Illinois Journal of Mathematics
Volume 56, Number 4, Winter 2012, Pages 1023–1084
S 0019-2082

CONTRACTION GROUPS, ERGODICITY, AND DISTAL PROPERTIES OF AUTOMORPHISMS OF COMPACT GROUPS

WOJCIECH JAWORSKI

Abstract. Given an automorphism τ of a compact group G, we study the factorization of $C(\tau,K)$, the contraction group of τ modulo a closed τ-invariant subgroup K, into the product $C(\tau)K$, of the contraction group $C(\tau)$ of τ, and K. We prove that the factorization $C(\tau,K)=C(\tau)K$ holds for every closed τ-invariant subgroup K if and only if G contains arbitrarily small closed normal τ-invariant subgroups N with finite-dimensional quotients G/N. For metrizable groups, we obtain that $C(\tau)K$ is a dense subgroup of $C(\tau,K)$, for every closed τ-invariant subgroup K. These results are used to link the contraction group to the properties of the dynamical system (G,τ). It follows that τ is distal if and only if $C(\tau)$ is trivial, while ergodicity of τ implies that $C(\tau)$ is nontrivial. When G is metrizable, the closure of $C(\tau)$ is the largest closed τ-invariant subgroup on which τ acts ergodically and, at the same time, it is the smallest among closed normal τ-invariant subgroups N such that τ acts distally on G/N. If τ is ergodic, then its restriction to any closed connected normal τ-invariant subgroup N with finite-dimensional quotient G/N is also ergodic. Moreover, when G is connected, the largest closed τ-invariant subgroup on which τ acts ergodically is necessarily connected.

Received August 7, 2011; received in final form June 12, 2012.
Supported by an NSERC grant.
2010 *Mathematics Subject Classification.* Primary 22C05, 22D40. Secondary 37B05, 37A05, 22D45.

1. Introduction

Let G be a (Hausdorff) topological group and τ an automorphism of G. The subgroup

$$C(\tau)=\left\{g\in G\,;\ \lim_{n\to\infty}\tau^n(g)=e\right\},$$

where e denotes the identity element of G, is called the *contraction group of* τ. When $K\le G$ is a closed subgroup invariant under τ, the formula $\tilde\tau(gK)=\tau(g)K$ $(g\in G)$, defines a homeomorphism of the homogeneous space G/K. The set

$$C(\tau,K)=\left\{g\in G\,;\ \lim_{n\to\infty}\tilde\tau^n(gK)=K\right\},$$

which contains the product $C(\tau)K$, is called the *contraction set of* τ *modulo* K. When K is compact, $C(\tau,K)$ is itself a subgroup and $C(\tau)$ is a normal subgroup of $C(\tau,K)$. We will say that τ has the *compact decomposition property*, if

$$C(\tau,K)=C(\tau)K \tag{1.1}$$

for every compact subgroup K invariant under τ.

It is known that all automorphisms of Lie groups and all automorphisms of totally disconnected locally compact groups have the compact decomposition property [6], [2], [13]. The main objective of the present article is to investigate whether and to what extent the compact decomposition property holds for automorphisms of compact groups, and to explore some of the consequences of this property.

The compact decomposition property has its origin in the theory of probability measures on groups [6], [5], [27], but contraction groups have been studied in variety of contexts beginning from the 1970s. When $C(\tau)=G$ (resp., $C(\tau,K)=G$), the automorphism τ is called contractive (resp., contractive modulo K) and a group admitting such an automorphism is called contractible (resp., contractible modulo K). It appears that the concept of a contractible locally compact group is due to Müller-Römer [23], [24], who introduced it in 1973 in a work motivated by certain questions in abstract harmonic analysis, in particular, the Wiener and Tauber properties of a group algebra. Subsequently, contractive automorhisms of locally compact groups were investigated by Wang [28] in a study of the Mautner phenomenon for p-adic Lie groups.

We note that $C(\tau)$ is itself a contractible group and τ is a contractive automorphism of $C(\tau)$. However, contraction groups are, as a rule, not closed subgroups of G and their study in, say, locally compact groups, cannot be reduced to a study of contractible locally compact groups.

Many fundamental results about contraction groups, contractive automorphisms, and contractible groups were discovered in the 1980s by Hazod and

Siebert [6], [27] in the course of their work on semistable convolution semigroups, in the theory of limit laws for products of independent random variables with values in a locally compact group. The main problem studied in this article for compact groups was first considered and solved by Hazod and Siebert for Lie groups, in an investigation prompted by the theory of the semistable convolution semigroups.

A convolution semigroup $(\mu_t)_{t\geq 0}$ is a continuous homomorphism $t \to \mu_t$ of the additive semigroup $[0,\infty)$ into the convolution semigroup $M_1(G)$ of probability measures on a locally compact group G (where $M_1(G)$ carries the usual weak topology). Such a semigroup is called semistable with respect to an automorphism $\tau \in \mathrm{Aut}(G)$, if there exists $c \in (0,1)$, such that $\tau\mu_t = \mu_{ct}$ for every $t \geq 0$ (where $(\tau\mu_t)(\cdot) = \mu(\tau^{-1}(\cdot))$). When $(\mu_t)_{t\geq 0}$ is a semistable convolution semigroup, μ_0 is necessarily the normalized Haar measure of some compact subgroup K, invariant under τ. The fundamental observation of Hazod and Siebert was that, when G is metrizable, then all the measures μ_t are supported on the contraction group $C(\tau,K)$. Then, when G is a Lie group, they proved the compact decomposition property $C(\tau,K)=C(\tau)K$ [6], while Siebert [27] previously described $C(\tau)$ as a simply connected nilpotent Lie subgroup, whose Lie algebra is $\{X \in \mathfrak{g}\,;\, \lim_{n\to\infty} \tau_*^n X = 0\}$, where $\mathfrak{g}$ is the Lie algebra of G and τ_*, the automorphism of $\mathfrak{g}$ induced by τ; when $C(\tau)$ happens to be closed, the factorization $C(\tau,K)=C(\tau)K$ is a semidirect product. Dani and Shah [5] showed that these results remain true also for p-adic Lie groups.

Contraction groups are prominent not only in the probabilistic problem studied by Hazod and Siebert but also in certain other questions involving probabilities on groups. Thus when μ is a probability measure on a locally compact group G, let μ^n denote the nth convolution power of μ. For the sake of simplicity let us assume that μ is adapted, that is, not supported on a proper closed subgroup. Let $\mathcal{K}$ denote the family of compact subsets of G and define functions $f_n : \mathcal{K} \to [0,1]$, $n=1,2,\ldots$, by $f_n(C)=\sup_{g\in G}\mu^n(Cg)$ $(C \in \mathcal{K})$. f_n provides a measure of the degree of concentration (or dissipation) of μ^n and is called the concentration function of μ^n. When G is noncompact, one generally expects that the convolution powers dissipate as $n \to \infty$. Thus, in terms of the concentration functions one expects that $\lim_{n\to\infty} f_n(C)=0$ for every $C \in \mathcal{K}$, unless μ is supported on a coset of a compact normal subgroup, a condition which obviously prevents the concentration functions from converging to zero. This natural conjecture, that the concentration functions converge to zero unless μ is supported on a coset of a compact normal subgroup is, in fact, close to the truth [15]. It can fail, but only for groups of a very special structure [12]: A noncompact locally compact group G supports an adapted probability measure whose concentration fail to converge to zero if and only if there exists an inner automorphism τ of G and a compact subgroup K, invariant under τ, such that G is isomorphic to the semidirect product $C(\tau,K) \times_\tau \mathbb{Z}$ (where $C(\tau,K)$ is necessarily closed).

Another question where contraction groups of inner automorphisms play a key role concerns the extension of the classical Choquet–Deny theorem to non-Abelian groups. Given a probability measure μ on a locally compact group G, a bounded continuous function $h : G \to \mathbb{C}$ is called μ-harmonic, if $h(g) = \int_G h(gg')\mu(dg')$ for every $g \in G$. The Choquet–Deny theorem asserts that when G is Abelian, then the only bounded continuous μ-harmonic functions possible, are the trivial ones, constant on the (left) cosets of the closed subgroup generated by the support of μ. It is well known that the theorem remains true for many non-Abelian groups, but is not true for all groups. However, the problem of determining the sufficient and necessary conditions under which the theorem holds remains an open one, despite many partial results in this direction. In particular, it is known that for groups that are either connected or totally disconnected, examples of measures for which the Choquet–Deny theorem fails can be constructed whenever the group admits an inner automorphism with a nontrivial contraction group [11], [14]. When G is a totally disconnected generalized $[FC]^-$-group (e.g., a compactly generated totally disconnected locally compact group of polynomial growth), the existence of an inner automorphism with a nontrivial contraction group turns out to be both sufficient and necessary for the existence of such measures, that is, the Choquet–Deny holds if and only if the contraction group of every inner automorphism is trivial [14].

Contraction groups have been also studied from a more fundamental point of view. In 2004, Baumgartner and Willis [2] demonstrated the significance of these subgroups in the theory of totally disconnected locally compact groups, by linking them with the theory of tidy subgroups and scales of automorphisms. The key to establishing this link is the compact decomposition property (1.1). In fact, automorphisms of totally disconnected locally compact groups not only have the compact decomposition property, but the factorization $C(\tau, K) = C(\tau)K$ remains true for any closed subgroup K, invariant under τ [2], [13]. We note that while for totally disconnected locally compact groups the latter generalized version of the compact decomposition property is a consequence of the compact decomposition property itself, the generalized version is not true for Lie groups.[1]

The proof of the compact decomposition property for Lie groups relies on the structure theory of such groups and the aforementioned description of the contraction group $C(\tau)$ as a simply connected nilpotent Lie subgroup. The proof of the compact decomposition property for totally disconnected locally compact groups makes use of properties specific to such groups (in particular, the fact that compact open subgroups form a neighbourhood base at e). As such, the proofs give no indication whether or to what extent the

[1] For example, $\mathrm{SL}(2, \mathbb{R})$ admits an inner automorphism τ and a proper closed subgroup K, such that $C(\tau) = \{e\}$, $\tau(K) = K$, and $C(\tau, K) = \mathrm{SL}(2, \mathbb{R})$.

compact decomposition property remains true for locally compact groups in general. However, the final part of the proof for totally disconnected groups, the extension from metrizable to nonmetrizable groups [13], uses a generally applicable projective limit technique. This is based on the theorem that if the dynamical system (G, τ), where G is a topological group and $\tau \in \mathrm{Aut}(G)$, is the projective limit of similar systems (G_j, τ_j), where each τ_j has the compact decomposition property, then τ itself has the compact decomposition property [13]. In the current work we use this theorem, along with certain results and techniques originating in ergodic theory [17], [26], [21], to study the compact decomposition property for automorphisms of compact groups.

When G is a compact group and $\tau \in \mathrm{Aut}(G)$, then the dynamical system (G, τ) is the projective limit of systems (G_j, τ_j), where each G_j is a closed shift-invariant subgroup of $L_j^{\mathbb{Z}}$ for some compact Lie group L_j, and τ_j is the restriction of the (Bernoulli) shift to G_j [17], [26]. This permits the reduction of our problem to the case of a closed shift-invariant subgroup of $L^{\mathbb{Z}}$, where L is a compact Lie group. The work of Miles and Thomas [21] suggests a further reduction to the following three subcases: I. G is a semisimple Lie group; II. τ is the shift on $G = L^{\mathbb{Z}}$ (where L is a compact Lie group); III. τ is a solenoidal automorphism. This reduction turns out to be indeed possible. In cases I and III, τ always has the compact decomposition property; in case II, the compact decomposition property holds if and only if G has finite dimension, that is, L is a finite group. It follows that an automorphism τ of a compact group G has the compact decomposition property if and only if no infinite-dimensional case II dynamical system arises as an equivariant homomorphic image of a closed τ-invariant subgroup of G. An equivalent sufficient and necessary condition is that the system (G, τ) be a projective limit of finite-dimensional systems. Thus, the compact decomposition property fails for compact groups in general. However, for metrizable groups we show that $C(\tau)K$ is a dense subgroup of $C(\tau, K)$, for every $\tau \in \mathrm{Aut}(G)$ and every τ-invariant compact subgroup K.

Our investigation of the compact decomposition property makes it possible to derive a number of interesting corollaries linking the contraction group to the properties of the dynamical system (G, τ). Recall that an automorphism τ of a topological group G acts distally on G if and only if for any $g \in G \setminus \{e\}$, e is not in the closure of the orbit $\{\tau^n(g); n \in \mathbb{Z}\}$. It is obvious that if τ acts distally, then $C(\tau) = C(\tau^{-1}) = \{e\}$. The converse is known to be true when G is a Lie group [1], or a totally disconnected locally compact group [14]. Notably, the proof for totally disconnected groups rests on the results of Baumgartner and Willis [2] obtained with the aid of the compact decomposition property. When τ is an automorphism of a compact group, we prove that τ acts distally if and only if $C(\tau) = \{e\}$. An equivalent result is that $C(\tau)$ is nontrivial whenever τ is ergodic and $G \neq \{e\}$. For metrizable compact groups we also show that the closure $[C(\tau)]^{-}$ of $C(\tau)$ is the largest closed

τ-invariant subgroup of G on which τ acts ergodically, and, at the same time, $[C(\tau)]^{-}$ is the smallest among closed normal τ-invariant subgroups N of G, such that τ acts distally on the quotients G/N. Moreover, $[C(\tau)]^{-}$ turns out to be connected whenever G is connected (while $C(\tau)$ itself can be totally disconnected). Another corollary concerns a generalization of the result that an ergodic automorphism of a compact group acts ergodically on the connected component of the identity [26]. Let N be a closed normal connected τ-invariant subgroup of G and τ_N denote the quotient of τ on G/N. We obtain that τ acts ergodically on N whenever τ acts ergodically on G and the system $(G/N, \tau_N)$ is a projective limit of finite-dimensional systems.

2. Ergodic and distal automorphisms of compact groups

Let G denote a compact group. Throughout the sequel, it will be convenient to work in the language of an action of $\mathbb{Z}$ on G by automorphisms, rather than with a single automorphism $\tau \in \mathrm{Aut}(G)$ (inducing the action). In this section, as this is helpful in the proof of Lemma 2.5 and does not lead to any extra difficulties, we consider an arbitrary group Γ acting on G by automorphisms. By an action, we always mean a left action. Let ω denote the normalized Haar measure of G. Recall that the action of Γ on G is called ergodic, if for every Γ-invariant Borel set $B \subseteq G$, we have $\omega(B) = 0$ or $\omega(B) = 1$. The following proposition is a restatement of a part of Lemma 1.2 in [26], except that we omit the assumption that G be metrizable and Γ countable, which is not used in the proof. While we are interested in actions of $\mathbb{Z}$, in the proof of Lemma 2.5 it will be necessary to consider an action of an uncountable group.

Throughout this paper, we do not assume, contrary to the prevailing custom in ergodic theory, that "compact group" means a metrizable (equivalently, second countable) compact group. By a Lie group, we mean a real Lie group of finite positive dimension, or a group equipped with the discrete topology.

Proposition 2.1. *The following conditions are equivalent for an action of a group Γ on the compact group G:*

(1) *Γ does not act ergodically.*

(2) *There exists a proper closed normal Γ-invariant subgroup $N \trianglelefteq G$, such that G/N is a Lie group whose topology is given by a 2-sided translation-invariant metric which is also invariant under the* (*quotient*) *action of Γ on G/N.*

(3) *There exists a compact Γ-invariant neighbourhood U of e in G, such that $U^2 \neq G$.*

Recall that the action of Γ on G is called topologically transitive, if Γ has a dense orbit in G, and that Γ acts distally on G if and only if for every $g \in G \setminus \{e\}$, e is not contained in the closure of the orbit Γg.

Corollary 2.2. *If $\mathit{\Gamma}$ acts topologically transitively, then it act ergodically. If G is metrizable, then $\mathit{\Gamma}$ acts topologically transitively if and only if it acts ergodically.*

Corollary 2.3. *Suppose that $\mathit{\Gamma}$ acts both ergodically and distally. If $\mathit{\Gamma}$ is countable or G is metrizable, then $G=\{e\}$.*

Proof. If G is metrizable, this is trivial by topological transitivity. If $\mathit{\Gamma}$ is countable, consider the semidirect product $G\rtimes\mathit{\Gamma}$, where $\mathit{\Gamma}$ is viewed as a discrete group. Then $G\rtimes\mathit{\Gamma}$ is a σ-compact locally compact group. By [7, Theorem 8.7] every neighbourhood of the identity in $G\rtimes\mathit{\Gamma}$ contains a compact normal subgroup H with metrizable quotient $(G\rtimes\mathit{\Gamma})/H$. Since $\mathit{\Gamma}$ is discrete, it easily follows that every neighbourhood of e in G contains a compact normal $\mathit{\Gamma}$-invariant subgroup N with metrizable quotient G/N. Since factors of ergodic actions are ergodic and factors of distal actions are distal [3, Corollary 6.10, p. 52], it follows that every such quotient is trivial. Hence, G must be trivial too. □

It is shown in [26, Theorem 1.4] that for every action of a countable group $\mathit{\Gamma}$ on a compact metrizable group G, there exists a closed normal $\mathit{\Gamma}$-invariant subgroup G_* which is the largest among closed $\mathit{\Gamma}$-invariant subgroups on which $\mathit{\Gamma}$ acts ergodically. The rather formidable proof of the key Lemma 1.3 in [26], needed to establish this result, uses metrizability of G in an essential way. Below we give a different, much simpler proof of this lemma (stated here as Lemma 2.5) which does not rely on metrizability, and, hence, obtain the existence of the largest ergodic subgroup without any restrictions on either G or $\mathit{\Gamma}$. (We will need this only for actions of $\mathbb{Z}$, but on a not necessarily metrizable group.)

Lemma 2.4. *Let A,B be subgroups of* $\mathrm{Aut}(G)$, *where A normalizes B and B is compact* (*in the usual topology of* $\mathrm{Aut}(G)$). *Then A acts ergodically on G if and only if AB acts ergodically on G.*

Proof. To prove the nontrivial "if" component of the "if and only" statement, we argue by contradiction. If A fails to act ergodically, then by Proposition 2.1(3) there exists a compact A-invariant neighbourhood U of e, such that $U\neq G$. Recall that a compact subgroup of $\mathrm{Aut}(G)$ acts equicontinuously on G. Hence, U contains a B-invariant neighbourhood of e. Therefore, $V=\bigcap_{b\in B} bU$ is a compact AB-invariant neighbourhood of e, and $V\subseteq U$. So AB fails to act ergodically. □

Lemma 2.5. *Let $\mathit{\Gamma}$ act on G and let H be a closed normal $\mathit{\Gamma}$-invariant subgroup. The following conditions are equivalent*:

(1) *The action of $\mathit{\Gamma}$ on H is not ergodic.*

(2) *There exists a proper closed Γ-invariant subgroup N of H, such that N is normal in G and H/N is a Lie group whose topology is given by 2-sided translation-invariant metric which is also invariant under the action of Γ on H/N.*

Proof. In view of Proposition 2.1, we only need to prove that (1) implies (2). Let A denote the image of Γ in $\mathrm{Aut}(H)$ and B the subgroup of $\mathrm{Aut}(H)$ consisting of restrictions of the inner automorphisms of G to H. Then A normalizes B and fails to act ergodically on H. Hence, by Lemma 2.4, AB does not act ergodically on H, and so by Proposition 2.1(2) there exists a proper closed AB-invariant normal subgroup N of H, such that H/N is a Lie group whose topology is given by a 2-sided translation-invariant metric which is also invariant under the action of AB. Since N is invariant under B, it is a normal subgroup of G. □

The next theorem is proven in [26] under the assumption that G be metrizable and Γ countable [26, Theorem 1.4]. With the aid of our Lemma 2.5, the proof can be redone, verbatim, without this assumption. For a given ordinal number α, we denote by $[0,\alpha)$ (resp., $[0,\alpha]$) the set of ordinals strictly smaller than α (resp., smaller than or equal to α).

THEOREM 2.6. *Given an action of Γ on G, there exists an ordinal α and a family $(G_\beta)_{\beta\in[0,\alpha]}$ of closed normal Γ-invariant subgroups of G, such that*:

(i) $G_0 = G$;

(ii) *for every $\beta\in[0,\alpha)$, $G_{\beta+1}$ is a proper closed Γ-invariant subgroup of G_β, and $G_\beta/G_{\beta+1}$ is a Lie group whose topology is given by a 2-sided translation-invariant metric which is also invariant under the action of Γ on $G_\beta/G_{\beta+1}$*;

(iii) *if $\beta\in[0,\alpha]$ is a nonzero limit ordinal, then $G_\beta = \bigcap_{\gamma\in[0,\beta)} G_\gamma$*;

(iv) *Γ acts ergodically on G_α and G_α is the largest closed subgroup of G on which Γ acts ergodically.*

The ordinal α is countable when G is metrizable.

The subgroup G_α will be denoted by G_{erg} and called the *ergodic component* of the dynamical system (G,Γ) (and of the action of Γ, as well as of G itself).

COROLLARY 2.7. *If Γ is countable or G is metrizable, then*:

(i) *G_{erg} is the smallest among closed normal Γ-invariant subgroups N of G, such that Γ acts distally on G/N.*

(ii) *If Γ acts on a compact group H and $\varphi: G\to H$ is a continuous equivariant surjective homomorphisms, then $\varphi(G_{\mathrm{erg}}) = H_{\mathrm{erg}}$.*

Proof. (i): Since $G\setminus G_\alpha = \bigcup_{\beta\in[0,\alpha)}(G_\beta\setminus G_{\beta+1})$, it follows from part (ii) of Theorem 2.6 that Γ acts distally on G/G_α. Let N be a closed normal Γ-invariant subgroup, such that Γ acts distally on G/N. Denote by

$\xi : G \to G/N$ the canonical homomorphism. Then Γ acts both ergodically and distally on $\xi(G_\alpha)$. Hence, by Corollary 2.3, N must contain G_α.

(ii) Since Γ acts ergodically on $\varphi(G_{\mathrm{erg}})$, $\varphi(G_{\mathrm{erg}}) \subseteq H_{\mathrm{erg}}$. On the other hand, $H/\varphi(G_{\mathrm{erg}})$ is a continuous equivariant image of G/G_{erg}. Hence, Γ acts distally on $H/\varphi(G_{\mathrm{erg}})$. Therefore, $H_{\mathrm{erg}} \subseteq \varphi(G_{\mathrm{erg}})$ by (i). □

The full extent of the connection between contraction groups of $\mathbb{Z}$-actions on G and ergodicity and distality will emerge only after a thorough investigation of the contraction groups themselves and the compact decomposition property. Standard examples of automorphisms suggest that ergodic automorphisms have dense contraction groups, while automorphisms with trivial contraction groups are distal. The general results collected above permit to draw a few preliminary conclusions. Let us denote the forward (resp., backward) contraction group of a $\mathbb{Z}$-action on G by $C_+(G)$ (resp., $C_-(G)$):

$$C_\pm(G) = \Big\{ g \in G\,;\ \lim_{n\to\infty} (\pm n)g = e \Big\}.$$

Proposition 2.8. *$C_+(G)$ and $C_-(G)$ are subgroups of G_{erg}. Moreover, $\mathbb{Z}$ acts ergodically on the closures $[C_+(G)]^-$ and $[C_-(G)]^-$. In particular, a $\mathbb{Z}$-action having a dense contraction group is ergodic.*

Proof. That $C_\pm(G) \subseteq G_{\mathrm{erg}}$ is an immediate consequence of Theorem 2.6. To see that $\mathbb{Z}$ acts ergodically on $[C_\pm(G)]^-$, observe that $C_\pm([C_\pm(G)]^-) = C_\pm(G)$. This yields $C_\pm(G) \subseteq ([C_\pm(G)]^-)_{\mathrm{erg}}$. □

3. General properties of contraction groups

Throughout this section, G is a (Hausdorff) topological group on which $\mathbb{Z}$ acts by automorphisms, and K is a closed $\mathbb{Z}$-invariant subgroup of G. Elements of the theory of contraction groups for general topological groups will prove useful in Section 8, where by retopologizing a compact group G we will be able to identify certain important subgroups of the contraction groups.

The forward (resp., backward) contraction set modulo K will be denoted by $C_+(G,K)$ (resp., $C_-(G,K)$):

$$C_\pm(G,K) = \Big\{ g \in G\,;\ \lim_{n\to\infty} \big((\pm n)g\big)K = K \Big\}.$$

It is clear that results about $C_-(G)$ and $C_-(G,K)$ can be obtained from those about $C_+(G)$ and $C_+(G,K)$, and vice versa, by using the reflection $k \to -k$ on $\mathbb{Z}$. We will therefore focus on studying $C_+(G)$ and $C_+(G,K)$. A few of our results will involve both $C_+(G)$ and $C_-(G)$.

We will say that the action of $\mathbb{Z}$ (or the dynamical system $(G,\mathbb{Z})$) has the (forward) compact decomposition property (cdp), if $C_+(G,K) = C_+(G)K$ for every compact $\mathbb{Z}$-invariant subgroup $K \le G$.

We note that

$$(3.1)\qquad g\in C_+(G,K) \quad\Leftrightarrow\quad \begin{array}{l}\text{for every neighbourhood } U \text{ of } e,\\ ng\in UK \text{ for large enough } n\in\mathbb{N}.\end{array}$$

It follows that $C_+(G,K)$ is a subgroup of G whenever for every neighbourhood U of e, one can find a neighbourhood V of e, such that $KV\subseteq UK$. In particular, $C_+(G,K)$ is a subgroup whenever K is compact, or $K\trianglelefteq G$. It is also easy to show that when K is compact, then $C_+(G)$ is a normal subgroup of $C_+(G,K)$, while $C_+(G)K$ is a subgroup of $C_+(G,K)$.

The next proposition results by a straightforward application of Lemma 3.9(3) in [2], but it is also not difficult to prove it directly.

PROPOSITION 3.1. *Let G be compact. Then $C_+(G,K)=\{g\in G; K$ contains every cluster point of the sequence $(ng)_{n=1}^{\infty}\}$. Moreover, if K is a normal subgroup of G, then so is $C_+(G,K)$; in particular, $C_+(G)\trianglelefteq G$.*

PROPOSITION 3.2. *Let K and H be closed $\mathbb{Z}$-invariant subgroups of G.*

(i) *If for every neighbourhood U of e in G there exists a neighbourhood V of e with $(VK)\cap H\subseteq U(K\cap H)$, then $C_+(G,K)\cap H=C_+(H,K\cap H)$.*

(ii) *If $K\subseteq H$ or K is compact, then $C_+(G,K)\cap H=C_+(H,K\cap H)$.*

(iii) *If the system $(G,\mathbb{Z})$ has the cdp then so does $(H,\mathbb{Z})$.*

Proof. (i) Using (3.1) and the neighbourhood condition, $C_+(G,K)\cap H\subseteq C_+(G,K\cap H)\cap H$. But (3.1) and the observation that if U is a neighbourhood of e in G, then $(U(K\cap H))\cap H=(U\cap H)(K\cap H)$, yield $C_+(G,K\cap H)\cap H=C_+(H,K\cap H)$. Since it is clear that $C_+(H,K\cap H)\subseteq C_+(G,K)\cap H$, the result follows.

(ii) If $K\subseteq H$, then the assumption of (i) is trivially satisfied. It is not difficult to see that compactness of K also makes this assumption true.

(iii) Let K be a compact $\mathbb{Z}$-invariant subgroup of H. Then using (ii), $C_+(H,K)=C_+(G,K)\cap H=(C_+(G)K)\cap H$, and it is easy to see that $(C_+(G)K)\cap H=C_+(H)K$. ☐

PROPOSITION 3.3. *Let $\mathbb{Z}$ act on a topological group G_1 and φ be a continuous open equivariant homomorphism of G onto G_1. Then:*

(i) *$\varphi(C_+(G,K))\subseteq C_+(G_1,[\varphi(K)]^{-})$ for every closed $\mathbb{Z}$-invariant subgroup $K\leq G$.*

(ii) *$\varphi^{-1}(C_+(G_1,K_1))=C_+(G,\varphi^{-1}(K_1))$ for every closed $\mathbb{Z}$-invariant subgroup $K_1\leq G_1$.*

(iii) *$\varphi(C_+(G))=C_+(G_1)$ if and only if $C_+(G,\operatorname{Ker}\varphi)=C_+(G)\operatorname{Ker}\varphi$.*

Proof. (i) and (ii) follow immediately from (3.1); (iii) is an immediate consequence of (ii). ☐

COROLLARY 3.4. *If $\operatorname{Ker}\varphi$ is compact, then the system $(G,\mathbb{Z})$ has the cdp if and only if both $(G_1,\mathbb{Z})$ and $(\operatorname{Ker}\varphi,\mathbb{Z})$ have the cdp, and $\varphi(C_+(G))=C_+(G_1)$.*

Proof. Let $(G,\mathbb{Z})$ have the cdp. By Proposition 3.3(iii), $\varphi(C_+(G)) = C_+(G_1)$. By Proposition 3.2, $(\operatorname{Ker}\varphi,\mathbb{Z})$ has the cdp. Given a compact $\mathbb{Z}$-invariant subgroup $K_1 \le G_1$, we have $\varphi^{-1}(C_+(G_1,K_1)) = C_+(G)\varphi^{-1}(K_1)$ by Proposition 3.3(ii). So $C_+(G_1,K_1) = \varphi(C_+(G)\varphi^{-1}(K_1)) = C_+(G_1)K_1$, that is, $(G_1,\mathbb{Z})$ has the cdp.

Conversely, suppose that $\varphi(C_+(G)) = C_+(G_1)$, and both $(G_1,\mathbb{Z})$ and $(\operatorname{Ker}\varphi,\mathbb{Z})$ have the cdp. Let $K \le G$ be a compact $\mathbb{Z}$-invariant subgroup. Then by Proposition 3.3(i), $\varphi(C_+(G,K)) \subseteq C_+(G_1,\varphi(K)) = C_+(G_1)\varphi(K) = \varphi(C_+(G))\varphi(K) = \varphi(C_+(G)K)$. So $C_+(G,K) \subseteq C_+(G)K\operatorname{Ker}\varphi$. Thus if $g \in C_+(G,K)$, then $g = xyz$ with $x \in C_+(G)$, $y \in K$, $z \in \operatorname{Ker}\varphi$. Since $C_+(G)K \le C_+(G,K)$ and $(\operatorname{Ker}\varphi,\mathbb{Z})$ has the cdp, using Proposition 3.2(ii) we conclude that $z \in C_+(G,K) \cap \operatorname{Ker}\varphi = C_+(\operatorname{Ker}\varphi, K \cap \operatorname{Ker}\varphi) = C_+(\operatorname{Ker}\varphi)(K \cap \operatorname{Ker}\varphi) \subseteq C_+(G)K$. Therefore, $g = xyz \in C_+(G)K$. Hence, $C_+(G)K \subseteq C_+(G,K) \subseteq C_+(G)K$. □

Corollary 3.5. *If* $\operatorname{Ker}\varphi$ *is compact and the system* $(G,\mathbb{Z})$ *has the cdp, then* $\varphi(C_+(G,K)) = C_+(G_1,\varphi(K))$ *for every compact* $\mathbb{Z}$*-invariant subgroup* $K \le G$. *In particular,* $\varphi(C_+(G)) = C_+(G_1)$.

Proof. Using Corollary 3.4: $\varphi(C_+(G,K)) = \varphi(C_+(G)K) = C_+(G_1)\varphi(K) = C_+(G_1,\varphi(K))$. □

We conclude this section by quoting the following three results.

Theorem 3.6 ([6, Theorem 2.4 and Lemma 1.8]). *Every* $\mathbb{Z}$*-action on a Lie group has the cdp. Moreover, if* $\mathbb{Z}$ *acts on a Lie group* G *and* $K \le G$ *is a closed normal* $\mathbb{Z}$*-invariant subgroup, then* $C_+(G,K) = C_+(G)K$. *If* $\mathbb{Z}$ *acts on Lie groups* G *and* G_1, *and* φ *is a continuous equivariant homomorphims of* G *onto* G_1, *then* $\varphi(C_+(G)) = C_+(G_1)$.

Theorem 3.7 ([2, Theorem 3.8], [13, Theorem 1]). *If* G *is a totally disconnected locally compact group, then* $C_+(G,K) = C_+(G)K$ *for every closed* $\mathbb{Z}$*-invariant subgroup* $K \le G$.

An independent proof of Theorem 3.7 for compact totally disconnected groups will be given in Section 5, Corollary 5.10.

