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ON CHOW GROUPS OF COMPLETE REGULAR LOCAL RINGS

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ABSTRACT. In this paper, we establish the validity of the Chow group problem for complete regular local rings R of dimension up to 4. For dimension n (> 4) over ramified regular local ring R, we have two results: (1) When I is an ideal of height 3 such that R/I is a Gorenstein ring, then $[I] = 0$ in $A_{n-3}(R)$. (2) We reduce any prime ideal of height i to an almost complete intersection ideal of height i and in some special cases of almost complete intersection ideal of height i, we show that all Chow groups except the top one vanish. A necessary and sufficient condition for the vanishing of Chow groups is also derived using Eisenstein extension.

1. Introduction

The main focus of this paper is to study some important cases of Chow groups of complete regular local rings. We define the ith Chow group $A_i(R)$ for a Noetherian Cohen–Macaulay ring R of dimension (henceforth dim) n by $Z_i(R)/\operatorname{Rat}_i(R)$, where $Z_i(R)$ is the free Abelian group generated by the prime ideals in R of height (henceforth ht) $n-i$ and $\operatorname{Rat}_i(R)$ is the subgroup of $Z_i(R)$ generated by the cycles of the form $\sum l(R_{P_i}/(q+(x))R_{P_i})[P_i]$, where q is a prime ideal of height $n-i-1, x \notin q$ and P_i ranges over the minimal associated prime ideals of $R/(q+(x))$ satisfying $\dim R/P = \dim R/(q+(x))$. When M is a finitely generated R-module, we denote by $[M]$ the cycle $\sum l(M_{P_i})[P_i]$, where P_i ranges over the minimal associated prime ideals of M satisfying $\dim R/P_i = \dim M$. From Claborn and Fossum [2], if R is a regular local ring, then the above definition is equivalent to the group $Z_i(R)/\langle R/(x_1, x_2, \dots, x_{n-i})\rangle$, where $\langle R/(x_1, x_2, \dots, x_{n-i})\rangle$ is the subgroup of

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$Z_i(R)$ generated by $\sum l(R/(x_1,x_2,\ldots,x_{n-i}))_P[P]$, P ranges over the minimal associated prime ideals of $R/(x_1,x_2,\ldots,x_{n-i})$, for all R-sequence $x_1,x_2,\ldots,x_{n-i}$. The Chow group problem asserts that for any regular local ring R of dimension n, $A_{n-i}(R)=0$ for $i<n$. The validity of the Chow group for equicharacteristic and unramified regular local rings is known due to works of Claborn and Fossum [2], Quillen [8] and Gillet and Levine [5].

In this paper in Section 2, we establish the vanishing of Chow groups for regular local rings of dimension ≤ 4. We utilize several results from Claborn and Fossum [2], Dutta [3] and Smoke [9] to prove our assertions. We also show that for any regular local ring of dimension n, $A_{n-2}(R)=0$. Section 3 contains a brief discussion on multi-linear algebra which can be applied to the structure theorem for a Gorenstein ideal of codimension 3. Using the structure theorem (Theorem 3.2) and a lifting theorem (Proposition 3.3), we prove that when I is a height 3 ideal in a ramified regular local ring R of dimension n such that R/I is a Gorenstein ring, then $[I]=0$ in $A_{n-3}(R)$. In Section 4, to calculate the Chow group, at first, we deal with some special cases of Chow groups of almost complete intersection ideals and next, we reduce any prime ideal of height i to an almost complete intersection ideal of height i. Finally, we show that $[R/P]=0$ in $A_{n-i}(R)\Leftrightarrow[H^0_f(A)/fH^0_f(A)]=0$ in $A_{n-i}(R)$, where the ramified regular local ring R is an Eisenstein extension of the form $R=B[X]/(f(X)), f(X)$ is an Eisenstein polynomial in $B[X]$, $B=V[[X_1,X_2,\ldots,X_{n-1}]]$ and V is a complete discrete valuation ring.

