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MIXING MULTILINEAR OPERATORS

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Abstract. As a natural extension of mixing linear operators we introduce the notion of mixing multilinear operators. We prove composition results for mixing multilinear operators extending those from the linear case.

Introduction and notation

The theory of operator ideals, as it was introduced by A. Pietsch in the linear case, is well established, as the reader can see in the excellent monographs: [5], [6], [11], [16]. In [12], A. Pietsch sketched an n-linear approach to the theory of absolutely summing operators and since then a large number of papers has followed this line, for example, [1], [2], [3], [4], [7], [8], [10], [13], [14], [15], where there are proven some extensions of the linear case to the multilinear one.

In this paper, as a natural extension of mixing linear operators, we introduce the notion of mixing multilinear operators. As far as we know, this is the first attempt in this regard. This extension was suggested by the results in our paper [14], see also Acknowledgement 1. We prove composition results for mixing multilinear operators extending those from the linear case, see [11, Chapter IV].

We fix some notations and notions.

Given $0 < p < \infty$, a Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, for a finite system $(x_i)_{1 \le i \le n} \subset X$ we define

$$l_p(x_i \mid 1 \le i \le n) = \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \quad \text{and}$$

$$w_p(x_i \mid 1 \le i \le n) = \sup_{\|x^*\| \le 1} \left(\sum_{i=1}^{n} \left| x^*(x_i) \right|^p \right)^{\frac{1}{p}}.$$

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If X is a Banach space, n a natural number, $1 \le p < \infty$, we define $w_p^n(X)$ as X^n on which we consider the norm defined by $w_p(x_i \mid 1 \le i \le n)$.

If n is a natural number, $0 < p < \infty$, then $l_p^n = (\mathbb{K}^n, \| \cdot \|_p)$, where $\|(\alpha_1, \dots, \alpha_n)\|_p = (\sum_{i=1}^n |\alpha_i|^p)^{\frac{1}{p}}$. If $a = (a_i)_{1 \le i \le n}$, $b = (b_i)_{1 \le i \le n}$ are two scalar sequences, by ab we denote their pointwise multiplication that is, $ab = (a_i b_i)_{1 \le i \le n}$.

Let X, Y be Banach spaces and $1 \le p < \infty$. A bounded linear operator $T : X \to Y$ is p-summing, if there exists a constant $C \ge 0$ such that for every $x_1, \dots, x_n \in X$ the following relation holds

$$\left(\sum_{i=1}^n \big\| T(x_i) \big\|^p \right)^{\frac{1}{p}} \le C w_p(x_i \mid 1 \le i \le n)$$

and the p-summing norm of T is $\pi_p(T) = \min\{C \mid C \text{ as above}\}$. We denote by $\Pi_p(X, Y)$ the class of p-summing operators, see [5], [6], [11], [16].

Besides the class of p-summing operators, in the linear case, there is the class of all (q, p)-mixing operators. The class of all (q, p)-mixing operators was first introduced by A. Pietsch in his monograph [11, Chapter IV, 20.4] and, as he said, this notion was implicitly used in Maurey's paper [9]. We recall now Pietsch's approach to (q, p)-mixing operators, see [11, Chapter IV, 20.4].

Let $1 \le p < q < \infty$ and define r by $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let X be a Banach space. For a finite system $(x_i)_{1 \le i \le n} \subset X$, we define

$$\mathfrak{m}_{q,p}(x_i \mid 1 \le i \le n) = \inf\big\{ \|\alpha\|_r w_q\big(x_i^0 \mid 1 \le i \le n\big) \big\},$$

where the infimum is taken over all systems $\alpha = (\alpha_i)_{1 \le i \le n} \subset \mathbb{K}$, $(x_i^0)_{1 \le i \le n} \subset X$ such that $x_i = \alpha_i x_i^0$ for each $1 \le i \le n$.