Theorem 3.8 ([13, Proposition 10]). *Let* $\mathcal{N}$ *be a nonempty family of compact normal* $\mathbb{Z}$*-invariant subgroups of* G, *such that for every* $N \in \mathcal{N}$, *the system* $(G/N,\mathbb{Z})$ *has the cdp. Then the system* $(G/\bigcap\mathcal{N},\mathbb{Z})$ *also has the cdp.*

4. Bernoulli systems

By the Bernoulli action on $L^{\mathbb{Z}}$, where L is a topological group, we mean the action generated by the left shift: when an element $x \in L^{\mathbb{Z}}$ is written as a sequence $x = (x_n)_{n\in\mathbb{Z}}$, the result of $k \in \mathbb{Z}$ acting on x is the sequence $kx = (x_{n+k})_{n\in\mathbb{Z}}$. The contraction group of the Bernoulli action reads $C_+(L^{\mathbb{Z}}) = \{x \in L^{\mathbb{Z}}\,;\, \lim_{n\to\infty} x_n = e\}$.

The following example of a Bernoulli system shows that the cdp, as well as the equality $\varphi(C_+(G)) = C_+(G_1)$ for a continuous equivariant surjective homomorphism $\varphi : G \to G_1$, can fail when G is a compact Abelian group.

EXAMPLE 4.1. Let $G = \mathbb{T}^{\mathbb{Z}}$ and $K = \{x \in \mathbb{T}^{\mathbb{Z}}\,;\, x = 1x\} = \{x \in \mathbb{T}^{\mathbb{Z}}\,;\, x_n = x_{n+1}$ for every $n \in \mathbb{Z}\}$. Suppose that $C_+(G,K) = C_+(G)K$, equivalently, that with $\varphi : G \to G/K$ denoting the canonical homomorphism, $\varphi(C_+(G)) = C_+(G/K)$ (cf. Proposition 3.3(iii)).

Now, if $x \in C_+(G,K)$, then $x = yh$ for some $y \in C_+(G)$ and $h \in K$. Hence, as $n \to \infty$, $nx = n(yh) = (ny)h \to h$. We conclude that for every $x \in C_+(G,K)$, the sequence $(x_n)_{n=1}^{\infty} = ((nx)_0)_{n=1}^{\infty}$ converges (in $\mathbb{T}$). However, define $x \in G$ by $x_n = 1$ for $n \le 0$ and $x_n = \exp(2\pi i \sum_{j=1}^{n} j^{-1})$ for $n \ge 1$. Let z be a cluster point of the sequence $(nx)_{n=1}^{\infty}$. Thus, there is a subsequence $(n_k x)_{k=1}^{\infty}$ convergent to z. If so, then for every $j \in \mathbb{Z}$, $(n_k x)_j = x_{n_k+j} \to z_j$ and $(n_k x)_{j+1} = x_{n_k+j+1} \to z_{j+1}$. But for large enough k, $x_{n_k+j+1} = x_{n_k+j} \exp(2\pi i (n_k + j + 1)^{-1})$. Therefore $z_{j+1} = z_j$, and so $z \in K$. Thus, every cluster point of $(nx)_{n=1}^{\infty}$ belongs to K. So, by Proposition 3.1, $x \in C_+(G,K)$. But we saw that this implies that the sequence $(x_n)_{n=1}^{\infty}$ converges in $\mathbb{T}$, that is, the infinite product $\prod_{j=1}^{\infty} \exp(2\pi i j^{-1})$ converges, which is a contradiction.

THEOREM 4.2. *The Bernoulli system* $(L^{\mathbb{Z}}, \mathbb{Z})$, *where* L *is a compact group, has the cdp if and only if* L *is totally disconnected.*

Proof. $\Leftarrow$: This is immediate by Theorem 3.7.

$\Rightarrow$: If L is not totally disconnected, it contains a compact normal subgroup N such that L/N is a compact Lie group of positive dimension. As such, L/N contains a closed subgroup S which is topologically isomorphic to $\mathbb{T}$. Then $S^{\mathbb{Z}}$ is a closed shift-invariant subgroup of $(L/N)^{\mathbb{Z}}$, and by Example 4.1 the system $(S^{\mathbb{Z}}, \mathbb{Z})$ fails to have the cdp. Hence, by Proposition 3.2(iii), $((L/N)^{\mathbb{Z}}, \mathbb{Z})$ fails to have the cdp. Corollary 3.4 shows that $(L^{\mathbb{Z}}, \mathbb{Z})$ fails to have the cdp. $\square$

In the remaining results of this section, we do not assume compactness of L. This setting will be useful in our study of Δ-contraction groups in Sections 8 and 11.

Note that if H is any shift-invariant subgroup of $L^{\mathbb{Z}}$, then

$$C_+(H) = \Big\{x \in H\,;\, \lim_{n\to\infty} x_n = e\Big\}. \tag{4.1}$$

It is also easy to see that if M is a closed subgroup of L, such that $M^{\mathbb{Z}} \subseteq H$, then

$$C_+\big(H, M^{\mathbb{Z}}\big) = \Big\{x \in H\,;\, \lim_{n\to\infty} x_n M = M \ (\text{in } L/M)\Big\}. \tag{4.2}$$

LEMMA 4.3. *Let* M *be a closed subgroup of a metrizable group* L, *and* H *a shift-invariant subgroup of* $L^{\mathbb{Z}}$, *containing* $M^{\mathbb{Z}}$. *Then* $C_+(H, M^{\mathbb{Z}}) = C_+(H)M^{\mathbb{Z}}$.

Proof. Let d be a right invariant metric on L. Then the topology of L/M is given by the metric $\bar{d}(\bar{x},\bar{y}) = \inf\{d(x,y)\,;\, x \in \bar{x}, y \in \bar{y}\}$ [7, 8.14]. Hence, for a sequence $(x_n)_{n=1}^{\infty}$ in L, we have $x_n M \to M$ if and only if $d(x_n, M) \to 0$, where $d(x_n, M)$ denotes the distance from x_n to M.

Given $x = (x_n)_{n\in\mathbb{Z}} \in C_+(H, M^{\mathbb{Z}})$, we can select a sequence $y = (y_n)_{n\in\mathbb{Z}} \in M^{\mathbb{Z}}$ with $d(x_n y_n^{-1}, e) = d(x_n, y_n) \le 2d(x_n, M)$ for every n. Then using (4.2) and (4.1) we conclude that $xy^{-1} \in C_+(H)$. So $x = xy^{-1}y \in C_+(H)M^{\mathbb{Z}}$. □

In the next lemma, $\pi_{\mathbb{N}}$ denotes the canonical projection of $L^{\mathbb{Z}}$ onto $L^{\mathbb{N}}$, mapping $(x_n)_{n\in\mathbb{Z}} \in L^{\mathbb{Z}}$ to $(x_n)_{n\in\mathbb{N}} \in L^{\mathbb{N}}$. The condition involving $\pi_{\mathbb{N}}$ will be a natural one in the context of the theory of Markov subgroups of $L^{\mathbb{Z}}$, to be reviewed in Section 5.

LEMMA 4.4. *Let L be a discrete group, M a subgroup of L, and G a shift-invariant subgroup of $L^{\mathbb{Z}}$. If $\pi_{\mathbb{N}}(G \cap M^{\mathbb{Z}}) = \pi_{\mathbb{N}}(G) \cap M^{\mathbb{N}}$, then $C_+(G, G\cap M^{\mathbb{Z}}) = C_+(G)(G \cap M^{\mathbb{Z}})$.*

Proof. Let $x \in C_+(G, G\cap M^{\mathbb{Z}})$. Since $C_+(G, G\cap M^{\mathbb{Z}}) \subseteq C_+(L^{\mathbb{Z}}, M^{\mathbb{Z}})$ and L has the discrete topology, (4.2) with $H = L^{\mathbb{Z}}$ implies that there exists $k \in \mathbb{Z}$, such that $x_n \in M$ for every $n \ge k$. Then our assumption involving $\pi_{\mathbb{N}}$ and the shift-invariance produce $y \in G \cap M^{\mathbb{Z}}$ with $y_n = x_n$ for every $n \ge k$. By (4.1), $xy^{-1} \in C_+(G)$. Hence, $x = xy^{-1}y \in C_+(G)(G\cap M^{\mathbb{Z}})$. □

5. The descending chain condition and Markov subgroups

Let a group Γ act by automorphisms on a compact group G. The dynamical system (G, Γ) (or the action of Γ) is said to satisfy the *descending chain condition* (dcc), if for every nonincreasing sequence $(G_n)_{n=1}^{\infty}$ of closed Γ-invariant subgroups there exists $N \in \mathbb{N}$, such that $G_n = G_N$ for every $n \ge N$ [17], [26]. It is not difficult to see that the dcc is equivalent to the formally stronger condition that every nonempty directed downward family $\mathcal{F}$ (filter base) of closed Γ-invariant subgroups has a minimal element. One immediate consequence of this observation is the following metrizability result.

PROPOSITION 5.1. *If an action of a countable group Γ on G satisfies the dcc, then G is metrizable.*

Proof. Indeed, the family $\mathcal{M}$ of closed normal Γ-invariant subgroups M with metrizable quotients G/M is directed downward (in fact, $M_1, M_2 \in \mathcal{M} \Rightarrow M_1 \cap M_2 \in \mathcal{M}$). But we saw in the proof of Corollary 2.3 that when Γ is countable, then every neighbourhood of e in G contains a member of $\mathcal{M}$. With the dcc in place, this implies that $\{e\} \in \mathcal{M}$. □

By an isomorphism between two dynamical systems (G, Γ) and (G', Γ) (where G, G' are topological groups and Γ acts by automorphisms), we will always mean an equivariant homeomorphims which is also a group isomorphism between G and G'. If L is any compact group, then Γ acts on the compact group L^{Γ} by shifts: $(\gamma x)_{\beta} = x_{\beta\gamma}$ $(\gamma, \beta \in \Gamma,\ x = (x_{\alpha})_{\alpha\in\Gamma} \in L^{\Gamma})$. A subgroup $V \le L^{\Gamma}$ is called *full* if $\{x_{\gamma}; x \in V\} = L$ for every $\gamma \in \Gamma$.

THEOREM 5.2 ([17, Theorem 3.2]). *An action of a finitely generated Abelian group Γ on a compact group G satisfies the dcc if and only if there exists a compact Lie group L and a full closed shift-invariant subgroup V of L^{Γ}, such that the system (G, Γ) is isomorphic to (V, Γ).*

Theorem 3.16 in [17] shows that every dynamical system (G, Γ), where G is a metrizable compact group and Γ is a finitely generated Abelian group, is the projective limit of a sequence (G_n, Γ) of systems satisfying the dcc. An obvious modification of the proof of this theorem yields a similar conclusion without the metrizability assumption.

THEOREM 5.3. *Let a finitely generated Abelian group Γ act on a compact group G and $\mathcal{C}$ denote the family of compact normal Γ-invariant subgroups C of G, such that the systems $(G/C, \Gamma)$ satisfy the dcc. Then $\mathcal{C}$ is directed downward and $\bigcap \mathcal{C} = \{e\}$.*

Our interest is in actions of $\Gamma = \mathbb{Z}$. We will use the following notations and terminology related to the Bernoulli dynamical system $(L^{\mathbb{Z}}, \mathbb{Z})$, where L is a compact group. The projection of $L^{\mathbb{Z}}$ onto the nth coordinate $(n \in \mathbb{Z})$ will be denoted by π_n^L, while $\hat{\pi}_k^L$ $(k \in \mathbb{N})$ will denote the projection of $L^{\mathbb{Z}}$ onto $L^{\{0,\dots,k\}}$. The elements of $L^{\{0,\dots,k\}}$ will be written as $(k+1)$-tuples $\langle x_0, \dots, x_k\rangle$; thus $\hat{\pi}_k((x_n)_{n\in\mathbb{Z}}) = \langle x_0, \dots, x_k\rangle$. A closed shift-invariant subgroup $V \le L^{\mathbb{Z}}$ will be called a *Markov subgroup*, if $V = \{x \in L^{\mathbb{Z}}; \hat{\pi}_1^L(kx) \in \hat{\pi}_1^L(V) \text{ for every } k \in \mathbb{Z}\} = \{x \in L^{\mathbb{Z}}; \langle x_k, x_{k+1}\rangle \in \hat{\pi}_1^L(V) \text{ for every } k \in \mathbb{Z}\}$. The subgroup $\hat{\pi}_1^L(V) \le L^{\{0,1\}}$ will be called the *transition subgroup* of V and denoted by T_V.

REMARK 5.4. If H is an arbitrary closed subgroup of $L^{\{0,1\}}$, then $V = \{x \in L^{\mathbb{Z}}; \langle x_k, x_{k+1}\rangle \in H \text{ for every } k \in \mathbb{Z}\}$ is a Markov subgroup of $L^{\mathbb{Z}}$ with transition subgroup $T_V = H \cap [\pi_0^L(V)]^{\{0,1\}}$. The equality $T_V = H$ holds if and only if $\{x_0 \in L; \langle x_0, x_1\rangle \in H \text{ for some } x_1 \in L\} = \{x_1 \in L; \langle x_0, x_1\rangle \in H \text{ for some } x_0 \in L\}$.

Let V be a Markov subgroup of $L^{\mathbb{Z}}$. Given $n \in \mathbb{Z}$ and $g \in \pi_0^L(V)$, we define, following [26, p. 83], sets

$$T_V(g, n) = \pi_n^L\big(V \cap \big(\pi_0^L\big)^{-1}(\{g\})\big), \qquad T_V(n) = T_V(e, n). \tag{5.1}$$

In particular,

$$T_V(g, 1) = \big\{x \in L; \langle g, x\rangle \in T_V\big\}, \qquad T_V(g, -1) = \big\{x \in L; \langle x, g\rangle \in T_V\big\}. \tag{5.2}$$

It follows that $(T_V(n))_{n=1}^{\infty}$ and $(T_V(-n))_{n=1}^{\infty}$ are nondecreasing sequences of closed normal subgroups of $\pi_0^L(V)$, while each $T_V(g, n)$ is a coset of $T_V(n)$.

We note that

$$T_V(\pm n \pm 1) = \bigcup_{g \in T_V(\pm 1)} T_V(g, \pm n) \quad (n \in \mathbb{N}). \tag{5.3}$$

The mapping $\Theta_n : \pi_0^L(V) \to \pi_0^L(V)/T_V(n)$, given by

$$\Theta_n(g) = T_V(g, n), \tag{5.4}$$

is a continuous homomorphism of $\pi_0^L(V)$ onto $\pi_0^L(V)/T_V(n)$, with kernel $\operatorname{Ker}\Theta_n = T_V(-n)$. It induces a topological isomorphism

$$\hat{\Theta}_n : \pi_0^L(V)/T_V(-n) \to \pi_0^L(V)/T_V(n). \tag{5.5}$$

Remark 5.5. Let L and L_1 be compact groups, and α and β, continuous homomorphisms of L onto L_1. Then $V = \{x \in L^{\mathbb{Z}} ; \alpha(x_n) = \beta(x_{n+1})$ for every $n \in \mathbb{Z}\}$ is a full Markov subgroup of $L^{\mathbb{Z}}$, with $T_V = \{\langle x_0, x_1\rangle \in L^{\{0,1\}} ; \alpha(x_0) = \beta(x_1)\}$. Moreover, $T_V(-1) = \operatorname{Ker}\alpha$, $T_V(1) = \operatorname{Ker}\beta$, and $L/T_V(-1)$ and $L/T_V(1)$ are canonically isomorphic to L_1. For every $k \in \mathbb{N}$, $\hat{\pi}_k^L(V) = \{\langle x_0, \dots, x_k\rangle \in L^{\{0,\dots,k\}} ; \alpha(x_j) = \beta(x_{j+1})$ for every $j = 0, \dots, k-1\}$. If α and β are both injective, then $\tau = \beta^{-1} \circ \alpha \in \operatorname{Aut}(L)$ and $\pi_0^L \restriction V$ is an isomorphism of $(V, \mathbb{Z})$ onto $(L, \mathbb{Z})$, where the action of $\mathbb{Z}$ on L is induced by τ, that is, $kx = \tau^k(x)$ ($k \in \mathbb{Z}$, $x \in L$).

We note that any full Markov subgroup V of $L^{\mathbb{Z}}$ can be defined in terms of homomorphims α and β, as in Remark 5.5. Indeed, let $\gamma : L \to L/T_V(1)$ denote the projection. Then using (5.2) and the facts that V is full and $T_V(g, 1)$ is a coset of $T_V(1)$, it follows that

$$\begin{aligned} T_V &= \bigl\{\langle x_0, x_1\rangle \in L^{\{0,1\}} ; \Theta_1(x_0) = \gamma(x_1)\bigr\}, \\ V &= \bigl\{x \in L^{\mathbb{Z}} ; \Theta_1(x_n) = \gamma(x_{n+1}) \text{ for every } n \in \mathbb{Z}\bigr\}. \end{aligned} \tag{5.6}$$

Below we collect a number of rather detailed technical results about Markov subgroups, which we will need in the sequel. Many of those are modified versions of similar results to be found or implicit in [17] and [26], however, the nature of the modifications and added details warrant not omitting the proofs. We denote by L an arbitrary compact group, unless explicitly stated otherwise.

Lemma 5.6. *Let V be a closed shift-invariant subgroup of $L^{\mathbb{Z}}$. Then $V = \bigcap_{k=1}^{\infty} H_k$, where $H_k = \{x \in L^{\mathbb{Z}} ; \hat{\pi}_k^L(nx) \in \hat{\pi}_k^L(V)$ for every $n \in \mathbb{Z}\}$. If L is a Lie group, then $V = H_k$ for some $k \in \mathbb{N}$.*

Proof. To prove the nontrivial inclusion $\bigcap_{k=1}^{\infty} H_k \subseteq V$, suppose that $\hat{\pi}_k^L(nx) \in \hat{\pi}_k^L(V)$ for every $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. Then for every $k \in \mathbb{N}$ there exists $y^{(k)} \in V$, such that $\hat{\pi}_{2k}^L((-k)x) = \langle x_{-k}, \dots, x_k\rangle = \hat{\pi}_{2k}^L(y^{(k)})$. Hence, given $j \in \mathbb{Z}$, $x_j = y_{j+k}^{(k)} = (ky^{(k)})_j$ for all $k \geq |j|$. Thus, the sequence $(ky^{(k)})_{k=1}^{\infty}$ of elements of V converges to x, and so $x \in V$.

Next, suppose that L is a Lie group. Then the system $(L^{\mathbb{Z}}, \mathbb{Z})$ satisfies the dcc (by Theorem 5.2). As $(H_k)_{k=1}^{\infty}$ is a nonincreasing sequence of closed shift-invariant subgroups of $L^{\mathbb{Z}}$, $V = \bigcap_{k=1}^{\infty} H_k = H_l$ when l is large enough. $\square$

LEMMA 5.7. *Suppose that* V *is a Markov subgroup of* $L^{\mathbb{Z}}$, $N \in \mathbb{N}$, *and* $M = L^{\{0,\dots,N\}} = \hat{\pi}_N^L(L^{\mathbb{Z}})$. *Define* $\Phi : L^{\mathbb{Z}} \to M^{\mathbb{Z}}$ *by* $\Phi(x)_n = \hat{\pi}_N^L(nx)$. *Then* Φ *is a continuous injective equivariant homomorphim and* $W = \Phi(V)$ *is a Markov subgroup of* $M^{\mathbb{Z}}$.

Proof. It is clear that Φ is a continuous injective equivariant homomorphism. Therefore W is a closed shift-invariant subgroup of $M^{\mathbb{Z}}$, obviously contained in $W' = \{y \in M^{\mathbb{Z}} ; \langle y_n, y_{n+1}\rangle \in \hat{\pi}_1^M(W)$ for every $n \in \mathbb{Z}\}$.

Let $y = (y_n)_{n\in\mathbb{Z}} \in W'$. Then $y_n = \langle y_{n0}, \dots, y_{nN}\rangle$ with $y_{nj} \in L$. Define $x = (x_n)_{n\in\mathbb{Z}} \in L^{\mathbb{Z}}$ by $x_n = y_{n0}$. The condition $\langle y_n, y_{n+1}\rangle \in \hat{\pi}_1^M(W) \subseteq \hat{\pi}_1^M(\Phi(L^{\mathbb{Z}}))$ implies that $y_{n\,j+k} = y_{n+k\,j}$ whenever $0 \le k \le N$ and $0 \le j \le N-k$. Hence, $\Phi(x) = y$. Moreover, for a given n, $\langle y_n, y_{n+1}\rangle = \hat{\pi}_1^M(\Phi(v)) = \langle \hat{\pi}_N^L(v), \hat{\pi}_N^L(1v)\rangle$ for some $v \in V$, and, hence, $\langle x_n, x_{n+1}\rangle = \langle y_{n0}, y_{n+10}\rangle = \langle v_0, v_1\rangle \in T_V$. So $x \in V$ and $y \in \Phi(V) = W$. Therefore $W = W'$, that is, W is indeed a Markov subgroup. $\square$

The next lemma is a strengthening of Corollary 3.11 in [17].

LEMMA 5.8. *Let* G *and* K *be closed shift-invariant subgroups of* $L^{\mathbb{Z}}$, *where* L *is a compact Lie group and* $G \supseteq K$. *Then there exists* $N \in \mathbb{N}$, *such that with* $M = L^{\{0,\dots,N\}} = \hat{\pi}_N^L(L^{\mathbb{Z}})$ *and* $\Phi : L^{\mathbb{Z}} \to M^{\mathbb{Z}}$ *defined by* $\Phi(x)_n = \hat{\pi}_N^L(nx)$, *the following statements hold true*:

(i) $\tilde{G} = \Phi(G)$ *and* $\tilde{K} = \Phi(K)$ *are Markov subgroups of* $M^{\mathbb{Z}}$;

(ii) $\tilde{K} = \tilde{G} \cap [\pi_0^M(\tilde{K})]^{\mathbb{Z}}$;

(iii) $T_{\tilde{K}} = T_{\tilde{G}} \cap [\pi_0^M(\tilde{K})]^{\{0,1\}}$.

Proof. By Lemma 5.6 there exists $N \in \mathbb{N}$, such that $G = \{x \in L^{\mathbb{Z}} ; \hat{\pi}_N^L(nx) \in \hat{\pi}_N^L(G)$ for every $n \in \mathbb{Z}\}$ and $K = \{x \in L^{\mathbb{Z}} ; \hat{\pi}_N^L(nx) \in \hat{\pi}_N^L(K)$ for every $n \in \mathbb{Z}\}$. Put $M = L^{\{0,\dots,N\}}$ and define $\Phi : L^{\mathbb{Z}} \to M^{\mathbb{Z}}$ by $\Phi(x)_n = \hat{\pi}_N^L(nx)$. Then by Lemma 5.6, $W = \Phi(L^{\mathbb{Z}})$ is a Markov subgroup of $M^{\mathbb{Z}}$, while the definition of N implies that $\tilde{G} = \Phi(G) = \{x \in W ; x_n \in \pi_n^M(\tilde{G})$ for every $n \in \mathbb{Z}\}$ and $\tilde{K} = \Phi(K) = \{x \in W ; x_n \in \pi_n^M(\tilde{K})$ for every $n \in \mathbb{Z}\}$. It follows that both $\tilde{G}$ and $\tilde{K}$ are Markov subgroups of $M^{\mathbb{Z}}$ with $T_{\tilde{G}} = T_W \cap [\pi_0^M(\tilde{G})]^{\{0,1\}}$ and $T_{\tilde{K}} = T_W \cap [\pi_0^M(\tilde{K})]^{\{0,1\}}$, resp. Clearly, $\tilde{K} = \tilde{G} \cap [\pi_0^M(\tilde{K})]^{\mathbb{Z}}$ and $T_{\tilde{K}} = T_{\tilde{G}} \cap [\pi_0^M(\tilde{K})]^{\{0,1\}}$. $\square$

THEOREM 5.9. *Let an action of* $\mathbb{Z}$ *on a compact group* G *satisfy the dcc and* $K \le G$ *be a closed* $\mathbb{Z}$*-invariant subgroup. Then there exists a compact Lie group* $\tilde{L}$ *and an injective continuous equivariant homomorphism* $\Psi : G \to \tilde{L}^{\mathbb{Z}}$, *such that*:

(i) $\tilde{G} = \Psi(G)$ *and* $\tilde{K} = \Psi(K)$ *are Markov subgroups of* $\tilde{L}^{\mathbb{Z}}$, *and* $\tilde{G}$ *is full*;

(ii) $\tilde{K} = \tilde{G} \cap [\pi_0^{\tilde{L}}(\tilde{K})]^{\mathbb{Z}}$;
(iii) $T_{\tilde{K}} = T_{\tilde{G}} \cap [\pi_0^{\tilde{L}}(\tilde{K})]^{\{0,1\}}$.

Proof. By Theorem 5.2, we may assume that G and K are a closed shift-invariant subgroups of $L^{\mathbb{Z}}$, where L is a compact Lie group. Then Lemma 5.8 applies. Replacing M of Lemma 5.8 with $\tilde{L} = \pi_0^M(\Phi(G))$ (to ensure that $\tilde{G}$ be full) and putting $\Psi = \Phi \restriction G$, completes the proof. □

As an application of Theorems 5.9, we will now give a proof of the cdp for totally disconnected compact groups.

COROLLARY 5.10. *Any $\mathbb{Z}$-action on a totally disconnected compact group G has the cdp.*

Proof. In view of Theorems 3.8 and 5.3, it suffices to prove this when the system $(G, \mathbb{Z})$ satisfies the dcc. Let K be a closed $\mathbb{Z}$-invariant subgroup of G. By Theorem 5.9 we may assume that G and K are Markov subgroups of $L^{\mathbb{Z}}$, where L is a compact Lie group, G is full, $K = G \cap [\pi_0^L(K)]^{\mathbb{Z}}$, and $T_K = T_G \cap [\pi_0^L(K)]^{\{0,1\}}$. But then L must be finite. Let $\pi_{\mathbb{N}} : L^{\mathbb{Z}} \to L^{\mathbb{N}}$ denote the natural projection, as in Lemma 4.4. The fact that K is a Markov subgroup with $T_K = T_G \cap [\pi_0^L(K)]^{\{0,1\}}$ easily implies that $\pi_{\mathbb{N}}(K) = \pi_{\mathbb{N}}(G) \cap [\pi_0^L(K)]^{\mathbb{N}}$. Hence, $C_+(G, K) = C_+(G)K$ by Lemma 4.4. □

LEMMA 5.11. *Let G be a Markov subgroup of $L^{\mathbb{Z}}$ and K a closed shift-invariant subgroup of G. Then $K = G \cap [\pi_0^L(K)]^{\mathbb{Z}}$ if and only if K is a Markov subgroup of $L^{\mathbb{Z}}$ with $T_K = T_G \cap [\pi_0^L(K)]^{\{0,1\}}$. Moreover, if $K = G \cap [\pi_0^L(K)]^{\mathbb{Z}}$, then for every $h \in \pi_0^L(K)$ and every $n \in \mathbb{Z}$, $T_G(h, n) \subseteq \pi_0^L(K)T_G(n)$.*

Proof. We omit an obvious proof of the first statement (partly based on Remark 5.4). The second statement is true because $h = \pi_0^L(x)$ for some $x = (x_n)_{n \in \mathbb{Z}} \in K \subseteq G$. Consequently, $x_n \in T_G(h, n) \cap \pi_0^L(K)$ and since $T_G(h, n)$ is a coset of $T_G(n)$, $T_G(h, n) = x_n T_G(n) \subseteq \pi_0^L(K)T_G(n)$. □

LEMMA 5.12. *Let L be a compact group and G a Markov subgroup of $L^{\mathbb{Z}}$, given by $G = \{x \in L^Z ; \alpha(x_n) = \beta(x_{n+1})$ for every $n \in \mathbb{Z}\}$, where α, β are continuous homomorphisms of L onto a compact group L_1. If $M \leq L$ is a closed subgroup with $\alpha(M) = \beta(M)$, then $K = G \cap M^{\mathbb{Z}}$ is a full Markov subgroup of $M^{\mathbb{Z}}$ with $T_K = T_G \cap M^{\{0,1\}}$ and $T_K(\pm 1) = T_G(\pm 1) \cap M$. Furthermore, if $T_G(1) \subseteq M$ (resp., $T_G(-1) \subseteq M$), then for every $x \in G$ and $k \in \mathbb{Z}$, the condition that $x_k \in M$ implies that $x_n \in M$ for every $n \geq k$ (resp., $n \leq k$).*

Proof. Clearly, $K = G \cap [\pi_0^L(K)]^{\mathbb{Z}}$ and, hence, by Lemma 5.11, K is a Markov subgroup of $L^{\mathbb{Z}}$ with $T_K = T_G \cap [\pi_0^L(K)]^{\{0,1\}}$. But as $\alpha(M) = \beta(M)$, given $h \in M$, we can construct a sequence $x = (x_n)_{n \in \mathbb{Z}} \in M^{\mathbb{Z}}$ with $x_0 = h$ and $\alpha(x_n) = \beta(x_{n+1})$ for every n. Thus $x \in K$, and it follows that $\pi_0^L(K) = M$. Therefore K is a full Markov subgroup of $M^{\mathbb{Z}}$ with $T_K = T_G \cap M^{\{0,1\}}$. The equality $T_K(\pm 1) = T_G(\pm 1) \cap M$ follows from (5.2).

If $T_G(1) \subseteq M$, then by the last statement of Lemma 5.11, given $h \in M$, $T_G(h,1) \subseteq M$. Hence, if $x \in G$ and $x_k \in M$, then $x_{k+1} \in T_G(x_k,1) \subseteq M$; by induction, $x_n \in M$ for all $n \geq k$. □

Given a function $\eta : L \to L_1$ and a nonempty set A, we will write η^A for the product function $\eta^A : L^A \to L_1^A$, mapping $(x_a)_{a \in A} \in L^A$ to $(\eta(x_a))_{a\in A} \in L_1^A$.

LEMMA 5.13. *Let G be a full Markov subgroup of $L^{\mathbb{Z}}$, and $p \in \mathbb{N}$. Put $L_p = L/T_G(p)$, and write $\eta_p : L \to L_p$ for the canonical projection. If $y \in L_p^{\mathbb{Z}}$ and $\langle y_n, y_{n+1}\rangle \in \eta_p^{\{0,1\}}(T_G)$ for every $n \in \mathbb{Z}$, then $y = \eta_p^{\mathbb{Z}}(x)$ for some $x \in G$.*

Proof. Let $S = \{y \in L_p^{\mathbb{Z}} ; \langle y_n, y_{n+1}\rangle \in \eta_p^{\{0,1\}}(T_G)$ for every $n \in \mathbb{Z}\}$. We need to prove that $S \subseteq \eta_p^{\mathbb{Z}}(G)$. We observe that this will be accomplished once we prove that for every $y \in S$ there exists a sequence $(z^{(n)})_{n=1}^{\infty}$ in $L^{\mathbb{Z}}$, such that $\langle z_i^{(n)}, z_{i+1}^{(n)}\rangle \in T_G$ for $i = -n, \dots, n-1$ and $y_i = \eta_p(z_i^{(n)})$ for $i = -n, \dots, n$. Indeed, it is clear that if x is any cluster point of such a sequence, then $x \in G$ and $y = \eta_p^{\mathbb{Z}}(x)$.

