theorem 2.4, we use the Artinian property of $\mathrm{H}_{\mathfrak{m}}^{\dim(R)}(R)$ and colon-capturing to give an alternative proof of the above Theorem 0.7.

Next, we review the definition of phantom projective dimension.

Definition 0.8 ([Ab1], [HH1] and [HH3]). Let R be a Noetherian ring of prime characteristic p. Let M be an R-module and

$$G_\bullet : \cdots \xrightarrow{\phi_{n+1}} G_n \xrightarrow{\phi_n} G_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} G_1 \xrightarrow{\phi_1} G_0 \longrightarrow 0$$

a complex of R-modules.

(1) We say that $G_\bullet$ is *stably phantom acyclic* if
$$\operatorname{Ker}\big(F^e(\phi_n)\big) \subseteq \big(\operatorname{Image}\big(F^e(\phi_{n+1})\big)\big)^*_{F^e(G_n)} \quad \text{for all } n \geq 1 \text{ and all } e \geq 0.$$
(2) If $G_\bullet$ is a stably phantom acyclic complex of finitely generated projective modules with $\mathrm{H}_0(G_\bullet) \cong M$ and $G_n = 0$ for all $n \geq r+1$ (for some given r), we say that $G_\bullet$ is a *phantom projective resolution* of M of length r.
(3) We say that the *phantom projective dimension* of M is r if there is a phantom projective resolution of M of length r and r is the minimum such number. In this case, we write $\operatorname{ppd}_R(M) = r$.

Here we remark that, by the "rank and height" phantom acyclicity theorem (cf. [HH1, Theorems (9.8) and (9.8)°] and [AHH, Theorem 5.3(c)]), $\operatorname{ppd}_R(R/(\underline{x})) < \infty$ for all (partial or full) systems of parameters (under certain assumptions on R, for example, if $(R, \mathfrak{m})$ is excellent and equidimensional).

If $(R, \mathfrak{m})$ is Cohen–Macaulay, then $\operatorname{pd}_R(M) = \operatorname{ppd}_R(M)$ for every finitely generated R-module M. (We need to make sure that the "rank and height" phantom acyclicity criterion holds, which is the case if $(R, \mathfrak{m})$ is excellent and equidimensional.)

Inspired by Sharp's result (Theorem 0.7), we then naturally ask whether there is a uniform test exponent for a given $c \in R^\circ$ and all finitely generated R-modules with (finite length and) finite phantom projective dimension. While this question remains unsettled, we can give an affirmative answer in case R

is Cohen–Macaulay or in case $\dim(R) \le 2$. Throughout this paper, we use $\lambda(M)$ to denote the length of an R-module M.

THEOREM (Corollary 3.3, Corollary 3.4). *Let $(R, \mathfrak{m})$ be an equidimensional Noetherian excellent local ring of prime characteristic p. Assume either that R is Cohen–Macaulay or $\dim(R) \le 2$. Then, for any $c \in R^\circ$, there is a test exponent for c and all R-modules M with $\lambda(M) < \infty$ and $\mathrm{ppd}(M) < \infty$.*

1. Some preliminary results about test exponents

We first observe the following easy lemma about test exponents, although it is not directly used in the sequel.

LEMMA 1.1. *Let R be a Noetherian ring of characteristic p. For any $b, c \in R^\circ$ and R-modules $N \subseteq M$, the following are true.*

(1) *If Q is a test exponent for bc and $N \subseteq M$, then Q is a test exponent for c and $N \subseteq M$.*
(2) *If, for some $q_0 = p^{e_0}$, Q is a test exponent for c^{q_0} and $N_M^{[q_0]} \subseteq F_R^{e_0}(M)$, then Q is a test exponent for c and $N \subseteq M$.*

Proof. (1) If $cx^q \in N_M^{[q]} \subseteq F_R^e(M)$ for some $x \in M$ and $p^e = q \ge Q$, then $bcx^q \in N_M^{[q]} \in F_R^e(M)$ and hence $x \in N_M^*$.

(2) Suppose $cx^q \in N_M^{[q]} \subseteq F_R^e(M)$ for some $x \in M$ and $p^e = q \ge Q$. Then $c^{q_0}x^{q_0 q} \in N_M^{[q_0 q]} \subseteq F_R^{e_0+e}(M)$, or, in other words, $c^{q_0}(x^{q_0})^q \in (N_M^{[q_0]})^{[q]}_{F_R^{e_0}(M)} \subseteq F_R^e(F_R^{e_0}(M))$. This implies $x^{q_0} \in (N_M^{[q_0]})^*_{F_R^{e_0}(M)}$, which forces $x \in N_M^*$. ☐

For simplicity, we state the next two results (Lemma 1.2 and Lemma 1.3) in terms of test exponent for c and $(0 \subseteq)$ M only. It is an easy task to give the corresponding statements in terms of test exponents for c and $N \subseteq M$.