2. Chow groups of complete regular local rings of dimension up to 4

Let R be a regular local ring and $\underline{R_i}$ be the category of all finitely generated R-modules M such that $M_P=0$ for all prime ideals of height less than i. Let $K_0(\underline{R_i})$ be the corresponding Grothendieck group. Then $K_0(\underline{R_i})$ has a generator $[M]$ for each isomorphic class of modules M, with a relation $[M]=[M']+[M'']$ for each exact sequence

$$0\longrightarrow M'\longrightarrow M\longrightarrow M''\longrightarrow 0$$

of modules.

Theorem 2.1. *Let R be a complete regular local ring of dimension n. Then $A_{n-2}(R)=0$.*

In order to prove Theorem 2.1, we need the following result from Smoke [9].

Theorem 2.2 (Smoke [9]). *Let R be a regular local ring of dimension n, then the group $K_0(\underline{R_2})$ is generated by the elements $[R/(x_1,x_2)]$, where x_1,x_2 form a regular sequence on R.*

Proof of Theorem 2.1. In the proof of Theorem 2.2, Smoke first showed that every $[M]$ in $K_0(\underline{R_2})$ is generated by $[R/I]$ and $[R/(x_1,x_2)]$, where x_1,x_2 in I form an R-sequence and I is a perfect ideal of codimension 2. Next, he showed that $[I/(x_1,x_2)]$ is generated by elements of the form $[R/(x_1,x_2)]$. We know that $[R/(x_1,x_2)]=0$ in $A_{n-2}(R)$. Thus, $[I/(x_1,x_2)]=0$ in $A_{n-2}(R)$. Now from the following exact sequence

$$0 \longrightarrow I/(x_1,x_2) \longrightarrow R/(x_1,x_2) \longrightarrow R/I \longrightarrow 0$$

we conclude that $[R/I]=0$ in $A_{n-2}(R)$. Hence, $[M]=0$ in $A_{n-2}(R)$ and therefore $A_{n-2}(R)=0$ by the choice of M. □

Remark. If P is a complete intersection ideal of height i in a regular local ring of dimension n, then, from the definition of the $(n-i)$th Chow group, $[P]=0$ in $A_{n-i}(R)$ for $0\le i<n$. Hence, $A_0(R)=0, A_{n-1}(R)=0$. Moreover, it is easy to see that $A_n(R)=Z$.

In order to characterize complete regular local rings of dimension up to 4, we need the following result due to Dutta [3].

Theorem 2.3 (Dutta [3]). *Let R be a complete regular local ring of dimension n, then $A_1(R)=0$.*

Now we have an immediate corollary which is stated as a theorem.

Theorem 2.4. *Let R be a complete regular local ring of dimension ≤ 4. Then $A_i(R)=0$ for $0\le i<\dim R$.*

Proof follows from Theorem 2.1, Theorem 2.3 and the above remark.

Remark. In a complete regular local ring dimension 5, we know that $A_i(R)=0$ for $i=0,1,3,4$ and $A_5(R)=Z$. Thus in this case, the determination of $A_2(R)$ remains open.

3. Lifting of Gorenstein rings

Our main purpose of this section is to prove Theorem 3.1. We use the lifting property of a Gorenstain ideal of codimension 3 and lifting theorem.

Theorem 3.1. *Let R be a regular local ring of dimension n, and let I be an ideal in R of height 3 such that R/I is a Gorenstein ring. Then $[I]=0$ in $A_{n-3}(R)$.*