We first show that for every $s \in S$ there exists a sequence $(x^{(m)})_{m=1}^{\infty}$ in $L^{\mathbb{Z}}$, such that $\langle x_i^{(m)}, x_{i+1}^{(m)}\rangle \in T_G$ for $i = -m, \dots, -1$ and $s_i = \eta_p(x_i^{(m)})$ for $i = -m, \dots, 0$. We proceed by induction. When $m = 1$, then $\langle s_{-1}, s_0\rangle = \eta^{\{0,1\}}(\langle z_0, z_1\rangle) = \langle \eta(z_0), \eta(z_1)\rangle$ for some $\langle z_0, z_1\rangle \in T_G$. We define $x^{(1)}$ by $x_{-1}^{(1)} = z_0$, $x_0^{(1)} = z_1$, and $x_i^{(1)} = e$ otherwise. Next, suppose that $x^{(1)}, \dots, x^{(m)}$ are already defined. Now, $\langle s_{-m-1}, s_{-m}\rangle = \eta_p^{\{0,1\}}(\langle z_0, z_1\rangle) = \langle \eta_p(z_0), \eta_p(z_1)\rangle$ for some $\langle z_0, z_1\rangle \in T_G$. Thus, $s_{-m} = \eta_p(z_1) = \eta_p(x_{-m}^{(m)})$ and so $x_{-m}^{(m)} z_1^{-1} \in \operatorname{Ker} \eta_p = T_G(p)$, that is, $x_{-m}^{(m)} z_1^{-1} = \pi_p^L(v)$, where $v \in G \cap \operatorname{Ker} \pi_0^L$. Note that $v_0, \dots, v_p \in T_G(p)$. We define $x^{(m+1)}$ by $x_{-m-1}^{(m+1)} = v_{p-1} z_0$ and $x_i^{(m+1)} = x_i^{(m)}$ otherwise. Then $\langle x_i^{(m+1)}, x_{i+1}^{(m+1)}\rangle \in T_G$ for $i = -m, \dots, -1$ and $s_i = \eta_p(x_i^{(m+1)})$ for $i = -m, \dots, 0$. For $i = -m-1$, we have $\langle x_{-m-1}^{(m+1)}, x_{-m}^{(m+1)}\rangle = \langle v_{p-1} z_0, x_{-m}^{(m)}\rangle = \langle v_{p-1} z_0, v_p z_1\rangle = \langle v_{p-1}, v_p\rangle\langle z_0, z_1\rangle \in T_G$ and $s_{-m-1} = \eta_p(z_0) = \eta_p(v_{p-1} z_0) = \eta_p(x_{-m-1}^{(m+1)})$. So the required sequence exists.

We can now conclude that when $y \in S$, then there exists a sequence $(z^{(n)})_{n=1}^{\infty}$ in $L^{\mathbb{Z}}$, such that $\langle z_i^{(n)}, z_{i+1}^{(n)}\rangle \in T_G$ for $i = -n, \dots, n-1$ and $y_i = \eta_p(z_i^{(n)})$ for $i = -n, \dots, n$. Indeed, given $n \in \mathbb{N}$, let $s = ny$ and let $(x^{(m)})_{m=1}^{\infty}$ be the sequence constructed in the preceding paragraph. Then $z^{(n)} = (-n)x^{(2n)}$ has the required properties. □

The case $p = 1$ of the following lemma is a part of Lemma 10.1 in [26, p. 83]. The lemma is also closely related to Proposition 5.7 in [17]. The term "semisimple Lie group" means a Lie group of positive dimension, whose Lie algebra is semisimple, or a finite group.

Lemma 5.14. *Let G be a full Markov subgroup of $L^{\mathbb{Z}}$ and let $p\in\mathbb{N}$. Put $L_p=L/T_G(p)$, $\Lambda_p=T_G(-p)\cap T_G(p)$, and write $\eta_p: L\to L_p$ for the canonical projection. Then*:

(1) *$G_p=\eta_p^{\mathbb{Z}}(G)$ is a full Markov subgroup of $L_p^{\mathbb{Z}}$ with $T_{G_p}=\eta_p^{\{0,1\}}(T_G)$.*
(2) *$\operatorname{Ker}(\eta_p^{\mathbb{Z}}\restriction G)=\Lambda_p^{\mathbb{Z}}\cap G$; $\operatorname{Ker}(\eta_p^{\mathbb{Z}}\restriction G)=\Lambda_p^{\mathbb{Z}}$ when $p=1$.*
(3) *$T_{G_p}(n)=\eta_p(T_G(n+p))$ for every $n\in\mathbb{N}$.*

Furthermore, if $T_G(p)=T_G(p+1)$ and L_p is a semisimple Lie group, then $\pi_0^{L_p}\restriction G_p$ is a topological isomorphism of G_p onto L_p.

Proof. It is clear that G_p is a full closed shift-invariant subgroup of $L_p^{\mathbb{Z}}$, that $\hat\pi_1^{L_p}(G_p)=\eta_p^{\{0,1\}}(\hat\pi_1^{L}(G))=\eta_p^{\{0,1\}}(T_G)$, and that $G_p\subseteq S=\{y\in L_p^{\mathbb{Z}};\ \langle y_n,y_{n+1}\rangle\in\eta_p^{\{0,1\}}(T_G)$ for every $n\in\mathbb{Z}\}$. Lemma 5.13 shows that $S\subseteq G_p$. This proves (1).

It is also clear that $\operatorname{Ker}(\eta_p^{\mathbb{Z}}\restriction G)=[T_G(p)]^{\mathbb{Z}}\cap G\supseteq\Lambda_p^{\mathbb{Z}}\cap G$. Let $x\in\operatorname{Ker}(\eta_p^{\mathbb{Z}}\restriction G)$. Then $x_n\in T_G(p)$ for every $n\in\mathbb{Z}$. Thus for a given n, $x_{n+p}\in T_G(p)$, so that $x_{n+p}=v_p$ for some $v\in G\cap\operatorname{Ker}\pi_0^L$. Put $w=p((nx)v^{-1})$. Then $w\in G\cap\operatorname{Ker}\pi_0^L$ and $x_n=w_{-p}$. Consequently, $x_n\in\Lambda_p$. It follows that $\operatorname{Ker}(\eta_p^{\mathbb{Z}}\restriction G)\subseteq\Lambda_p^{\mathbb{Z}}\cap G$.

To complete the proof of (2), it sufficies to show that $\Lambda_p^{\mathbb{Z}}\subseteq G$ when $p=1$. But if $x\in\Lambda_1^{\mathbb{Z}}$, then for a given $n\in\mathbb{Z}$, $\langle e,x_{n+1}\rangle,\langle x_n,e\rangle\in T_G$, and so $\langle x_n,x_{n+1}\rangle=\langle e,x_{n+1}\rangle\langle x_n,e\rangle\in T_G$. Therefore, $x\in G$.

We proceed to the proof of (3). Suppose $g\in T_G(n+p)$, that is, $g=v_{n+p}$, where $v\in G\cap\operatorname{Ker}\pi_0^L$. Then $\eta_p(g)=\eta_p(v_{n+p})=(\eta_p^{\mathbb{Z}}(v))_{n+p}=(\eta_p^{\mathbb{Z}}(pv))_n$, and $(\eta_p^{\mathbb{Z}}(pv))_0=\eta_p(v_p)=e$, where $\eta_p^{\mathbb{Z}}(pv)\in G_p$. Thus, $\eta_p(g)\in T_{G_p}(n)$. So $\eta_p(T_G(n+p))\subseteq T_{G_p}(n)$. Conversely, let $h\in T_{G_p}(n)$, that is, $h=w_n$ where $w\in G_p\cap\operatorname{Ker}\pi_0^{L_p}$. If $v\in G$ is such that $w=\eta_p^{\mathbb{Z}}(v)$, then $v_0\in T_G(p)$, and this easily implies that $v_n\in T_G(n+p)$. Then $h=w_n=(\eta_p^{\mathbb{Z}}(v))_n=\eta_p(x_n)\in\eta_p(T_G(n+p))$. Consequently, $T_{G_p}(n)=\eta_p(T_G(n+p))$.

If $T_V(p)=T_V(p+1)$, then, by (3), $T_{G_p}(1)$ is trivial. Hence, the homomorphism $L_p\ni g\to\Theta_1(g)=T_{G_p}(g,1)\in L_p/T_{G_p}(1)$ (cf. (5.4)), can be viewed as a surjective homomorphism of the semisimple Lie group L_p onto itself. But (due the finiteness of the centre of the connected component of e in L_p) such a homomorphism is necessarily an automorphism. So $T_{G_p}(-1)$ is trivial too. Then G_p is isomorphic to L_p by the last statement of Remark 5.5. ☐

Corollary 5.15. *Let G be a full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact semisimple Lie group. Then there exists $p\in\mathbb{N}$, such that, with the notations of Lemma 5.14, $\pi_0^{L_p}\restriction G_p$ is a topological isomorphism of G_p onto L_p.*

Proof. $(T_G(n))_{n=1}^{\infty}$ is a nondecreasing sequence of closed normal subgroups of the semisimple Lie group L. Since such a group can have only a finite number of closed normal subgroups, there exists $p \in \mathbb{N}$ with $T_G(p) = T_G(p+1)$. □

The final three results of this section concern the dimension of a shift-invariant subgroup of $L^{\mathbb{Z}}$. Since we will only need to distinguish between finite and infinite-dimensional groups, and between finite-dimensional groups of different dimensions, the following working definition will do the job. We define the dimension $\dim G = 0, 1, \ldots, \infty$ of a compact group G as the supremum of the dimensions of all Lie groups of the form G/N, where $N \leq G$ is a compact normal subgroup. It follows that $\dim G < \infty$ if and only if there exists a closed totally disconnected normal subgroup N, such that G/N is a Lie group (and then $\dim G$ is the dimension of the Lie group G/N). A well-known reformulation of this criterion is that $\dim G < \infty$ if and only G admits a neighbourhood of e which contains no nontrivial connected subgroups [22, p. 182]. For a thorough discussion of the concept of dimension of a compact group, we refer to [8].

Proposition 5.16. *The following conditions are equivalent for a full Markov subgroup V of $L^{\mathbb{Z}}$, where L is a compact Lie group*:

(i) *One of the subgroups $T_V(-1), T_V(1)$ is finite.*
(ii) *$T_V(n)$ is finite for every $n \in \mathbb{Z}$.*
(iii) $\dim V = \dim L$.
(iv) $\dim V < \infty$.

Proof. (i)⇒(ii): Since L is a compact Lie group, it follows from (5.5) that if one of $T_V(-1)$, $T_V(1)$ is finite, then both are finite. Then an inductive argument using (5.3) shows that $T_V(n)$ is finite for every $n \in \mathbb{Z}$.

(ii)⇒(iii): Clearly, $\operatorname{Ker}(\pi_0^L \upharpoonright V) = V \cap \prod_{n\in\mathbb{Z}} T_V(n)$, and so $\operatorname{Ker}(\pi_0^L \upharpoonright V)$ is totally disconnected. As $\pi_0^L(V) = L$, $\dim V = \dim L$.

(iv)⇒(i): Let U be a neighbourhood of e in V which contains no nontrivial connected subgroups. Using $\mathbb{Z}$-invariance of V we may assume that there is $N \in \mathbb{N}$ and a neighbourhood Ω of e in L, such that $V \cap \bigcap_{j=-N}^{0} (\pi_j^L)^{-1}(\Omega) \subseteq U$. Then the closed subgroup $H = V \cap \bigcap_{j=-N}^{0} \operatorname{Ker} \pi_j^L$ is contained in U, and so is totally disconnected. Consequently, $\pi_1^L(H)$ is finite. But using the fact that V is a Markov subgroup, it is easy to see that $\pi_1^L(H) = T_V(1)$. □

Corollary 5.17. *Let G be a connected finite-dimensional full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact semisimple Lie group. Then there exists $p \in \mathbb{N}$, such that, with the notations of Lemma 5.14, $\eta_p \circ \pi_0^L \upharpoonright G$ is a topological isomorphism of G onto $L/T_G(p)$.*

Proof. In view of Corollary 5.15, it suffices to show that for every $p \in \mathbb{N}$, $\eta_p^{\mathbb{Z}} \upharpoonright G$ is an isomorphism of G onto G_p (because $\pi_0^{L_p} \circ \eta_p^{\mathbb{Z}} = \eta_p \circ \pi_0^L$). We will first show that G has finite centre.

Since G is full, L is connected. Let $\tilde{L}$ be the universal covering of L, and N denote the (finite) number of elements in the centre $Z(\tilde{L})$ of $\tilde{L}$. Observe that if K is a (compact) connected Lie group whose Lie algebra is isomorphic to that of L, then $\tilde{L}$ is the universal covering of K and $Z(K)$ is a homomorphic image of $Z(\tilde{L})$. Thus, $|Z(K)| \leq N$.

Clearly, for every $k \in \mathbb{N}$, $\dim(\hat{\pi}_k^L(G)) \leq \dim G = \dim L$. But since G is full, L is a continuous homomorphic image of $\hat{\pi}_k^L(G)$. Therefore, $\dim(\hat{\pi}_k^L(G)) = \dim L$. Thus $\hat{\pi}_k^L(G)$ is a connected Lie group whose Lie algebra is isomorphic to that of L. Consequently, $|Z(\hat{\pi}_k^L(G))| \leq N$. Therefore, $|\hat{\pi}_k^L(Z(G))| \leq N$ for every $k \in \mathbb{N}$. This and the $\mathbb{Z}$-invariance of $Z(G)$ imply that $|Z(G)| \leq N$.[2]

We will now show by induction on p that $\eta_p^Z \restriction G$ is an isomorphism. When $p = 1$, $\mathrm{Ker}(\eta_1^{\mathbb{Z}} \restriction G) = \Lambda_1^{\mathbb{Z}}$ (cf. Lemma 5.14), where Λ_1 is finite by Proposition 5.16. Hence, $\mathrm{Ker}(\eta_1^{\mathbb{Z}} \restriction G)$ is either trivial or infinite, and it is a totally disconnected closed normal subgroup of G, thus contained in $Z(G)$. So $\mathrm{Ker}(\eta_1^{\mathbb{Z}} \restriction G)$ is finite and therefore trivial.

Next, let us suppose that we already proved that $\eta_p^{\mathbb{Z}} \restriction G$ is an isomorphism. Let $\tilde{\eta}_1 : L_p \to L_p/T_{G_p}(1)$ denote the projection. Then the argument in the case $p = 1$ demonstrates that $\mathrm{Ker}(\tilde{\eta}_1^{\mathbb{Z}} \restriction G_p) = \{e\}$. But using Lemma 5.14(3), $\mathrm{Ker}(\tilde{\eta}_1 \circ \eta_p) = T_G(p+1) = \mathrm{Ker}\, \eta_{p+1}$. It follows that there exists an isomorphism $\varphi : L_{p+1} \to L_p/T_{G_p}(1)$, such that $\varphi \circ \eta_{p+1} = \tilde{\eta}_1 \circ \eta_p$. Hence, $\varphi^{\mathbb{Z}} \circ \eta_{p+1}^{\mathbb{Z}} \restriction G = \tilde{\eta}_1^{\mathbb{Z}} \circ \eta_p^{\mathbb{Z}} \restriction G$, and by induction, $\mathrm{Ker}(\eta_{p+1}^{\mathbb{Z}} \restriction G) = \{e\}$. $\square$

PROPOSITION 5.18. *Let K be a proper closed normal $\mathbb{Z}$-invariant subgroup of $L^{\mathbb{Z}}$. Then $L^{\mathbb{Z}}/K$ is uncountable, and either 0-dimensional or infinite-dimensional.*

Proof. Let $\zeta : L^{\mathbb{Z}} \to L^{\mathbb{Z}}/K$ and $\zeta_k : L^{\{0,\ldots,k\}} \to L^{\{0,\ldots,k\}}/\hat{\pi}_k^L(K)$ ($k \in \mathbb{N}$) denote the canonical homomorphisms. For every $k \in \mathbb{N}$ there exists a surjective homomorphism $\tilde{\pi}_k : L^{\mathbb{Z}}/K \to L^{\{0,\ldots,k\}}/\hat{\pi}_k^L(K)$ with $\tilde{\pi}_k \circ \zeta = \zeta_k \circ \hat{\pi}_k^L$. For $l \in \mathbb{N}$, let Φ_k^l denote the canonical isomorphism of $L^{\{0,\ldots,(k+1)l-1\}}$ onto $[L^{\{0,\ldots,k\}}]^l$, mapping $\langle x_0, \ldots, x_{(k+1)l-1}\rangle$ to $(\langle 0, \ldots, x_k\rangle, \ldots, \langle x_{(k+1)(l-1)}, \ldots, x_{(k+1)l-1}\rangle)$. Using the fact that K is $\mathbb{Z}$-invariant, one concludes that $\Phi_k^l(\hat{\pi}_{(k+1)l-1}^L(K)) \subseteq [\hat{\pi}_k^L(K)]^l$. It follows that $|\tilde{\pi}_{(k+1)l-1}(L^{\mathbb{Z}}/K)| \geq |L^{\{0,\ldots,k\}}/\pi_k^L(K)|^l$, and that $\dim(\tilde{\pi}_{(k+1)-1}(L^{\mathbb{Z}}/K)) \geq l \dim(L^{\{0,\ldots,k\}}/\pi_k^L(K))$. Therefore:

(1) $\sup_{l\in\mathbb{N}} |\tilde{\pi}_{(k+1)l-1}(L^{\mathbb{Z}}/K)| = \infty$, unless $\hat{\pi}_k^L(K) = L^{\{0,\ldots,k\}}$;

(2) $\sup_{l\in\mathbb{N}} \dim(\tilde{\pi}_{(k+1)-1}(L^{\mathbb{Z}}/K)) = \infty$, unless $\dim(L^{\{0,\ldots,k\}}/\pi_k^L(K)) = 0$.

The first of the above conclusions yields that $L^{\mathbb{Z}}/K$ is infinite, because by Lemma 5.6 there exists $k \in \mathbb{N}$ with $\hat{\pi}_k^L(K) \neq L^{\{0,\ldots,k\}}$. By the Baire category theorem and compactness, $L^{\mathbb{Z}}/K$ is then uncountable.

[2] Finiteness of $Z(G)$ can be also deduced by using Theorems 9.24, 9.26, and 9.55 in [8] to conclude that G is a semisimple Lie group.

Next, if $d = \dim(L^{\mathbb{Z}}/K) < \infty$, then (2) yields that $\dim(L^{\{0,\dots,k\}}/\hat{\pi}_k^L(K)) = 0$ for every $k \in \mathbb{N}$. This means that $\hat{\pi}_k^L((L^{\mathbb{Z}})_e) = (L^{\{0,\dots,k\}})_e = [\hat{\pi}_k^L(K)]_e = \hat{\pi}_k^L(K_e)$, where by H_e we denote the connected component of the identity in a group H. Therefore by Lemma 5.6, $(L^{\mathbb{Z}})_e = K_e$, and so $L^{\mathbb{Z}}/K$ must be 0-dimensional. □

6. Solenoids in $(\mathbb{T}^m)^{\mathbb{Z}}$

The solenoid plays a special role in ergodic theory of automorphisms of compact groups and presents challenges of its own, see, for example, [17], [19], [20], [21], [26], [29]. The situation is no different in the theory of contraction groups, where a detailed technical knowledge of the solenoid is called for. Even though the solenoid is a well known classical object, the relevant information is not readily available in the literature. Moreover, there appears to be some confusion, including outright mistakes, regarding certain facts which will play a key role in our argument. Hence, we include a rather detailed review of the solenoid, supplemented by the Appendix containing counterexamples.

Given $m \in \mathbb{N}$, we will denote by $\exp_m : \mathbb{R}^m \to \mathbb{T}^m$ the exponential function

$$\exp_m(x_1, \dots, x_m) = \big(\exp(2\pi i x_1), \dots, \exp(2\pi i x_m)\big). \tag{6.1}$$

When U is a matrix in $\mathrm{GL}(m, \mathbb{Q})$, we will write $E_U : \mathbb{R}^m \to (\mathbb{T}^m)^{\mathbb{Z}}$ for the homomorphism

$$E_U(x) = \big(\exp_m\big(U^n x\big)\big)_{n\in\mathbb{Z}} \tag{6.2}$$

(where x is treated as a column vector when multiplied by U^n). The closure of $E_U(\mathbb{R}^m)$ in $(\mathbb{T}^m)^{\mathbb{Z}}$ is a full shift-invariant subgroup of $(\mathbb{T}^m)^{\mathbb{Z}}$, which will be called the *solenoid generated by* U, or just *solenoid*, if U does not need to be mentioned. A Markov subgroup of this form will be called a *Markov solenoid*. For the sake of conciseness, a dynamical system $(V, \mathbb{Z})$ in which V is a solenoid and $\mathbb{Z}$ acts by the Bernoulli shifts, will be also referred to as a solenoid.

Another common way of introducing the solenoid is via duality theory. Recall that the dual group of $(\mathbb{T}^m)^{\mathbb{Z}}$ is canonically identified with the (weak) direct product $\prod^*_{k\in\mathbb{Z}} \mathbb{Z}^m$. Given $U \in \mathrm{GL}(m, \mathbb{Q})$, let $\Xi_U \le \prod^*_{k\in\mathbb{Z}} \mathbb{Z}^m$ denote the subgroup $\Xi_U = \{\xi \in \prod^*_{k\in\mathbb{Z}} \mathbb{Z}^m ; \sum_{k\in\mathbb{Z}} \xi_k U^k = 0\}$ (where ξ_k is treated as a row vector when multiplied by U^k). Standard duality theory yields that the solenoid V generated by U is precisely the anihilator $\Xi_U^{\perp}$ of Ξ_U in $(\mathbb{T}^m)^{\mathbb{Z}}$. The dual group of V can be identified with the subgroup of $\mathbb{Q}^m$ generated by $\bigcup_{k\in\mathbb{Z}} \mathbb{Z}^m U^k$.

The solenoid is an m-dimensional compact connected group which is locally connected if and only if it is isomorphic to $\mathbb{T}^m$. A sufficient and necessary condition in order that the solenoid V generated by U be isomorphic to $\mathbb{T}^m$ is that $U^p \in \mathrm{GL}(m, \mathbb{Z})$ for some $p \in \mathbb{N}$. When V is a Markov solenoid, it is isomorphic to $\mathbb{T}^m$ if and only if $U \in \mathrm{GL}(m, \mathbb{Z})$.

When V is the solenoid generated by U, then for every $k \in \mathbb{N}$,

$$\hat{\pi}_k^{\mathbb{T}^m}(V) = \big\{\big\langle \exp_m(x), \ldots, \exp_m\big(U^k x\big)\big\rangle\,;\, x \in \mathbb{R}^m\big\} \tag{6.3}$$

is a closed subgroup of $(\mathbb{T}^m)^{\{0,\ldots,k\}}$, isomorphic to $\mathbb{T}^m$. Let B be any nonsingular matrix in $\mathrm{M}(m,\mathbb{Z})$, such that $A = BU \in \mathrm{M}(m,\mathbb{Z})$ (e.g., $B = bI$ where $b \in \mathbb{N}$ is such that $bU \in \mathrm{M}(m,\mathbb{Z})$). Recall that the formula $\gamma_C(\exp_m(x)) = \exp_m(Cx)$ defines a one-to-one correspondence between matrices $C \in \mathrm{M}(m,\mathbb{Z})$ and continuous homomorphisms $\gamma_C : \mathbb{T}^m \to \mathbb{T}^m$; nonsingular matrices in $\mathrm{M}(m,\mathbb{Z})$ are in one-to-one correspondence with surjective homomorphisms, and matrices in $\mathrm{GL}(m,\mathbb{Z})$, with automorphisms of $\mathbb{T}^m$. Hence, A and B define surjective homomorphims $\alpha, \beta : \mathbb{T}^m \to \mathbb{T}^m$. It is clear that

$$\hat{\pi}_1^{\mathbb{T}^m}(V) \subseteq \big\{\langle x_0, x_1\rangle \in \big(\mathbb{T}^m\big)^{\{0,1\}}\,;\, \alpha(x_0) = \beta(x_1)\big\}, \tag{6.4}$$

and, hence, V is contained in the Markov subgroup

$$W = \big\{x \in \big(\mathbb{T}^m\big)^{\mathbb{Z}}\,;\, \alpha(x_n) = \beta(x_{n+1}) \text{ for every } n \in \mathbb{Z}\big\}. \tag{6.5}$$

It is not difficult to show that when $m = 1$ and $\gcd(A,B) = 1$, then $V = W$. Thus in dimension $m = 1$ every solenoid is a Markov solenoid. However, while every solenoid is isomorphic to a Markov solenoid (cf. Proposition 6.2), solenoids in dimension $m > 1$ are, in general, not Markov solenoids, see Example A.1 in the Appendix. When V is a Markov solenoid, then

$$\begin{gathered} T_V = \big\{\big\langle \exp_m(x), \exp_m(Ux)\big\rangle\,;\, x \in \mathbb{R}^m\big\}, \\ T_V(1) = \big\{\exp_m(Ux)\,;\, x \in \mathbb{Z}^m\big\}, \qquad T_V(-1) = \big\{\exp_m\big(U^{-1}x\big)\,;\, x \in \mathbb{Z}^m\big\}, \end{gathered} \tag{6.6}$$

in particular, $T_V(\pm 1)$ are finite subgroups of $\mathbb{T}^m$.

The question when a solenoid is a Markov solenoid is closely related to a question which will play a key role in the sequel. In the proof of Theorem 7.1 we will need to conclude that certain Markov subgroups of $(\mathbb{T}^m)^{\mathbb{Z}}$ are solenoids. Example A.2 in the Appendix shows that the criteria used in [4, p. 701] and [21, p. 214] are in error (unless $m = 1$).

Let V be any full Markov subgroup of $(\mathbb{T}^m)^{\mathbb{Z}}$, where $T_V(1)$ and $T_V(-1)$ are finite. Then $\mathbb{T}^m/T_V(1)$ and $\mathbb{T}^m/T_V(-1)$ are isomorphic to $\mathbb{T}^m$, and by (5.6) and Remark 5.5 one can find continuous surjective homomorphisms $\alpha, \beta : \mathbb{T}^m \to \mathbb{T}^m$, determined by nonsingular matrices $A, B \in \mathrm{M}(m,\mathbb{Z})$, such that

$$\begin{gathered} V = \big\{x \in \big(\mathbb{T}^m\big)^{\mathbb{Z}}\,;\, \alpha(x_n) = \beta(x_{n+1}) \text{ for every } n \in \mathbb{Z}\big\}, \\ T_V = \big\{\langle x_0, x_1\rangle \in \big(\mathbb{T}^m\big)^{\{0,1\}}\,;\, \alpha(x_0) = \beta(x_1)\big\}, \\ T_V(-1) = \operatorname{Ker}\alpha, \qquad T_V(1) = \operatorname{Ker}\beta. \end{gathered} \tag{6.7}$$

It is clear that the solenoid generated by $B^{-1}A$ is a subgroup of V. When V is the (Markov) solenoid generated by U, then comparing the descriptions of T_V in (6.6) and (6.7), one concludes that $U = B^{-1}A$.

We denote by H_e the connected component of the identity in a group H.

Proposition 6.1. *Let V be a full Markov subgroup of $(\mathbb{T}^m)^{\mathbb{Z}}$. Then*:

(1) *$T_V(1)$ is finite if and only if $T_V(-1)$ is finite.*

(2) *V is a Markov solenoid if and only if V is connected and $T_V(1)$ is finite.*

(3) *If V is disconnected, then V/V_e is uncountable.*

Proof. (1) is a special case Proposition 5.16. Equation (6.6) explains that $T_V(\pm 1)$ are finite when V is a Markov solenoid. To complete the proof of (2) it remains to show that if V is connected and $T_V(\pm 1)$ are finite, then V is a Markov solenoid. The paragraph containing Equation (6.7) applies in the current situtation. Since the solenoid W generated by $U = B^{-1}A$ is contained in V, it remains to show that $V \subseteq W$. Using Lemma 5.6, this will be accomplished once we showed that $\hat{\pi}_k^{\mathbb{T}^m}(V) \subseteq \hat{\pi}_k^{\mathbb{T}^m}(W)$ for every $k \in \mathbb{N}$.

By (6.3), $\hat{\pi}_k^{\mathbb{T}^m}(W) = \{\langle \exp_m(x), \ldots, \exp_m(U^n x)\rangle\,;\, x \in \mathbb{R}^m\}$. On the other hand, $\hat{\pi}_k^{\mathbb{T}^m}(V)$ is a closed connected subgroup of the torus $(\mathbb{T}^m)^{\{0,\ldots,k\}}$. Therefore if $v \in \pi_k^{\mathbb{T}^m}(V)$, then there exist $x_0, \ldots, x_k \in \mathbb{R}^m$, such that $v = \langle \exp_m(x_0), \ldots, \exp_m(x_m)\rangle$ and $\langle \exp_m(tx_0), \ldots, \exp_m(tx_m)\rangle \in \pi_k^{\mathbb{T}^m}(V)$ for every $t \in \mathbb{R}$. But $\hat{\pi}_k^{\mathbb{T}^m}(V) = \{\langle y_0, \ldots, y_k\rangle\,;\, \alpha(y_j) = \beta(y_{j+1})$ for every $j = 0, \ldots, k-1\}$ (cf. Remark 5.5). So $\alpha(\exp_m(tx_j)) = \exp_m(tAx_j) = \beta(\exp_m(tx_{j+1})) = \exp_m(tBx_{j+1})$ for all $j = 0, \ldots, k-1$ and $t \in \mathbb{R}$. Hence, $Ax_j = Bx_{j+1}$, that is, $x_{j+1} = Ux_j$. So $v = \langle \exp_m(x_0), \ldots, \exp_m(U^k x_0)\rangle \in \hat{\pi}_k^{\mathbb{T}^m}(W)$.

By the Baire category theorem, to prove (3) it is enough to show that V/V_e is infinite. Let $x \in V \setminus V_e$. By Lemma 5.6 there exist $n \in \mathbb{Z}$ and $N \in \mathbb{N}$, such that $\hat{\pi}_N^{\mathbb{T}^m}(nx) \notin \hat{\pi}_N^{\mathbb{T}^m}(V_e)$. Observe that V_e is a full subgroup of $(\mathbb{T}^m)^{\mathbb{Z}}$. Hence, $x_n = v_0$ for some $v \in V_e$. Put $y = (nx)v^{-1}$. Then $\hat{\pi}_N^{\mathbb{T}^m}(y) \notin \hat{\pi}_N^{\mathbb{T}^m}(V_e)$ and $y_0 = e$. Define $z \in (\mathbb{T}^m)^{\mathbb{Z}}$ by $z_j = e$ for $j \leq 0$ and $z_j = y_j$ for $j \geq 1$. Since V is a Markov subgroup, $z \in V$, and, hence, $(-kN)z \in V$ for every $k = 0, 1, \ldots$. Each of the latter elements can be seen to belong to a distinct coset of V_e. $\square$

Proposition 6.1 combined with Theorem 5.9, and Proposition 5.16 results in the following well known characterization of the solenoid [19, Theorem 19], [17, p. 712], [26, Chapter 3].

Proposition 6.2. *The following conditions are equivalent for an action of $\mathbb{Z}$ on a compact group G*:

(1) *The dynamical system $(G, \mathbb{Z})$ is isomorphic to a Markov solenoid in $(\mathbb{T}^m)^{\mathbb{Z}}$.*

(2) *G is an m-dimensional connected Abelian group and the system $(G, \mathbb{Z})$ satisfies the dcc.*

Remark 6.3. When V is any full Markov subgroup of $(\mathbb{T}^m)^{\mathbb{Z}}$ with finite $T_V(\pm 1)$, the proof of Proposition 6.1(2) shows that the connected component of the identity in V coincides with the solenoid generated by $B^{-1}A$ (where A and B determine homomorphisms α and β satisfying (6.7)).