Let $1 \le p < q < \infty$, X, Y be Banach spaces. A bounded linear operator $U : X \to Y$ is called (q, p)-mixing if there exists $C \ge 0$ such that for all finite systems $(x_i)_{1 \le i \le n} \subset X$ we have

$$\mathfrak{m}_{q,p}\big(U(x_i) \mid 1 \le i \le n\big) \le C w_p(x_i \mid 1 \le i \le n)$$

and the (q, p)-mixing norm of U is $M_{q,p}(U) = \inf\{C \mid C \text{ as above}\}$.

We denote by $\mathcal{M}_{q,p}(X, Y)$ the class of all (q, p)-mixing operators from X into Y. When we say that an operator is (q, p)-mixing we always understand that $1 \le p < q < \infty$ and r is defined by $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

The following basic results can be found in [11, Chapter IV, 20.4]

(PI) $$\Pi_q \circ \mathcal{M}_{q,p} \subset \Pi_p;$$

(PII) $$\mathcal{M}_{q,p} = \Pi_q^{-1} \circ \Pi_p;$$

(PIII) $$\Pi_r \subset \mathcal{M}_{q,p};$$

(PIV) $$\mathcal{M}_{b,q} \circ \mathcal{M}_{q,p} \subset \mathcal{M}_{b,p} \quad \text{whenever } 1 \le p < q < b.$$

We mention that regarding to the formula (PI), A. Pietsch in [11, Chapter IV, 20.2.1, page 285] writes: "This formula was the starting-point of the theory of (q,p)-mixing operators". More interesting details on the (q,p)-mixing operators the reader can find also in A. Defant and K. Floret [5].

A special importance has

$$\mathrm{Space}(\mathcal{M}_{q,p}) = \{X \mid I_X \in \mathcal{M}_{q,p}(X,X)\},$$

where $I_X : X \to X$ is the identity operator that is, $I_X(x) = x$.

We observe that from (PII), X is a (q,p)-mixing space that is, $X \in \mathrm{Space}(\mathcal{M}_{q,p})$ (and $M_{q,p}(X) = M_{q,p}(I_X)$) if and only if for all Banach spaces Y we have the equality $\Pi_q(X,Y) = \Pi_p(X,Y)$.

For $1 \le p \le \infty$, we denote by p^* the conjugate of p that is, $\frac{1}{p} + \frac{1}{p^*} = 1$.

We recall some possible extensions of the concept of p-summing operator in the multilinear case (see [1], [2], [3], [4], [7], [8], [10], [12], [13], [14], [15]).

We introduce first two useful functions.

Definition 1. Let k be a natural number. We define $v_k : [1,\infty)^k \to [\frac{1}{k},\infty)$ by

$$\frac{1}{v_k(p_1,\dots,p_k)} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$$

and $d_k : A_k \to (1,\infty)$ by

$$\frac{1}{[d_k(p_1,\dots,p_k)]^*} = \frac{1}{p_1^*} + \cdots + \frac{1}{p_k^*},$$

where

$$A_k = \left\{(p_1,\dots,p_k) \in (1,\infty)^k \;\middle|\; \frac{1}{p_1} + \cdots + \frac{1}{p_k} > k-1\right\}.$$

Definition 2. Let $p_1,\dots,p_k \in [1,\infty)$ and $t \in (0,\infty)$ be such that $v_k(p_1,\dots,p_k) \le t$. A bounded k-linear operator $U : X_1 \times \cdots \times X_k \to Y$ is called $(t;p_1,\dots,p_k)$-summing if and only if there exists $C \ge 0$ such that for each $(x_i^j)_{1\le i\le n} \subset X_j$ $(1 \le j \le k)$ the following hold

$$\left(\sum_{i=1}^{n} \left\|U\left(x_i^1,\dots,x_i^k\right)\right\|^t\right)^{\frac{1}{t}} \le C w_{p_1}\left(x_i^1 \mid 1 \le i \le n\right) \cdots w_{p_k}\left(x_i^k \mid 1 \le i \le n\right)$$

and $\pi^k_{t;p_1,\dots,p_k}(U) = \inf\{C \mid C \text{ as above}\}$.