LEMMA 1.2. *Let R be a Noetherian ring of characteristic p with the set of minimal primes $\min(R) = \{P_1, P_2, \dots, P_r\}$ so that $\sqrt{0} = \bigcap_{i=1}^r P_i$. For any $c \in R^\circ$ (or simply $c \in R$) and any (finitely generated) R-module M, the following statements are true.*

(1) *If Q is a test exponent for $c + P_i$ and M/P_iM over R/P_i for all $i = 1, 2, \dots, r$, then Q is a test exponent for c and M.*
(2) *If Q is a test exponent for $c + \sqrt{0}$ and $M/\sqrt{0}M$ over $R/\sqrt{0}$, then Q is a test exponent for c and M.*

Proof. (1) Suppose $cx^q = 0 \in F_R^e(M)$ for some $x \in M$ and $p^e = q \ge Q$. Then, $(c + P_i)(x + P_iM)^q_{M/P_iM} = 0 \in F^e_{R/P_i}(M/P_iM)$, which implies $x + P_iM \in 0^*_{M/P_iM}$ for every $i = 1, 2, \dots, r$. This forces $x \in 0_M^*$ (see [HH1]).

(2) This follows similarly. ☐

The next lemma deals with module-finite and pure ring extensions. In particular, the lemma applies to any reduced Nagata (e.g., excellent) ring and its integral closure in its total quotient ring.

LEMMA 1.3. *Let $R \subseteq S$ be an extension of Noetherian rings of characteristic p, $c \in R$, and let M be a finitely generated R-module. Assume either* (1) *$R \subseteq S$ is module-finite, or* (2) *$R \subseteq S$ is a pure extension with a common weak test element in R. If Q is a test exponent for c and $0 \subseteq M \otimes_R S$ over S, then Q is a test exponent for c and $0 \subseteq M$.*

Proof. Suppose $cx^q = 0 \in F_R^e(M)$ for some $x \in M$ and $p^e = q \geq Q$. Then $c(x \otimes 1)^q = 0 \in F_S^e(M \otimes_R S)$ and hence $x \otimes 1 \in 0^*_{M \otimes_R S}$, which implies $x \in 0^*_M$. □

The next lemma relies on the "colon-capturing" property of tight closure, which is systematically studied in [HH1, Section 7].

LEMMA 1.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p, $\dim(R) = d$, and $\underline{x} = x_1, x_2, \ldots, x_d$ and $\underline{y} = y_1, y_2, \ldots, y_d$ be two systems of parameters such that $(\underline{y}) \subseteq (\underline{x})$. For each $j = 1, 2, \ldots, d$, say $y_j = \sum_{i=1}^d x_i a_{ij}$ with $a_{ij} \in R$. Denote the resulting $d \times d$ matrix $(a_{ij})_{d \times d}$ by A. Then*

(1) $(\underline{y})^* :_R (\underline{x}) \supseteq ((\underline{y}) + (\det(A)))^*$ *and* $(\underline{y})^* :_R \det(A) \supseteq (\underline{x})^*$.

Further assume that $(R, \mathfrak{m})$ is equidimensional and, moreover, that R is either excellent or a homomorphic image of a Cohen–Macaulay ring. Then

(2) *If R is Cohen–Macaulay, then we have* $(\underline{y}) :_R (\underline{x}) = (\underline{y}) + (\det(A))$ *and* $(\underline{y}) :_R \det(A) = (\underline{x})$

(2°) $(\underline{y})^* :_R (\underline{x}) = ((\underline{y}) + (\det(A)))^*$ *and* $(\underline{y})^* :_R \det(A) = (\underline{x})^*$.

(3) *For any $c \in R$, if Q is a test exponent for c and $(\underline{y}) \subseteq R$, then Q is a test exponent for c and $(\underline{x}) \subseteq R$.*

Proof. (1) This is straightforward (cf. [HH1, Proposition 4.1(b)(k)]).