It is obvious that the connectedness of a Markov subgroup V of $(\mathbb{T}^m)^{\mathbb{Z}}$ implies that of the transition subgroup $T_V = \hat{\pi}_1^{\mathbb{T}^m}(V)$. It is also not difficult to see that when V is full, the condition that T_V be connected and $T_V(\pm 1)$ finite, is equivalent to the condition that T_V be topologically isomorphic to $\mathbb{T}^m$. However, contrary to the claim made in [4, p. 701], the connectedness of T_V is not sufficient in order that V be connected, and a Markov solenoid cannot be defined as a full Markov subgroup $V \leq (\mathbb{T}^m)^{\mathbb{Z}}$ such that T_V is isomorphic to $\mathbb{T}^m$; see Example A.2 in the Appendix.[3]

Proposition 6.4. *Every closed totally disconnected $\mathbb{Z}$-invariant subgroup of a solenoid is finite.*

Proof. Indeed, if D is such a subgroup of a solenoid $V \leq (\mathbb{T}^m)^{\mathbb{Z}}$, then $\pi_0^{\mathbb{T}^m}(D)$ is a finite subgroup of $\mathbb{T}^m$, hence, contained in $\Omega_p^m = \{x \in \mathbb{T}^m\,;\, x^p = e\}$ for some $p \in \mathbb{N}$. Therefore $D \subseteq (\Omega_p^m)^{\mathbb{Z}}$, and so $x^p = e$ for every $x \in D$. Since for every $k \in \mathbb{N}$, $\hat{\pi}_k^{\mathbb{T}^m}(V)$ is isomorphic to $\mathbb{T}^m$ (cf. (6.3)), it follows that for every $k \in \mathbb{N}$, $\hat{\pi}_k^{\mathbb{T}^m}(D)$ has at most p^m elements. This forces D to have at most p^m elements. $\square$

Corollary 6.5. *Every closed $\mathbb{Z}$-invariant subgroup of a solenoid has finitely many connected components.*

Proof. If H is such a subgroup of a solenoid V, then H/H_e is a closed totally disconnected $\mathbb{Z}$-invariant subgroup of V/H_e. But V/H_e is either trivial, or, by Proposition 6.2, it can be regarded as a solenoid, hence, Proposition 6.4 applies. $\square$

One consequence of Proposition 6.4 is that if V is a Markov solenoid, then $T_V(1) \cap T_V(-1)$ is trivial. This is because $T_V(\pm 1)$ are finite and V contains $[T_V(1) \cap T_V(-1)]^{\mathbb{Z}}$ (cf. Lemma 5.14(2)). The condition $T_V(1) \cap T_V(-1) = \{e\}$ is therefore necessary in order that a Markov subgroup of $(\mathbb{T}^m)^{\mathbb{Z}}$ be a solenoid. In [21] the authors define a "generalized torus" as a full Markov subgroup V of $(\mathbb{T}^m)^{\mathbb{Z}}$, such that $T_V(-1)$ and $T_V(1)$ are finite, and $T_V(-1) \cap T_V(1) = \{e\}$ [21, p. 214]. It is taken for granted [21, p. 216] that the generalized torus is connected (and therefore coincides with what we call a Markov solenoid). This is indeed the case when $m = 1$: for a full Markov subgroup $V \leq \mathbb{T}^{\mathbb{Z}}$ with finite $T_V(\pm 1)$ the three conditions: V is a solenoid; T_V is connected; and, $T_V(-1) \cap T_V(1) = \{e\}$, are equivalent. However, the situation turns out to be entirely different when $m > 1$. In general, the generalized torus need not be a solenoid, even if the original definition is strengthened by additionally requiring that the transition subgroup T_V be connected, see Example A.2 in the Appendix.

[3] The source of the error in [4] appears to be the final statement of Proposition 5.4 in [17] or Lemma 9.9 in [26], which can be misunderstood as a claim that the connectedness of T_V implies that of V.

The second of the next two interrelated results about Markov subgroups of $(\mathbb{T}^m)^{\mathbb{Z}}$ will have a direct application in the proof of Theorem 7.1. The first is a version of Lee's supplement theorem [8, Theorem 9.41], which asserts that every compact group V contains a closed totally disconnected subgroup D with $V = DV_e$. We denote by Ω_p^m the subgroup of elements of order p in the torus $\mathbb{T}^m$.

PROPOSITION 6.6. *Let V be a full Markov subgroup of $(\mathbb{T}^m)^{\mathbb{Z}}$, where $T_V(\pm 1)$ are finite. Then there exists $p \in \mathbb{N}$, such that $V = [(\Omega_p^m)^{\mathbb{Z}} \cap V]V_e$.*

Proof. Due to Theorem 5.9, there exists a compact Lie group L and a continuous injective equivariant homomorphism $\Psi : V \to L^{\mathbb{Z}}$, such that $\Psi(V)$ and $\Psi(V_e)$ are Markov subgroups of $L^{\mathbb{Z}}$, $\Psi(V)$ is full, and $\Psi(V_e) = \Psi(V) \cap [\pi_0^L(\Psi(V_e))]^{\mathbb{Z}}$. But $\Psi(V_e) = [\Psi(V)]_e$, $\pi_0^L([\Psi(V)]_e) = L_e$, and L/L_e is a group of finite order p. Hence, for every $w \in \Psi(V)$, $w^p \in \Psi(V_e)$. Therefore, $v^p \in V_e$ for every $v \in V$.

Let $D = \{x \in V\,;\, x^p = e\} = (\Omega_p^m)^{\mathbb{Z}} \cap V$. Clearly, D is closed, shift-invariant, and totally disconnected. It remains to show that $V = DV_e$. But if $v \in V$ then $v^p \in V_e$. As V_e is divisible [8, Corollary 8.5], we can find $x \in V_e$ with $v^p = x^p$. Then $v = (vx^{-1})x$, where $(vx^{-1}) \in D$ and $x \in V_e$. □

COROLLARY 6.7. *There exists $N \in \mathbb{N}$, such that with $\eta_N : \mathbb{T}^m \to \mathbb{T}^m/T_V(N)$ denoting the canonical projection, the system $(\eta_N^{\mathbb{Z}}(V), \mathbb{Z})$ is isomorphic to a Markov solenoid in $(\mathbb{T}^m)^{\mathbb{Z}}$. In particular, $V \subseteq [T_V(N)]^{\mathbb{Z}} V_e$.*

Proof. Let $K = (\Omega_p^m)^{\mathbb{Z}} \cap V$ where p is as in Proposition 6.6. It is clear that $K = V \cap [\pi_0^{\mathbb{T}^m}(K)]^{\mathbb{Z}}$. So by Lemma 5.11, K is a Markov subgroup of $(\mathbb{T}^m)^{\mathbb{Z}}$ with $T_K = T_V \cap [\pi_0^{\mathbb{T}^m}(K)]^{\{0,1\}}$. K is then a full Markov subgroup of $[\pi_0^{\mathbb{T}^m}(K)]^{\mathbb{Z}}$, where $\pi_0^{\mathbb{T}^m}(K)$ is finite. Therefore by Corollary 5.15, there exists $N \in \mathbb{N}$, such that with $\kappa_N : \pi_0^{\mathbb{T}^m}(K) \to \pi_0^{\mathbb{T}^m}(K)/T_K(N)$ denoting the canonical homomorphism, $\kappa_N^{\mathbb{Z}}(K)$ is finite. Recall that $\operatorname{Ker}(\kappa_N^{\mathbb{Z}} \restriction K) = [T_K(N) \cap T_K(-N)]^{\mathbb{Z}} \cap K$ (Lemma 5.14). So $[T_K(N) \cap T_K(-N)]^{\mathbb{Z}} \cap K$ has finite index in K.

Clearly, $[T_K(N) \cap T_K(-N)]^{\mathbb{Z}} \cap K \subseteq [T_V(N) \cap T_V(-N)]^{\mathbb{Z}} \cap V = \operatorname{Ker}(\eta_N^{\mathbb{Z}} \restriction V)$. It follows that $\eta_N^{\mathbb{Z}}(K)$ is finite.

Now, $V = KV_e$, so that $\tilde{V} = \eta_N^{\mathbb{Z}}(V) = \eta_N^{\mathbb{Z}}(K)\eta_N^{\mathbb{Z}}(V_e) = \eta_N^{\mathbb{Z}}(K)\tilde{V}_e$. Thus, $\tilde{V}_e$ has finite index in $\tilde{V}$. Furthermore, by Proposition 5.16, $T_V(N)$ and $T_V(N+1)$ are finite. Hence, by Lemma 5.14, $\tilde{V}$ is a full Markov subgroup of $[\mathbb{T}^m/T_V(N)]^{\mathbb{Z}}$, where $\mathbb{T}^m/T_V(N)$ is isomorphic to $\mathbb{T}^m$ and $T_{\tilde{V}}(1)$ is finite. Using first part (3) and then part (2) of Proposition 6.1, one concludes that $\tilde{V}$ is indeed isomorphic to a Markov solenoid. □

We conclude this section with a description of the contraction group of the solenoid generated by $U \in \mathrm{GL}(m, \mathbb{Q})$. This involves the contraction group $C(U)$ of the automorphism $x \to Ux$ of $\mathbb{R}^m$, which we will denote also by U.

PROPOSITION 6.8. *Let $g \in G$, where G is the solenoid generated by $U \in \mathrm{GL}(m,\mathbb{Q})$. Then $g \in C_+(G)$ if and only if there exists $x \in C(U)$, such that $g_n = \exp_m(U^n x)$ for large enough n. Moreover, $[C_+(G)]^{-}$ is connected.*

Proof. By virtue of (4.1), it suffices to show that if $g \in C_+(G)$, then for some $x \in C(U)$ and all large enough n, $g_n = \exp_m(U^n x)$. Choose matrices $A, B \in \mathrm{M}(m,\mathbb{Z})$, such that $U = B^{-1}A$, and let $\alpha, \beta : \mathbb{T}^m \to \mathbb{T}^m$ be the corresponding homomorphims. Then G is contained in the Markov subgroup $\{g \in (\mathbb{T}^m)^{\mathbb{Z}}\,; \alpha(g_n) = \beta(g_{n+1})$ for every $n \in \mathbb{Z}\}$, cf. (6.4), (6.5). Let Ω be a neighbourhood of $0 \in \mathbb{R}^m$, such that $\exp_m \restriction \Omega$ is a homeomorphism onto a neighbourhood of e in $\mathbb{T}^m$, and set $\Omega_1 = \Omega \cap (A^{-1}\Omega) \cap (B^{-1}\Omega)$. If $g \in C_+(G)$, then there exists $N \in \mathbb{N}$, such that $g_n \in \exp_m(\Omega_1)$ for all $n \geq N$. Thus for $n \geq N$, $g_n = \exp_m(x_n)$ with $x_n \in \Omega_1$. But $\exp_m(Ax_n) = \alpha(g_n) = \beta(g_{n+1}) = \exp_m(Bx_{n+1})$. Hence, due to our choice of Ω_1, $Ax_n = Bx_{n+1}$, or $x_{n+1} = Ux_n$. Thus if $x = U^{-N}x_N$, then $x_n = U^n x$ for all $n \geq N$. As $\lim_{n\to\infty} g_n = e$ and $U^n x \in \Omega$ for $n \geq N$, $\lim_{n\to\infty} U^n x = 0$, that is, $x \in C(U)$.

To see that $[C_+(G)]^{-}$ is connected, recall that by Corollary 6.5, $[C_+(G)]^{-}$ has finitely many connected components. Therefore, $([C_+(G)]^{-})_e$ is open in $[C_+(G)]^{-}$. This implies that $C_+([C_+(G)]^{-}) \subseteq ([C_+(G)]^{-})_e$. But $C_+(G) = C_+([C_+(G)]^{-})$. So $[C_+(G)]^{-} = ([C_+(G)]^{-})_e$. $\square$

REMARK 6.9. While $[C_+(G)]^{-}$ is connected when G is a solenoid, $C_+(G)$ itself need not be connected. In fact, $C_+(G)$ is nontrivial and totally disconnected whenever U has only unimodular eigenvalues and not all of them are roots of 1, see Example 9.12.

7. The structure of an automorphism of a compact group

The following structure theorem for automorphisms of compact groups will serve as a main tool in our investigation of the cdp for general compact groups. Readers familiar with the work of Miles and Thomas [21] will immediately recognize here a version of Theorem A in [21], with “nontoroidal” and “generalized torus” in Theorem A replaced by “semisimple” and “Markov solenoid,” resp., and two new statements added, one about contraction groups, and one about uniqueness. It is not clear to us whether the statement about contraction groups can be derived from Theorem A without invoking the extra information contained in its proof. In addition, as we indicated in Section 6 and demonstrated in Example A.2 in the Appendix, the generalized torus, as defined in [21], need not be a connected group (despite what is claimed on p. 216 in [21]). The proof of Theorem A in [21] (in particular, of Lemma 7) seems to have overlooked the connectedness question of the generalized torus arrived at,[4] a question which appears to require some nontrival technical effort

[4] This question does not arise at all when the compact group in the theorem is either connected or totally disconnected. In these two cases the proof in [21] is complete.

to resolve and which prompted much of the lengthy review of the solenoid in Section 6. This warrants giving a detailed proof of Theorem 7.1, rather than to refer to [21].

The solenoids defined in Section 6 do not include the trivial group $\{e\}$. Throughout the remainder of this paper, the term (Markov) solenoid will also mean the trivial group. An action of $\mathbb{Z}$ on a compact group G (and the dynamical system $(G,\mathbb{Z})$, as well as G itself) will be called *solenoidal*, if the system $(G,\mathbb{Z})$ is isomorphic to a solenoid. It will be called *Bernoullian* (resp., *Bernoullian of Lie type*), if $(G,\mathbb{Z})$ is isomorphic to a Bernoulli system $(L^{\mathbb{Z}},\mathbb{Z})$, where L is a compact group (resp., compact Lie group). Recall that by a compact semisimple Lie group we mean a Lie group of positive dimension whose Lie algebra is semisimple, or a finite group.

THEOREM 7.1. *Let $\mathbb{Z}$ act on a compact group G. If the system $(G,\mathbb{Z})$ satisfies the dcc, then there exist $k\in\mathbb{N}$ and closed normal $\mathbb{Z}$-invariant subgroups $G=G_0\geq G_1\geq\cdots\geq G_k\geq G_{k+1}=\{e\}$, such that*:

(1) *G/G_1 is a compact semisimple Lie group*;

(2) *$(G_1/G_2,\mathbb{Z})$ is a solenoidal system*;

(3) *when $j=2,3,\dots,k$, $(G_j/G_{j+1},\mathbb{Z})$ is a Bernoullian system of Lie type*;

(4) *when $j=0,\dots,k$, the canonical homomorphism $\xi_j: G_j\to G_j/G_{j+1}$ maps $C_+(G_j)$ onto $C_+(G_j/G_{j+1})$.*

Furthermore, G_1 and G_2 are uniquely determined by conditions (1)–(3): *in any two sequences, $G\geq G_1\geq\cdots\geq G_k\geq G_{k+1}=\{e\}$ and $G\geq G_1'\geq\cdots\geq G_{k'}'\geq G_{k'+1}'=\{e\}$, of closed normal $\mathbb{Z}$-invariant subgroups satisfying* (1)–(3), *$G_1=G_1'$ and $G_2=G_2'$.*

Our proof of Theorem 7.1 will require three lemmas. We will denote by $Z(G)$, the center of a group G, and by $Z_e(G)$, the connected component of the identity in $Z(G)$. We note that any locally compact group contains the largest connected normal solvable subgroup, called the radical [10, Theorem 15], [25, Proposition 3.7]. When G is a compact group, $Z_e(G_e)$ is the radical of G. An important property of the radical, which will be used repeatedly in our argument, is that $\varphi(Z_e(G_e))=Z_e(H_e)$ whenever $\varphi: G\to H$ is a continuous surjective homomorphism between compact groups G and H [8, Proposition 9.26].

REMARK 7.2. The quotient G_1/G_2 in Theorem 7.1 is the radical of G/G_2, hence, $G_1=Z_e(G_e)G_2$. (Indeed, since $G/G_1\cong(G/G_2)/(G_1/G_2)$ is semisimple, $G_1/G_2\supseteq Z_e((G/G_2)_e)$; on the other hand, as a compact connected normal Abelian subgroup of $(G/G_2)_e$, G_1/G_2 is contained in $Z_e((G/G_2)_e)$ [10, Theorem 4].)

LEMMA 7.3. *Let L and M be compact Lie groups, and $\alpha,\beta: L\to M$, continuous surjective homomorphisms. Then $\operatorname{Ker}\alpha\subseteq Z_e(L_e)$ if and only if $\operatorname{Ker}\beta\subseteq Z_e(L_e)$.*

Proof. As pointed out above, $\alpha(Z_e(L_e)) = \beta(Z_e(L_e)) = Z_e(M_e)$. Hence, α and β induce surjective homomorphisms $\tilde{\alpha}, \tilde{\beta} : L/Z_e(L_e) \to M/Z_e(M_e)$, in the canonical way. If (say) $\operatorname{Ker}\alpha \subseteq Z_e(L_e)$, then $\tilde{\alpha}$ is an isomorphism and so $\tilde{\beta} \circ \tilde{\alpha}^{-1}$ is a surjective homomorphism of $M/Z_e(M_e)$ onto itself. But a continuous homomorphism φ of a compact Lie group K onto itself has finite kernel. Since K/K_e is finite, $\operatorname{Ker}\varphi$ is contained in K_e, and therefore in the center of K_e. Since $\varphi(K_e) = K_e$, one easily concludes that if $Z(K_e)$ is finite, then $\varphi(Z(K_e)) = Z(K_e)$,[5] and, hence, $\operatorname{Ker}\varphi = \{e\}$. In our case $K = M/Z_e(M_e)$ is semisimple, so that $Z(K_e)$ is finite. Consequently, $\operatorname{Ker}(\tilde{\beta} \circ \tilde{\alpha}^{-1}) = \{e\}$. This forces $\tilde{\beta}$ to be injective, therefore $\operatorname{Ker}\beta \subseteq Z_e(L_e)$. □

LEMMA 7.4. *Let G be a full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact Lie group, and let $\zeta : L \to L/Z_e(L_e)$ denote the canonical homomorphism. Suppose that $T_G(1) \subseteq Z_e(L_e)$, $T_G(1)$ is finite, and $G \cap [Z_e(L_e)]^{\mathbb{Z}}$ is connected. Then $G \cap [Z_e(L_e)]^{\mathbb{Z}}$ is a full Markov subgroup of $[Z_e(L_e)]^{\mathbb{Z}}$, $(G \cap [Z_e(L_e)]^{\mathbb{Z}}, \mathbb{Z})$ is a solenoidal system, and $\zeta \circ \pi_0^L \restriction G$ is a homomorphism of G onto $L/Z_e(L_e)$, with kernel $\operatorname{Ker}(\zeta \circ \pi_0^L \restriction G) = G \cap [Z_e(L_e)]^{\mathbb{Z}}$.*

Proof. Recall that $G = \{x \in L^{\mathbb{Z}}\,; \Theta_1(x_n) = \gamma(x_{n+1})$ for every $n \in \mathbb{Z}\}$, where Θ_1 and γ are defined in (5.4) and (5.6). Let $H = G \cap [Z_e(L_e)]^{\mathbb{Z}}$. By Lemma 5.12 (with $\alpha = \Theta_1$, $\beta = \gamma$, $M = Z_e(L_e)$), H is a full Markov subgroup of $[Z_e(L_e)]^{\mathbb{Z}}$ with $T_H(1) = T_G(1)$. As H is connected, it follows from Proposition 6.1 that the system $(H, \mathbb{Z})$ is isomorphic to a Markov solenoid. Obviously, the homomorphism $\zeta \circ \pi_0^L \restriction G$ maps G onto $L/Z_e(L_e)$ and its kernel contains H. But as $\operatorname{Ker}\gamma = T_G(1) \subseteq Z_e(L_e)$, it follows from Lemma 7.3 that $T_G(-1) = \operatorname{Ker}\Theta_1 \subseteq Z_e(L_e)$. Hence, the last statement of Lemma 5.12 can be used to conclude that $\operatorname{Ker}(\zeta \circ \pi_0^L \restriction G) \subseteq H$. □

LEMMA 7.5. *Let G be a full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact Lie group. Then there exists $N \in \mathbb{N}$, such that for every $n \geq N-1$:*

(i) $\dim T_G(n) = \dim T_G(N-1)$;
(ii) $T_G(n) \subseteq T_G(N-1) Z_e(L_e)$;
(iii) $G \cap [T_G(N-1) Z_e(L_e)]^{\mathbb{Z}} \subseteq [G \cap [T_G(N-1) Z_e(L_e)]^{\mathbb{Z}}]_e [T_G(N-1)]^{\mathbb{Z}}$.

Proof. Since $T_G(n)$, $n = 1, 2, \ldots$ is a nondecreasing sequence of closed subgroup of the Lie group L, there exists $N_1 \in \mathbb{N}$, such that $\dim T_G(n) = \dim T_G(N_1 - 1)$ for every $n \geq N_1 - 1$. Let $\zeta : L \to L/Z_e(L_e)$ denote the projection. Then $\zeta(T_G(n))$, $n = 1, 2, \ldots$ is a nondecreasing sequence of closed normal subgroups of the semisimple Lie group $L/Z_e(L_e)$. Since such a group has only a finite number of closed normal subgroups, we can find $N_2 > N_1$, such that $T_G(n) \subseteq T_G(N_2 - 1) Z_e(L_e)$ for every $n \geq N_2 - 1$.

Let $\tilde{L} = L/T_G(N_2 - 1)$ and $\eta : L \to \tilde{L}$ be the projection. By Lemma 5.14, $\tilde{G} = \eta^{\mathbb{Z}}(G)$ is a full Markov subgroup of $\tilde{L}^{\mathbb{Z}}$, with $T_{\tilde{G}} = \eta^{\{0,1\}}(T_G)$ and $T_{\tilde{G}}(n) =$

[5] See also [8, Theorem 9.28].

$\eta(T_G(n+N_2-1))$ for all $n\in\mathbb{N}$. Since $\dim T_G(n+N_2-1)=\dim T_G(N_2-1)$, each $T_{\tilde G}(n)$ is finite.

By Lemma 5.12, $W=\tilde G\cap[Z_e(\tilde L_e)]^{\mathbb{Z}}$ is a full Markov subgroup of $[Z_e(\tilde L_e)]^{\mathbb{Z}}$ with finite $T_W(1)$. As $Z_e(\tilde L_e)$ is isomorphic to $\mathbb{T}^m$ for some $m\ge 0$, it follows from Proposition 6.1(a) and Corollary 6.7 that for some $M\in\mathbb{N}$, $W\subseteq[T_W(M)]^{\mathbb{Z}}W_e$. As $T_W(M)\subseteq T_{\tilde G}(M)$, we obtain that $W\subseteq[T_{\tilde G}(M)]^{\mathbb{Z}}W_e=[\eta(T_G(M+N_2-1))]^{\mathbb{Z}}W_e$.

Let $N=M+N_2-1$. Clearly, (i) and (ii) will hold with this choice of N. We claim that (iii) will also hold. Indeed, $\eta^{\mathbb{Z}}(G\cap[T_G(N_2-1)Z_e(L_e)]^{\mathbb{Z}})\subseteq\tilde G\cap[Z_e(\tilde L_e)]^{\mathbb{Z}}=W$. But if $w\in W$, then $w=\eta^{\mathbb{Z}}(g)$, where $g\in G$ and $\eta(g_n)\in Z_e(\tilde L_e)$ for every $n\in\mathbb{Z}$. So $g_n\in T_G(N_2-1)Z_e(L_e)$ and thus $g\in G\cap[T_G(N_2-1)Z_e(L_e)]^{\mathbb{Z}}$. Hence, $\eta^{\mathbb{Z}}(G\cap[T_G(N_2-1)Z_e(L_e)]^{\mathbb{Z}})=W$, and $\eta^{\mathbb{Z}}([G\cap[T_G(N_2-1)Z_e(L_e)]^{\mathbb{Z}}]_e)=W_e$. So $\eta^{\mathbb{Z}}(G\cap[T_G(N_2-1)Z_e(L_e)]^{\mathbb{Z}})=W\subseteq[\eta(T_G(N))]^{\mathbb{Z}}W_e=\eta^{\mathbb{Z}}([T_G(N)]^{\mathbb{Z}}[G\cap[T_G(N_2-1)Z_e(L_e)]^{\mathbb{Z}}]_e)\subseteq\eta^{\mathbb{Z}}([T_G(N)]^{\mathbb{Z}}[G\cap[T_G(N)Z_e(L_e)]^{\mathbb{Z}}]_e)$. Since $\operatorname{Ker}\eta^Z=[T_G(N_2-1)]^{\mathbb{Z}}\subseteq[T_G(N)]^{\mathbb{Z}}$, while $T_G(N-1)Z_e(L_e)=T_G(N_2-1)Z_e(L_e)$ by the definition of N_2, (iii) will be satisfied. □

Proof of Theorem 7.1. By Theorem 5.9, it suffices to consider the case that G is a full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact Lie group. Given such G, let $N_*=N_*(G)$ denote the smallest positive integer N with properties (i)–(iii) of Lemma 7.5. We proceed by induction on $N_*(G)$.

If $N_*(G)=1$, we put $k=1$ and $G_1=G\cap[Z_e(L_e)]^{\mathbb{Z}}$. By Lemma 7.4, $(G_1,\mathbb{Z})$ is a solenoidal system. Moreover, with $\zeta: L\to L/Z_e(L_e)$ denoting the projection, $\zeta\circ\pi_0^L$ is a homomorphism onto the semisimple Lie group $L/Z_e(L_e)$, with $\operatorname{Ker}(\pi_0^L\circ\zeta)=G_1$. Hence, our theorem is true when $N_*(G)=1$.

Next, assume that the theorem is already proven when $N_*(G)\le N$. Let G be a full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact Lie group, and suppose that $N_*(G)=N+1$. Let $\tilde L$, Λ, η, and $\tilde G$ be the objects which in Lemma 5.14 are denoted by L_1, Λ_1, η_1, and G_1, resp. Then for $n\ge N-1$, $\dim T_{\tilde G}(n)=\dim\eta(T_G(n+1))=\dim\eta(T_G(N))=\dim T_{\tilde G}(N-1)$, and $T_{\tilde G}(n)=\eta(T_G(n+1))\subseteq\eta(T_G(N)Z_e(L_e))=T_{\tilde G}(N-1)Z_e(\tilde L_e)$. Moreover, since $\operatorname{Ker}\eta^{\mathbb{Z}}=[T_G(1)]^{\mathbb{Z}}\subseteq[T_G(N)Z_e(L_e)]^{\mathbb{Z}}$, we obtain

$$\begin{aligned}\tilde G\cap\big[T_{\tilde G}(N-1)Z_e(\tilde L_e)\big]^{\mathbb{Z}}&=\eta^{\mathbb{Z}}(G)\cap\eta^{\mathbb{Z}}\big(\big[T_G(N)Z_e(L_e)\big]^{\mathbb{Z}}\big)\\&=\eta^{\mathbb{Z}}\big(G\cap\big[T_G(N)Z_e(L_e)\big]^{\mathbb{Z}}\big)\\&\subseteq\eta^{\mathbb{Z}}\big(\big[G\cap\big[T_G(N)Z_e(L_e)\big]^{\mathbb{Z}}\big]_e\big[T_G(N)\big]^{\mathbb{Z}}\big)\\&=\big[\tilde G\cap\big[T_{\tilde G}(N-1)Z_e(\tilde L_e)\big]^{\mathbb{Z}}\big]_e\big[T_{\tilde G}(N-1)\big]^{\mathbb{Z}}.\end{aligned}$$

Hence, $N_*(\tilde G)\le N$. Applying our inductive assumption to $\tilde G$, we can find a sequence of closed normal $\mathbb{Z}$-invariant subgroups $\tilde G=\tilde G_0\ge\tilde G_1\ge\cdots\ge\tilde G_{\tilde k}\ge\tilde G_{\tilde k+1}=\{e\}$, such that statements (1)–(4) of our theorem are true. We let

$k = \tilde{k}+1$ and define $G_j = (\eta^{\mathbb{Z}} \restriction G)^{-1}(\tilde{G}_j)$ for $j = 0, 1, \ldots, k$, and $G_{k+1} = \{e\}$. These are closed normal $\mathbb{Z}$-invariant subgroups of G with $G = G_0 \geq G_1 \geq \cdots \geq G_k = \Lambda^{\mathbb{Z}} \geq G_{k+1} = \{e\}$. For each $j = 0, \ldots, k-1$, the system $(G_j/G_{j+1}, \mathbb{Z})$ is canonically isomorphic to $(\tilde{G}_j/\tilde{G}_{j+1}, \mathbb{Z})$, while $(G_k/G_{k+1}, \mathbb{Z})$ is trivially isomorphic to $(\Lambda^{\mathbb{Z}}, \mathbb{Z})$. This shows that statements (1)–(3) hold for G. We proceed to verify (4).

That $\xi_k(C_+(G_k)) = C_+(G_{k+1})$ is trivial. That $\xi_0(C_+(G)) = C_+(G_1)$ is also trivial, because any action of $\mathbb{Z}$ on a compact semisimple Lie group is equicontinuous, so that $C_+(G/G_1) = \{e\}$. When $j = 1, \ldots, k-1 = \tilde{k}$, let $\tilde{\xi}_j : \tilde{G}_j \to \tilde{G}_j/\tilde{G}_{j+1}$ denote the projection, $\tilde{\varphi}_j : G_j/G_{j+1} \to \tilde{G}_j/\tilde{G}_{j+1}$, the canonical (equivariant) isomorphism, and let $\varphi_j = \eta^{\mathbb{Z}} \restriction G_j$, which is a surjective equivariant homomorphism onto $\tilde{G}_j$. Thus $\tilde{\varphi}_j \circ \xi_j = \tilde{\xi}_j \circ \varphi_j$. Then $\tilde{\varphi}_j(C_+(G_j/G_{j+1})) = C_+(\tilde{G}_j/\tilde{G}_{j+1}) = \tilde{\xi}_j(C_+(\tilde{G}_j))$. But $\operatorname{Ker}\varphi_j = \Lambda^{\mathbb{Z}}$ and therefore by Lemma 4.3, $C_+(G_j, \operatorname{Ker}\varphi_j) = C_+(G_j)\operatorname{Ker}\varphi_j$. So $C_+(\tilde{G}_j) = \varphi_j(C_+(G_j))$ by Proposition 3.3(iii). We thus obtain $\tilde{\varphi}_j(C_+(G_j/G_{j+1})) = \tilde{\xi}_j\big(\varphi_j(C_+(G_j)) = \tilde{\varphi}_j(\xi_j(C_+(G_j))$. Therefore, $C_+(G_j/G_{j+1}) = \xi_j(C_+(G_j))$.

In view of Remark 7.2, to prove the last statement of the theorem it is enough to prove that G_2 is unique. The latter is an immediate consequence of the characterization of G_2 which we obtain in Corollary 7.11 below. □

An action of $\mathbb{Z}$ on a compact group G (and the dynamical system $(G, \mathbb{Z})$, as well as G itself) will be called *poly-Bernoullian*, if there exists a finite sequence $G = G_0 \geq G_1 \geq \cdots \geq G_l \geq G_{l+1} = \{e\}$ of closed normal $\mathbb{Z}$-invariant subgroups, such that for every $j = 0, \ldots, l$, $(G_j/G_{j+1}, \mathbb{Z})$ is a Bernoullian system. It is easy to see that a poly-Bernoullian system $(G, \mathbb{Z})$ satisfies the dcc if and only if each $(G_j/G_{j+1}, \mathbb{Z})$ is a Bernoullian system of Lie type. The action of $\mathbb{Z}$ on the subgroup G_2 of Theorem 7.1 is poly-Bernoullian.

We note that our use of the term "Bernoullian" differs from its standard use in ergodic theory, where "Bernoullian" means measure isomorphic to Bernoulli shifts (the isomorphism being an equivariant Borel isomorphism which does not need to be a group homomorphism). In fact, every ergodic action of $\mathbb{Z}$ on a compact metrizable group, and therefore every poly-Bernoullian action, is Bernoullian in the sense of ergodic theory [20], [21]. The next example, adapted from [16, Example 4], shows that not every poly-Bernoullian system is Bernoullian in our sense.

Example 7.6. We will use the following two facts about Bernoullian systems: (1) A Markov subgroup G of $L^{\mathbb{Z}}$ is equal to $L^{\mathbb{Z}}$ if and only if $T_G(1) = L$. (2) If the system $(G, \mathbb{Z})$ is isomorphic to a Bernoulli system $(L^{\mathbb{Z}}, \mathbb{Z})$, then L is isomorphic to the fixed point subgroup $G_{\text{fix}} = \{x \in G\,;\, 1x = x\}$.

Let $\alpha, \beta : \mathbb{Z}_4 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ be the homomorphims given by $\alpha(a,b) = b$ and $\beta(a,b) = (a+b)(\operatorname{mod} 2)$. Then $G = \{x \in (Z_4 \times \mathbb{Z}_2)^{\mathbb{Z}}\,;\, \alpha(x_n) = \beta(x_{n+1})$ for every $n \in \mathbb{Z}\}$ is a full Markov subgroup of $(Z_4 \times \mathbb{Z}_2)^{\mathbb{Z}}$ with $T_G(-1) = \mathbb{Z}_4 \times \{0\}$ and

$T_G(1) = \{(0,0),(1,1),(2,0),(3,1)\} \cong \mathbb{Z}_4$. $[T_G(-1) \cap T_G(1)]^{\mathbb{Z}} = \{(0,0),(2,0)\}^{\mathbb{Z}}$ is a $\mathbb{Z}$-invariant subgroup of G. Using Lemma 5.14 with $p = 1$, we conclude that the system $(G/\{(0,0),(2,0)\}^{\mathbb{Z}}, \mathbb{Z})$ is isomorphic to a Markov subgroup $\tilde{G}$ of $[(\mathbb{Z}_4 \times \mathbb{Z}_2)/T_G(1)]^{\mathbb{Z}}$, where $T_{\tilde{G}}(1) = T_G(2)/T_G(1)$. But $T_G(2) = \mathbb{Z}_4 \times \mathbb{Z}_2$, and so $\tilde{G} = [(\mathbb{Z}_4 \times \mathbb{Z}_2)/T_G(1)]^{\mathbb{Z}}$. Therefore, $(G, \mathbb{Z})$ is a poly-Bernoullian system.