Before proving Theorem 3.1, we describe some terminology: Let R be a commutative ring with 1 and F be a finitely generated free R-module. Then a map $g: F^* \to F$ is said to be alternating with respect to some (and therefore every) basis and dual basis of F and F^*, if the matrix representation of g is skew symmetric and all its diagonal entries are 0. More invariantly, let $\sigma \in F\otimes F$ be the element corresponding to g under the isomorphism

$$\operatorname{Hom}\big(F^*,F\big) \cong \big(F^*\big)^* \otimes F \cong F$$

and identify $\wedge^2 F$ as a submodule of $F \otimes F$. Then g is alternating if and only if $\sigma \in \wedge^2 F$. If F has even rank and $g : F^* \to F$ is alternating, then it turns out that $\det(g)$ is the square of a polynomial function of the entries of a matrix for g, called the Pfaffian of g. We will write $\det(g) = (\mathrm{Pf}(g))^2$. More generally, if F has any rank, and G is a free summand of F of even rank n, with projection $\pi : F \to G$, then the composite

$$h : G^* \xrightarrow{\pi^*} F^* \xrightarrow{g} F \xrightarrow{\pi} G$$

is an alternating map, represented in $\wedge^2 G$ by $\wedge^2 \pi(\sigma)$, and we say that $\mathrm{Pf}(h)$ is a Pfaffian of order n of g. We define $\mathrm{Pf}_n(g)$ to be the ideal generated by all the nth order Pfaffians of g.

In the proof of Theorem 3.1, we need the following result from Buchsbaum and Eisenbud [1].

THEOREM 3.2 (Buchsbaum and Eisenbud [1]). *Let (R, m, k) be a noetherian local ring.*

(1) *Let $n \geq 3$ be an odd integer, and F be a free R-module of rank n. Let $g : F^* \to F$ be an alternating map whose image is contained in mF. Suppose* $\mathrm{Pf}_{n-1}(g)$ *has grade* 3. *Then* $\mathrm{Pf}_{n-1}(g)$ *is a Gorenstein ideal, minimally generated by n elements.*

(2) *Every Gorenstein ideal of grade* 3 *arises as in* (1).

Moreover, they showed that when g is alternating, then $\mathrm{grade}(\mathrm{Pf}_{n-1}(g)) \leq 3$ and $\mathrm{grade}(\mathrm{Pf}_{n-1}(g)) = 3$ if and only if $\mathrm{Pf}_{n-1}(g)$ is a Gorenstein ideal. If $\mathrm{Pf}_{n-1}(g)$ is a Gorenstein ideal, then the following sequence is exact.

$$0 \longrightarrow R \xrightarrow{l^*} F^* \xrightarrow{g} F \xrightarrow{l} R \longrightarrow R/\mathrm{Pf}_{n-1}(g) \longrightarrow 0.$$

In the course of our proof of Theorem 3.1, we will also need the following result due to Dutta [4].

PROPOSITION 3.3 (Dutta [2]). *Let $(T, m, k), k$ infinite, be a Cohen–Macaulay local ring of dimension n and f be a nonzero-divisor in m. Let I be an ideal in T of height i such that $[I] = 0$ in $A_{n-i}(T)$ and $f \notin I$. Then $[I + fT/fT] = 0$ in $A_{n-1-i}(T/fT)$.*

Proof of Theorem 3.1. Let $R = V[[X_1, X_2, \ldots, X_n]]/(f) = T/(f)$, where $T = V[[X_1, X_2, \ldots, X_n]]$ and f is an Eisenstein polynomial in X_n over the ring $V[[X_1, X_2, \ldots, X_{n-1}]]$. Since I is a Gorenstein ideal of height 3, by Theorem 3.2, we have a minimal resolution of R/I as follows:

$$C_1 : 0 \longrightarrow R \xrightarrow{l^*} F^* \xrightarrow{g} F \xrightarrow{l} R \longrightarrow R/I \longrightarrow 0.$$

Let $n =$ rank of $F =$ minimal number of generators of I. In C_1, g is an alternating map and $I = \mathrm{Pf}_{n-1}(g)$. Write $G = T^n$.

Let $\tilde{g} : G^* \longrightarrow G$ be a lift of g such that $\tilde{g}$ is also alternating (i.e., $\tilde{g} \otimes id_R = g$).

Now consider the following sequence

$$C_2 : 0 \longrightarrow T \xrightarrow{\tilde{l}} G^* \xrightarrow{\tilde{g}} G \xrightarrow{\tilde{l}} T \longrightarrow T/J \longrightarrow 0,$$

where $J = \mathrm{Pf}_{n-1}(\tilde{g})$.