(2) Follows from the fact that $\mathrm{H}^d_{\mathfrak{m}}(R)$ may be viewed as the direct limit of the modules $R/(\underline{x})R$ as the system of parameters $\underline{x}$ varies, that when R is Cohen–Macaulay the maps $R/(\underline{x})R \to \mathrm{H}^d_{\mathfrak{m}}(R)$ are injective, and that under our hypotheses there is a factorization $R/(\underline{x})R \to R/(\underline{y})R \to \mathrm{H}^d_{\mathfrak{m}}(R)$ in which the first map is given on the numerators by multiplication by $\det(A)$, so that multiplication by $\det(A)$ yields an injective map $R/(\underline{x})R \to R/(\underline{y})R$. This is equivalent to the second statement in (2). The annihilator W of $(\underline{x})$ in $\mathrm{H}^d_{\mathfrak{m}}(R)$, thought of as the directed union of the modules $H_t = R/(x_1^t, \ldots, x_d^t)$, is the union of the annihilators W_t in the various H_t. In a given H_t, W_t is generated by w_t, the image of $(x_1 \cdots x_d)^{t-1}$, each $w_t R \cong R/(\underline{x})$, and each w_t maps to w_{t+1} in the direct limit system. It follows that $W \cong R/(\underline{x})$. Since the image W' of $R/(\underline{x}) \to R/(\underline{y}) \subseteq \mathrm{H}^d_{\mathfrak{m}}(R)$ is already $\cong R/(\underline{x})$, it follows $W' = W$. Since the annihilator of $(\underline{x})R$ in $R/(\underline{y})R$ is between W' and W, it is equal to W'.

To prove (2°) and (3), we may assume $(R, \mathfrak{m})$ is an equidimensional homomorphic image of a Cohen–Macaulay ring S without loss of generality. (Indeed, in case R is equidimensional and excellent, it suffices to prove (2°) and (3) for $\widehat{R}$.)

(2°) By killing a maximal regular sequence in the kernel of the surjection $S \to R$, where S is Cohen–Macaulay local, we may assume that R and S have the same dimension: we will have that $R = S/I$ with I of pure height 0. We can choose $\widetilde{c}$ precisely in those minimal primes of S that do not contain I, so that its image c in R is in R°. Then $\widetilde{c}I$ is nilpotent, and after replacing $\widetilde{c}$ by a suitable power we can choose an integer $q_0 = p^{e_0}$ such that $\widetilde{c}I^{[q_0]} = 0$. By Lemma 1.5 below, we can choose a system of parameters $\underline{\widetilde{x}}$ for S that lifts $\underline{x}$ and a matrix $\widetilde{A} = (\widetilde{a}_{ij})$ over S that lifts $A = (a_{ij})$ such that if we define $\widetilde{y}_j = \sum_{i=1}^{d} \widetilde{x}_i \widetilde{a}_{ij}$, $1 \le j \le d$, then $\underline{\widetilde{y}}$ is also a system of parameters for S.

For both statements, "$\supseteq$" has already been proved in (1). Now suppose that $u \in R$ is such that $u(\underline{x}) \subseteq (\underline{y})^*$ (respectively, $u \det(A) \in (\underline{y})^*$). Then there exists q_1 and $b \in R^{\circ}$ such that $bu^q(\underline{x})^{[q]}$ (respectively, $b(u \det(A))^q$) is contained in $(\underline{y})^{[q]}$ for all $q \ge q_1$. We can lift $\underline{x}$, A and $\underline{y}$ as above to $\underline{\widetilde{x}}$, $\widetilde{A}$ and $\underline{\widetilde{y}}$. By a standard prime avoidance argument we can also lift b to an element $\widetilde{b} \in S^{\circ}$, and u to an element $\widetilde{u}$ of S. Then for all $q \ge q_1$, $\widetilde{b}\widetilde{u}^q(\underline{\widetilde{x}})^{[q]}$ (respectively, $\widetilde{b}(\widetilde{u}(\det(\widetilde{A}))^q)$ is contained in $(\underline{\widetilde{y}})^{[q]} + I$. Raise both sides to the q_0 power and multiply by $\widetilde{c}$. The contribution from I becomes 0, and, with $q' = qq_0$, we obtain that $\widetilde{c}\widetilde{b}^{q_0}\widetilde{u}^{q'}(\underline{\widetilde{x}})^{[q']}$ (respectively, $\widetilde{c}\widetilde{b}^{q_0}\widetilde{u}^{q'}(\det(\widetilde{A}))^{q'}$) is contained in $(\underline{\widetilde{y}})^{[q']}$. Since S is Cohen–Macaulay, we may apply part (2) to the systems of parameters and matrix arising from $\underline{\widetilde{x}}$, $\underline{\widetilde{y}}$ and $(\widetilde{a}_{ij})$ by taking qth powers of all elements to conclude that $\widetilde{c}\widetilde{b}^{q_0}\widetilde{u}^{q'} \subseteq ((\underline{\widetilde{y}}) + (\det(\widetilde{A})))^{[q']}$ (respectively, $\subseteq (\underline{\widetilde{x}})^{[q']}$) for all $q' \gg 0$. The required result now follows by taking images in R and applying the definition of tight closure.