Next, G_{fix} consists of the constant sequences $(x)_{n\in\mathbb{Z}}$, where $x \in \{(a,b) \in \mathbb{Z}_4 \times \mathbb{Z}_2\,;\, \alpha(a,b) = \beta(a,b)\} = \{0,2\} \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus if $(G,\mathbb{Z})$ were a Bernoullian system, it would be isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\mathbb{Z}}$. In particular, every $g \in G \setminus \{e\}$ would have order 2. However, G contains the sequence $\dots(1,1),(1,0),(1,1),(1,0),(1,1),\dots$ which has order 4. Therefore, $(G,\mathbb{Z})$ cannot be a Bernoullian system.

Lemma 7.7. *Let K be a proper closed normal $\mathbb{Z}$-invariant subgroup of a poly-Bernoullian system $(H,\mathbb{Z})$. Then H/K is uncountable, and either* 0*-dimensional or infinite-dimensional.*

Proof. Let $H = H_0 \geq H_1 \geq \cdots \geq H_l \geq H_{l+1} = \{e\}$ be as in the definition of a poly-Bernoullian system, and $\zeta : H \to H/K$ denote the projection. Let $I = \{i = 0,\dots,l+1\,;\, \zeta(H_i) = \{e\}$, or $\zeta(H_i)$ is uncountable and $\dim(\zeta(H_i)) \in \{0,\infty\}\}$. Since $l+1$ is trivially a member of I, it suffices to show that whenever $i+1 \in I$ for some $i = 0,\dots,l$, then $i \in I$.

Let $\xi_i : H_i \to H_i/H_{i+1}$ and $\eta_i : \zeta(H_i) \to \zeta(H_i)/\zeta(H_{i+1})$ be the projections. It follows that there exists an equivariant surjective homomorphism $\tilde{\zeta}_i : H_i/H_{i+1} \to \zeta(H_i)/\zeta(H_{i+1})$ with $\tilde{\zeta}_i \circ \xi_i = \eta_i \circ \zeta$. Hence, by Proposition 5.18, $\zeta(H_i)/\zeta(H_{i+1})$ is either trivial, or is uncountable and $\dim(\zeta(H_i)/\zeta(H_{i+1})) \in \{0,\infty\}$. If $\zeta(H_i)/\zeta(H_{i+1}) = \{e\}$, then $\zeta(H_i) = \zeta(H_{i+1})$ and so $i \in I$. Otherwise, $\zeta(H_i)$ is necessarily uncountable, while the identity $\dim(\zeta(H_i)) = \dim(\zeta(H_{i+1})) + \dim(\zeta(H_i)/\zeta(H_{i+1}))$ implies that $\dim(\zeta(H_i)) \in \{0,\infty\}$. Thus $i \in I$ in any case. □

The next lemma results by an elementary application of Proposition 6.4.

Lemma 7.8. *Let $\mathbb{Z}$ act on a compact group G. Suppose that $G_1 \leq G$ is a closed normal $\mathbb{Z}$-invariant subgroup, such that G/G_1 is a Lie group and $(G_1,\mathbb{Z})$ is a solenoidal system. Then G has finitely many connected components and every closed totally disconnected $\mathbb{Z}$-invariant subgroup of G is finite.*

Lemma 7.9. *Let $\mathbb{Z}$ act on compact groups H and G, and $\varphi : H \to G$ be a continuous equivariant homomorphism. Suppose that $G \geq G_1 \geq \cdots \geq G_k \geq G_{k+1} = \{e\}$ is a sequence of closed normal $\mathbb{Z}$-invariant subgroups which satisfies conditions* (1)–(3) *of Theorem* 7.1*. If $(H,\mathbb{Z})$ is a poly-Bernoullian system, then $\varphi(H) \subseteq G_2$.*

Proof. Let $\xi : G \to G/G_2$ denote the projection. Since $\dim(G/G_2) < \infty$, $\dim(\xi(\varphi(H))) < \infty$, and so by Lemma 7.7, $\xi(\varphi(H))$ is either trivial or is an uncountable closed totally disconnected $\mathbb{Z}$-invariant subgroup of G/G_2. By Lemma 7.8, $\xi(\varphi(H))$ must be trivial, that is, $\varphi(H) \subseteq G_2$. $\square$

COROLLARY 7.10. *Let $\mathbb{Z}$ act on compact groups H and G and $\varphi : H \to G$ be a continuous equivariant surjective homomorphism. If $(H, \mathbb{Z})$ is a poly-Bernoullian system while $(G, \mathbb{Z})$ satisfies the dcc, then $(G, \mathbb{Z})$ is also poly-Bernoullian.*

COROLLARY 7.11. *Let $\mathbb{Z}$ act on a compact group G and $G \geq G_1 \geq \cdots \geq G_k \geq G_{k+1} = \{e\}$ be a sequence of closed normal $\mathbb{Z}$-invariant subgroups which satisfies conditions (1)–(3) of Theorem 7.1. Then G_2 is the largest poly-Bernoullian subgroup of G.*

Proof. Apply Lemma 7.9 with $H =$ a poly-Bernoullian subgroup, and $\varphi =$ the inclusion. $\square$

The largest poly-Bernoullian subgroup (if exists) will be denoted by G_{pB} and called the *poly-Bernoullian component* of the system $(G, \mathbb{Z})$.

REMARK 7.12. If the system $(G, \mathbb{Z})$ satisfies the dcc, then $G_{pB} \trianglelefteq G$ and $\dim(G/G_{pB}) < \infty$.

LEMMA 7.13. *Let actions of $\mathbb{Z}$ on compact groups H and G satisfy the dcc. If $\varphi : H \to G$ is a continuous equivariant surjective homomorphism, then $\varphi(H_{pB}) = G_{pB}$.*

Proof. By Lemma 7.9, $\varphi(H_{pB}) \subseteq G_{pB}$. To prove the opposite inclusion, let $H \geq H_1 \geq \cdots \geq H_{k'} \geq H_{k'+1} = \{e\}$ and $G \geq G_1 \geq \cdots \geq G_k \geq G_{k+1} = \{e\}$ be the sequences of subgroups described in Theorem 7.1 ($H_2 = H_{pB}$, $G_2 = G_{pB}$).

Consider the quotient $G/\varphi(H_2)$, which is an equivariant continuous image of H/H_2. Hence, $\dim(G/\varphi(H_2)) < \infty$. Therefore $\dim(G_2/\varphi(H_2)) < \infty$, and so by Lemma 7.7, $G_2/\varphi(H_2)$ is either trivial or an uncountable closed totally disconnected $\mathbb{Z}$-invariant subgroup of $G/\varphi(H_2)$. But $\varphi(H_1)/\varphi(H_2)$ is solenoidal as an equivariant image of H_1/H_2, while $(G/\varphi(H_2))/(\varphi(H_1)/\varphi(H_2))$ is a Lie group. Hence, by Lemma 7.8, $G_2/\varphi(H_2)$ is trivial, that is, $G_{pB} = G_2 = \varphi(H_2) = \varphi(H_{pB})$. $\square$

An action of $\mathbb{Z}$ on a compact group G (and the dynamical system $(G, \mathbb{Z})$, as well as G itself) will be called *Bernoullizable*, if there exists a directed downward family $\mathcal{P}$ of closed normal $\mathbb{Z}$-invariant subgroups $P \leq G$, such that the systems $(G/P, \mathbb{Z})$ are poly-Bernoullian and $\bigcap \mathcal{P} = \{e\}$.[6] It obvious that every poly-Bernoullian system is Bernoullizable and that a Bernoullizable system satisfying the dcc is poly-Bernoullian.

[6] The term *pro-poly-Bernoullian* would give a more exact description to this concept.

Lemma 7.14. *Let $\mathcal{M}$ be a directed downward family of closed subgroups of a compact group K, and ξ, a continuous homomorphism of K onto a compact group L. Suppose $M_* \in \mathcal{M}$ is such that $\xi(M) = \xi(M_*)$ for every $M \in \mathcal{M}$ with $M \subseteq M_*$. Then $M_* \subseteq (\operatorname{Ker}\xi)(\bigcap \mathcal{M})$.*

Proof. Let Ω be an open subset of L containing $\xi(\bigcap \mathcal{M})$. Using the finite intersection property and the fact that $\mathcal{M}$ is directed downward, we conclude that there is some $M \in \mathcal{M}$ with $M \subseteq M_*$ and $M \subseteq \xi^{-1}(\Omega)$. Consequently, $\xi(M_*) = \xi(M) \subseteq \Omega$. As Ω is arbitrary, it follows that $\xi(M_*) \subseteq \xi(\bigcap \mathcal{M})$. So $M_* \subseteq (\operatorname{Ker}\xi)(\bigcap \mathcal{M})$. □

Proposition 7.15. *Let $\mathbb{Z}$ act on compact groups H and G, and $\varphi : H \to G$ be a continuous equivariant surjective homomorphism. If H is Bernoullizable, then so is G.*

Proof. Let $\mathcal{P}$ be a directed downward family of closed normal $\mathbb{Z}$-invariant subgroups $P \le H$, such that the systems $(H/P, \mathbb{Z})$ are poly-Bernoullian and $\bigcap \mathcal{P} = \{e\}$. By Theorem 5.3, it suffices to show that the system $(G/C, \mathbb{Z})$ is poly-Bernoullian whenever $C \le G$ is a closed normal $\mathbb{Z}$-invariant subgroup, such that $(G/C, \mathbb{Z})$ satisfies the dcc.

Let $\xi : G \to G/C$ denote the projection. Then $\{\xi(\varphi(P))\,;\, P \in \mathcal{P}\}$ is a directed downward family of closed normal $\mathbb{Z}$-invariant subgroups of G/C. Hence, by the dcc there exists $P_* \in \mathcal{P}$, such that $\xi(\varphi(P)) = \xi(\varphi(P_*))$ for every $P \in \mathcal{P}$ with $P \subseteq P_*$. So by Lemma 7.14, $P_* \subseteq \operatorname{Ker}(\xi \circ \varphi)$, that is, $\varphi(P_*) \subseteq C$. If so, then there exists a continuous equivariant surjective homorphism $\eta : H/P_* \to G/C$, such that $\xi \circ \varphi = \eta \circ \xi_*$, where $\xi_* : H \to H/P_*$ is the projection. Then $G/C = \xi(\varphi(H)) = \eta(H/P_*)$, and so $(G/C, \mathbb{Z})$ is poly-Bernoullian by Corollary 7.10. □

Proposition 7.16. *Every action of $\mathbb{Z}$ on a compact group admits the largest Bernoullizable subgroup. The largest Bernoullizable subgroup is a normal subgroup.*

Proof. Let $\mathcal{C}$ denote the family of closed normal $\mathbb{Z}$-invariant subgroups C, such that the systems $(G/C, \mathbb{Z})$ satisfy the dcc. By Theorem 5.3, the system $(G, \mathbb{Z})$ is the projective limit of the systems $(G/C, \mathbb{Z})$, $C \in \mathcal{C}$. Denote by $\xi_C : G \to G/C$ the projection and by B_C the poly-Bernoullian component of G/C. Then $B = \bigcap_{C \in \mathcal{C}} \xi_C^{-1}(B_C)$ is a closed normal $\mathbb{Z}$-invariant subgroup of G. We claim that B is the largest Bernoullizable subgroup of G.

Given $C_1, C_2 \in \mathcal{C}$ with $C_1 \supseteq C_2$, write $\xi_{C_1C_2} : G/C_2 \to G/C_1$ for the homomorphism satisfying $\xi_{C_1} = \xi_{C_1C_2} \circ \xi_{C_2}$. By Lemma 7.13, $\xi_{C_1C_2}(B_{C_2}) = B_{C_1}$. Hence, by the basic theory of projective limits, B is the projective limit of the system $(B_C)_{C \in \mathcal{C}}$, and $\xi_C(B) = B_C$ for every $C \in \mathcal{C}$. Therefore, B is a Bernoullizable subgroup. That B is the largest Bernoullizable subgroup follows from Proposition 7.15 applied to the projections ξ_C, and from the observation that a Bernoullizable subgroup of a system satisfying the dcc is poly-Bernoullian. □

The largest Bernoullizable subgroup will be denoted by G_B and called the *Bernoullizable component* of the system $(G,\mathbb{Z})$. Note that for a system satisfying the dcc, $G_B = G_{pB}$ is the subgroup G_2 of Theorem 7.1.

PROPOSITION 7.17. *Let $\mathbb{Z}$ act on compact groups H and G and $\varphi : H \to G$ be a continuous equivariant surjective homomorphism. Then $\varphi(H_B) = G_B$.*

Proof. Let $\mathcal{C}$ (resp., $\mathcal{C}'$) denote the family of closed normal $\mathbb{Z}$-invariant subgroups C, such that the systems $(G/C,\mathbb{Z})$ (resp., $(H/C,\mathbb{Z})$) satisfy the dcc. As shown in the proof of Proposition 7.16, $\xi_C(G_B) = B_C$ (resp., $\xi'_C(H_B) = B'_C$) for every $C \in \mathcal{C}$ (resp., $C \in \mathcal{C}'$), where B_C (resp., B'_C) is the poly-Bernoullian component of G/C (resp., H/C), and $\xi_C : G \to G/C$ (resp., $\xi'_C : H \to G'/C$), the projection. Consequently, it suffices to show that $\xi_C(\varphi(H_B)) = B_C$ for every $C \in \mathcal{C}$.

But given $C \in \mathcal{C}$ there exists $C' \in \mathcal{C}'$ with $\varphi(C') \subseteq C$. This can be established using Lemma 7.14 as in the proof of Proposition 7.15. It follows that there exists a surjective homorphism $\eta : H/C' \to G/C$, such that $\eta \circ \xi'_{C'} = \xi_C \circ \varphi$. Then $\xi_C(\varphi(H_B)) = \eta(\xi'_{C'}(H_B)) = \eta(B'_{C'})$. But by Lemma 7.13, $\eta(B'_{C'}) = B_C$. □

An action of $\mathbb{Z}$ on a compact group G (as well as the dynamical system $(G,\mathbb{Z})$) will be called *pro-finite-dimensional*, if there exists a directed downward family $\mathcal{P}$ of closed normal $\mathbb{Z}$-invariant subgroups $P \leq G$, such that the groups G/P are finite-dimensional and $\bigcap \mathcal{P} = \{e\}$. A pro-finite-dimensional system satisfying the dcc is finite-dimensional.

COROLLARY 7.18. *If $\mathbb{Z}$ acts on a compact group G, then*:

(i) *$(G,\mathbb{Z})$ is a pro-finite-dimensional system if and only if G_B is totally disconnected.*

(ii) *$(G/G_B)_B = \{e\}$.*

(iii) *$G_B = \{e\}$ if an only if $(G/C)_{pB} = \{e\}$ for every closed normal $\mathbb{Z}$-invariant subgroup C, such that $(G/C,\mathbb{Z})$ satisfies the dcc.*

Proof. (i) $\Rightarrow$: Let $C \leq G$ be a closed normal $\mathbb{Z}$-invariant subgroup, such that the system $(G/C,\mathbb{Z})$ satisfies the dcc. Then $(G/C,\mathbb{Z})$ is a pro-finite-dimensional system, and therefore finite-dimensional. Lemma 7.7 forces $(G/C)_{pB}$ to be totally disconnected. By Proposition 7.17 and Theorem 5.3, G_{pB} must be totally disconnected.

$\Leftarrow$: Let $\mathcal{C}$ denote the family of closed normal $\mathbb{Z}$-invariant subgroups C, such that the systems $(G/C,\mathbb{Z})$ satisfy the dcc, and let $\xi_C : G \to G/C$ denote the projection. Then $\xi_C(G_B) = (G/C)_{pB}$ is totally disconnected, and so by Remark 7.12, $\dim(G/C) < \infty$. Theorem 5.3 yields that $(G,\mathbb{Z})$ is pro-finite-dimensional. □

Corollary 7.19. *If $\mathbb{Z}$ acts on a connected compact group G, then G_B is connected.*

Proof. Due to Theorem 5.3 and Proposition 7.17, it suffices to consider the case that the system $(G,\mathbb{Z})$ satisfies the dcc. Let $\tilde{G}=G/(G_{pB})_e$ and denote by $\varphi: G\to\tilde{G}$ the projection. By Proposition 7.17, $\tilde{G}_{pB}$ is totally disconnected, and so by Remark 7.12 (or Corollary 7.18 and the dcc), $\dim\tilde{G}<\infty$.

Let $H=\tilde{G}/Z_e(\tilde{G})$. Then H is a finite-dimensional compact connected group with $Z_e(H)=\{e\}$. Hence, by Theorem 5.9, there exists an equivariant isomorphism of H onto a full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact connected semisimple Lie group. Then Corollary 5.17 yields that H itself is a semisimple Lie group. Hence, H_{pB} being, by Proposition 7.17, a closed normal totally disconnected subgroup of H, is finite, and therefore trivial by Lemma 7.7. Thus, $\tilde{G}_{pB}\subseteq Z_e(\tilde{G})$. But $\dim Z_e(\tilde{G})<\infty$, and so by Proposition 6.2 $(Z_e(\tilde{G}),\mathbb{Z})$ is a solenoidal system. Therefore, $\tilde{G}_{pB}$, being totally disconnected, is finite by Proposition 6.4, and so trivial according to Lemma 7.7. Consequently, $G_{pB}=(G_{pB})_e$. $\square$

Corollary 7.20. *If $\mathbb{Z}$ acts on a compact connected group G, then $(G,\mathbb{Z})$ is a pro-finite-dimensional system if and only if $G_B=\{e\}$.*

The final result of this section relates the Bernoullizable component G_B of the dynamical system $(G,\mathbb{Z})$ to the ergodic component G_{erg}. Recall that G_{erg} is the largest among closed $\mathbb{Z}$-invariant subgroups on which $\mathbb{Z}$ acts ergodically.

Proposition 7.21. *If $\mathbb{Z}$ acts on a compact group G, then $G_B\le G_{\mathrm{erg}}\le Z_e(G_e)G_B$.*

Proof. Using Proposition 2.1 it is easy to see that projective limits of ergodic actions on compact groups are ergodic, and that if $\mathbb{Z}$ acts ergodically on a closed normal $\mathbb{Z}$-invariant subgroup $N\le G$ and on the quotient G/N, then it acts ergodically on G. Hence, every Bernoulizable action is ergodic. Thus $G_B\le G_{\mathrm{erg}}$.

Thanks to Corollary 2.7(ii) and Theorem 5.3, to demonstrate that $G_{\mathrm{erg}}\le Z_e(G_e)G_B$, it suffices to consider the case that the system $(G,\mathbb{Z})$ satisfies the dcc. But then $Z_e(G_e)G_B$ is the subgroup G_1 of Theorem 7.1. As G/G_1 is a semisimple Lie group, $(G/G_1)_{\mathrm{erg}}=\{e\}$. By Corollary 2.7(ii), $G_{\mathrm{erg}}\le G_1$. $\square$

Corollary 7.22. *If G is totally disconnected, then $G_{\mathrm{erg}}=G_B$.*

8. The semidiscrete topology and Δ-contraction groups

Let G be a group and $\mathcal{N}$ a directed downward family of normal subgroups of G, such that $\bigcap\mathcal{N}=\{e\}$. Then there exists a unique topology $\sigma(\mathcal{N})$, making G into a (Hausdorff) topological group in which $\mathcal{N}$ is a neighbourhood base at e. In this topology every member of $\mathcal{N}$ is a closed-open subgroup and so G is a 0-dimensional group.

Let G be a compact group. We will denote by $\mathcal{N}(G)$ the set of closed normal subgroups of G, such that G/N is a Lie group. It is well known that $\mathcal{N}(G)$ is closed under finite intersections [22, p. 177], [9, p. 148]. The topology $\Delta = \sigma(\mathcal{N}(G))$ will be called the *semidiscrete topology*, or the Δ-*topology.* In the next remark, we collect a few readily verifiable properties Δ.

REMARK 8.1.

(a) Δ is no weaker than the original compact topology.

(b) Δ coincides with the original topology if and only if the latter is totally disconnected.

(c) Δ is discrete if and only if G is a Lie group.

PROPOSITION 8.2. *Let $\mathcal{M}$ be a directed downward subset of $\mathcal{N}(G)$, such that $\bigcap \mathcal{M} = \{e\}$. Then $\mathcal{M}$ is neighbourhood base at e for the Δ-topology.*

Proof. We need to show that every $N \in \mathcal{N}(G)$ contains some $M \in \mathcal{M}$. But with $\xi : G \to G/N$ denoting the canonical projection, $\{\xi(M)\,;\, M \in \mathcal{M}\}$ is a family of compact subgroups of the Lie group G/N. Hence, there exists $M_* \in \mathcal{M}$, such that $\xi(M) = \xi(M_*)$ for every $M \in \mathcal{M}$ with $M \subseteq M_*$. Then Lemma 7.14 yields the desired result. $\square$

COROLLARY 8.3. *Let $(f_a)_{a\in A}$ be a family of homomorphisms, each mapping G into a Lie group L_a, and each continuous with respect to the original compact topology on G. If $(f_a)_{a\in A}$ separates points and each L_a is given the discrete topology, then Δ coincides with the weak topology generated $(f_a)_{a\in A}$.*

Proof. Clearly, $\operatorname{Ker} f_a \in \mathcal{N}(G)$ (because $f_a(G)$ is a Lie group canonically isomorphic to $G/\operatorname{Ker} f_a$). Let $\mathcal{M}$ denote the family of finite intersections of the kernels of the homomorphisms f_a, $a \in A$. Then $\mathcal{M}$ satisfies the assumptions of Proposition 8.2, and so is a neighbourhood base at e for Δ. But the weak topology Δ' generated $(f_a)_{a\in A}$ also makes G into a topological group and $\mathcal{M}$ is a neighbourhood base at e for Δ'. Therefore, $\Delta = \Delta'$. $\square$

COROLLARY 8.4. *Let $(L_a)_{a\in A}$ be a family of compact Lie groups and G a closed subgroup of $\prod_{a\in A} L_a$. Denote by $L_a^{(d)}$ the set L_a equipped with the discrete topology. Then the Δ-topology on G is the restriction of the Tychonoff topology of $\prod_{a\in A} L_a^{(d)}$ to G.*

Henceforth we will adopt the convention that whenever we say "open," "closed," "continuous," "dense," etc., we refer to the original compact topology of a compact group and not to the Δ-topology, unless explicitly stated otherwise.

COROLLARY 8.5. *Let G and H be compact groups.*

(a) *If $\varphi : G \to H$ is a continuous homomorphism into* (*resp.*, *onto*) *H, then φ is continuous* (*resp.*, *continuous and open*) *with respect to the Δ-topologies of G and H.*

(b) *If H is a closed subgroup of G, then the Δ-topology of H is the restriction of the Δ-topology of G to H.*

(c) *The Δ-topology of $G \times H$ is the product of the Δ-topology of G and the Δ-topology of H.*

Proof. (a) If $N \in \mathcal{N}(H)$, then the image of $\varphi(G)$ in H/N is a Lie group isomorphic to $G/\varphi^{-1}(N)$. Hence, $\varphi^{-1}(N) \in \mathcal{N}(G)$, and so φ is Δ-continuous. If φ is surjective and $N \in \mathcal{N}(G)$, then $H/\varphi(N)$, being a homomorphic image of G/N, is a Lie group. So $\varphi(N) \in \mathcal{N}(H)$, and, hence, φ is Δ-open.

(b) Let Δ' be the restriction of the Δ-topology of G to H. Then Δ' makes H into a topological group in which $\mathcal{M} = \{N \cap H\,; N \in \mathcal{N}(G)\}$ is a neighbourhood base at e. But $\mathcal{M} \subseteq \mathcal{N}(H)$ and $\mathcal{M}$ satisfies the assumptions of Proposition 8.2.

(c) The product of the Δ-topologies of G and H makes $G \times H$ into a topological group in which $\mathcal{M} = \{N' \times N''\,; N' \in \mathcal{N}(G), N'' \in \mathcal{N}(H)\}$ is a neighbourhood base at e. One concludes using Proposition 8.2 as in (b). $\square$

Suppose now that $\mathbb{Z}$ acts on the compact group G, and K is a closed $\mathbb{Z}$-invariant subgroup. When G is topologized using the Δ-topology, then owing to Corollary 8.5(a), $\mathbb{Z}$ remains acting by topological automorphisms and so the general theory of contraction groups outlined in Section 3 is applicable in this new setting. When the Δ-topology is used, the contraction group and the contraction group modulo K will be denoted by $C_+^{\Delta}(G)$ and $C_+^{\Delta}(G,K)$, resp. They will be referred to as the Δ-*contraction group* and the Δ-*contraction group modulo* K. We emphasize that in our discussion of $C_+^{\Delta}(G)$ and $C_+^{\Delta}(G,K)$, we will always assume that $\mathbb{Z}$ acts by automorphisms of the original compact group structure and that K is closed in the original compact topology. It is clear that $C_+^{\Delta}(G) \subseteq C_+(G)$ and $C_+^{\Delta}(G,K) \subseteq C_+(G,K)$, with the inclusions becoming equalities when G is totally disconnected. Moreover, since the members of $\mathcal{N}(G)$ are normal subgroups, it follows that $C_+^{\Delta}(G,K) \trianglelefteq G$ whenever $K \trianglelefteq G$, in particular, $C_+^{\Delta}(G) \trianglelefteq G$.

It is obvious that $g \in C_+^{\Delta}(G)$ (resp., $g \in C_+^{\Delta}(G,K)$) if and only for every continuous homomorphism f from G into a Lie group there exists $n_f \in \mathbb{N}$, such that $f(ng) = e$ (resp., $f(ng) \in f(K)$) for every $n \geq n_f$. By Corollary 8.3, to verify that $g \in C_+(G)$ it suffices to work with a family of homomorphisms which separates points.

LEMMA 8.6. *Let $(f_a)_{a\in A}$ be a family of homomorphisms, each mapping G continuously into a Lie group L_a, and let $(f_a)_{a\in A}$ separate points. If $K \subseteq G$ is a closed $\mathbb{Z}$-invariant subgroup with the property that $K = \bigcap_{a\in A} f_a^{-1}(f_a(K))$, then $g \in C_+^{\Delta}(G,K)$ if and only if for every $a \in A$ there exists $n_a \in \mathbb{N}$, such that $f_a(ng) \in f_a(K)$ for every $n \geq n_a$.*

Proof. It suffices to show that if $N \in \mathcal{N}(G)$, then there exists a finite set $A_N \subseteq A$ with $\bigcap_{a\in A_N} f_a^{-1}(f_a(K)) \subseteq NK$. But this can be established by an argument analogous to that used in the proof of Proposition 8.2. $\square$

PROPOSITION 8.7. *Let G and K be closed $\mathbb{Z}$-invariant subgroups of $L^{\mathbb{Z}}$, where L is a compact Lie group and $K = G \cap [\pi_0^L(K)]^{\mathbb{Z}}$. Then $g \in C_+^\Delta(G)$ (resp., $g \in C_+^\Delta(G,K)$) if and only if $g_n = e$ (resp., $g_n \in \pi_0^L(K)$) for large enough n.*

COROLLARY 8.8. *If L is a compact Lie group, then $C_+^\Delta(L^{\mathbb{Z}})$ is dense in $L^{\mathbb{Z}}$.*

COROLLARY 8.9. *Let G be a full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact Lie group. Then $C_\pm^\Delta(G) = \{e\}$ if and only if $T_G(\mp 1) = \{e\}$. Moreover, the following conditions are equivalent:*

(i) *G is a Lie group.*
(ii) *$C_+^\Delta(G) = C_-^\Delta(G) = \{e\}$.*
(iii) *$\pi_0^L \restriction G$ is a topological isomorphism of G onto L.*

Proof. The "if and only if" statement follows immediately from the definition of $T_G(\pm 1)$. The implication (iii) $\Rightarrow$ (i) is trivial, while (i) $\Rightarrow$ (ii) is true because the Δ-topology on a Lie group is discrete. Finally, if (ii) holds, then $T_G(-1) = T_G(1) = \{e\}$. Applying the last statement of Remark 5.5 yields (iii). $\square$

COROLLARY 8.10. *Suppose that the system $(G,\mathbb{Z})$, where G is a compact group, satisfies the dcc.*

(1) *If $C_+^\Delta(G) = \{e\}$, then $G_{pB} = \{e\}$, in particular, G is a finite-dimensional group.*

(2) *G is a Lie group if and only if $C_+^\Delta(G) = C_-^\Delta(G) = \{e\}$.*

Proof. (1) Let $G \geq G_1 \geq \cdots \geq G_k \geq G_{k+1} = \{e\}$ be the closed normal $\mathbb{Z}$-invariant subgroups described in Theorem 7.1. Since $(G_j/G_{j+1},\mathbb{Z})$ is a Bernoullian system of Lie type when $j = 2,\ldots,k$, triviality of $C_+^\Delta(G)$ implies, inductively, that $G_{k+1} = G_k = \cdots = G_2$ (using Corollaries 8.5 and 8.8). Hence, $G_{pB} = G_2 = \{e\}$.

(2) By Theorem 5.9 and Corollary 8.5(a) we may assume that G is a full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact Lie group. Then (2) is an immediate consequence of Corollary 8.9. $\square$

EXAMPLE 8.11. Let G be the Markov solenoid generated by $U \in \mathrm{GL}(m,\mathbb{Q})$. Since $T_G(\pm 1) = \{\exp_m(U^{\pm 1}x)\,;\, x \in \mathbb{Z}^m\}$ (cf. (6.6)), it follows that $C_\pm^\Delta(G) = \{e\}$ if and only if $U^{\mp 1} \in \mathrm{M}(m,\mathbb{Z})$.

COROLLARY 8.12. *If $(G,\mathbb{Z})$ is a pro-finite-dimensional system, then $C_\pm^\Delta(G)$ are totally disconnected.*

Proof. Let C be a compact normal $\mathbb{Z}$-invariant subgroup, such that the system $(G/C,\mathbb{Z})$ satisfies the dcc. Then $C^{\Delta}_{\pm}(G/C) \supseteq C^{\Delta}_{\pm}(G)/C$ and so if $C^{\Delta}_{\pm}(G/C)$ is totally disconnected, then C must contain the connected component of the identity in $C^{\Delta}_{\pm}(G)$. Thus by Theorem 5.3, it suffices to consider the case that the system $(G,\mathbb{Z})$ satisfies the dcc.

But when the dcc holds, then $\dim G < \infty$ and we may assume that G is a full Markov subgroup of $L^{\mathbb{Z}}$, where L is a compact Lie group. Then by Proposition 5.16, $T_G(n)$ is finite for every $n \in \mathbb{Z}$. Hence, $L_{\pm} = \bigcup_{n=1}^{\infty} T_G(\pm n)$ are countable and therefore totally disconnected subgroups of L. Since $C^{\Delta}_{\pm}(G) \subseteq (L_{\mp})^{\mathbb{Z}}$, the result follows. ☐

We will say that the action of $\mathbb{Z}$ on a compact group G (and the dynamical system $(G,\mathbb{Z})$) has the Δ-*decomposition property* (Δdp), if $C^{\Delta}_{+}(G,K) = C^{\Delta}_{+}(G)K$ for every closed $\mathbb{Z}$-invariant subgroup $K \leq G$. In Section 11, we will prove that any action of $\mathbb{Z}$ on a compact group has the Δdp. The following is a preliminary result in this direction.

LEMMA 8.13. *If an action of $\mathbb{Z}$ on a compact group G satisfies the dcc, then the action has the* Δdp.