Note that $C_2 \otimes id_R = C_1$.

From the proof of Theorem 3.2, we know that (a) C_2 is a complex and (b) C_2 is exact if $\mathrm{ht}\, J = 3$.

First, we want to show that $\mathrm{ht}\, J = 3$.

Note that $J + (f)/(f) \cong I$. We also know that $\mathrm{ht}\, J \leq 3$. If $\mathrm{ht}\, J \leq 2$ and Q is a minimal prime ideal of J such that $\mathrm{ht}\, Q = \mathrm{ht}\, J$, then $\mathrm{ht}(Q + (f)/(f)) \leq 2$. But this would imply $\mathrm{ht}\, I = \mathrm{ht}(J + (f)/(f)) \leq \mathrm{ht}(Q + (f)/(f)) \leq 2$—which is a contradiction. Hence, C_2 is exact and since $C_2 \otimes id_R = C_1$ we derive that $J \cap (f) = fJ$, that is, f is a nonzero-divisor on T/J. This shows that J is a lifting of I to T. Moreover, by Theorem 2.1 $[J] = 0$ in $A_{n-2}(T)$. Hence, by Proposition 3.3, we have $[I] = 0$ in $A_{n-3}(R)$. □

Remark. In [7], we have a completely different approach to Theorem 3.1. Instead of lifting criteria, we use the idea of linkage of ideals for height 3 Gorenstein ideals. For details, refer to Lee [7].

4. Reduction to an almost complete intersection ideal

Let $P = (x_1, x_2, \ldots, x_{i+1})$ be an almost complete intersection prime ideal of height i in a regular local ring R of dimension n. Then we have two cases to consider:

Case 1. There are i elements in a minimal generating set of P which do not form an R-sequence.

Case 2. Any set of i elements in a minimal generating set of P form an R-sequence.

Proposition 4.1. *Let R, P be as above. Then $[P] = 0$ in $A_{n-i}(R)$ in Case* 1.

Proof. Suppose that $x_1, x_2, \ldots, x_i$ is not an R-sequence. Let $I = (x_1, x_2, \ldots, x_i)$ and Q be a minimal prime ideal containing I such that $\mathrm{ht}\, I = \mathrm{ht}\, Q = i - 1$. Then $x_{i+1} \notin Q$. By the definition of $(n-i)$th Chow group, we have $[Q + (x_{i+1})] = 0$ in $A_{n-i}(R)$. Since P is contained in $Q + (x_{i+1})$ and $\mathrm{ht}(Q + (x_{i+1})) = i$, we have $Q + (x_{i+1}) = P$ and therefore $[P] = 0$ in $A_{n-i}(R)$. □

Corollary 4.2. *Let R, P be as above. If there exists a subset of i elements, say, $x_1, x_2, \ldots, \check{x_j}, \ldots, x_{i+1}$ such that it forms an R-sequence, but $x_1, x_2, \ldots, \check{x_j}, \ldots, x_i, ax_{i+1} + x_j$ do not form an R-sequence for some $a \in R$ (where $\check{x_j}$ means deleting x_j), then $[P] = 0$ in $A_{n-i}(R)$.*

Proof. Let $I = (x_1, x_2, \dots, \check{x_j}, \dots, x_i, ax_{i+1} + x_j)$ and Q be a minimal prime ideal containing I such that $\operatorname{ht} I = \operatorname{ht} Q = i - 1$. Then, by assumption on the R-sequence, $x_{i+1} \notin Q$. Thus $Q + (x_{i+1}) = P$ and $[P] = [Q + (x_{i+1})] = 0$ in $A_{n-i}(R)$. □

In view of Proposition 4.1 and Corollary 4.2, it would be interesting to find out whether the Chow group problem for arbitrary prime ideals can be reduced to the same for almost complete intersection ideals. We have the following.