We denote by $\Pi^k_{t;p_1,\dots,p_k}(X_1,\dots,X_k;Y)$ the class of all $(t;p_1,\dots,p_k)$-summing operators from $X_1 \times \cdots \times X_k$ into Y on which $\pi^k_{t;p_1,\dots,p_k}$ is a norm if $t \ge 1$ (t-norm, if $t < 1$). In case when $p_1 = \cdots = p_k = p$, we write simply $\Pi^k_{t;p}$ instead of $\Pi^k_{t;p,\dots,p}$. Also we write Π^k_p instead of $\Pi^k_{p;p,\dots,p}$.

The situation when $v_k(p_1,\dots,p_k) = t$ is an important particular case of this general definition.

DEFINITION 3. Let $p_1, \dots, p_k \in [1, \infty)$. A bounded k-linear operator $U : X_1 \times \cdots \times X_k \to Y$ is called $(p_1, \dots, p_k)$-dominated if and only if there exists $C \geq 0$ such that for each $(x_i^j)_{1 \leq i \leq n} \subset X_j$ $(1 \leq j \leq k)$ the following hold

$$\left(\sum_{i=1}^{n} \left\| U(x_i^1, \dots, x_i^k) \right\|^{v_k(p_1, \dots, p_k)} \right)^{\frac{1}{v_k(p_1, \dots, p_k)}} \leq C w_{p_1}\left(x_i^1 \mid 1 \leq i \leq n\right) \cdots w_{p_k}\left(x_i^k \mid 1 \leq i \leq n\right)$$

and $\Delta_{p_1, \dots, p_k}(U) = \inf\{C \mid C \text{ as above}\}$.

We denote by $\Delta_{p_1, \dots, p_k}(X_1, \dots, X_k; Y)$ the class of all $(p_1, \dots, p_k)$-dominated operators from $X_1 \times \cdots \times X_k$ into Y on which $\Delta_{p_1, \dots, p_k}$ is a norm if $v_k(p_1, \dots, p_k) \geq 1$ ($v_k(p_1, \dots, p_k)$-norm, if $v_k(p_1, \dots, p_k) < 1$). In case when $p_1 = \cdots = p_k = p$, we write simply Δ_p^k instead of $\Delta_{p, \dots, p}$.

We mention that the class of $(p_1, \dots, p_k)$-dominated operators is characterized by a Grothendieck–Pietsch type domination theorem, see [8], [10].

The results

In the definition of an ideal of multilinear (k-linear) bounded operators, see [7], [12], appear two natural kind of compositions

$$X_1 \times \cdots \times X_k \overset{(A_1, \dots, A_k)}{\to} Y_1 \times \cdots \times Y_k \overset{T}{\to} Z; \qquad Y_1 \times \cdots \times Y_k \overset{T}{\to} Y \overset{S}{\to} Z$$

which can be written under the form $\mathcal{L}_k \circ \underbrace{(\mathcal{L}_1, \dots, \mathcal{L}_1)}_{k\text{-times}} \subset \mathcal{L}_k$; $\mathcal{L}_1 \circ \mathcal{L}_k \subset \mathcal{L}_k$.

The first kind of composition is used in the so called "Factorization method", which is the one of the most natural way to construct (λ-) Banach ideals of bounded multilinear operators, see [3, page 75], [4, Definition 3.1]. We recall this general definition.

DEFINITION 4. Let k be a natural number and $(\mathcal{J}_1, \|\cdot\|_{\mathcal{J}_1}), \dots, (\mathcal{J}_k, \|\cdot\|_{\mathcal{J}_k})$ Banach ideals of bounded linear operators.