(3) Suppose $cx^q \in (\underline{x})^{[q]}$ for some $x \in R$ and $q \ge Q$. Then $c(\det(A)x)^q = \det(A)^q cx^q \in (\underline{y})^{[q]}$ and hence $\det(A)x \in (\underline{y})^*$, which implies $x \in (\underline{y})^* :_R \det(A) = (\underline{x})^*$ by part (2°) above. □

Lemma 1.5. *Let S be a Cohen–Macaulay ring of dimension d, let I be an ideal of height 0, let $R = S/I$, let $\underline{x}$ and $\underline{y}$ be systems of parameters for R, and let $A = (a_{ij})$ be a matrix over R such that for all j, $1 \le j \le d$, $y_j = \sum_{i=1}^{d} a_{ij}x_i$. Then we can choose liftings $\underline{\widetilde{x}}$ and $\widetilde{A} = (\widetilde{a}_{ij})$ of the matrix A to S such that if we define $\widetilde{y}_j = \sum_{i=1}^{d} \widetilde{a}_{ij}\widetilde{x}_i$ for all j, $1 \le j \le d$, then $\underline{\widetilde{y}}$ is also a system of parameters for S.*

Proof. We may lift $\underline{x}$ to a system of parameters $\underline{\widetilde{x}}$ by [HH1, Lemma 7.10], and we assume this has been done. We prove by induction on k, $1 \le k \le d$, that we can choose the lifts $\widetilde{a}_{ij}$ for all i and for $1 \le j \le k$, the elements $\widetilde{y}_1, \dots, \widetilde{y}_k$ are part of a system of parameters for S. We assume that this has been done for $1 \le j \le k-1$ (we allow $k-1=0$), and we construct the elements $\widetilde{a}_{ik}$. First, choose elements $b_{ik} \in S$ arbitrarily that lift the a_{ik}. We will show that we can choose $\delta_1, \dots, \delta_d \in I$ such that the choice $\widetilde{a}_{ik} = b_{ik} + \delta_i$ for all i produces an element $\widetilde{y}_k$ not in any minimal prime of $(\widetilde{y}_1, \dots, \widetilde{y}_{k-1})$. Let $z = \sum_{i=1}^d b_{ik}\widetilde{x}_i$. Let $Q_1, \dots, Q_s$ be the minimal primes of $(\widetilde{y}_1, \dots, \widetilde{y}_{k-1})$, which will all have height $k-1$. We may assume these are numbered so that $Q_1, \dots, Q_h$ contain z and $Q_{h+1}, \dots, Q_s$ do not. Note that all of the Q_v that contain I occur for $v \ge h+1$, or else we would have y_k in a minimal prime of $(y_1, \dots, y_{k-1})$. Choose $\Delta \in I \cap (\bigcap_{v \ge h+1} Q_v) - (\bigcup_{t \le h} Q_t)$. This is possible, or else $I \cap (\bigcap_{v \ge h+1} Q_v) \subseteq \bigcup_{t \le h} Q_t$, and then $I \cap (\bigcap_{v \ge h+1} Q_v) \subseteq Q_t$ for some $t \le h$, which is impossible, since neither I nor any Q_v for $v \ge h+1$ is contained in Q_t for $t \le h$. Choose m so that $\Delta^m \in (\underline{\widetilde{x}})S$, which is possible because $\underline{\widetilde{x}}$ is a system of parameters for S. Then replace Δ by Δ^{m+1}, which is in $I(\underline{\widetilde{x}})$, so that we may assume that $\Delta = \sum_{i=1}^d \delta_i \widetilde{x}_i$ with the δ_i in I. These choices for the δ_i give what we need, since then $\widetilde{y}_k = z + \Delta$ and this element is not in any of the minimal prime Q_t of $(\widetilde{y}_1, \dots, \widetilde{y}_{k-1})$: we have that $z \in Q_t$ if and only if $\Delta \notin Q_t$. $\square$

2. Test exponents for Artinian modules and an alternative proof of Sharp's theorem

We first prove a result about the existence of a test exponent for Artinian modules. Although the argument can be traced back to [HH4] (for modules of finite length), we include a proof here for the sake of convenience and completeness.

Proposition 2.1 (Compare with [HH4, Proposition 2.6]). *Let R be a Noetherian ring of prime characteristic p and $N \subseteq M$ be R-modules such that M/N is Artinian. Assume there exists $d \in R^\circ$ that is a q_0-weak test element for $N_M^{[q]} \subseteq F_R^e(M)$ for all $q \gg 0$. Then, for any $c \in R^\circ$, there exists a test exponent for c and $N \subseteq M$.*

Proof. For every e, let $N_e = \{u \in M \mid cu^q \in (N_M^{[q]})^F_{F^e(M)}\}$. Then, as shown in the proof of [HH4, Proposition 2.6], $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_e \supseteq N_{e+1} \supseteq \cdots \supseteq N$ and hence there exists $Q = p^E$ such that $N_e = N_E$ for all $e \ge E$.