Let P be a prime ideal of height r in a regular local ring (R, m) of dimension n. Then we can find $x_1, x_2, \dots, x_r \in P$ an R-sequence such that $(x_1, x_2, \dots, x_r)R_P = PR_P$. Let's denote $(x_1, x_2, \dots, x_r)$ be $(\underline{x})$.

If $(\underline{x}) = P \cap q_2 \cap \cdots \cap q_t$ be a primary decomposition such that q_i being P_i primary for $i = 2, 3, \dots, t$. Let

$$\Omega = H_m^{n-r}(R/P)^\vee = \operatorname{Hom}_{R/(\underline{x})}\bigl(R/P, R/(\underline{x})\bigr)$$

denote the canonical module for R/P. Then Ω can be viewed as an ideal of $R/(\underline{x})$.

LEMMA 4.3. *Let $\Omega, q_2, \dots, q_t, (\underline{x})$ be as above. Then $\Omega = q_2 \cap \cdots \cap q_t/(\underline{x})$.*

Proof. Since R/q_i is P_i-primary for each i, for any $a \in P - P_i$, the map $R/q_i \longrightarrow R/q_i$ is injective. Thus, for any $\mu \in \Omega$, $a\mu = 0$ implies $\mu \in q_i$, for each i. Hence, $\Omega \subseteq q_2 \cap \cdots \cap q_t/(\underline{x})$. Conversely, for any $\eta \in q_2 \cap \cdots \cap q_t/(\underline{x})$, $\eta P \subset (\underline{x})$. Hence, we have $\Omega \supseteq q_2 \cap \cdots \cap q_t/(\underline{x})$. □

DEFINITION. Let $\underline{R_i}$ be the category of finitely generated R-modules M such that $P \in \operatorname{Supp} M$ if and only if $\operatorname{ht} P \geq i$.

REMARK. (1) From the definition, we know that $\underline{R_i} = \{M :$ finitely generated R-module$| M_P \neq 0$ if $\operatorname{ht} P \geq i\}$. Let P be a minimal prime ideal containing $\operatorname{ann} M$. Then $M \in \underline{R_i}$ if and only if $\dim M = \dim R/(\operatorname{ann} M) = \dim R - \operatorname{ht}(\operatorname{ann} M) = \dim R - \operatorname{ht} P \leq n - i$. Hence, we have $M \in \underline{R_i}$ if and only if $\dim M \leq n - i$, where $n = \dim R$.

(2) Let $K_0(\underline{R_i})$ be the corresponding Grothendieck group. Then $K_0(\underline{R_i})$ has a generator $[M]$ for each isomorphic class of modules of M in $\underline{R_i}$, with a relation $[M] = [M'] + [M'']$ for each exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of modules.

(3) For $M \in \underline{R_i}$, we define $\chi_i(M) = \sum_{\operatorname{ht} P = i} \ell_{R_P}(M_P)$, then we have the followings:

(i) Since $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence in $\underline{R_i}$, we have $\chi_i(M) = \chi_i(M') + \chi_i(M'')$.

(ii) $M \in \underline{R_{i+1}}$ if and only if $\chi_i(M) = 0$ (Claborn and Fossum [2]).

Combine (i) and (ii) we conclude that if $\dim M \le n-(i+1)$, then $[M]=0$ in $A_{n-i}(R)$.

In regards to the Chow group problem, now we are ready to reduce any prime ideal to an almost complete intersection ideal case.

THEOREM 4.4. *Let* (R,m) *be a regular local ring of dimension* n *and* $P \in \operatorname{Spec} R$ *such that* $\operatorname{ht} P = i$. *Then we can find* $x_1, x_2, \dots, x_i, \lambda \in P$ *such that* $[R/P] = [R/(x_1,x_2,\dots,x_i,\lambda)]$ *in* $A_{n-i}(R)$.

Proof. Since R is a regular local ring, we can find an R-sequence $x_1, x_2, \dots, x_i \in P$ such that $(x_1,x_2,\dots,x_i)R_P = PR_P$. Set $(x_1,x_2,\dots,x_i) = (\underline{x})$.