We say that $U \in \mathcal{L}_k(X_1, \dots, X_k; Y)$ belongs to $\mathcal{L}_k \circ (\mathcal{J}_1, \dots, \mathcal{J}_k)(X_1, \dots, X_k; Y)$ if and only if there exists Banach spaces $Y_1, \dots, Y_k$ and $A_1 \in \mathcal{J}_1(X_1, Y_1), \dots, A_k \in \mathcal{J}_k(X_k, Y_k)$, $T \in \mathcal{L}_k(Y_1, \dots, Y_k; Y)$ such that $U = T \circ (A_1, \dots, A_k)$. In this case, we define

$$\|U\|_{\mathcal{L}_k \circ (\mathcal{J}_1, \dots, \mathcal{J}_k)} = \inf\left\{ \|A_1\|_{\mathcal{J}_1} \cdots \|A_k\|_{\mathcal{J}_k} \|T\| \right\},$$

where the infimum is taken over all factorizations of U as above.

It can be proved, see [4, Proposition 3.1], that $(\mathcal{L}_k \circ (\mathcal{J}_1, \dots, \mathcal{J}_k), \|\cdot\|_{\mathcal{L}_k \circ (\mathcal{J}_1, \dots, \mathcal{J}_k)})$ is a $\frac{1}{k}$-Banach ideal of bounded k-linear operators. We state now a particular case of this general definition.

DEFINITION 5. Let k be a natural number, $1 \le p_j < q_j < \infty$, $1 < r_j < \infty$ such that $\frac{1}{p_j} = \frac{1}{q_j} + \frac{1}{r_j}$ for each $1 \le j \le k$.

We say that $U \in \mathcal{L}_k(X_1, \dots, X_k; Y)$ belongs to $\mathcal{L}_k \circ (\mathcal{M}_{q_1,p_1}, \dots, \mathcal{M}_{q_k,p_k})(X_1, \dots, X_k; Y)$ if and only if there exists Banach spaces $Y_1, \dots, Y_k$ and $A_1 \in \mathcal{M}_{q_1,p_1}(X_1, Y_1), \dots, A_k \in \mathcal{M}_{q_k,p_k}(X_k, Y_k)$, $T \in \mathcal{L}_k(Y_1, \dots, Y_k; Y)$ such that $U = T \circ (A_1, \dots, A_k)$. In this case, we define

$$\|U\|_{\mathcal{L}_k \circ (\mathcal{M}_{q_1,p_1}, \dots, \mathcal{M}_{q_k,p_k})} = \inf\{M_{q_1,p_1}(A_1) \cdots M_{q_k,p_k}(A_k)\|T\|\},$$

where the infimum is taken over all factorizations of U as above.

The following result is natural.

THEOREM 6. *Let k be a natural number, $1 \le p_j < q_j < \infty$, $1 < r_j < \infty$ such that $\frac{1}{p_j} = \frac{1}{q_j} + \frac{1}{r_j}$ for each $1 \le j \le k$. Then*

$$\Delta_{r_1,\dots,r_k} \subset \mathcal{L}_k \circ (\mathcal{M}_{q_1,p_1}, \dots, \mathcal{M}_{q_k,p_k}),$$
$$\|\cdot\|_{\mathcal{L}_k \circ (\mathcal{M}_{q_1,p_1}, \dots, \mathcal{M}_{q_k,p_k})} \le \Delta_{r_1,\dots,r_k}(\cdot).$$

Proof. Since by Pietsch's basic formula, (PIII), $\Pi_r \subset \mathcal{M}_{q,p}$ and $M_{q,p}(\cdot) \le \pi_r(\cdot)$ whenever $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, we get

$$\mathcal{L}_k \circ (\Pi_{r_1}, \dots, \Pi_{r_k}) \subset \mathcal{L}_k \circ (\mathcal{M}_{q_1,p_1}, \dots, \mathcal{M}_{q_k,p_k}),$$
$$\|\cdot\|_{\mathcal{L}_k \circ (\mathcal{M}_{q_1,p_1}, \dots, \mathcal{M}_{q_k,p_k})}(\cdot) \le \|\cdot\|_{\mathcal{L}_k \circ (\Pi_{r_1}, \dots, \Pi_{r_k})}.$$