Suppose $cx^{q'} \in N_M^{[q']}$ for some $x \in M$ and $q' \ge Q$. Then $x \in N_{e'}$ and thus $x \in N_e$ for all $e \ge E$. This means $cx^q \in (N_M^{[q]})^F_{F^e(M)} \subseteq (N_M^{[q]})^*_{F^e(M)}$ for all $q \ge Q$. Consequently, $dc^{q_0}x^{qq_0} = d(cx^q)^{q_0} \in (N_M^{[q]})^{[q_0]}_{F^e(M)} = N_M^{[qq_0]}$ for all $q \gg Q$, which implies $x \in N_M^*$. $\square$

In the light of Theorem 0.5, we get the following consequence of Proposition 2.1.

THEOREM 2.2. *Let R be an algebra essentially of finite type over an excellent local ring of characteristic p, $c \in R^\circ$, and M an Artinian R-module. Then there exists a test exponent for c and M.*

Proof. This follows immediately from Theorem 0.5(2) and Proposition 2.1. $\square$

As an immediate consequence, we see that if $(R, \mathfrak{m})$ is an excellent local ring of characteristic p and $c \in R^\circ$, then there exists a test exponent for c and $\mathrm{H}^i_{\mathfrak{m}}(R)$ for all $i = 0, \ldots, \dim(R)$.

In fact, we can prove the existence of a test exponent for $\mathrm{H}^{\dim(R)}_{\mathfrak{m}}(R)$ under a weaker condition as follows.

PROPOSITION 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic p and $c \in R^\circ$. Assume $(R, \mathfrak{m})$ has the colon-capturing property and there exists a q_0-weak test element $b \in R^\circ$ for all parameter ideals of R.*

Then there exists a test exponent for c and $0 \subset \mathrm{H}^{\dim(R)}_{\mathfrak{m}}(R)$.

Proof. Say $\dim(R) = d$. Then $\mathrm{H}^d_{\mathfrak{m}}(R) = \varinjlim_{\underline{x}} \frac{R}{(\underline{x})R}$, in which $\underline{x}$ runs through all systems of parameters of R. For any $u \in R$ and any system of parameters $\underline{x} = x_1, \ldots, x_d$ of R, denote the image of $\frac{u}{(x_1,\ldots,x_d)}$ in $\mathrm{H}^d_{\mathfrak{m}}(R)$ by $[\frac{u}{(x_1,\ldots,x_d)}]$. Recall that, for any $e \in \mathbb{N}$, there is a canonical isomorphism $F^e_R(\mathrm{H}^d_{\mathfrak{m}}(R)) \cong \mathrm{H}^d_{\mathfrak{m}}(R)$, under which we may simply write $[\frac{u}{(x_1,\ldots,x_d)}]^q_{\mathrm{H}^d_{\mathfrak{m}}(R)} = [\frac{u^q}{(x_1^q,\ldots,x_d^q)}]$. By colon-capturing, we see that $[\frac{u}{(x_1,\ldots,x_d)}] \in 0^*_{\mathrm{H}^d_{\mathfrak{m}}(R)}$ if and only if $u \in (x_1, \ldots, x_d)^*_R$ (cf. [Sm, Proposition 2.5]). This implies that b is a weak test element for $0 \subset \mathrm{H}^d_{\mathfrak{m}}(R)$. (Indeed, for any $[\frac{u}{(x_1,\ldots,x_d)}] \in 0^*_{\mathrm{H}^d_{\mathfrak{m}}(R)}$, we have $u \in (x_1, \ldots, x_d)^*_R$. Then $bu^q \in (x_1, \ldots, x_d)^{[q]}$ for all $q \geq q_0$, which implies $b[\frac{u}{(x_1,\ldots,x_d)}]^q_{\mathrm{H}^d_{\mathfrak{m}}(R)} = [\frac{bu^q}{(x_1^q,\ldots,x_d^q)}] = 0 \in F^e(\mathrm{H}^d_{\mathfrak{m}}(R))$ for all $q \geq q_0$.) Consequently, b is a weak test element for $0 \subset F^e(\mathrm{H}^d_{\mathfrak{m}}(R))$ for all $e \in \mathbb{N}$. Thus, by Proposition 2.1, there exists a test exponent, say $Q = p^E$, for c and $\mathrm{H}^d_{\mathfrak{m}}(R)$. $\square$

Now we are ready to give a new proof of R. Y. Sharp's result about a uniform test exponent for $c \in R^\circ$ and all ideals generated by systems of parameters.