Proof. Let $K \leq G$ be a closed $\mathbb{Z}$-invariant subgroup. By Theorem 5.9 and Corollary 8.5(a), we may assume that G and K are Markov subgroups of $L^{\mathbb{Z}}$, where L is a compact Lie group, G is full, $K = G \cap [\pi_0^L(K)]^{\mathbb{Z}}$, and $T_K = T_G \cap [\pi_0^L(K)]^{\{0,1\}}$. By Corollary 8.4, L needs to be considered as a discrete group. We are then in the setting of the main part of the proof of Corollary 5.10, except that L can now be infinite. That $C^{\Delta}_{+}(G,K) = C^{\Delta}_{+}(G)K$ follows from Lemma 4.4. ☐

9. Analytic elements in contraction groups

Throughout this section, G will denote a compact Abelian group. We will make use of elements of the theory the exponential function for such groups [8, Chapter 7]. Let $\mathcal{L}(G)$ denote the set $\mathrm{Hom}(\mathbb{R},G)$ of continuous homomorphisms from $\mathbb{R}$ to G (1-parameter subgroups). $\mathcal{L}(G)$ is in a natural way a real vector space: the scalar multiplication is given by $(a \cdot f)(t) = f(at)$ ($f \in \mathcal{L}(G)$, $a,t \in \mathbb{R}$), and the addition $f+g$ ($f,g \in \mathcal{L}(G)$) is just the ordinary addition of homomorphisms in $\mathcal{L}(G)$ when additive notation is used for the group operation in G. To conform with the rest of this paper, we will continue to use multiplicative notation for the group operation, so that $(f+g)(t) = f(t)g(t)$. We note that $\mathcal{L}(G)$ is, in general, an infinite-dimensional vector space; $\dim \mathcal{L}(G) < \infty$ if and only if G is a finite-dimensional group, in which case $\dim \mathcal{L}(G) = \dim G$ [8].

The exponential function of G, denoted EXP_G, is the function

$$\mathrm{EXP}_G : \mathcal{L}(G) \to G, \quad \mathrm{EXP}_G(f) = f(1).$$

It is a homomorphism of the additive group $(\mathcal{L}(G),+)$ into G. We note that $\mathcal{L}(G)$ equipped with the topology of uniform convergence on compacta is a topological vector space and EXP_G becomes then a continuous mapping. However, for our applications it will be sufficient to know that the restriction of EXP_G to any finite-dimensional subspace S of $\mathcal{L}(G)$ is continuous (with respect to the unique topology making S into a topological vector space). When G is a Lie group, we have the canonical identification of its Lie algebra with $\mathcal{L}(G)$. With this identification in place, EXP_G becomes the exponential function as defined in the standard theory of Lie groups.

We will continue to denote by $\mathcal{N}(G)$ the family of closed normal subgroups N of a compact group G, such that G/N is a Lie group.

LEMMA 9.1. *Suppose that S is a finite-dimensional subspace of $\mathcal{L}(G)$. Then* $\mathrm{Ker}(\mathrm{EXP}_G \restriction S)$ *is a discrete subgroup of S. Moreover, there exists $N_* \in \mathcal{N}(G)$, such that, with $\xi_N : G \to G/N$ denoting the projection,* $\mathrm{Ker}(\xi_N \circ \mathrm{EXP}_G \restriction S)$ *is discrete for every $N \in \mathcal{N}(G)$ with $N \subseteq N_*$.*

Proof. The first statement follows from the fact that $\mathrm{Ker}(\mathrm{EXP}_G)$ is a totally disconnected subgroup of $\mathcal{L}(G)$ [8, Theorem 7.66]. It is also not difficult to give a self-contained proof using only the continuity of $\mathrm{EXP}_G \restriction S$: Let $K = \mathrm{Ker}(\mathrm{EXP}_G \restriction S)$. Since S is a vector (Lie) group, K_0, the connected component of 0 in K, is a vector subspace of S, and it suffices to show that $K_0 = \{0\}$. But if $f \in K_0$ then for every $t \in \mathbb{R}$, $tf \in K_0$, so that $f(t) = \mathrm{EXP}_G(tf) = \{e\}$. Hence, $f = 0$.

To prove the second statement, observe that $\mathcal{F} = \{(\mathrm{EXP}_G \restriction S)^{-1}(N);\ N \in \mathcal{N}(G)\}$ is a directed downward family of closed subgroups of S, and $\bigcap \mathcal{F} = \mathrm{Ker}(\mathrm{EXP}_G \restriction S)$. Since $\dim S < \infty$, we can find $N_* \in \mathcal{N}(G)$, such that $\dim((\mathrm{EXP}_G \restriction S)^{-1}(N)) = \dim((\mathrm{EXP}_G \restriction S)^{-1}(N_*))$ for every $N \in \mathcal{N}(G)$ with $N \subseteq N_*$. Since closed subgroups of S are Lie groups, it follows that $((\mathrm{EXP}_G \restriction S)^{-1}(N))_0 = ((\mathrm{EXP}_G \restriction S)^{-1}(N_*))_0$ for every $N \in \mathcal{N}(G)$ with $N \subseteq N_*$. The fact that $\mathrm{Ker}(\mathrm{EXP}_G \restriction S) = \bigcap \mathcal{F}$ is discrete, forces $(\mathrm{EXP}_G \restriction S)^{-1}(N)$ to be totally disconnected, and therefore discrete. ☐

Let φ be a continuous homomorphism of G into a compact Abelian group H. We will denote by φ_* the mapping

$$\varphi_* : \mathcal{L}(G) \to \mathcal{L}(H), \quad \varphi_*(f) = \varphi \circ f.$$

Several basic properties of φ_* are obvious:

(i) φ_* is a linear mapping;
(ii) $\varphi \circ \mathrm{EXP}_G = \mathrm{EXP}_H \circ \varphi_*$;
(iii) φ_* is injective whenever φ is injective;
(iv) $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ when $\psi : H \to K$ is a continuous homomorphism into a compact Abelian group K.

The following is also true but far from obvious [8, Lemma 7.41], [9, Lemma 4.19], [22, Theorem 1, p. 192]:

(v) φ_* is surjective whenever φ is surjective.

It follows from (iii) and (v) that when $\tau \in \mathrm{Aut}(G)$, then τ_* is an automorphism of the vector space $\mathcal{L}(G)$. Thus when $\mathbb{Z}$ acts on G, there is the corresponding action on $\mathcal{L}(G)$ which makes EXP_G into an equivariant function. When $\mathbb{Z}$ acts also on the compact Abelian group H and $\varphi : G \to H$ is a continuous equivariant homomorphism, then $\varphi_* : \mathcal{L}(G) \to \mathcal{L}(H)$ is an equivariant linear mapping.

Let $\mathbb{Z}$ act on the compact Abelian group G. We will now define a subgroup of $C_+(G)$ which for finite-dimensional groups will be later shown to complement the Δ-contraction group $C_+^\Delta(G)$ (Corollary 11.7). Let us call an element $g \in C_+(G)$, an *analytic element*, if $g \in \mathrm{EXP}_G(C_+(A))$, where A is a finite-dimensional $\mathbb{Z}$-invariant subspace of $\mathcal{L}(G)$. Let $C_+^a(G) = \{g \in C_+(G); g \text{ is analytic}\}$. Note that $C_+^a(G) \subseteq G_e$, so that $C_+^a(G) = C_+^a(G_e)$.

PROPOSITION 9.2. *$C_+^a(G)$ is a path connected subgroup of $C_+(G_e)$, and $C_+^a(G) \cap C_+^\Delta(G) = \{e\}$. Moreover, if $\mathbb{Z}$ acts on a compact Abelian group H and $\varphi : G \to H$ is a continuous equivariant homomorphism, then $\varphi(C_+^a(G)) \subseteq C_+^a(H)$.*

Proof. That $C_+^a(G)$ is a path connected subgroup is obvious. By virtue of the equality $\varphi \circ \mathrm{EXP}_G = \mathrm{EXP}_H \circ \varphi_*$, the final statement is also obvious.

Suppose $g \in C_+^a(G) \cap C_+^\Delta(G)$. Thus $g = \mathrm{EXP}_G(x)$, where $x \in C_+(A)$ and $A \subseteq \mathcal{L}(G)$ is a finite-dimensional $\mathbb{Z}$-invariant subspace. By Lemma 9.1, there exists $N \in \mathcal{N}(G)$, such that, with $\xi_N : G \to G/N$ denoting the projection, $D = \mathrm{Ker}(\xi_N \circ \mathrm{EXP}_G \restriction A)$ is discrete. But by the definition of $C_+^\Delta(G)$, $ng = \mathrm{EXP}_G(nx) \in N$ for large enough n. Thus for such n, $nx \in D$. As $x \in C_+(A)$, this implies that $nx = 0$ for large enough n. So $x = 0$, that is, $g = e$. □

EXAMPLE 9.3. Let G be the solenoid generated by $U \in \mathrm{GL}(m, \mathbb{Q})$. Given $x \in \mathbb{R}^m$, the formula $f_x(t) = E_U(tx)$ defines an element of $\mathcal{L}(G)$. It is easy to see that the mapping $\mathbb{R}^m \ni x \to \Phi(x) = f_x \in \mathcal{L}(G)$ is linear. Φ is also injective, which follows from the fact that $\mathrm{Ker}(E_U)$ is discrete. Finally, Φ is surjective because $\dim G = \dim \mathcal{L}(G) = m$. We also have $\mathrm{EXP}_G \circ \Phi = E_U$. Φ is the natural isomorphism between $\mathbb{R}^m$ and $\mathcal{L}(G)$, and it is an equivariant isomorphism when the action of $\mathbb{Z}$ on $\mathbb{R}^m$ is the one induced by U (i.e., $nx = U^n x$). It follows that $C_+^a(G) = E_U(C(U))$, while Proposition 8.7 and the description of $C_+(G)$ obtained in Proposition 6.8 yield the factorization $C_+(G) = C_+^a(G) C_+^\Delta(G)$ (which is, algebraically, a direct product by Proposition 9.2).

We proceed to investigate the property $\varphi(C_+^a(G)) = C_+^a(H)$, given a continuous equivariant surjective homomorphism $\varphi : G \to H$ between compact Abelian groups.

Lemma 9.4. *If* $\dim(\operatorname{Ker}\varphi) < \infty$, *then for every finite-dimensional* $\mathbb{Z}$*-invariant subspace* $A \subseteq \mathcal{L}(H)$ *there exists a finite-dimensional* $\mathbb{Z}$*-invariant subspace* $\tilde{A} \subseteq \mathcal{L}(G)$, *such that* $A = \varphi_*(\tilde{A})$.

Proof. Since $\operatorname{Ker}\varphi_* = \mathcal{L}(\operatorname{Ker}\varphi)$ has finite dimension and φ_* is equivariant, $\tilde{A} = \varphi_*^{-1}(A)$ is the desired finite-dimensional $\mathbb{Z}$-invariant subspace. □

Lemma 9.5. *Let* A *and* B *be Abelian topological groups,* $\xi \in \operatorname{Hom}(A, B^{\mathbb{Z}})$, $\alpha \in \operatorname{Aut}(A)$, *and let* σ *denote the shift* $\sigma((b_n)_{n\in\mathbb{Z}}) = (b_{n+1})_{n\in\mathbb{Z}}$ *on* $B^{\mathbb{Z}}$. *Then there exists* $\eta \in \operatorname{Hom}(A, B^{\mathbb{Z}})$, *such that* $\xi(a)\sigma(\eta(a)) = \eta(\alpha(a))$ *for every* $a \in A$.

Proof. Writing $\xi = (\xi_n)_{n\in\mathbb{Z}}$ ($\xi_n \in \operatorname{Hom}(A,B)$), we need to find $\eta = (\eta_n)_{n\in\mathbb{Z}}$ ($\eta_n \in \operatorname{Hom}(A,B)$), such that $\xi_n(a)\eta_{n+1}(a) = \eta_n(\alpha(a))$ for every $n \in \mathbb{Z}$ and $a \in A$. One possible solution is to define

$$\eta_n(a) = \begin{cases} \prod_{j=1}^{n} [\xi_{j-1}(\alpha^{n-j}(a))]^{-1}, & \text{when } n \geq 1, a \in A; \\ \prod_{j=1}^{-n} \xi_{-j}(\alpha^{n+j-1}(a)), & \text{when } n \leq -1, a \in A; \end{cases}$$

and $\eta_0(a) = e$ for every $a \in A$. □

Lemma 9.6. *If* $(\operatorname{Ker}\varphi, \mathbb{Z})$ *is a Bernoullian system, then for every finite-dimensional* $\mathbb{Z}$*-invariant subspace* $A \subseteq \mathcal{L}(H)$ *there exists a finite-dimensional* $\mathbb{Z}$*-invariant subspace* $\tilde{A} \subseteq \mathcal{L}(G)$, *such that* $A = \varphi_*(\tilde{A})$ *and* $\dim A = \dim \tilde{A}$.

Proof. Since φ_* is a linear surjection, there is a subspace $A' \subseteq \mathcal{L}(G)$, such that $\varphi_* \restriction A'$ is a linear isomorphism of A' onto A. Let $\zeta = \operatorname{EXP}_H \restriction A$ and $\zeta' = \operatorname{EXP}_G \circ F$, where $F = (\varphi_* \restriction A')^{-1}$. Then $\zeta \in \operatorname{Hom}(A, H)$, $\zeta' \in \operatorname{Hom}(A, G)$, and $\varphi \circ \zeta' = \zeta$. In the next step of the proof, we modify ζ' to obtain an equivariant homomorphim $\tilde{\zeta} \in \operatorname{Hom}(A, G)$ with $\varphi \circ \tilde{\zeta} = \zeta$.

Let $\tau_G \in \operatorname{Aut}(G)$, $\tau_H \in \operatorname{Aut}(H)$, and $\alpha \in \operatorname{GL}(A)$ denote the automorphisms inducing the $\mathbb{Z}$-actions (i.e., corresponding to $1 \in \mathbb{Z}$). Then $\varphi \circ \tau_G \circ \zeta' = \tau_H \circ \varphi \circ \zeta' = \tau_H \circ \zeta = \zeta \circ \alpha = \varphi \circ \zeta' \circ \alpha$. Hence, for every $a \in A$, $\xi(a) = \tau_G(\zeta'(a))[\zeta'(\alpha(a))]^{-1} \in \operatorname{Ker}\varphi$. Thus when $\sigma = \tau_G \restriction \operatorname{Ker}\varphi$ and $\operatorname{Ker}\varphi$ is identified with $B^{\mathbb{Z}}$, we are in the situation to which Lemma 9.5 applies. Consequently, there exists $\eta \in \operatorname{Hom}(A, \operatorname{Ker}\varphi)$, such that $\xi(a)\tau_G(\eta(a)) = \eta(\alpha(a))$ for every $a \in A$. Invoking the definition of ξ, the latter identity can be rewritten as $\tau_G(\zeta'(a)\eta(a)) = \zeta'(\alpha(a))\eta(\alpha(a))$. Thus, the formula $\tilde{\zeta}(a) = \zeta'(a)\eta(a)$ defines a continuous equivariant homomorphism from A into G, such that $\varphi \circ \tilde{\zeta} = \zeta$.

We will now make use of the following fact about the exponential function EXP_G: if V is a finite-dimensional real vector space, then every homomorphism $h \in \operatorname{Hom}(V, G)$ has the form $h = \operatorname{EXP}_G \circ T$, where $T : V \to \mathcal{L}(G)$ is a linear mapping, uniquely determined by h. Let $T : A \to \mathcal{L}(G)$ be such a linear mapping corresponding to $\tilde{\zeta}$. It easily follows that $\operatorname{EXP}_H \circ \varphi_* \circ T = \operatorname{EXP}_H \restriction A$ and $\operatorname{EXP}_G \circ \tau_{G*} \circ T = \operatorname{EXP}_G \circ T \circ \alpha$. Hence, $\varphi_* \circ T = \operatorname{id}_A$ and $\tau_{G*} \circ T = T \circ \alpha$.

Thus T is an equivariant linear mapping of A into $\mathcal{L}(G)$. $\tilde{A}=T(A)$ is a finite-dimensional $\mathbb{Z}$-invariant subspace of $\mathcal{L}(G)$ with $\varphi_*(\tilde{A})=A$. It is clear that $\dim A=\dim\tilde{A}$. □

LEMMA 9.7. *If the system* $(\operatorname{Ker}\varphi,\mathbb{Z})$ *satisfies the dcc, then for every finite-dimensional* $\mathbb{Z}$*-invariant subspace* $A\subseteq\mathcal{L}(H)$, *there exists a finite-dimensional* $\mathbb{Z}$*-invariant subspace* $\tilde{A}\subseteq\mathcal{L}(G)$, *such that* $A=\varphi_*(\tilde{A})$.

Proof. Let $K=\operatorname{Ker}\varphi$. By Theorem 7.1, there exist closed $\mathbb{Z}$-invariant subgroups $K\geq K_1\geq\cdots\geq K_k\geq K_{k+1}=\{e\}$, such that K/K_1 is finite, $(K_1/K_2,\mathbb{Z})$ is a solenoidal system, and for $j=2,\dots,k$, $(K_j/K_{j+1},\mathbb{Z})$ is a Bernoullian system. For $j=1,\dots,k+1$, let $\xi_j:G\to G/K_j$ denote the canonical homomorphism and $\varphi_j:G/K_j\to H$, the homomorphism satisfying $\varphi_j\circ\xi_j=\varphi$. Furthermore, when $j=1,\dots,k$, let $\eta_j:G/K_{j+1}\to G/K_j$ be the homomorphism with $\eta_j\circ\xi_{j+1}=\xi_j$. Note that $\varphi_{j+1}=\varphi_j\circ\eta_j$.

We will show by finite induction that for every $j=2,\dots,k+1$ there exists a finite-dimensional $\mathbb{Z}$-invariant subspace $\tilde{A}_j\subseteq\mathcal{L}(G/K_j)$, such that $\varphi_{j*}(\tilde{A}_j)=A$. Since φ_{k+1} becomes φ when G/K_{k+1} is identified with G, this will prove the lemma.

Since $\operatorname{Ker}\varphi_2=K/K_2$ is a finite-dimensional group, our claim is true for $j=2$, due to Lemma 9.4. Suppose $j=2,\dots,k$ and that we already obtained a finite-dimensional $\mathbb{Z}$-invariant subspace $\tilde{A}_j\subseteq\mathcal{L}(G/K_j)$ with $\varphi_{j*}(\tilde{A}_j)=A$. Note that $\operatorname{Ker}\eta_j=K_j/K_{j+1}$ satisfies the assumptions of Lemma 9.6. Hence, there exists a finite-dimensional $\mathbb{Z}$-invariant subspace $\tilde{A}_{j+1}\subseteq\mathcal{L}(G/K_{j+1})$, such that $\eta_{j*}(\tilde{A}_{j+1})=\tilde{A}_j$. Then $\varphi_{j+1*}(\tilde{A}_{j+1})=(\varphi_{j*}\circ\eta_{j*})(\tilde{A}_{j+1})=A$. □

PROPOSITION 9.8. *Suppose that* $\mathbb{Z}$ *acts on compact Abelian groups* G *and* H, *and* $\varphi:G\to H$ *is a continuous equivariant surjective homomorphism. If* $\dim\operatorname{Ker}\varphi<\infty$ *or the system* $(\operatorname{Ker}\varphi,\mathbb{Z})$ *satisfies the dcc, then* $\varphi(C_+^a(G))=C_+^a(H)$.

Proof. We already know that $\varphi(C_+^a(G))\subseteq C_+^a(H)$ (Proposition 9.2). Suppose $g\in C_+^a(H)$, that is, $g\in\operatorname{EXP}_H(C_+(A))$ where $A\subseteq\mathcal{L}(H)$ is a finite-dimensional $\mathbb{Z}$-invariant subspace. By Lemma 9.4 or 9.7 there exists a finite-dimensional $\mathbb{Z}$-invariant subspace $\tilde{A}\subseteq\mathcal{L}(G)$ with $\varphi_*(\tilde{A})=A$. But using Theorem 3.6 or finite-dimensional linear algebra, $\varphi_*(C_+(\tilde{A}))=C_+(A)$. Hence, $g\in\operatorname{EXP}_H(\varphi_*(C_+(\tilde{A})))=\varphi(\operatorname{EXP}_G(C_+(\tilde{A})))\subseteq\varphi(C_+^a(G))$. □

COROLLARY 9.9. *Every solenoidal system has the cdp.*

Proof. Let $(G,\mathbb{Z})$ be a solenoidal system. Due to Proposition 3.3(iii) and the fact that G is Abelian, it suffices to show that whenever $\mathbb{Z}$ acts on a compact group G_1 and $\varphi:G\to G_1$ is a continuous equivariant surjective homomorphism, then $\varphi(C_+(G))=C_+(G_1)$. But by Proposition 6.2, $(G_1,\mathbb{Z})$ is also a solenoidal system. Then by Example 9.3, we have $C_+(G)=C_+^a(G)C_+^\Delta(G)$

and $C_+(G_1) = C_+^a(G_1)C_+^\Delta(G_1)$. Proposition 9.8 yields $\varphi(C_+^a(G)) = C_+^a(G_1)$ while Lemma 8.13, Corollary 8.5(a), and Proposition 3.3(iii) allow to conclude that $\varphi(C_+^\Delta(G)) = C_+^\Delta(G_1)$. So $\varphi(C_+(G)) = C_+(G_1)$. $\square$

COROLLARY 9.10. *Let $\mathbb{Z}$ act on compact Abelian groups G and H and let $\varphi : G \to H$ be a continuous equivariant surjective homomorphism, whose kernel is metrizable. Then $C_+^a(H) \subseteq \varphi(C_+(G))$.*

Proof. Since $\operatorname{Ker}\varphi$ is metrizable, it follows from Theorem 5.3 that there exists a nonincreasing sequence $(C_n)_{n=1}^\infty$ of closed $\mathbb{Z}$-invariant subgroups of $\operatorname{Ker}\varphi$, such that for every $n \in \mathbb{N}$, the system $((\operatorname{Ker}\varphi)/C_n, \mathbb{Z})$ satisfies the dcc and $\bigcap_{n=1}^\infty C_n = \{e\}$. Let $\xi_n : G \to G/C_n$ denote the projection and $\eta_n : G/C_{n+1} \to G/C_n$, the homomorphism satisfying $\eta_n \circ \xi_{n+1} = \xi_n$. There also exists a homomorphism $\varphi_1 : G/C_1 \to H$, such that $\varphi_1 \circ \xi_1 = \varphi$.

Let $h \in C_+^a(H)$. We claim that there exists a sequence $g_j \in C_+^a(G/C_j)$, $j \in \mathbb{N}$, such that $\varphi_1(g_1) = h$, and $g_j = \eta_j(g_{j+1})$ for every $j \in \mathbb{N}$. Indeed since $\operatorname{Ker}\varphi_1 = (\operatorname{Ker}\varphi)/C_1$, by Proposition 9.8 there exists $g_1 \in C_+^a(G/C_1)$ with $\varphi_1(g_1) = h$. But for every $j \in \mathbb{N}$, $\operatorname{Ker}\eta_j = C_j/C_{j+1} \le (\operatorname{Ker}\varphi)/C_{j+1}$, so that $(\operatorname{Ker}\eta_j, \mathbb{Z})$ satisfies the dcc. Using Proposition 9.8 the required sequence can be constructed by induction.

Note that G is the projective limit of the sequence $(G/C_j)_{j=1}^\infty$. Hence, there exists $g \in G$ with $\xi_j(g) = g_j$ for every $j \in \mathbb{N}$. Since $g_j \in C_+(G/C_j)$, it follows that $g \in C_+(G)$. Finally $\varphi(g) = (\varphi_1 \circ \xi_1)(g) = h$, so $h \in \varphi(C_+(G))$. $\square$

PROPOSITION 9.11. *Let G denote the solenoid generated by $U \in \mathrm{GL}(m, \mathbb{Q})$. Then*:

(i) *$C_+(G) = \{e\}$ if and only if every eigenvalue of U is a root of* 1.
(ii) *If $C_+(G) = \{e\}$, then G is topologically isomorphic to $\mathbb{T}^m$.*
(iii) *$[C_+(G)]^{-} = G$ if and only if no eigenvalue of U is a root of* 1.
(iv) *If $[C_+(G)]^{-} = G$, then $[C_+(H)]^{-} = H$ for every closed connected $\mathbb{Z}$-invariant subgroup $H \le G$.*

Proof. (i) Recall that there exists an equivariant isomorphism Ψ of G onto a Markov solenoid G' generated by some $U' \in \mathrm{GL}(m, \mathbb{Q})$. Then Ψ_* is an equivariant isomorphism of $\mathcal{L}(G)$ onto $\mathcal{L}(G')$. Identifying $\mathcal{L}(G)$ and $\mathcal{L}(G')$ with $\mathbb{R}^m$, as in Example 9.3, it follows that U and U' are similar matrices. Hence, it is enough to prove our claim when G is a Markov solenoid, which we will now assume.

$\Rightarrow$: Since $C_+^a(G) = \{e\}$, it is clear from the description of $C_+^a(G)$ given in Example 9.3 that if λ is an eigenvalue of U, then $|\lambda| \ge 1$. But as $C_+^\Delta(G) = \{e\}$, $U^{-1} \in \mathrm{M}(m, \mathbb{Z})$, as seen in Example 8.11. Therefore, $|\det U| \le 1$. Hence, U has only unimodular eigenvalues. If so, then $\det(U^{-1}) = \pm 1$ and, hence, $U \in \mathrm{M}(m, \mathbb{Z})$. Thus $U \in \mathrm{GL}(m, \mathbb{Z})$. But a matrix in $\mathrm{GL}(m, \mathbb{Z})$ has only unimodular eigenvalues if and only all its eigenvalues are roots of 1.[7] Since $U \in \mathrm{GL}(m,\mathbb{Z})$, we also obtain that G is topologically isomorphic to $\mathbb{T}^m$, that is, (ii) is true.

$\Leftarrow$: Since every eigenvalue of U is a root of 1, $C_+^a(G) = \{e\}$. Moreover, we can find $p \in \mathbb{N}$, such that 1 is the only eigenvalue of U^p and U^{-p}. Then the Cayley–Hamilton theorem yields $(U^p - 1)^m = 0 = (U^{-p} - 1)^m$. This can be used to express every power of U as a linear combination, with integral coefficients, of the powers U^j, where $|j| < pm$. Hence, there exists $d \in \mathbb{N}$, such that $dU^j \in \mathrm{M}(m,\mathbb{Z})$ for every $j \in \mathbb{Z}$. It follows that $(d\mathbb{Z})^m \subseteq \mathrm{Ker}(E_U)$, and so $E_U(\mathbb{R}^m) = E_U([0,d]^m)$ is closed in $(\mathbb{T}^m)^{\mathbb{Z}}$. Therefore, $G = E_U(\mathbb{R}^m)$ and thus G is a Lie group (isomorphic to $\mathbb{T}^m$).[8] Consequently, $C_+^{\Delta}(G) = \{e\}$. So $C_+(G) = C_+^a(G) C_+^{\Delta}(G) = \{e\}$ (cf. Example 9.3).

(iii) $\Rightarrow$: We argue by contradiction. Suppose that U has a root of 1 among its eigenvalues. Then there exists $p \in \mathbb{N}$, and a nonzero $\xi \in \mathbb{Z}^m$ with $\xi U^p = \xi$. Let $\chi \in (\mathbb{T}^m)^\frown$ be the character defined by ξ and let $\tilde{\chi} = \chi \circ \pi_0^{\mathbb{T}^m} \restriction G$. Then $\tilde{\chi}$ is a nontrivial character on G and $\tilde{\chi}(E_U(x)) = \exp(2\pi i \xi \cdot x)$ for every $x \in \mathbb{R}^m$, where $\cdot$ is the dot product in $\mathbb{R}^m$. It follows that for every $n \in \mathbb{Z}$ and $x \in \mathbb{R}^m$, $\tilde{\chi}((np)E_U(x)) = \tilde{\chi}(E_U(x))$. Hence, $\tilde{\chi}((np)g) = \tilde{\chi}(g)$ for every $n \in \mathbb{Z}$ and $g \in G$. Consequently, $C_+(G) \subseteq \mathrm{Ker}\,\tilde{\chi}$, which contradicts the density of $C_+(G)$ in G.

$\Leftarrow$: We again argue by contradiction. If $[C_+(G)]^- \neq G$, then there is an equivariant isomorphism of $G/[C_+(G)]^-$ onto a solenoid G' generated by $U' \in \mathrm{GL}(m',\mathbb{Q})$, where $m' \leq m$. By Corollaries 9.9 and 3.4, $C_+(G') = \{e\}$, and so all eigenvalues of U' are roots of 1. But $G' = \varphi(G)$, where $\varphi : G \to G'$ is a continuous equivariant surjective homomorphism. φ induces a linear surjection $F : \mathbb{R}^m \to \mathbb{R}^{m'}$, such that $U' \circ F = F \circ U$. But then every eigenvalue of U' is an eigenvalue of U, contradicting the absence roots of 1 among the eigenvalues of U.

(iv) If H is nontrivial, then it is isomorphic to a solenoid generated by $U' \in \mathrm{GL}(m',\mathbb{Q})$. The inclusion mapping $\mathrm{id}_H : H \to G$ induces a linear injection $F : \mathbb{R}^{m'} \to \mathbb{R}^m$ with $U \circ F = F \circ U'$. Hence, U' has no roots of 1 among its eigenvalues. $\square$

EXAMPLE 9.12. When $U \in \mathrm{GL}(m,\mathbb{Q})$ and $U^p \in \mathrm{GL}(m,\mathbb{Z})$ for some $p \in \mathbb{N}$, that is, when G is isomorphic to $\mathbb{T}^m$, then U has only unimodular eigenvalues

[7] This follows from Kronecker's result [18] on polynomial equations with integral coefficients. Alternatively, one can argue as follows: Induction on m shows that if $\sum_{j=0}^m b_j x^j$ is a monic polynomial in $\mathbb{C}[x]$, with all roots on the unit circle, then $|b_0| = 1$ and $|b_j| \leq 2^{m-1}$ for $j = 1, \dots, m-1$. Hence, there are only finitely many possibilities for the characteristic polynomial of a matrix $U \in \mathrm{GL}(m,\mathbb{Z})$, whose eigenvalues are unimodular. Thus the set $\mathcal{U}$ consisting of the eigenvalues of all such matrices is finite. Since for every $\lambda \in \mathcal{U}$ and $n \in \mathbb{Z}$, $\lambda^n \in \mathcal{U}$, λ must be a root of 1 (then there even exists $l \in \mathbb{N}$ such that each $\lambda \in \mathcal{U}$ is an lth root of 1).

[8] This argument shows that if for a matrix $U \in \mathrm{GL}(m,\mathbb{Q})$ one can find $d \in \mathbb{N}$, such that $dU^j \in \mathrm{M}(m,\mathbb{Z})$ for every $j \in \mathbb{Z}$, then $U^p \in \mathrm{GL}(m,\mathbb{Z})$ for some $p \in \mathbb{N}$. A purely algebraic proof of this fact is elementary, although nontrivial.

if and only if all its eigenvalues are roots of 1. However, when G is not isomorphic to $\mathbb{T}^m$, it is possible for $C_+(G)$ to be nontrivial when U has only unimodular eigenvalues. For example, $U = \left[\begin{smallmatrix} \frac{1}{2} & -1 \\ 1 & 0 \end{smallmatrix}\right]$ has unimodular eigenvalues $\frac{1}{4}(1 \pm i\sqrt{15})$ which are not roots of 1. If U has only unimodular eigenvalues but not all of them are roots of 1, Proposition 9.11(ii), Example 9.3, and Corollary 8.12 show that $C_+(G) = C_+^{\Delta}(G)$ is a nontrivial totally disconnected group.

10. The compact decomposition property for compact groups

We are now in a position to obtain a characterization of those $\mathbb{Z}$-actions on compact groups which have the cdp. As Theorem 4.2 may suggest, the Bernoullizable and poly-Bernoullian components G_B and G_{pB} play a key role in this characterization.

THEOREM 10.1. *If an action of $\mathbb{Z}$ on a compact group G satisfies the dcc, then the following conditions are equivalent*:

(i) *The action has the cdp.*
(ii) *G_{pB} is totally disconnected.*
(iii) $\dim G < \infty$.
(iv) *If $H \leq G$ is a closed $\mathbb{Z}$-invariant subgroup and φ a continuous equivariant homomorphism of H onto a Bernoulli system $(K^{\mathbb{Z}}, \mathbb{Z})$, then K is finite.*

Proof. (i) $\Rightarrow$ (ii) Let $G \geq G_1 \geq \cdots \geq G_k \geq G_{k+1} = \{e\}$ be the closed normal $\mathbb{Z}$-invariant subgroups described in Theorem 7.1. By Proposition 3.2(iii) and Corollary 3.4, each of the systems $(G_j/G_{j+1}, \mathbb{Z})$, $j = 2, \ldots, k$, has the cdp. Hence, by Theorem 4.2, G_j/G_{j+1} is totally disconnected for every $j = 2, \ldots, k$. This implies that $G_{pB} = G_2$ is totally disconnected.

(ii) $\Rightarrow$ (iii) This is obvious because $\dim G/G_{pB} < \infty$ by Remark 7.12.