Further, if we use the well-known characterization of $(r_1, \dots, r_k)$-dominated operators, see [8], [10],

$$\Delta_{r_1,\dots,r_k} = \mathcal{L}_k \circ (\Pi_{r_1}, \dots, \Pi_{r_k}), \qquad \Delta_{r_1,\dots,r_k}(\cdot) = \|\cdot\|_{\mathcal{L}_k \circ (\Pi_{r_1}, \dots, \Pi_{r_k})}(\cdot)$$

we get the statement. □

The following definition was suggested by Corollary 5 in [14], see also Acknowledgement 1.

DEFINITION 7. Let k be a natural number, $1 < q_1, \dots, q_k < \infty$ such that $\frac{1}{q_1} + \cdots + \frac{1}{q_k} > k - 1$ and $1 \le p_j < q_j < \infty$, $1 < r_j < \infty$ such that $\frac{1}{p_j} = \frac{1}{q_j} + \frac{1}{r_j}$ for each $1 \le j \le k$. Let $U \in \mathcal{L}_k(X_1, \dots, X_k; Y)$.

For each $(x_i^1)_{1 \le i \le n} \subset X_1, \dots, (x_i^k)_{1 \le i \le n} \subset X_k$ we define

$$\begin{aligned}&\mathfrak{m}^k_{(q_1,p_1),\dots,(q_k,p_k)}\big(U(x_i^1, \dots, x_i^k) \mid 1 \le i \le n\big)\\ &\quad = \inf\big\{\|\alpha_1\|_{r_1} \cdots \|\alpha_k\|_{r_k} w_{d_k(q_1,\dots,q_k)}(y_i \mid 1 \le i \le n)\big\},\end{aligned}$$

where the infimum is taken over all $\alpha_1 = (\alpha_i^1)_{1 \le i \le n} \subset \mathbb{K}, \dots, \alpha_k = (\alpha_i^k)_{1 \le i \le n} \subset \mathbb{K}$, $(y_i)_{1 \le i \le n} \subset Y$ such that

$$U(x_i^1, \dots, x_i^k) = \alpha_i^1 \cdots \alpha_i^k y_i \quad \text{for each } 1 \le i \le n.$$

We say that U is $((q_1,p_1),\dots,(q_k,p_k))$-mixing if and only if there exists $C\geq 0$ such that for each $(x_i^1)_{1\leq i\leq n}\subset X_1,\dots,(x_i^k)_{1\leq i\leq n}\subset X_k$ the following relation hold

$$\begin{aligned}&\mathfrak{m}^k_{(q_1,p_1),\dots,(q_k,p_k)}\big(U\big(x_i^1,\dots,x_i^k\big)\mid 1\leq i\leq n\big)\\&\quad\leq Cw_{p_1}\big(x_i^1\mid 1\leq i\leq n\big)\cdots w_{p_k}\big(x_i^k\mid 1\leq i\leq n\big)\end{aligned}$$

and $M^k_{(q_1,p_1),\dots,(q_k,p_k)}(U)=\inf\{C\geq 0\mid C \text{ as above}\}$.

We denote by $\mathcal{M}^k_{(q_1,p_1),\dots,(q_k,p_k)}(X_1,\dots,X_k;Y)$ the class of all $((q_1,p_1),\dots,(q_k,p_k))$-mixing operators $U:X_1\times\cdots\times X_k\to Y$.