THEOREM 2.4 (Sharp [Sh, Theorem 3.2]). *Let $(R, \mathfrak{m})$ be an equidimensional excellent local ring of prime characteristic p and $c \in R^\circ$. Then there exists a uniform test exponent for c and all ideals generated by (partial or full) systems of parameters of R.*

Proof. Say $\dim(R) = d$. By Proposition 2.3, there is a test exponent Q for c and $\mathrm{H}^d_{\mathfrak{m}}(R)$. Here we keep the same usage of $[\frac{u}{(x_1,\ldots,x_d)}]$ as in the above proof of Proposition 2.3.

Now, it suffices to show that Q is a test exponent for c and $(x_1, \ldots, x_i) \subseteq R$ for any (partial or full) system of parameters $\underline{x} = x_1, \ldots, x_i$ of R. But, then, it suffices to verify the case where $\underline{x} = x_1, \dots, x_d$ is any full system of parameters, since for any q, $cu^q \in (x_1^q, \dots, x_i^q, x_{i+1}^{qt}, \dots, x_d^{qt})$ for all t if and only if $cu^q \in (x_1^q, \dots, x_i^q)$.

Finally, for any $u \in R$ and $q \ge Q$, suppose $cu^q \in (\underline{x})^{[q]} = (x_1^q, \dots, x_d^q)$. This implies $c[\frac{u}{(x_1,\dots,x_d)}]^q_{\mathrm{H}^d_{\mathfrak{m}}(R)} = 0 \in F^e_R(\mathrm{H}^d_{\mathfrak{m}}(R))$. Thus, by the choice of Q, $[\frac{u}{(x_1,\dots,x_d)}] \in 0^*_{\mathrm{H}^d_{\mathfrak{m}}(R)}$, which forces $u \in (x_1, \dots, x_d)^*_R$ by colon-capturing as in Proposition 2.3 (cf. [Sm, Proposition 2.5]). □

Next, we state a corollary of the theorem above. In a Noetherian ring A, a sequence of elements $x_1, \dots, x_n$ is called a *sequence of parameters* if their images form part of a system of parameters in every local ring A_P for all prime ideals P containing the ideal $I = (x_1, \dots, x_n)A$. In this case, we refer to I as an *ideal generated by parameters* or as a *parameter ideal.* See [HH2, §2].

Corollary 2.5. *Let $(R, \mathfrak{m})$ be an equidimensional excellent local ring of prime characteristic p and $c \in R^\circ$. Then there exists a uniform test exponent for $c/1$ and all ideals generated by parameters of $S^{-1}R$ (over the ring $S^{-1}R$) for all multiplicatively closed subset $S \subset R$.*

In particular, there exists a uniform test exponent for $c/1$ and all ideals generated by (partial or full) systems of parameters of R_P (over R_P) for all $P \in \operatorname{Spec}(R)$.

Proof. By Theorem 2.4, there exists a test exponent $Q = p^E$ for c and all ideals generated by parts of systems of parameters of R. Fix an arbitrary multiplicative subset $S \subset R$. Let $J = (y_1, \dots, y_h)S^{-1}R$ be a parameter ideal of $S^{-1}R$ generated by a sequence of parameters $y_1, \dots, y_h \in S^{-1}R$. It sufices to show that Q is a test exponent for $c/1$ and $J = (y_1, \dots, y_h)$ over $S^{-1}R$.

By [AHH, Lemma 3.3(a)], there exists a sequence of parameters $x_1, \dots, x_h$ of R such that $(x_1, \dots, x_h)S^{-1}R = J$. (We apply [AHH, Lemma 3.3(a)] to find $x_1 \in R^\circ$ such that $S^{-1}(x_1) = (y_1)S^{-1}R$. Then apply [AHH, Lemma 3.3(a)] to $R/(x_1)$ to find x_2 whose image is in $(R/(x_1))^\circ$; and so on.)

Now suppose $(c/1)v^q \in J^{[q]}$ for some $v \in S^{-1}R$ and $q \ge Q$. Without loss of generality, we may assume $v = u/1$ with $u \in R$. That is, there exists $s \in S$ such that $scu^q \in (x_1, \dots, x_h)^{[q]}R$. Hence $c(su)^q \in (x_1, \dots, x_h)^{[q]}R$, which implies $su \in (x_1, \dots, x_h)^*_R$. Therefore, $v = u/1 \in (x_1, \dots, x_h)^*_R S^{-1}R \subseteq (S^{-1}(x_1, \dots, x_h))^*_{S^{-1}R} = J^*_{S^{-1}R}$. This completes the proof. □

3. Modules with finite projective dimension

Question 3.1. Assume $(R, \mathfrak{m})$ is an equidimensional local ring of prime characteristic p such that R is either excellent or a homomorphic image of a

Cohen–Macaulay ring. For a given $c \in R^{\circ}$, does there exist a test exponent for c and all finitely generated R-modules of finite phantom projective dimension?