(iii) $\Rightarrow$ (iv) Clearly, $\dim K^{\mathbb{Z}} < \infty$, and so K is totally disconnected. But $(K^{\mathbb{Z}}, \mathbb{Z})$ inherits the dcc from $(G, \mathbb{Z})$. Since compact open subgroups of K form a neighbourhood base at e in K, the dcc forces $\{e\}$ to be a neighbourhood of e. Hence, K is discrete, and so finite.

(iv) $\Rightarrow$ (i) Let $G \geq G_1 \geq \cdots \geq G_k \geq G_{k+1} = \{e\}$ be the closed normal $\mathbb{Z}$-invariant subgroups described in Theorem 7.1. Then each of the Bernoullian systems $(G_j/G_{j+1}, \mathbb{Z})$, $j = 2, \ldots, k$, is totally disconnected, and, hence, by Theorem 4.2, has the cdp. By Corollary 9.9, the solenoidal system $(G_1/G_2, \mathbb{Z})$ also has the cdp. Finally, the system $(G/G_1, \mathbb{Z})$ trivially has the cdp, because any action of $\mathbb{Z}$ on a semisimple Lie group is equicontinuous (so that $C_+(G/G_1, K) = K$). Then the cdp of $(G, \mathbb{Z})$ can be established by finite induction, starting from $(G_k, \mathbb{Z})$ and repeatedly using Corollary 3.4 and Theorem 7.1(4) to arrive at $(G, \mathbb{Z})$. ∎

Theorem 10.2. *The following conditions are equivalent for an action of $\mathbb{Z}$ on a compact group G:*

(i) *The action has the cdp.*
(ii) *G_B is totally disconnected.*
(iii) *$(G,\mathbb{Z})$ is a pro-finite-dimensional system.*
(iv) *If $H \le G$ is a closed $\mathbb{Z}$-invariant subgroup and φ a continuous equivariant homomorphism of H onto a Bernoulli system $(K^{\mathbb{Z}},\mathbb{Z})$, then K is totally disconnected.*

Proof. (i) ⇒ (ii) If $(G,\mathbb{Z})$ has the cdp, then so does $(G_B,\mathbb{Z})$. Then for every closed normal $\mathbb{Z}$-invariant subgroup C of G_B, such that $(G_B/C,\mathbb{Z})$ satisfies the dcc, $(G_B/C,\mathbb{Z})$ also has the cdp. Hence, by Proposition 7.17 and Theorem 10.1, $G_B/C = (G_B/C)_{pB}$ is totally disconnected. Theorem 5.3 implies that G_B is totally disconnected.

(ii) ⇔ (iii) Corollary 7.18(i).

(iii) ⇒ (iv) The system $(H,\mathbb{Z})$ is also pro-finite-dimensional. Hence, so is $(K^{\mathbb{Z}},\mathbb{Z})$. By Corollary 7.18(i), $K^{\mathbb{Z}}$, and therefore K, is totally disconnected.

(iv) ⇒ (i) Let $C \le G$ be a closed normal $\mathbb{Z}$-invariant subgroup, such that $(G/C,\mathbb{Z})$ satisfies the dcc. Then condition (iv) for our system $(G,\mathbb{Z})$ implies condition (iv) of Theorem 10.1 for the system $(G/C,\mathbb{Z})$. Hence, $(G/C,\mathbb{Z})$ has the cdp. An application of Theorems 3.8 and 5.3 yields (i). □

Corollary 10.3. *An action of $\mathbb{Z}$ on a compact connected group G has the cdp if and only if $G_B = \{e\}$.*

Proof. Corollary 7.20. □

While the cdp fails for every system $(G,\mathbb{Z})$ which fails to be pro-finite-dimensional, the question remains about the extent of this failure. When $K \le G$ is a closed $\mathbb{Z}$-invariant subgroup, such that $C_+(G)K$ is a proper subgroup of $C_+(G,K)$, one would like to know how different $C_+(G)K$ and $C_+(G,K)$ can be. In particular, is it possible for $C_+(G,K)$ to be strictly larger than K when $C_+(G) = \{e\}$? A special case of this question is whether $C_+(G/K)$ can be nontrivial when K is normal and $C_+(G) = \{e\}$ (cf. Proposition 3.3(iii)). These questions will be answered in the negative using Theorem 10.2 and the Δ-contraction groups together with the Δ-decomposition property, to be proven in the next section. Subsequently, we will show that for a metrizable group, $C_+(G)K$ is always a dense subgroup of $C_+(G,K)$ (Theorem 12.2).

11. The Δ-decomposition property

Our main objective in this section is to prove the following theorem, extending Lemma 8.13 to arbitrary $\mathbb{Z}$-actions on compact groups.

Theorem 11.1. *Any action of $\mathbb{Z}$ on a compact group has the* Δdp.

Before embarking on a lengthy proof of the theorem, let us explore a few of its consequences.

PROPOSITION 11.2. *For any action of $\mathbb{Z}$ on a compact group G, $G_B \subseteq [C_+^\Delta(G)]^-$.*

Proof. First, assume that the dcc holds. Let $G \geq G_1 \geq \cdots \geq G_k \geq G_{k+1} = \{e\}$ be the subgroups described in Theorem 7.1 and $\xi_j : G_j \to G_j/G_{j+1}$ denote the projections. Since for $j = 2, \ldots, k$, $(G_j/G_{j+1}, \mathbb{Z})$ is a Bernoullian system of Lie type, $[C_+^\Delta(G_j/G_{j+1})]^- = G_j/G_{j+1}$ by Corollary 8.8. Then thanks to Corollary 8.5(a), Proposition 3.3(iii), and Theorem 11.1, $\xi_j([C_+^\Delta(G_j)]^-) = G_j/G_{j+1}$. Therefore, $G_j = [C_+^\Delta(G_j)]^- G_{j+1} \subseteq [C_+^\Delta(G)]^- G_{j+1}$ for $j = 2, \ldots, k$. Since $G_{k+1} = \{e\}$, this implies that $G_B = G_2 \subseteq [C_+^\Delta(G)]^-$.

When the dcc does not hold, then by Theorem 5.3 it suffices to show that for every closed normal $\mathbb{Z}$-invariant subgroup C, such that the dcc holds for $(G/C, \mathbb{Z})$, we have $\xi_C(G_B) \subseteq \xi_C([C_+^\Delta(G)]^-)$, where $\xi_C : G \to G/C$ is the projection. But $\xi_C([C_+^\Delta(G)]^-) = [C_+^\Delta(G/C)]^-$ by Theorem 11.1 and Proposition 3.3(iii), while $\xi_C(G_B) = (G/C)_B$ by Proposition 7.17. □

COROLLARY 11.3. *If $C_+^\Delta(G) = \{e\}$, then the system $(G, \mathbb{Z})$ has the cdp.*

Proof. Combine Proposition 11.2 and Theorem 10.2. □

COROLLARY 11.4. *If $C_+(G) = \{e\}$, then $C_+(G, K) = K$ for every closed $\mathbb{Z}$-invariant subgroup $K \leq G$; in particular, $C_+(H) = \{e\}$ whenever $\mathbb{Z}$ acts on a compact group H and $\varphi : G \to H$ is a continuous equivariant surjective homomorphism.*

COROLLARY 11.5. $C_+(G) = C_+^\Delta(G) C_+(G_e)$.

Proof. Let $\xi : G \to G/G_e$ denote the projection. Since G/G_e is totally disconnected, using Proposition 3.3(iii), Corollary 8.5(a), and Theorem 11.1, we obtain $C_+(G/G_e) = C_+^\Delta(G/G_e) = \xi(C_+^\Delta(G)) \subseteq \xi(C_+(G)) \subseteq C_+(G/G_e)$. Thus, $\xi(C_+^\Delta(G)) = \xi(C_+(G))$ and, hence, $C_+(G) \subseteq C_+^\Delta(G) G_e$. But as $C_+^\Delta(G) \subseteq C_+(G)$ and $C_+(G) \cap G_e = C_+(G_e)$, we obtain $C_+(G) \subseteq C_+^\Delta(G) C_+(G_e) \subseteq C_+(G)$. □

PROPOSITION 11.6. *If $\mathbb{Z}$ acts on a finite-dimensional compact group G, then $C_+(G)$ is, algebraically, the direct product of $C_+^a(Z_e(G_e))$ and $C_+^\Delta(G)$.*

Proof. Since $C_+^a(Z_e(G_e)) \cap C_+^\Delta(G) = C_+^a(Z_e(G_e)) \cap Z_e(G_e) \cap C_+^\Delta(G) = C_+^a(Z_e(G_e)) \cap C_+^\Delta(Z_e(G_e))$, Proposition 9.2 yields $C_+^a(Z_e(G_e)) \cap C_+^\Delta(G) = \{e\}$. Moreover, since $C_+^\Delta(G)$ is totally disconnected (Corollary 8.12) and normal in G, while $C_+^a(Z_e(G_e))$ is connected (Proposition 9.2), it follows that $C_+^a(Z_e(G_e))$ and $C_+^\Delta(G)$ commute elementwise. Hence, it remains to show that $C_+(G) = C_+^a(Z_e(G_e)) C_+^\Delta(G)$.

Let us first consider the case that the dcc holds. Since $\dim G < \infty$, G_{pB} is totally disconnected (cf. Theorem 10.1). Moreover, with $G_1 = Z_e(G_e)G_{pB}$, G/G_1 is a semisimple Lie group while $(G_1/G_{pB}, \mathbb{Z})$ is a solenoidal system. Thus, $C_+(G) = C_+(G_1)$, and by Example 9.3, $C_+(G_1/G_{pB}) = C_+^a(G_1/G_{pB})C_+^\Delta(G_1/G_{pB})$.

Let $\xi : G \to G/G_{pB}$ denote the projection. Using Theorem 11.1, and Propositions 3.3(iii) and 9.8, we obtain $\xi(C_+(G)) = \xi(C_+(G_1)) \subseteq C_+(G_1/G_{pB}) = \xi(C_+^a(Z_e(G_e)))\xi(C_+^\Delta(G_1))$. Therefore $C_+(G) \subseteq C_+^a(Z_e(G_e))C_+^\Delta(G_1)G_{pB} = C_+^a(Z_e(G_e))C_+^\Delta(G)G_{pB}$. Hence, $C_+(G) \subseteq C_+^a(Z_e(G_e))C_+^\Delta(G)C_+(G_{pB}) = C_+^a(Z_e(G_e))C_+^\Delta(G)C_+^\Delta(G_{pB}) = C_+^a(Z_e(G_e))C_+^\Delta(G) \subseteq G_+(G)$, because G_{pB} is totally disconnected. This finishes the proof in the case that the dcc holds.

In the general case, since $\dim G < \infty$, there is a neighbourhood of e in G which contains no nontrivial connected subgroups. This neighbourhood contains a closed normal $\mathbb{Z}$-invariant subgroup C, such that the system $(G/C, \mathbb{Z})$ satisfies the dcc. But then C is necessarily totally disconnected and we can complete the proof by using Theorem 11.1, and Propositions 3.3(iii) and 9.8, similarly as in the dcc case (with C assuming the role of G_{pB}). □

Corollary 11.7. *If $\mathbb{Z}$ acts on a finite-dimensional compact Abelian group, then $C_+(G)$ is, algebraically, the direct product of $C_+^a(G)$ and $C_+^\Delta(G)$.*

Corollary 11.8. *Let $\mathbb{Z}$ act on compact groups G and H, and $\varphi : G \to H$ be a continuous equivariant surjective homomorphism. If $Z_e(G_e) \cap \operatorname{Ker}\varphi$ is finite-dimensional or metrizable, and $\dim H < \infty$, then $C_+(H) = \varphi(C_+(G))$.*

Proof. Using Proposition 9.8 and Corollary 9.10, we obtain $C_+^a(Z_e(H_e)) \subseteq \varphi(C_+(Z_e(G_e))) \subseteq \varphi(C_+(G))$. By Theorem 11.1 and Proposition 3.3(iii), $C_+^\Delta(H) = \varphi(C_+^\Delta(G)) \subseteq \varphi(C_+(G))$. Hence, $C_+(H) = C_+^a(Z_e(H_e))C_+^\Delta(H) \subseteq \varphi(C_+(G)) \subseteq C_+(H)$. □

Corollary 11.9. *Let $\mathbb{Z}$ act on a compact group G, where $Z_e(G_e)$ is finite-dimensional or metrizable. If a closed $\mathbb{Z}$-invariant subgroup $K \leq G$ contains a closed normal $\mathbb{Z}$-invariant subgroup $N \trianglelefteq G$, such that $\dim(G/N) < \infty$, then $C_+(G,K) = C_+(G)K$.*

Proof. Let $\xi : G \to G/N$ denote the projection. Then by Theorem 10.2 and Corollary 11.8, $C_+(G/N, \xi(K)) = C_+(G/N)\xi(K) = \xi(C_+(G)K)$. Hence, $C_+(G,K) = C_+(G)K$ by Proposition 3.3(ii). □

We now turn to the proof of Theorem 11.1. In view of Lemma 8.13 and Theorem 5.3, Theorem 11.1 will be an immediate consequence of the following projective limit result, whose proof closely follows that in [13] of Proposition 3.8.

LEMMA 11.10. *Let $\mathcal{C}$ be a nonempty family of closed normal $\mathbb{Z}$-invariant subgroups of a compact group G, such that for every $C \in \mathcal{C}$, the system $(G/C, \mathbb{Z})$ has the* Δdp. *Then the system $(G/\bigcap \mathcal{C}, \mathbb{Z})$ also has the* Δdp.

The proof of Lemma 11.10 occupies the remainder of Section 11.

LEMMA 11.11. *Let K and H be closed $\mathbb{Z}$-invariant subgroups of G. Then $C_+^\Delta(G,K) \cap H = C_+^\Delta(H, K \cap H)$. Moreover, if the system $(G,\mathbb{Z})$ has the* Δdp, *then so does $(H,\mathbb{Z})$.*

Proof. According to Proposition 3.2(i), to prove the first statement, it suffices to verify that for every $N \in \mathcal{N}(G)$ there exists $M \in \mathcal{N}(G)$, such that $(MK) \cap H \subseteq N(K \cap H)$. But $\mathcal{M} = \{(MK) \cap H; M \in \mathcal{N}(G)\}$ is a directed downward family of closed subgroups with $\bigcap \mathcal{M} = K \cap H$. Then Lemma 7.14 can be applied similarly as in the proof of Proposition 8.2.

The proof of the last statement is analogous to the proof of Proposition 3.2(iii). □

LEMMA 11.12. *Let actions of $\mathbb{Z}$ on compact groups G_1 and G_2 have the* Δdp. *Then the action of $\mathbb{Z}$ on $(G_1 \times G_2)$ also has the* Δdp.

Proof. Let $\xi_i : G_1 \times G_2 \to G_i$ be the projections $(i = 1,2)$. Suppose $K \subseteq G_1 \times G_2$ is a compact $\mathbb{Z}$-invariant subgroup and $g = (g_1, g_2) \in C_+^\Delta(G_1 \times G_2, K)$. Put $K_1 = \xi_1(K)$ and $K_2 = \xi_2(K \cap (\{e\} \times G_2))$. Then K_i is a compact $\mathbb{Z}$-invariant subgroup. Moreover, $K \cap (\{e\} \times G_2) = \{e\} \times K_2$.

Since $g_1 \in C_+^\Delta(G_1, K_1)$ (cf. Proposition 3.3(i) and Corollary 8.5), there exists $k_1 \in K_1$ with $g_1 k_1^{-1} \in C_+^\Delta(G_1)$. We can also find $k_2 \in G_2$ with $k = (k_1, k_2) \in K$. Now, $gk^{-1} = (g_1 k_1^{-1}, e)(e, g_2 k_2^{-1}) \in C_+^\Delta(G_1 \times G_2, K)$. But $(g_1 k_1^{-1}, e) \in C_+^\Delta(G_1 \times G_2)$. Hence, we obtain $(e, g_2 k_2^{-1}) \in C_+^\Delta(G_1 \times G_2, K) \cap (\{e\} \times G_2) = C_+^\Delta(\{e\} \times G_2, \{e\} \times K_2)$, by Lemma 11.11. This implies that $g_2 k_2^{-1} \in C_+^\Delta(G_2, K_2)$. Therefore $g_2 k_2^{-1} l^{-1} \in C_+^\Delta(G_2)$ for some $l \in K_2$. Consequently, $g[(e,l)k]^{-1} = (g_1 k_1^{-1}, e)(e, g_2 k_2^{-1} l^{-1}) \in C_+^\Delta(G_1 \times G_2)$ and $(e,l)k \in K$. So $g \in C_+^\Delta(G_1 \times G_2)K$. □

LEMMA 11.13. *Let V_1 and V_2 be closed normal $\mathbb{Z}$-invariant subgroups of G. If the actions of $\mathbb{Z}$ on G/V_1 and G/V_2 have the* Δdp, *then the action on $G/(V_1 \cap V_2)$ also has the* Δdp.

Proof. Let $\varphi_i : G \to G/V_i$ denote the canonical homomorphisms $(i = 1,2)$. Then $\varphi_1 \times \varphi_2 : G \to (G/V_1) \times (G/V_2)$ is a continuous equivariant homomorphism with $\mathrm{Ker}(\varphi_1 \times \varphi_2) = V_1 \cap V_2$. $H = (\varphi_1 \times \varphi_2)(G)$ is a closed $\mathbb{Z}$-invariant subgroup of $(G/V_1) \times (G/V_2)$. The systems $(H, \mathbb{Z})$ and $(G/(V_1 \cap V_2), \mathbb{Z})$ are canonically isomorphic. But by Lemma 11.12, $((G/V_1) \times (G/V_2), \mathbb{Z})$ has the Δdp. By Lemma 11.11, the same is true about $(H, \mathbb{Z})$. Consequently, the $\mathbb{Z}$-action on $G/(V_1 \cap V_2)$ has the Δdp (cf. Corollary 8.5(a)). □

LEMMA 11.14. *Let B be a directed set and $(V_\beta)_{\beta\in B}$, a nonincreasing family of closed normal $\mathbb{Z}$-invariant subgroups of G. Put $\tilde{V}=\bigcap_{\beta\in B}V_\beta$ and denote by $\xi_\beta : G\to G/V_\beta$ $(\beta\in B)$, and $\tilde{\xi} : G\to G/\widetilde{V}$, the canonical homomorphisms.*

Let $g\in C_+^\Delta(G,K)$, where $K\subseteq G$ is a compact $\mathbb{Z}$-invariant subgroup. Suppose there exists a function $f=(f_1,f_2) : B\to G\times K$, such that for every $\beta\in B$:

(1) $\xi_\beta(f_1(\beta)f_2(\beta))=\xi_\beta(g)$;
(2) $\xi_\beta(f_1(\beta))\in C_+^\Delta(G/V_\beta)$;
(3) $\xi_\gamma(f_1(\beta))=\xi_\gamma(f_1(\gamma))$ *and* $\xi_\gamma(f_2(\beta))=\xi_\gamma(f_2(\gamma))$ *for every* $\gamma\le\beta$.

Then there exist $x\in G$ and $y\in K$, such that:

(1′) $\tilde{\xi}(xy)=\tilde{\xi}(g)$;
(2′) $\tilde{\xi}(x)\in C_+^\Delta(G/\tilde{V})$;
(3′) $\xi_\gamma(x)=\xi_\gamma(f_1(\gamma))$ *and* $\xi_\gamma(y)=\xi_\gamma(f_2(\gamma))$ *for every* $\gamma\in B$.

Proof. Let β_0 be any element of B. By Properties (1) and (3), for every $\beta\ge\beta_0$, $\xi_{\beta_0}(f_1(\beta)f_2(\beta))=\xi_{\beta_0}(f_1(\beta_0)f_2(\beta_0))=\xi_{\beta_0}(g)$. Hence, $f_1(\beta)f_2(\beta)\in gV_{\beta_0}$ and so $f(\beta)=(f_1(\beta),f_2(\beta))\in gV_{\beta_0}K\times K$. By compactness, the net $(f(\beta))_{\beta\in B}$ has a convergent subnet $(f(\beta_j))$. Put $x=\lim_j f_1(\beta_j)$ and $y=\lim_j f_2(\beta_j)$.

To verify Property (1′), we use Properties (3) and (1). When $\gamma\in B$, then, for large enough j, we have $\xi_\gamma(f_1(\beta_j)f_2(\beta_j))=\xi_\gamma(f_1(\gamma)f_2(\gamma))=\xi_\gamma(g)$. Hence, $f_1(\beta_j)f_2(\beta_j)\in gV_\gamma$, and so $xy\in gV_\gamma$. Since this holds for every $\gamma\in B$, $xy\in\bigcap_{\gamma\in B}gV_\gamma=g\tilde{V}$. Therefore, $\tilde{\xi}(xy)=\tilde{\xi}(g)$.

Property (3′) follows using Property (3) and the continuity of ξ_γ.

To verify Property (2′), let $N\in\mathcal{N}(G/\tilde{V})$. Then $\tilde{\xi}^{-1}(N)\in\mathcal{N}(G)$ and using Lemma 7.14 similarly as in the proof of Proposition 8.2, we can find j with $V_{\beta_j}\subseteq\tilde{\xi}^{-1}(N)\tilde{V}=\tilde{\xi}^{-1}(N)$. Now, $\xi_{\beta_j}(x)=\xi_{\beta_j}(f_1(\beta_j))\in C_+^\Delta(G/V_{\beta_j})$. So $x\in C_+^\Delta(G,V_{\beta_j})$ by Proposition 3.3(ii) (applied with the Δ-topologies). Hence, $nx\in\tilde{\xi}^{-1}(N)V_{\beta_j}=\tilde{\xi}^{-1}(N)$ for sufficiently large n. But then $n\tilde{\xi}(x)=\tilde{\xi}(nx)\in N$ for sufficiently large n. Consequently, $\tilde{\xi}(x)\in C_+^\Delta(G/\tilde{V})$, as required. ☐

Recall that given an ordinal α, we denote by $[0,\alpha)$ (resp., $[0,\alpha]$) the set of ordinals strictly smaller than α (resp., smaller or equal to α).

LEMMA 11.15. *Let α be a nonzero ordinal and $(V_\beta)_{\beta\in[0,\alpha)}$, a nonincreasing family of closed normal $\mathbb{Z}$-invariant subgroups of G, such that for every nonzero limit ordinal $\beta\in[0,\alpha)$, $V_\beta=\bigcap_{\gamma\in[0,\beta)}V_\gamma$. For every $\beta\in[0,\alpha)$, denote by ξ_β the projection $\xi_\beta : G\to G/V_\beta$. Suppose that for every $\beta\in[0,\alpha)$, the system $(G/V_\beta,\mathbb{Z})$ has the* Δdp *and let $g\in C_+^\Delta(G,K)$, where $K\subseteq G$ is a compact $\mathbb{Z}$-invariant subgroup. Then there is a function $f=(f_1,f_2) : [0,\alpha)\to G\times K$, such that for every $\beta\in[0,\alpha)$*:

(1) $\xi_\beta(f_1(\beta)f_2(\beta)) = \xi_\beta(g)$;
(2) $\xi_\beta(f_1(\beta)) \in C_+^\Delta(G/V_\beta)$;
(3) $\xi_\gamma(f_1(\beta)) = \xi_\gamma(f_1(\gamma))$ *and* $\xi_\gamma(f_2(\beta)) = \xi_\gamma(f_2(\gamma))$ *for every* $\gamma \in [0,\beta]$.

Proof. Let Ω denote the set of functions $\varphi = (\varphi_1, \varphi_2)$, such that:

(i) the range of φ is contained in $G \times K$;
(ii) the domain of φ has the form $[0, \alpha_\varphi)$ where $\alpha_\varphi \in [0, \alpha]$;
(iii) Properties (1), (2), (3) hold with $f = \varphi$, for every $\beta \in [0, \alpha_\varphi)$.

We order Ω by inclusion (i.e., $\varphi \subseteq \varphi'$ if φ' is an extension of φ). An application of Zorn's lemma shows that Ω has a maximal element, f. It remains to prove that $\alpha_f = \alpha$. This will follow if we show that every element $\varphi \in \Omega$ with $\alpha_\varphi < \alpha$ has a proper extension in Ω.

Case I. $\alpha_\varphi = 0$ (φ is the empty function). By Proposition 3.3(i), $\xi_0(g) \in C_+^\Delta(G/V_0, \xi_0(K))$. Since $(G/V_0, \mathbb{Z})$ has the Δdp, $\xi_0(g) = ab$, where $a \in C_+^\Delta(G/V_0)$ and $b \in \xi_0(K)$. Choosing $x \in G$ and $y \in K$ with $\xi_0(x) = a$ and $\xi_0(y) = b$, we define $\varphi' \in \Omega$ with domain $\{0\} = [0,1)$, by $\varphi'(0) = (x, y)$. φ' trivially extends φ.

Case II. α_φ *is a nonzero limit ordinal.* We are in a position to apply Lemma 11.14 with $f = \varphi$ and $B = [0, \alpha_\varphi)$. This yields elements $x \in G$ and $y \in K$ with Properties ($1'$), ($2'$), ($3'$), where $\tilde{\xi} = \xi_{\alpha_\varphi}$. We define $\varphi' : [0, \alpha_\varphi + 1) \to G \times K$ by $\varphi'(\beta) = \varphi(\beta)$ for $\beta \in [0, \alpha_\varphi)$ and $\varphi'(\alpha_\varphi) = (x, y)$. Using Properties ($1'$), ($2'$), ($3'$), it is easy to see that $\varphi' \in \Omega$.

Case III. α_φ *has an immediate predecessor,* σ. Let $\xi_{\sigma\alpha_\varphi} : G/V_{\alpha_\varphi} \to G/V_\sigma$ denote the homomorphism with $\xi_{\sigma\alpha_\varphi} \circ \xi_{\alpha_\varphi} = \xi_\sigma$. Since $\xi_{\sigma\alpha_\varphi}$ is a surjection and $\xi_{\sigma\alpha_\varphi}(\xi_{\alpha_\varphi}(K)) = \xi_\sigma(K)$, there exist $s \in G/V_{\alpha_\varphi}$ and $t \in \xi_{\alpha_\varphi}(K)$, such that $\xi_\sigma(\varphi_1(\sigma)) = \xi_{\sigma\alpha_\varphi}(s)$ and $\xi_\sigma(\varphi_2(\sigma)) = \xi_{\sigma\alpha_\varphi}(t)$. By Proposition 3.3 and Property (2), $s \in C_+^\Delta(G/V_{\alpha_\varphi}, \operatorname{Ker}\xi_{\sigma\alpha_\varphi})$. Since $(G/V_{\alpha_\varphi}, \mathbb{Z})$ has the Δdp, there exists $p \in \operatorname{Ker}\xi_{\sigma\alpha_\varphi}$, such that $sp^{-1} \in C_+^\Delta(G/V_{\alpha_\varphi})$.

Next, $\xi_{\sigma\alpha_\varphi}(\xi_{\alpha_\varphi}(g)) = \xi_\sigma(g) = \xi_\sigma(\varphi_1(\sigma)\varphi_2(\sigma)) = \xi_{\sigma\alpha_\varphi}(st)$. So $\xi_{\alpha_\varphi}(g) = str$ where $r \in \operatorname{Ker}\xi_{\sigma\alpha_\varphi}$. As $\xi_{\alpha_\varphi}(g) \in C_+^\Delta(G/V_{\alpha_\varphi}, \xi_{\alpha_\varphi}(K))$ (by Proposition 3.3), and $\xi_{\alpha_\varphi}(g) = (sp^{-1})(ptrt^{-1})t$, where $(sp^{-1}) \in C_+^\Delta(G/V_{\alpha_\varphi})$ and $t \in \xi_{\alpha_\varphi}(K)$, we conclude that $ptrt^{-1} \in C_+^\Delta(G/V_{\alpha_\varphi}, \xi_{\alpha_\varphi}(K)) \cap \operatorname{Ker}\xi_{\sigma\alpha_\varphi} \subseteq C_+^\Delta(G/V_{\alpha_\varphi}, \xi_{\alpha_\varphi}(K) \cap \operatorname{Ker}\xi_{\sigma\alpha_\varphi})$ (by Lemma 11.11). Hence, by the Δdp of $(G/V_{\alpha_\varphi}, \mathbb{Z})$ there exists $q \in \xi_{\alpha_\varphi}(K) \cap \operatorname{Ker}\xi_{\sigma\alpha_\varphi}$ with $ptrt^{-1}q^{-1} \in C_+^\Delta(G/V_{\alpha_\varphi})$.

Thus, $\xi_{\alpha_\varphi}(g) = (strt^{-1}q^{-1})qt$ where $strt^{-1}q^{-1} = (sp^{-1})(ptrt^{-1}q^{-1}) \in C_+^\Delta(G/V_{\alpha_\varphi})$, $qt \in \xi_{\alpha_\varphi}(K)$, and $q, trt^{-1}q^{-1} \in \operatorname{Ker}\xi_{\sigma\alpha_\varphi}$.

We can find $x \in G$ and $y \in K$ with $\xi_{\alpha_\varphi}(x) = strt^{-1}q^{-1}$ and $\xi_{\alpha_\varphi}(y) = qt$. Then $\xi_{\alpha_\varphi}(xy) = strt^{-1}q^{-1}qt = str = \xi_{\alpha_\varphi}(g)$, $\xi_{\alpha_\varphi}(x) \in C(G/V_{\alpha_\varphi})$, $\xi_\sigma(x) = \xi_{\sigma\alpha_\varphi}(\xi_{\alpha_\varphi}(x)) = \xi_{\sigma\alpha_\varphi}(strt^{-1}q^{-1}) = \xi_{\sigma\alpha_\varphi}(s) = \xi_\sigma(\varphi_1(\sigma))$, and $\xi_\sigma(y) = \xi_{\sigma\alpha_\varphi}(\xi_{\alpha_\varphi}(y)) = \xi_{\sigma\alpha_\varphi}(qt) = \xi_{\sigma\alpha_\varphi}(t) = \xi_\sigma(\varphi_2(\sigma))$. Let $\varphi' : [0, \alpha_\varphi + 1) \to G \times K$ be defined by $\varphi'(\gamma) = \varphi(\gamma)$ for $\gamma \in [0, \alpha_\varphi)$ and $\varphi'(\alpha_\varphi) = (x, y)$. Then φ' is a proper extension of φ in Ω. $\square$

COROLLARY 11.16. *Let α be a nonzero ordinal and $(V_\beta)_{\beta\in[0,\alpha)}$, a nonincreasing family of compact normal $\mathbb{Z}$-invariant subgroups of G, such that for every nonzero limit ordinal $\beta\in[0,\alpha)$, $V_\beta=\bigcap_{\gamma\in[0,\beta)}V_\gamma$. Suppose that for every $\beta\in[0,\alpha)$, the system $(G/V_\beta,\mathbb{Z})$ has the* Δdp. *Then $(G/\bigcap_{\beta\in[0,\alpha)}V_\beta,\mathbb{Z})$ also has the* Δdp.

Proof. The claim is trivial when α has an immediate predecessor, so assume that α is a limit ordinal. Put $\tilde V=\bigcap_{\beta\in[0,\alpha)}V_\beta$ and write $\tilde\xi$ for the canonical homomorphism $\tilde\xi: G\to G/\tilde V$.

Let $\tilde g\in C_+^\Delta(G/\tilde V,\tilde K)$, where $\tilde K\subseteq G/\tilde V$ is a compact $\mathbb{Z}$-invariant subgroup. Let $K=\tilde\xi^{-1}(\tilde K)$ and $g\in\tilde\xi^{-1}(\{\tilde g\})\subseteq\tilde\xi^{-1}(C_+^\Delta(G/\tilde V,\tilde K))=C_+^\Delta(G,K)$ (cf. Proposition 3.3). An application of Lemma 11.15 followed by Lemma 11.14 yields elements $x\in G$ and $y\in K$, such that $\tilde\xi(x)\tilde\xi(y)=\tilde g$, $\tilde\xi(x)\in C_+^\Delta(G/\tilde V)$, and $\tilde\xi(y)\in\tilde K$. □

Proof of Lemma 11.10. Let A denote the set of those nonzero cardinal numbers γ for which there exists a family $\mathcal{V}$, of cardinality γ, consisting of compact normal $\mathbb{Z}$-invariant subgroups of G, such that for every $V\in\mathcal{V}$, the quotient $(G/V,\mathbb{Z})$ has the Δdp but $(G/\bigcap\mathcal{V},\mathbb{Z})$ fails to have the Δdp. It suffices to show that $A=\varnothing$.