Observe that in case $k=1$, we get the class of linear mixing operators. Unfortunately, we have no proof for the fact that the class $(\mathcal{M}^k_{(q_1,p_1),\dots,(q_k,p_k)}, M^k_{(q_1,p_1),\dots,(q_k,p_k)})$ is a Banach ideal (or λ-Banach ideal, for some $0<\lambda<1$) of bounded k-linear operators. However, we do not need in our paper of this result.

Since Definition 7 depends on the values of the function d_k we give two situations in which the values of the function d_k is reasonably simple, see [14, Observation 6, proof of Theorem 4].

EXAMPLE 8. (i) Given a natural number k, $1<q<k^*$, we have $d_k(q,\dots,q)=(\frac{q^*}{k})^*$.

(ii) Given a natural number k, $1<q<\infty$, we have $d_k((kq^*)^*,\dots,(kq^*)^*)=q$.

From Definition 7 and Example 8(i), we get

DEFINITION 9. Let k be a natural number, $1\leq p<q<k^*$, $k<r<\infty$ such that $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$. Let $U\in\mathcal{L}_k(X_1,\dots,X_k;Y)$.

For each $(x_i^1)_{1\leq i\leq n}\subset X_1,\dots,(x_i^k)_{1\leq i\leq n}\subset X_k$ we define

$$\mathfrak{m}^k_{(q,p)}\big(U\big(x_i^1,\dots,x_i^k\big)\mid 1\leq i\leq n\big)=\inf\big\{\|\alpha_1\|_r\cdots\|\alpha_k\|_r w_{(\frac{q^*}{k})^*}(y_i\mid 1\leq i\leq n)\big\},$$

where the infimum is taken over all $\alpha_1=(\alpha_i^1)_{1\leq i\leq n}\subset\mathbb{K},\dots,\alpha_k=(\alpha_i^k)_{1\leq i\leq n}\subset\mathbb{K}$, $(y_i)_{1\leq i\leq n}\subset Y$ such that

$$U\big(x_i^1,\dots,x_i^k\big)=\alpha_i^1\cdots\alpha_i^k y_i\quad\text{for each } 1\leq i\leq n.$$

We say that U is (q,p)-mixing if and only if there exists $C\geq 0$ such that for each $(x_i^1)_{1\leq i\leq n}\subset X_1,\dots,(x_i^k)_{1\leq i\leq n}\subset X_k$ the following relation hold

$$\mathfrak{m}^k_{(q,p)}\big(U\big(x_i^1,\dots,x_i^k\big)\mid 1\leq i\leq n\big)\leq Cw_p\big(x_i^1\mid 1\leq i\leq n\big)\cdots w_p\big(x_i^k\mid 1\leq i\leq n\big)$$

and $M^k_{(q,p)}(U)=\inf\{C\geq 0\mid C \text{ as above}\}$.

We denote by $\mathcal{M}^k_{(q,p)}(X_1,\dots,X_k;Y)$ the class of all (q,p)-mixing operators $U:X_1\times\cdots\times X_k\to Y$.

If in Definition 9, we take $k=2$, which forces $1\leq p<q<2$, we get the following possible concept for mixing bounded bilinear operators.

DEFINITION 10. Let $1 \le p < q < 2 < r < \infty$ be such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Let $U : X \times Y \to Z$ be a bounded bilinear operator. For each $(x_i)_{1\le i\le n} \subset X$, $(y_i)_{1\le i\le n} \subset Y$ we define

$$\mathfrak{m}^2_{(q,p)}\big(U(x_i, y_i) \mid 1 \le i \le n\big) = \inf\big\{\|\alpha\|_r\|\beta\|_r w_{(\frac{q^*}{2})^*}(z_i \mid 1 \le i \le n)\big\},$$

where the infimum is taken over all $\alpha = (\alpha_i)_{1\le i\le n} \subset \mathbb{K}$, $\beta = (\beta_i)_{1\le i\le n} \subset \mathbb{K}$, $(z_i)_{1\le i\le n} \subset Z$ such that