We say that a finitely generated R-module M is *perfect* if $\mathrm{pd}_R(M) = \mathrm{grade}(\mathrm{Ann}_R(M))$, with $\mathrm{grade}(I)$ being the length of any maximal R-regular sequence contained in the ideal I. For example, over a local ring $(R, \mathfrak{m})$, every R-module of finite length and finite projective dimension is perfect.

If R is Cohen–Macaulay, it is known that phantom projective dimension is the same as projective dimension. For this reason, the following theorem may be viewed as a partial answer to Question 3.1.

THEOREM 3.2. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay Noetherian local ring of prime characteristic p. For $c \in R$, if $Q = p^E$ is a uniform test exponent for c and all ideals generated by (full) systems of parameters of R, then Q is a uniform test exponent for c and all perfect R-modules.*

Proof. Let $M \neq 0$ be a perfect R-module; say $\mathrm{pd}(M) = \mathrm{grade}(\mathrm{Ann}(M)) = d$. Fix an R-regular sequence $\underline{x} = x_1, \ldots, x_d$ in $\mathrm{Ann}_R(M)$.

Suppose $cu^{q'} = 0 \in F^{e'}(M)$ for some $u \in M$ and $e' \geq E$ (i.e., $q' \geq Q$). All we need to show is $u \in 0^*_M$.

Fix a minimal projective resolution $G_\bullet$ of M as follows

$$G_\bullet : 0 \longrightarrow G_d \xrightarrow{\phi_d} G_{d-1} \xrightarrow{\phi_{d-1}} \cdots \xrightarrow{\phi_2} G_1 \xrightarrow{\phi_1} G_0 \longrightarrow 0.$$

Also construct the Koszul complex $K_\bullet(\underline{x}, R)$ as follows

$$K_\bullet(\underline{x}, R) : 0 \longrightarrow K_d \xrightarrow{\psi_d} K_{d-1} \xrightarrow{\psi_{d-1}} \cdots \xrightarrow{\psi_2} K_1 \xrightarrow{\psi_1} K_0 \longrightarrow 0,$$

where $K_i = R^{\binom{d}{i}}$. In particular, ψ_d is represented by matrix $(x_1, x_2, \ldots, x_d)$ and the 0th homology of $K_\bullet(\underline{x}, R)$ is $R/(\underline{x})$. Since $(\underline{x}) \subseteq \mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(u)$, there exists an R-linear map $h : R/(\underline{x}) \to M = H_0(G_\bullet)$ sending the class of 1 to u. This map h can be lifted to a chain map

$$g_\bullet : K_\bullet(\underline{x}, R) \longrightarrow G_\bullet.$$

Denote $g_0(1) = y$; so $cy^{q'} \in (\mathrm{Image}(\phi_1))^{[q']}_{G_0}$. Now we only need to show $y \in (\mathrm{Image}(\phi_1))^*_{G_0}$.

For every q, there is an induced R-linear chain map $g_\bullet^{[q]} : F^e(K_\bullet(\underline{x}, R)) \to F^e(G_\bullet)$. Now the fact that $cy^{q'} \in (\mathrm{Image}(\phi_1))^{[q']}_{G_0}$ (i.e., $cu^{q'} = 0$) implies that the chain map $cg_\bullet^{[q']}$ is homotopic to the zero chain map. In particular, there exists $\delta_{d-1} \in \mathrm{Hom}_R(F^{e'}(K_{d-1}), F^{e'}(G_d))$ such that $cg_d^{[q']} = \delta_{d-1} \circ \psi_d^{[q']}$. Applying $\mathrm{Hom}_R(-, R)$, we get

$$\begin{aligned} c\big(\mathrm{Image}\big(\mathrm{Hom}(g_d, R)\big)\big)^{[q']}_{K_d} &= \mathrm{Image}\big(\mathrm{Hom}\big(cg_d^{[q']}, R\big)\big) \\ &\subseteq \mathrm{Image}\big(\mathrm{Hom}\big(\psi_d^{[q']}, R\big)\big) = (\underline{x})^{[q']}, \end{aligned}$$

which implies $\mathrm{Image}(\mathrm{Hom}(g_d, R)) \subseteq (\underline{x})^*_R$ since $q' \geq Q$. That is to say that there exists $b \in R^\circ$ such that

$$\begin{aligned}\mathrm{Image}\big(\mathrm{Hom}\big(bg_d^{[q]}, R\big)\big) &= b\,\mathrm{Image}\big(\mathrm{Hom}\big(g_d^{[q]}, R\big)\big)\\ &= b\big(\mathrm{Image}\big(\mathrm{Hom}(g_d, R)\big)\big)^{[q]}_R \subseteq (\underline{x})^{[q]}\\ &= \mathrm{Image}\big(\mathrm{Hom}\big(\psi_d^{[q]}, R\big)\big)\end{aligned}$$