Let $(\underline{x}) = P \cap q_2 \cap \cdots \cap q_t$ be a primary decomposition such that q_j is P_j primary for $j = 2,3,\dots,t$. Choose $\lambda \in P - \bigcup_{j=2}^{t} P_j$. Then we have a short exact sequence

$$0 \longrightarrow P/(\underline{x},\lambda) \longrightarrow R/(\underline{x},\lambda) \longrightarrow R/P \longrightarrow 0.$$

Now we claim that $\dim P/(\underline{x},\lambda) \le n-(i+1)$.

By Lemma 4.3, $q_2 \cap \cdots \cap q_t \subset \operatorname{ann}_R(P/(\underline{x},\lambda))$ and $\lambda \in \operatorname{ann}_R(P/(\underline{x},\lambda))$.

By our choice, λ is a nonzero-divisor on $R/q_2 \cap \cdots \cap q_t$. Since $R/(\underline{x})$ is Cohen–Macaulay, we have

$$\begin{aligned} &\dim\bigl(R/\operatorname{ann}_R\bigl(P/(\underline{x},\lambda)\bigr)\bigr) \\ &\quad \le \dim(R/\lambda R + q_2 \cap \cdots \cap q_t) \\ &\quad \le \dim(R/q_2 \cap \cdots \cap q_t) - 1 = \dim\bigl(R/(\underline{x})\bigr) - 1 = n-i-1. \end{aligned}$$

Hence by the above remark, we have $[R/P] = [R/(x_1,x_2,\dots,x_i,\lambda)]$ in $A_{n-i}(R)$. □

From Theorem 4.4, any prime ideal P can be assumed to be an almost complete intersection ideal while calculating Chow groups. We also have the following.

COROLLARY 4.5. *Let* R, P, Ω *be as above, then* $[R/P] = [\Omega]$.

Proof. Let $S = R/(\underline{x})$. Then since $\Omega \subset S$, we have two exact sequences

$$\begin{aligned} &0 \longrightarrow S/\Omega \longrightarrow R/(\underline{x}) = S \longrightarrow R/(\underline{x},\lambda) = S/\lambda S \longrightarrow 0, \\ &0 \longrightarrow \Omega \longrightarrow S \longrightarrow S/\Omega \longrightarrow 0. \end{aligned}$$

Hence, we have $[S/\lambda S] = -[S/\Omega] = [\Omega]$ and by Theorem 4.4, we have $[R/P] = [S/\lambda S] = [\Omega]$. □

REMARK. Since $\operatorname{ht}(\lambda,\underline{x}) = \operatorname{ht}(\underline{x})$, we notice that λ is a zero-divisor on S.

Next, we utilize Theorem 4.4 to introduce a necessary and sufficient condition for vanishing of Chow groups.

First, we state the following lemma.

LEMMA 4.6 (Kaplansky [6]). *Let R be any ring and $P_1, P_2, \dots, P_n$ be any prime ideals in R, no two of which are comparable. Let $x \in R$, J an ideal in R such that $(x, J) \nsubseteq P_1 \cup P_2 \cup \cdots \cup P_n$ then there exists $j \in J$ such that $x + j \notin P_1 \cup P_2 \cup \cdots \cup P_n$.*

From Lemma 4.6, we have an immediate corollary which will be used to prove Proposition 4.8.