$$U(x_i, y_i) = \alpha_i\beta_i z_i \quad \text{for each } 1 \le i \le n.$$

We say that U is (q,p)-mixing if and only if there exists $C \ge 0$ such that for each $(x_i)_{1\le i\le n} \subset X$, $(y_i)_{1\le i\le n} \subset Y$ the following relation hold

$$\mathfrak{m}^2_{(q,p)}\big(U(x_i, y_i) \mid 1 \le i \le n\big) \le C w_p(x_i \mid 1 \le i \le n) w_p(y_i \mid 1 \le i \le n)$$

and $M^2_{(q,p)}(U) = \inf\{C \ge 0 \mid C \text{ as above}\}$.

From Definition 7 and Example 8(ii), we get the following.

DEFINITION 11. Let k be a natural number, $1 < q < \infty$, $1 \le p < (kq^*)^*$ i.e. $k < \frac{p^*}{q^*}$ and $1 < r < \infty$ such that $\frac{1}{p} = \frac{1}{(kq^*)^*} + \frac{1}{r}$. Let $U \in \mathcal{L}_k(X_1, \ldots, X_k; Y)$.

For each $(x^1_i)_{1\le i\le n} \subset X_1, \ldots, (x^k_i)_{1\le i\le n} \subset X_k$ we define

$$\mathfrak{m}^k_{(kq^*)^*,p}\big(U\big(x^1_i, \ldots, x^k_i\big) \mid 1 \le i \le n\big) = \inf\big\{\|\alpha_1\|_r \cdots \|\alpha_k\|_r w_q(y_i \mid 1 \le i \le n)\big\},$$

where the infimum is taken over all $\alpha_1 = (\alpha^1_i)_{1\le i\le n} \subset \mathbb{K}, \ldots, \alpha_k = (\alpha^k_i)_{1\le i\le n} \subset \mathbb{K}$, $(y_i)_{1\le i\le n} \subset Y$ such that

$$U\big(x^1_i, \ldots, x^k_i\big) = \alpha^1_i \cdots \alpha^k_i y_i \quad \text{for each } 1 \le i \le n.$$

We say that U is $((kq^*)^*, p)$-mixing if and only if there exists $C \ge 0$ such that for each $(x^1_i)_{1\le i\le n} \subset X_1, \ldots, (x^k_i)_{1\le i\le n} \subset X_k$ the following relation hold

$$\mathfrak{m}^k_{(kq^*)^*,p}\big(U\big(x^1_i, \ldots, x^k_i\big) \mid 1 \le i \le n\big) \le C w_p\big(x^1_i \mid 1 \le i \le n\big) \cdots w_p\big(x^k_i \mid 1 \le i \le n\big)$$

and $M^k_{(kq^*)^*,p}(U) = \inf\{C \ge 0 \mid C \text{ as above}\}$.

We denote by $\mathcal{M}^k_{(kq^*)^*,p}(X_1, \ldots, X_k; Y)$ the class of all $((kq^*)^*, p)$-mixing operators $U : X_1 \times \cdots \times X_k \to Y$.

Taking $r = kq^*$ in Definition 11 we get Definition 12.

DEFINITION 12. Let k be a natural number, $1 < q < \infty$. Let $U \in \mathcal{L}_k(X_1, \ldots, X_k; Y)$. For each $(x^1_i)_{1\le i\le n} \subset X_1, \ldots, (x^k_i)_{1\le i\le n} \subset X_k$ we define

$$\mathfrak{m}^k_{(kq^*)^*,1}\big(U\big(x^1_i, \ldots, x^k_i\big) \mid 1 \le i \le n\big) = \inf\big\{\|\alpha_1\|_{kq^*} \cdots \|\alpha_k\|_{kq^*} w_q(y_i \mid 1 \le i \le n)\big\},$$

where the infimum is taken over all $\alpha_1 = (\alpha_i^1)_{1\le i\le n} \subset \mathbb{K}, \dots, \alpha_k = (\alpha_i^k)_{1\le i\le n} \subset \mathbb{K}$, $(y_i)_{1\le i\le n} \subset Y$ such that

$$U\big(x_i^1, \dots, x_i^k\big) = \alpha_i^1 \cdots \alpha_i^k y_i \quad \text{for each } 1 \le i \le n.$$