for all $q \gg 0$. Therefore, the chain maps

$$\mathrm{Hom}\big(bg_\bullet^{[q]}, R\big) : \mathrm{Hom}\big(F^e(G_\bullet), R\big) \to \mathrm{Hom}\big(F^e\big(K_\bullet(\underline{x}, R)\big), R\big)$$

are homotopic to 0 for all $q \gg 0$. So there exist $\varepsilon_1^{[q]} \in \mathrm{Hom}_R(F^e G_1), F^e(K_0))$ such that $\mathrm{Hom}(bg_0^{[q]}, R) = \varepsilon_1^{[q]} \circ \mathrm{Hom}(\phi_1^{[q]}, R)$ for all $q \gg 0$. This, after going through $\mathrm{Hom}(-, R)$ again, would in turn imply

$$by^q \in b\big(\mathrm{Image}(g_0)\big)^{[q]}_{G_0} = \mathrm{Image}\big(bg_0^{[q]}\big) \subseteq \mathrm{Image}\big(\phi_1^{[q]}\big) = \big(\mathrm{Image}(\phi_1)\big)^{[q]}_{G_0}$$

for all $q \gg 0$. This allows us to conclude $y \in (\mathrm{Image}(\phi_1))^*_{G_0}$ by definition, and the proof is complete. ☐

We remark that the above argument of using homotopy to determine membership in the tight closure has appeared in [Ab2].

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay Noetherian excellent local ring of prime characteristic p. Then, for any $c \in R^\circ$, there is a test exponent for c and all R-modules of finite length and of finite (phantom) projective dimension.*

Proof. This follows from Theorem 0.7 and Theorem 3.2. ☐

We also notice that Question 3.1 reduces to the Cohen–Macaulay case if $\dim(R) \leq 2$.

Corollary 3.4. *Let $(R, \mathfrak{m})$ be an equidimensional excellent Noetherian local ring of prime characteristic p with $\dim(R) \leq 2$. Then, for any given $c \in R^\circ$, there exists a test exponent for c and all R-modules of finite length and of finite phantom projective dimension.*

Proof. By [HH1, Definition 9.1], we observe that any R-module of finite length and of finite phantom projective dimension over R remains so after we extend the scalar to the integral closure of R/P in its fraction field for every $P \in \min(R)$. Therefore, by Lemma 1.2 and Lemma 1.3, we may assume that R is normal without loss of generality. (We may assume that R is complete as well.) But now R is excellent Cohen–Macaulay and the claim follows from Corollary 3.3. ☐

Lastly, we remark that Corollary 3.3 plays an important role in an upcoming paper [HY], where the F-rational signature is defined and studied. To be specific, the existence of a uniform test exponent allows us to characterize F-rationality in terms of the (phantom) F-rational signature being positive.

References

[Ab1] I. Aberbach, *Finite phantom projective dimension*, Amer. J. Math. **116** (1994), no. 2, 447–477. MR 1269611

[Ab2] I. Aberbach, *Tight closure in F-rational rings*, Nagoya Math. J. **135** (1994), 43–54. MR 1295816

[AHH] I. Aberbach, M. Hochster and C. Huneke, *Localization of tight closure and modules of finite phantom projective dimension*, J. Reine Angew. Math. **434** (1993), 67–114. MR 1195691

[El] H. Elitzur, *Tight closure in Artinian modules*, Ph.D. thesis, University of Michigan, Ann Arbor, MI, 2003. MR 2704806

[HH1] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon–Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no. 1, 31–116. MR 1017784

[HH2] M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62. MR 1273534

[HH3] M. Hochster and C. Huneke, *Phantom homology*, Mem. Amer. Math. Soc. **103** (1993), no. 490, 1–91. MR 1144758

[HH4] M. Hochster and C. Huneke, *Localization and test exponents for tight closure*, Michigan Math. J. **48** (2000), 305–329. MR 1786493

[HY] M. Hochster and Y. Yao, *The F-rational signature and drops in the Hilbert–Kunz multiplicity*, preprint.

[McD] M. McDermott, *Tight closure, plus closure and Frobenius closure in cubical cones*, Ph.D. thesis, University of Michigan, Ann Arbor, MI, 1996. MR 2694487

[Sh] R. Y. Sharp, *Tight closure test exponents for certain parameter ideals*, Michigan Math. J. **54** (2006), no. 2, 307–317. MR 2252761

[Sm] K. Smith, *F-rational rings have rational singularities*, Amer. J. Math. **119** (1997), no. 1, 159–180. MR 1428062

Melvin Hochster, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

E-mail address: `hochster@umich.edu`

Yongwei Yao, Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, USA

E-mail address: `yyao@gsu.edu`