COROLLARY 4.7. *Let $I = (a_1, a_2, \dots, a_t)$ be an ideal of a Cohen–Macaulay local ring R with $\operatorname{depth}_I R \geq 1$, then there exist $d_2, d_3, \dots, d_t \in R$ such that $a_1 + d_2a_2 + \cdots + d_ta_t$ is a nonzero-divisor on R.*

The following discussion would lead us to our final proposition. We continue with the notation in Theorem 4.4. Since a complete ramified regular local ring R is an Eisenstein extension, we can write $R = T/(f)$ where $T = V[[X_1, X_2, \dots, X_n]], V$ is a complete discrete valuation ring, f is an Eisenstein polynomial in X_n over $V[[X_1, X_2, \dots, X_{n-1}]]$. From our discussion in Corollary 4.5, we have $S = R/(\underline{x})$. We notice that

$$\dim T/(\underline{x}, \lambda, f) = \dim R/(\underline{x}, \lambda) = n - i$$

and in T, $\operatorname{ht}(\underline{x}, \lambda, f) = i + 1$ and in $T/(\underline{x})$, $\operatorname{depth}_{(\lambda, f)} T/(\underline{x}) \geq 1$.

By Corollary 4.7, $\lambda + af$ is a nonzero-divisor on $T/(\underline{x})$, for some $a \in T/(\underline{x})$. We can change λ to $\lambda + af = \mu$ such that $T/(\underline{x}, \mu)$ is a complete intersection and f is a zero-divisor in $T/(\underline{x}, \mu)$. Then

$$T/(\underline{x}, \mu) \otimes T/(f) = R/(\underline{x}, \lambda).$$

Let $A = T/(\underline{x}, \mu)$. Then since A is Noetherian, there exists a d such that

$$\big(0 : f^d\big)A = \big(0 : f^{d+1}\big)A = \cdots .$$

Now from the exact sequence

$$0 \longrightarrow \big(0 : f^d\big)A \longrightarrow A \longrightarrow f^dA \longrightarrow 0$$

we get an exact sequence

$$0 \longrightarrow \frac{(0 : f^d)A}{f(0 : f^d)A} \longrightarrow A/fA \longrightarrow f^dA/f^{d+1}A \longrightarrow 0.$$

Note that $f^dA/f^{d+1}A = (\underline{x}, \mu, f^d)/(\underline{x}, \mu, f^{d+1})$. From the lifting property of Proposition 3.3, since f is a nonzero-divisor on f^dA and T, we have $[f^dA/f^{d+1}A] = 0$ in $A_{n-i}(R)$. Thus, we have

$$\left[\frac{(0 : f^d)A}{f(0 : f^d)A}\right] = [A/fA] = [R/P].$$

We also have:

$$\big(0 : f^d\big)A = H^0_f(A) = \operatorname{Tor}^T_1\big(T/f^dT, A\big).$$

The above discussion leads us to our final proposition.

PROPOSITION 4.8. *Let R, P, A, f be as above. Then $[R/P] = [A/fA] = 0$ in $A_{n-i}(R) \Leftrightarrow [H^0_f(A)/fH^0_f(A)] = 0$ in $A_{n-i}(R)$.*

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REFERENCES

[1] D. A. Buchsbaum and D. Eisenbud, *Lifting modules and a theorem on finite free resolutions*, Ring theory (R. Gorden, ed.), Acdemic Press, New York, 1987. MR 0340343

[2] L. Claborn and R. Fossum, *Generalization of the notion of class group*, Illinois J. Math. **12** (1968), 228–253. MR 0224601

[3] S. P. Dutta, *A note on Chow groups and intersection multiplicity of modules*, J. Algebra **161** (1993), 186–198. MR 1245850

[4] S. P. Dutta, *On Chow groups and intersection multiplicity of modules II*, J. Algebra **171** (1995), 370–382. MR 1315901

[5] H. Gillet and M. Levine, *The relative form of Gersten's conjecture over a discrete valuation ring: The smooth case*, J. Pure Appl. Algebra **46** (1987), no. 1, 59–71. MR 0894392

[6] I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, 1970. MR 0254021

[7] S. Lee, *Chow groups of complete regular local rings III*, Commun. Korean Math. Soc. **17** (2002), no. 2, 221–227. MR 1909327

[8] D. Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I—Higher K-theories, Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin, 1973, pp. 85–147. MR 0338129

[9] W. Smoke, *Perfect modules over Cohen–Macaulay local rings*, J. Algebra **106** (1987), 367–375. MR 0880963

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