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ERRATA FOR *SYZYGIES OF SEMI-REGULAR SEQUENCES*

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There are some problems in [3] that lead to unnecessary confusion. The main problems include a simple minded (but easy to fix) error in the proof of Theorem 3.6, our failure to expose the main point in that proof, notational problems in the proof of Theorem 4.4 and a gap in that proof. Fortunately, these theorems are true, but the paper is not up to our standards. We accept responsibility for our lapse and regret the confusion that our readers have suffered. We are grateful to C. Diem and J. Shan for bringing some of these problems to our attention and to P. Roberts for telling us the lemma that we give below.

We address the problems in the order in which they appear. In the Introduction, we give an incorrect reference for Stanley's theorem that $x_1^{d_1}, \dots, x_n^{d_n}$, $(x_1 + \cdots + x_n)^{d_{n+1}}$ is a semiregular sequence in characteristic 0. The most common reference for this theorem is Stanley's [5], and we have given this reference elsewhere. But this reference is also incorrect. Indeed, although Stanley discovered this theorem, he never published it. The first appearance of this theorem in print is on page 367 of Iarrobino's [2], where Iarrobino gives credit to Stanley.

We now turn to the proof of Theorem 3.6. In the sixth paragraph, we show that the multiplication by f map from $(S/\tilde{I})_{j-d}$ to $(S/\tilde{I})_j$ is injective for all $j \le \rho + \varepsilon$. While this is correct, we start the seventh paragraph by asserting that it follows that $(\tilde{I} : f)_j = \tilde{I}_j$ for $j \le \rho + \varepsilon$. This is false. What is true is that $(\tilde{I} : f)_j = \tilde{I}_j$ for $j \le \rho + \varepsilon - d$. We are surprised that we published this error. Correcting the error leads to two changes in the eighth paragraph. First, $G_\bullet^{(\rho+\varepsilon-1-d)} \cong H_\bullet^{(\rho+\varepsilon-1-d)}$ replaces $G_\bullet^{(\rho+\varepsilon-1)} \cong H_\bullet^{(\rho+\varepsilon-1)}$. Second, $H_\bullet^{(\rho+\varepsilon-2)}$ must be removed from the displayed chain of isomorphisms.

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With these changes, the rest of the proof is correct. But it is difficult to follow, so we add some more details here. Note first that if $C_\bullet$ is a complex satisfying the hypothesis of Definition 3.3, then $C[a](b)^{(\tau)} = C^{(\tau-a+b)}[a](b)$, where $[a]$ indicates a shift in homological degree and (b) indicates a shift in the grading of each module. Then for the mapping cone $M = G \oplus H(-d)[-1]$ we have that

$$\begin{aligned} M^{(\rho+\varepsilon-2)} &\cong \big(G \oplus H(-d)[-1]\big)^{(\rho+\varepsilon-2)} \cong G^{(\rho+\varepsilon-2)} \oplus H^{(\rho+\varepsilon-1-d)}(-d)[-1] \\ &\cong G^{(\rho+\varepsilon-2)} \oplus G^{(\rho+\varepsilon-1-d)}(-d)[-1] \\ &\cong L^{(\rho+\varepsilon-2)} \oplus L^{(\rho+\varepsilon-1-d)}(-d)[-1] \\ &\cong \big(L \oplus L(-d)[-1]\big)^{(\rho+\varepsilon-2)} \cong K^{(\rho+\varepsilon-2)}. \end{aligned}$$

These isomorphisms are isomorphisms of complexes. Indeed, since $d > 0$ we have a multiplication by f map $G^{(\rho+\varepsilon-1-d)}(-d) \to G^{(\rho+\varepsilon-2)}$ and this map is clearly compatible with the multiplication by f map $S/(\tilde{I} : f)(-d) \to S/\tilde{I}$. This induces a compatible map $H^{(\rho+\varepsilon-1-d)}(-d) \to G^{(\rho+\varepsilon-2)}$ through the isomorphism of $H^{(\rho+\varepsilon-1-d)}$ with $G^{(\rho+\varepsilon-1-d)}$. Thus, $H^{(\rho+\varepsilon-1-d)}(-d) \to G^{(\rho+\varepsilon-2)}$ is a partial lifting of the map $S/(\tilde{I} : f)(-d) \to S/\tilde{I}$ to a map on their minimal free resolutions $H(-d) \to G$. Since the terms of $H^{(\rho+\varepsilon-1-d)}(-d)$ and $G^{(\rho+\varepsilon-2)}$ are summands of the terms of $H(-d)$ and G generated by the basis elements of degree below a certain bound, we may choose the map $H(-d) \to G$ to be compatible with the map on the subcomplexes. With this lifting, the isomorphisms above are maps of complexes.

The reader may also consult [1], where Diem proves a more general form of Theorem 3.6. While our argument also proves Diem's theorem, we were unaware of the more general statement until Diem explained it to us.

We now turn to Section 4. In the first paragraph, the inequality "$\leq \rho - 1$" should be "$\leq \rho$". This puts the introduction of the section in agreement with the statement of Theorem 4.4.

In the proof of Theorem 4.4, every instance of the subscript ρ on a module (Γ_ρ, Ω_ρ, etc.) should instead be the subscript (ρ) ($\Gamma_{(\rho)}$, $\Omega_{(\rho)}$, etc.) indicating the submodule generated in degrees ρ and less, as in Definition 3.1, rather than the graded component of degree ρ.

In the end of the proof, we have produced h such that $hf_i \in (f_1, \dots, f_{i-1})$ where hf_i has degree $a \leq \rho$ and we must argue that this implies that $h \in (f_1, \dots, f_{i-1})$. We argue that we have already shown that all syzygies in that degree are Koszul syzygies. But, what we have shown is that all syzygies are generated by Koszul syzygies in $f_1, \dots, f_r$, while what we require is that the syzygy $h_1 f_1 + \cdots + h_{i-1} f_{i-1} + hf_i = 0$ is generated by Koszul syzygies in just $f_1, \dots, f_i$.

This is not completely obvious, but it is true as the following lemma shows.

LEMMA ([4]). *Let $f_1, \dots, f_r \in S = k[x_1, \dots, x_n]$ be homogenous polynomials of positive degree, q be a natural number less than r, K be the Koszul complex on $f_1, \dots, f_r$ and L be the Koszul complex on $f_1, \dots, f_q$. For natural numbers D and i, if $H_i(K)_j = 0$ for all $j \le D$, then $H_i(L)_j = 0$ for all $j \le D$.*

Proof. It suffices to treat the case of $r = q+1$, after which the general case follows by induction on $r - q$. In the case of $r = q+1$, K is given by the mapping cone for $L(-d) \to L$, where the map is given by multiplication by f, which has degree $d > 0$. If $H_i(L) = 0$, then we are done. Otherwise, let e be the lowest degree for which $H_i(L)_e \neq 0$. Then the long exact sequence for the mapping cone gives that

$$0 = H_i\big(L(-d)\big)_e \to H_i(L)_e \to H_i(K)_e.$$

So, $H_i(K)_e \neq 0$ and thus $e > D$. □

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