We argue by contradiction, supposing that $A\neq\varnothing$. Since every set of cardinal numbers is well ordered, A has the smallest element, α. Let $\mathcal{V}$ be the corresponding family of compact normal $\mathbb{Z}$-invariant subgroups.

Recall that if γ is a cardinal number, then γ is an ordinal, such that $[0,\gamma)$ has cardinality γ while $[0,\delta)$ has cardinality strictly less than γ for every $\delta\in[0,\gamma)$. Let $[0,\alpha)\ni\gamma\to W_\gamma\in\mathcal{V}$ be any bijection of $[0,\alpha)$ onto $\mathcal{V}$. Put $V_0=W_0$ and $V_\beta=\bigcap_{\gamma\in[0,\beta)}W_\gamma$ for every nonzero ordinal $\beta<\alpha$. Then $(V_\beta)_{\beta\in[0,\alpha)}$ is a nonincreasing family of compact normal $\mathbb{Z}$-invariant subgroups of G and it is easy to see that for every nonzero limit ordinal $\beta\in[0,\alpha)$, $V_\beta=\bigcap_{\gamma\in[0,\beta)}V_\gamma$. Since for every $\beta\in[0,\alpha)$, $[0,\beta)$ has cardinality strictly less than α, it follows that for every $\beta\in[0,\alpha)$, the system $(G/V_\beta,\mathbb{Z})$ has the Δdp. Hence, by Corollary 11.16, the quotient $(G/\bigcap_{\beta\in[0,\alpha)}V_\beta,\mathbb{Z})$ also has the Δdp. Therefore, $\bigcap_{\beta\in[0,\alpha)}V_\beta\neq\bigcap\mathcal{V}$. This forces α to be finite, for otherwise α would be a limit ordinal and this easily implies that $\bigcap_{\beta\in[0,\alpha)}V_\beta=\bigcap\mathcal{V}$. But when α is finite, then $\bigcap\mathcal{V}=W_{\alpha-1}\cap V_{\alpha-1}=W_{\alpha-1}\cap\bigcap_{\beta\in[0,\alpha)}V_\beta$ and Lemma 11.13 yields a contradiction. □

12. Closures of contraction groups and the weak cdp

We will say that an action of $\mathbb{Z}$ on a compact group G has the *weak compact decomposition property*, if for every closed $\mathbb{Z}$-invariant subgroup $K\leq G$, $C_+(G)K$ is dense in $C_+(G,K)$, that is, $C_+(G)K\subseteq C_+(G,K)\subseteq[C_+(G)]^{-}K$.

LEMMA 12.1. *Let $\mathbb{Z}$ act on compact groups G and H, and $\varphi: G \to H$ be a continuous equivariant surjective homomorphism. If $(H,\mathbb{Z})$ is a pro-finite-dimensional system and $Z_e(G_e)$ is metrizable, then $C_+(H) \subseteq [\varphi(C_+(G))]^-$ and $C_+(G, \operatorname{Ker}\varphi) \subseteq [C_+(G)]^- \operatorname{Ker}\varphi$.*

Proof. Let $\mathcal{M}$ be a directed downward family of closed normal $\mathbb{Z}$-invariant subgroups $M \leq H$, such that $\dim(H/M) < \infty$ and $\bigcap \mathcal{M} = \{e\}$. Then Corollary 11.8 yields $\xi_M(\varphi(C_+(G))) = C_+(H/M)$, where $\xi_M : H \to H/M$ is the projection. Using Proposition 3.3(ii), we then obtain $C_+(H) \subseteq C_+(H,M) = \varphi(C_+(G))M$, for every $M \in \mathcal{M}$. This implies that $C_+(H) \subseteq [\varphi(C_+(G))]^-$. Another application of Proposition 3.3(ii), together with the fact that φ is a closed mapping, yields $C_+(G, \operatorname{Ker}\varphi) \subseteq [C_+(G)]^- \operatorname{Ker}\varphi$. □

THEOREM 12.2. *Let the radical $Z_e(G_e)$ of a compact group G be metrizable. Then every action of $\mathbb{Z}$ on G has the weak cdp. Moreover, if $\mathbb{Z}$ acts on a compact group H and $\varphi : G \to H$ is a continuous equivariant surjective homomorphism, then $[C_+(H)]^- = [\varphi(C_+(G))]^-$.*

Proof. Let $K \leq G$ be a closed $\mathbb{Z}$-invariant subgroup. Let $\varphi : G \to G/G_B$ denote the projection. As $(G/G_B, \mathbb{Z})$ is a pro-finite-dimensional system (Corollary 7.18), $C_+(G/G_B, \varphi(K)) = C_+(G/G_B)\varphi(K)$ by Theorem 10.2. Hence, by Proposition 3.3(ii), $C_+(G,K) \subseteq C_+(G, KG_B) = C_+(G,G_B)KG_B$. Then by Lemma 12.1, $C_+(G,K) \subseteq [C_+(G)]^- G_B K G_B$. But G_B is a normal subgroup contained in $[C_+(G)]^-$ (Proposition 11.2). So $C_+(G,K) \subseteq [C_+(G)]^- K$.

The second statement is an immediate consequence of the first one and of Proposition 3.3(ii). □

The remaining results in this section are motivated by the concept of the ergodic component G_{erg} and by Theorem 4.11 in [26], according to which the restriction of an ergodic action on a compact metrizable group G to the connected component G_e, is ergodic. As will be shown in Section 13, for metrizable groups ergodicity is equivalent to the density of $C_+(G)$ in G; thus if $C_+(G)$ is dense in G, then $C_+(G_e)$ is dense in G_e. Proposition 12.5 shows that the latter implication remains true when G_e is replaced by any closed connected normal $\mathbb{Z}$-invariant subgroup H with a pro-finite-dimensional $\mathbb{Z}$-action on G/H. Proposition 12.4, stating that the connectedness of G forces the closure of $C_+(G)$ to be connected, will be used in Section 13 to prove that the ergodic component G_{erg} of a connected system $(G,\mathbb{Z})$ is connected.

LEMMA 12.3. *Let $\mathbb{Z}$ act on a compact group G and $H \leq G$ be a closed connected normal $\mathbb{Z}$-invariant subgroup, such that $(G/H,\mathbb{Z})$ is a pro-finite-dimensional system. If $[C_+(G)]^- = G$ and $C_+(H) = \{e\}$, then $H = \{e\}$.*

Proof. We will first consider the case that the system $(G,\mathbb{Z})$ satisfies the dcc. Then $\dim G/H < \infty$. But the system $(H,\mathbb{Z})$ also satisfies the dcc, and

$H_{pB} = \{e\}$, because $C_+(H) = \{e\}$ (Proposition 11.2). Therefore, $\dim H < \infty$ (Theorem 10.1). Concluding, $\dim G < \infty$ and G_{pB} is totally disconnected.

Recall that the quotient system $((Z_e(G_e)G_{pB})/G_{pB}, \mathbb{Z})$ is solenoidal, while $G/(Z_e(G_e)G_{pB})$ is a semisimple Lie group. The latter implies that $C_+(G) \subseteq Z_e(G_e)G_{pB}$. Therefore, $G = Z_e(G_e)G_{pB}$. Thus, $(G/G_{pB}, \mathbb{Z})$ is a solenoidal system and $[C_+(G/G_{pB})]^{-} = G/G_{pB}$. Let $\varphi : G \to G/G_{pB}$ denote the projection. Then $\varphi(H)$ is a closed connected $\mathbb{Z}$-invariant subgroup of G/G_{pB}. By Proposition 9.11, $[C_+(\varphi(H))]^{-} = \varphi(H)$. But $C_+(\varphi(H)) = \{e\}$ (by Corollary 11.4, or Theorem 10.1 and Corollary 3.5). Therefore $H \subseteq G_{pB}$, and as G_{pB} is totally disconnected, $H = \{e\}$.

We are ready to consider the general case. By Proposition 11.2, $H_B = \{e\}$ and, hence, $(H, \mathbb{Z})$ is a pro-finite-dimensional system (Theorem 10.2). But it is not difficult to see that then $(G, \mathbb{Z})$ is itself pro-finite-dimensional. Let $C \le G$ be a closed normal $\mathbb{Z}$-invariant subgroup, such that the system $(G/C, \mathbb{Z})$ satisfies the dcc and denote by $\xi_C : G \to G/C$ the projection. Clearly, $[C_+(G/C)]^{-} = G/C$, while $C_+(\xi_C(H)) = \{e\}$ by Corollary 11.4. Moreover, $(G/C)/\xi_C(H)$ is an equivariant image of G/H and therefore the system $((G/C)/\xi_C(H), \mathbb{Z})$ is pro-finite-dimensional. Hence, we can apply what we proved assuming the dcc to conclude that $\xi_C(H) = \{e\}$. Since this holds for every C, such that $(G/C, \mathbb{Z})$ satisfies the dcc, $H = \{e\}$. □

PROPOSITION 12.4. *Let $\mathbb{Z}$ act on a compact connected group G. If $(G, \mathbb{Z})$ is a pro-finite-dimensional system or $Z_e(G)$ is metrizable, then $[C_+(G)]^{-}$ is connected.*

Proof. Let $(G, \mathbb{Z})$ be a pro-finite-dimensional system. By Theorems 10.2 and 5.3, it suffices to consider the case that $(G, \mathbb{Z})$ satisfies the dcc. But then $G_{pB} = \{e\}$ by Corollary 10.3. Consequently, $(Z_e(G), \mathbb{Z})$ is a solenoidal system while $G/Z_e(G)$ is a semisimple Lie group. Thus $C_+(G) \subseteq Z_e(G)$, and so $[C_+(G)]^{-}$ is connected by Proposition 6.8.

Suppose that $(G, \mathbb{Z})$ fails to be pro-finite-dimensional but $Z_e(G)$ is metrizable. By Corollary 7.19, G_B is connected and by Proposition 11.2, $G_B \subseteq [C_+(G)]^{-}$. Hence, it suffices to show that $[C_+(G)]^{-}/G_B$ is connected. But by Theorem 12.2, $[C_+(G)]^{-}/G_B = [C_+(G/G_B)]^{-}$. Since $(G/G_B, \mathbb{Z})$ is a pro-finite-dimensional system (Corollary 7.18), the first part of the proof yields the desired conclusion. □

PROPOSITION 12.5. *Let $\mathbb{Z}$ act on a compact group G and $H \le G$ be a closed connected normal $\mathbb{Z}$-invariant subgroup. If $(G, \mathbb{Z})$ is a pro-finite-dimensional system, or $Z_e(G_e)$ is metrizable and $(G/H, \mathbb{Z})$ is a pro-finite-dimensional system, then $[C_+(H)]^{-} = (H \cap [C_+(G)]^{-})_e$, in particular, $[C_+(H)]^{-} = H$ whenever $[C_+(G)]^{-} = G$.*

Proof. We will give a proof in the case that $Z_e(G_e)$ is metrizable and $(G/H, \mathbb{Z})$ is a pro-finite-dimensional system. When $(G, \mathbb{Z})$ is assumed to be a pro-finite-dimensional system but metrizability of $Z_e(G_e)$ is not assumed, the pro-finite-dimensionality of $(G/H,\mathbb{Z})$ follows from that of $(G,\mathbb{Z})$ and the proof can be completed along the same lines, making use of the cdp instead of the weak cdp.

We will first prove that $[C_+(H)]^- = H$ whenever $[C_+(G)]^- = G$. To this end, observe that $C_+(H) \trianglelefteq G$, because $H \trianglelefteq G$. Let $\xi : G \to G/[C_+(H)]^-$ denote the projection. As $[C_+(G)]^- = G$, it is clear that $[C_+(G/[C_+(H)]^-)]^- = G/[C_+(H)]^-$. On the other hand, as $Z_e(H) \subseteq Z_e(G_e)$, Theorem 12.2 implies that $C_+(\xi(H)) = \{e\}$. Moreover, $\xi(H)$ is a closed connected normal $\mathbb{Z}$-invariant subgroup of $G/[C_+(H)]^-$, and the system $((G/[C_+(H)]^-)/\xi(H),\mathbb{Z})$ is pro-finite-dimensional, as an equivariant image of $(G/H,\mathbb{Z})$. Hence, Lemma 12.3 applies. We obtain $\xi(H) = \{e\}$, and so $H = [C_+(H)]^-$.

Turning to the general case note that $[C_+(H)]^-$ is connected by Proposition 12.4 (as $Z_e(H) \subseteq Z_e(G_e)$). Hence, $C_+(H) \subseteq (H \cap [C_+(G)]^-)_e$, and so $C_+(H) = C_+((H \cap [C_+(G)]^-)_e)$. The system $([C_+(G)]^-/(H \cap [C_+(G)]^-),\mathbb{Z})$ is pro-finite-dimensional since it is isomorphic to the canonical image of $[C_+(G)]^-$ in G/H, while the system $((H \cap [C_+(G)]^-)/(H \cap [C_+(G)]^-)_e,\mathbb{Z})$ is 0-dimensional. It follows that $([C_+(G)]^-/(H \cap [C_+(G)]^-)_e,\mathbb{Z})$ is a pro-finite-dimensional system. Since the radical of $[C_+(G)]^-$ is contained in the radical of G, the assumptions in the first part of the proof will be satisfied for the system $([C_+(G)]^-,\mathbb{Z})$ and the subgroup $(H \cap [C_+(G)]^-)_e$. So $(H \cap [C_+(G)]^-)_e = [C_+((H \cap [C_+(G)]^-)_e)]^- = [C_+(H)]^-$. $\square$

13. Contraction groups versus ergodic and distal properties

Since $[C_+(G)]^- \supseteq G_B$, our first lemma is significant only when $G_B = \{e\}$. Under an additional assumption, the lemma will be superseded by a much stronger Theorem 13.4.

Lemma 13.1. *If $\mathbb{Z}$ acts ergodically on a nontrivial compact group G, then $C_+(G) \neq \{e\}$.*

Proof. Suppose that the dcc holds. Since $[C_+(G)]^- \supseteq G_{pB}$ (cf. Proposition 11.2), it suffices to consider the case that $G_{pB} = \{e\}$. Then $(Z_e(G_e),\mathbb{Z})$ is a solenoidal system, while $G/Z_e(G_e)$ is a semisimple Lie group. The latter must be trivial by ergodicity. Therefore, the system $(G,\mathbb{Z})$ is isomorphic to a solenoid generated by some $U \in \mathrm{GL}(m,\mathbb{Z})$. But it is well known that a $\mathbb{Z}$-action on such a solenoid is ergodic if and only if U has no roots of 1 among its eigenvalues. Proposition 9.11 yields $C_+(G) \neq \{e\}$.

To prove the lemma without the dcc assumption, observe that there exists a closed normal $\mathbb{Z}$-invariant subgroup $C \leq G$, such that $(G/C,\mathbb{Z})$ is a nontrivial system satisfying the dcc. Since $\mathbb{Z}$ acts ergodically on G/C, $C_+(G/C) \neq \{e\}$, and then Corollary 11.4 shows that $C_+(G) \neq \{e\}$. $\square$

We turn to one of our final results. It concerns the connection between contraction groups and distal properties of $\mathbb{Z}$-actions on compact groups, and is motivated by certain known results about automorphisms of (not necessary compact) Lie groups [1] and totally disconnected locally compact groups [14]. By the work of Abels [1], a group Γ of automorphims of a Lie group acts distally on G if and only if $C(\gamma)=\{e\}$ for every $\gamma\in\Gamma$. Corollary 2.4 in [14] shows that the same is true when G is a totally disconnected locally compact group and Γ contains a polycyclic subgroup of finite index. In particular, if G is a Lie group or a totally disconnected locally compact group, then an action of $\mathbb{Z}$ on G is distal if and only if $C_+(G)=C_-(G)=\{e\}$.

THEOREM 13.2. *The following conditions are equivalent for an action of $\mathbb{Z}$ on a compact group G*:

(i) *$\mathbb{Z}$ acts distally on G.*
(ii) $C_+(G)=\{e\}$.
(iii) $C_-(G)=\{e\}$.
(iv) $\mathbb{N}$ (*as a subsemigroup of* $\mathbb{Z}$) *acts distally on G.*

Proof. It is obvious that (i) implies (ii), (iii), and (iv), and that (iv) implies (ii). To prove that (ii) implies (i), argue by contradiction. If the action is not distal, then by Corollary 2.7(i) the ergodic component G_{erg} is nontrivial. But then Lemma 13.1 yields $C_+(G_{\mathrm{erg}})\neq\{e\}$, which contradicts the inclusion $C_+(G_{\mathrm{erg}})\subseteq C_+(G)$. That (iii) implies (i) follows by using the reflection $k\to -k$ on $\mathbb{Z}$. □

REMARK 13.3. By Proposition 11.2 and Theorem 10.2, a distal action of $\mathbb{Z}$ on a compact group G is necessarily pro-finite-dimensional; furthermore, when the action satisfies the dcc, then, by Corollary 8.10, G is necessarily a Lie group.

We note that Theorem 13.2, derived here as a consequence of Lemma 13.1, is, by virtue of Corollary 2.3, equivalent to Lemma 13.1. The next theorem can therefore be regarded as a strengthening of both Lemma 13.1 and Theorem 13.2 in the case that $Z_e(G_e)$ is metrizable or $(G,\mathbb{Z})$ is a pro-finite-dimensional system.

THEOREM 13.4. *Let $\mathbb{Z}$ act on a compact group G. If the action is pro-finite-dimensional or $Z_e(G_e)$ is metrizable, then $[C_+(G)]^- = [C_-(G)]^- = G_{\mathrm{erg}}$.*

Proof. It is enough to show that $[C_+(G)]^- = G_{\mathrm{erg}}$. By Proposition 2.8, $C_+(G)\subseteq G_{\mathrm{erg}}$. But by Corollary 3.5 or Theorem 12.2, $C_+(G/[C_+(G)]^-)=\{e\}$. Hence, $\mathbb{Z}$ acts distally on $G/[C_+(G)]^-$. Then Corollary 2.7(i) yields $G_{\mathrm{erg}}\subseteq [C_+(G)]^-$. □

COROLLARY 13.5. *If $\mathbb{Z}$ acts on a compact connected group G, then G_{erg} is connected.*

Proof. When $Z_e(G)$ is metrizable, this is immediate by combining Theorem 13.4 and Proposition 12.4. When $Z_e(G)$ is not necessarily metrizable, let C be a closed normal $\mathbb{Z}$-invariant subgroup, such that the dcc holds for the system $(G/C,\mathbb{Z})$. Then Proposition 5.1, Theorem 13.4, and Proposition 12.4 yield that $(G/C)_{\mathrm{erg}}$ is connected. But by Corollary 2.7(ii), $(G/C)_{\mathrm{erg}}$ is the image of G_{erg} under the canonical projection. Hence, the result follows using Theorem 5.3. □

Schmidt [26, Theorem 4.11] showed that ergodic actions of certain countable groups, including $\mathbb{Z}$, on compact metrizable groups remain ergodic when restricted to the connected component of the identity. For actions of $\mathbb{Z}$, this result is a special case of our last theorem.

THEOREM 13.6. *Let $\mathbb{Z}$ act on a compact group G and $H \leq G$ be a closed connected normal $\mathbb{Z}$-invariant subgroup, such that $(G/H,\mathbb{Z})$ is a pro-finite-dimensional system. Then $H_{\mathrm{erg}} = (H \cap G_{\mathrm{erg}})_e$. In particular, if $\mathbb{Z}$ acts ergodically on G, then it acts ergodically H.*

Proof. If $Z_e(G_e)$ is metrizable, the result is an immediate consequence of Theorem 13.4 and Proposition 12.5. Otherwise, Proposition 5.1, Theorem 13.4, Proposition 12.5, and Corollary 2.7(ii) yield that whenever $C \leq G$ is a closed normal $\mathbb{Z}$-invariant subgroup such that the dcc holds for the system $(G/C,\mathbb{Z})$, and $\xi_C : G \to G/C$ denotes the projection, then $\xi_C(H_{\mathrm{erg}}) = (\xi_C(H) \cap \xi_C(G_{\mathrm{erg}}))_e \supseteq \xi_C((H \cap G_{\mathrm{erg}})_e)$, so that $H_{\mathrm{erg}}C \supseteq (H \cap G_{\mathrm{erg}})_e$. Since C can be arbitrarily small, $H_{\mathrm{erg}} \supseteq (H \cap G_{\mathrm{erg}})_e$. On the other hand, it is clear that $H_{\mathrm{erg}} \subseteq (H \cap G_{\mathrm{erg}})_e$. □

Appendix: Solenoids versus Markov subgroups

We mentioned in Section 6 that every solenoid in $\mathbb{T}^{\mathbb{Z}}$ is a Markov solenoid. Example A.1 shows solenoids which are not Markov solenoids.

EXAMPLE A.1. Let U be any matrix in $\mathrm{GL}(m,\mathbb{Q}) \setminus \mathrm{GL}(m,\mathbb{Z})$, such that $U^p \in \mathrm{GL}(m,\mathbb{Z})$ for some $p \geq 2$ (e.g., $U = \left[\begin{smallmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{smallmatrix}\right]$). Then the solenoid generated by U is isomorphic to $\mathbb{T}^m$, and so cannot be a Markov solenoid because that would require that $U \in \mathrm{GL}(m,\mathbb{Z})$. The following is a 2-dimensional example of a non-Markov solenoid V which is not isomorphic to $\mathbb{T}^2$.

Let $V \subseteq (\mathbb{T}^2)^{\mathbb{Z}}$ be generated $U = \left[\begin{smallmatrix} 0 & \frac{1}{6} \\ \frac{2}{3} & 0 \end{smallmatrix}\right]$. Then $U^2 = \frac{1}{9}I$, which forces E_U to be injective. By elementary Lie theory, V cannot be isomorphic to $\mathbb{T}^2$. Next,

$$\hat{\pi}_1^{\mathbb{T}^2}(V) = \{\langle (s_0,t_0),(s_1,t_1)\rangle \in (\mathbb{T}^2)^{\{0,1\}}\,;\, s_0^2 = t_1^3, t_0 = s_1^6\},$$
$$\hat{\pi}_2^{\mathbb{T}^2}(V) \subseteq \{\langle (s_0,t_0),(s_1,t_1),(s_2,t_2)\rangle \in (\mathbb{T}^2)^{\{0,1,2\}}\,;\, s_0 = s_2^9, t_0 = t_2^9\}.$$

Hence, $\langle (1,1),(-1,1)\rangle, \langle (-1,1),(-1,1)\rangle \in \hat{\pi}_1^{\mathbb{T}^2}(V)$, and so if V were a Markov solenoid, one could find $x = (x_n)_{n\in\mathbb{Z}} \in V$ with $x_0 = (1,1)$ and $x_1 = x_2 = (-1,1)$. But such x cannot belong to V, because $\langle (1,1),(-1,1),(-1,1)\rangle \notin \hat{\pi}_2^{\mathbb{T}^2}(V)$. Therefore, V is not a Markov solenoid.

We note that using duality theory one can prove that each of the following two conditions is both necessary and sufficient in order that the solenoid generated by $U \in \mathrm{GL}(m,\mathbb{Z})$ be a Markov solenoid:

(1) $S = \{\xi \in \prod^*_{k\in\mathbb{Z}} \mathbb{Z}^m\,;\, (\exists l \in \mathbb{Z})\ (\forall k \neq l, l+1)\ \xi_k = 0 \text{ and } \xi_l + \xi_{l+1}U = 0\}$ generates the subgroup $\Xi_U = \{\xi \in \prod^*_{k\in\mathbb{Z}} \mathbb{Z}^m\,;\, \sum_{k\in\mathbb{Z}} \xi_k U^k = 0\}$.

(2) $\mathbb{Z}^m \cap (\mathbb{Z}^m U + \cdots + \mathbb{Z}^m U^k) = \mathbb{Z}^m \cap (\mathbb{Z}^m U)$ for every $k \in \mathbb{N}$.

Our second example concerns a characterization of solenoids among full Markov subgroups V of $(\mathbb{T}^m)^{\mathbb{Z}}$. Suppose $T_V(\pm 1)$ are finite. As noted in Section 6, the three statements: (a) *V is a solenoid*; (b) *T_V is connected*; and, (c) $T_V(-1) \cap T_V(1) = \{e\}$, are equivalent when $m = 1$, while regardless of the value of m, (a) implies both (b) and (c). Example A.2 shows that when $m > 1$, then (b) $\not\Rightarrow$ (c), (c) $\not\Rightarrow$ (b), and [(b) and (c)] $\not\Rightarrow$ (a).

Example A.2. Observe that Remark 6.3 implies that if V is a non-Markov solenoid, then the Markov subgroup W, whose transition subgroup is $T_W = \hat{\pi}_1^{\mathbb{T}^m}(V)$, cannot be connected. Hence, Example A.1 demonstrates that there are disconnected full Markov subgroups W of $(\mathbb{T}^m)^{\mathbb{Z}}$ for which $T_W(\pm 1)$ are finite and T_W is connected. Thus, (b) $\not\Rightarrow$ (a). In the second part of Example A.1, V is the solenoid generated by $U = \left[\begin{smallmatrix} 0 & \frac{1}{6} \\ \frac{2}{3} & 0 \end{smallmatrix}\right]$. Now, $U = B^{-1}A$, where $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 3 \\ 6 & 0 \end{smallmatrix}\right]$. The Markov subgroup W defined by A and B (i.e., by the corresponding homomorphisms $\alpha, \beta : \mathbb{T}^m \to \mathbb{T}^m$, cf. Remark 5.5) is then disconnected by Remark 6.3, but one can easily verify that $T_W = \hat{\pi}_1^{\mathbb{T}^m}(V)$ and $T_W(1) \cap T_W(-1) \supseteq \{(1,1), (-1,1)\}$. Therefore, (b) implies neither (a) nor (c).

On the other hand, $A = \left[\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 3 \end{smallmatrix}\right]$ define a Markov subgroup $V \le (\mathbb{T}^2)^{\mathbb{Z}}$, such that $T_V(1) \cap T_V(-1) = \{(1,1), (-i,i), (-1,-1), (i,-i)\} \cap \{(1,1), (-1,1), (i,-1), (-i,-1)\} = \{(1,1)\}$, and $\langle(-1,-1),(1,-1)\rangle \in T_V \setminus (T_V)_e$. Hence, (c) implies neither (b) nor (a).

We did not succeed in finding a 2-dimensional example in which T_V is connected, $T_V(1) \cap T_V(-1) = \{e\}$, but V is not a solenoid. The following is a 3-dimensional example of this situation. Let

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad U = B^{-1}A = \begin{bmatrix} 0 & 0 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $T_V = \{\langle (r_0, s_0, t_0), (r_1, s_1, t_1)\rangle \in (\mathbb{T}^3)^{\{0,1\}}\,;\, r_0 = s_1^2, s_0 = t_1, t_0^2 = r_1\}$ is connected and $T_V(-1) \cap T_V(1) = \{(1,1,1), (1,1,-1)\} \cap \{(1,1,1), (1,-1,1)\} = \{(1,1,1)\}$. However, as $U^3 = I$ and $U \notin \mathrm{GL}(3,\mathbb{Z})$, the solenoid generated by U $(= V_e)$ is isomorphic to $\mathbb{T}^3$ and is not a Markov solenoid. Therefore, V cannot be a solenoid. A similar example in which V_e is not isomorphic to $\mathbb{T}^3$ can be obtained by replacing B with $3B$ and arguing as in the second part of Example A.1.

We note that Example A.2 also shows that Proposition 3.10 in [4], claiming that the condition $A\mathbb{Z}^m + B\mathbb{Z}^m = \mathbb{Z}^m$ (where A, B are as in (6.7)) is both necessary and sufficient in order that the Markov subgroup V defined by A and B be connected (i.e., be a solenoid), is false: the first and third pair (A, B) in Example A.2 satisfy the stated condition. The condition $A\mathbb{Z}^m + B\mathbb{Z}^m = \mathbb{Z}^m$ is equivalent to the connectedness of the transition subgroup T_V of the resulting Markov subgroup, but not to the connectedness of V.

References

[1] H. Abels, *Distal automorphism groups of Lie groups*, J. Reine Angew. Math. **329** (1981), 82–87. MR 0636446

[2] U. Baumgartner and G. A. Willis, *Contraction groups and scales of automorphisms of totally disconnected locally compact groups*, Israel J. Math. **142** (2004), 221–248. MR 2085717

[3] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on semigroups*, Wiley, New York, 1989. MR 0999922

[4] B. Brenken, *The local product structure of expansive automorphisms of solenoids and their associated C^*-algebras*, Canadian J. Math. **48** (1996), 692–709. MR 1407604

[5] S. G. Dani and R. Shah, *Contraction subgroups and semistable measures on p-adic Lie groups*, Math. Proc. Cambridge Philos. Soc. **110** (1991), 299–306. MR 1113428

[6] W. Hazod and E. Siebert, *Automorphisms on a Lie group contracting modulo a compact subgroup and applications to semistable convolutions semigroups*, J. Theoret. Probab. **1** (1988), 211–225. MR 0938260

[7] E. Hewitt and K. Ross, *Abstract harmonic analysis*, 2nd ed., vol. I, Springer-Verlag, New York, 1979. MR 0551496

[8] K. H. Hofmann and S. A. Morris, *The structure of compact groups*, 2nd ed., de Gruyter, Berlin, 2006. MR 2261490

[9] K. H. Hofmann and S. A. Morris, *The Lie theory of connected pro-Lie groups*, European Mathematical Society, Zürich, 2007. MR 2337107

[10] K. Iwasawa, *On some types of topological groups*, Ann. Math. **50** (1949), 507–558. MR 0029911

[11] W. Jaworski, *Strong approximate transitivity, polynomial growth, and spread out random walks on locally compact groups*, Pacific J. Math. **170** (1995), 517–533. MR 1363877

[12] W. Jaworski, *Contractive automorphisms of locally compact groups and the concentration function problem*, J. Theoret. Probab. **10** (1997), 967–989. MR 1481656

[13] W. Jaworski, *On contraction groups of automorphisms of totally disconnected locally compact groups*, Israel J. Math. **172** (2009), 1–8. MR 2534235

[14] W. Jaworski and C. R. E. Raja, *The Choquet–Deny theorem and distal properties of totally disconnected locally compact groups*, New York J. Math. **13** (2007), 159–174. MR 2336237

[15] W. Jaworski, J. Rosenblatt and G. Willis, *Concentration functions in locally compact groups*, Math. Ann. **305** (1996), 673–691. MR 1399711

[16] B. Kitchens, *Expansive dynamics of zero-dimensional groups*, Ergodic Theory Dynam. Systems **7** (1987), 249–261. MR 0896796

[17] B. Kitchens and K. Schmidt, *Automorphisms of compact groups*, Ergodic Theory Dynam. Systems **9** (1989), 691–735. MR 1036904

[18] L. Kronecker, *Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten*, J. Reine Angew. Math. **53** (1857), 173–175.

[19] W. Lawton, *The structure of compact connected groups which admit an expansive automorphism*, Recent advances in topological dynamics (A. Beck, ed.), Lecture Notes in Mathematics, vol. 318, Springer, Berlin, 1973, pp. 182–196. MR 0391051

[20] D. A. Lind, *The structure of skew products with ergodic group automorphisms*, Israel J. Math. **28** (1977), 205–248. MR 0460593

[21] G. Miles and R. K. Thomas, *The breakdown of automorphisms of compact topological groups*, Studies in probability and ergodic theory (G.-C. Rota, ed.), Advances in Mathematics, Supplementary Studies, vol. 2, Academic Press, New York, 1978, pp. 207–218. MR 0517262

[22] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955. MR 0073104

[23] P. R. Müller-Römer, *Contracting extensions and contractible groups*, Bull. Amer. Math. Soc. (N.S.) **79** (1973), 1264–1269. MR 0328475

[24] P. R. Müller-Römer, *Kontrahierende Erweiterungen und kontrahierbare Gruppen*, J. Reine. Angew. Math. **283/284** (1976), 238–264. MR 0407194

[25] A. L. Paterson, *Amenability*, AMS, Providence, RI, 1988. MR 0961261

[26] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser, Basel, 1995. MR 1345152

[27] E. Siebert, *Contractive automorphisms of locally compact groups*, Math. Z. **191** (1986), 73–90. MR 0812604

[28] J. S. P. Wang, *The Mauter phenomenon for p-adic Lie groups*, Math. Z. **185** (1984), 403–412. MR 0731685

[29] S. A. Yuzvinskii, *Computing the entropy of a group endomorphism*, Siberian Math. J. **8** (1967), 172–178. MR 0214726

Wojciech Jaworski, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6

E-mail address: wjaworsk@math.carleton.ca