We say that U is $((kq^*)^*, 1)$-mixing if and only if there exists $C \ge 0$ such that for each $(x_i^1)_{1\le i\le n} \subset X_1, \dots, (x_i^k)_{1\le i\le n} \subset X_k$ the following relation hold

$$\begin{aligned}&\mathfrak{m}^k_{(kq^*)^*,1}\big(U\big(x_i^1, \dots, x_i^k\big) \mid 1 \le i \le n\big)\\ &\quad\le C w_1\big(x_i^1 \mid 1 \le i \le n\big) \cdots w_1\big(x_i^k \mid 1 \le i \le n\big)\end{aligned}$$

and $M^k_{(kq^*)^*,1}(U) = \inf\{C \ge 0 \mid C \text{ as above}\}$.

We denote by $\mathcal{M}^k_{(kq^*)^*,1}(X_1, \dots, X_k; Y)$ the class of all $((kq^*)^*, 1)$-mixing operators $U : X_1 \times \cdots \times X_k \to Y$.

Our next objective is to prove that between these two classes of bounded multilinear operators, that is, $\mathcal{L}_k \circ (\mathcal{M}_{q_1,p_1}, \dots, \mathcal{M}_{q_k,p_k})$ and $\mathcal{M}^k_{(q_1,p_1),\dots,(q_k,p_k)}$, both seeming to be the natural extensions of linear mixing operators introduced by A. Pietsch, there is a predictable connection, see Theorem 14 below. To do this, we need the following result, perhaps well known.

Proposition 13. *Let k be a natural number, $1 < q_1, \dots, q_k < \infty$ such that $\frac{1}{q_1} + \cdots + \frac{1}{q_k} > k - 1$ and $T : X_1 \times \cdots \times X_k \to Y$ a bounded k-linear operator. Then for each natural number n, the operator*

$$h_T : w^n_{q_1}(X_1) \times \cdots \times w^n_{q_k}(X_k) \to w^n_{d_k(q_1,\dots,q_k)}(Y)$$

defined by

$$h_T\big(\big(x_i^1\big)_{1\le i\le n}, \dots, \big(x_i^k\big)_{1\le i\le n}\big) = \big(T\big(x_i^1, \dots, x_i^k\big)\big)_{1\le i\le n}$$

is bounded k-linear and $\|h_T\| = \|T\|$.

Proof. We have

$$\begin{aligned}&\big\|h_T\big(\big(x_i^1\big)_{1\le i\le n}, \dots, \big(x_i^k\big)_{1\le i\le n}\big)\big\|\\ &\quad= \sup_{\|y^*\|\le 1}\ \sup_{\|\alpha\|_{[d_k(q_1,\dots,q_k)]^*}\le 1} \left|\sum_{i=1}^n \alpha_i y^* T\big(x_i^1, \dots, x_i^k\big)\right|.\end{aligned}$$

Let $\|y^*\| \le 1$ and $\alpha = (\alpha_i)_{1\le i\le n}$ be such that $\|\alpha\|_{[d_k(q_1,\dots,q_k)]^*} \le 1$. From $\frac{1}{[d_k(q_1,\dots,q_k)]^*} = \frac{1}{q_1^*} + \cdots + \frac{1}{q_k^*}$ there exists

$$\begin{aligned}&\beta_1 = \big(\beta_i^1\big)_{1\le i\le n} \quad \text{with } \|\beta_1\|_{q_1^*} \le 1; \qquad \dots;\\ &\beta_k = \big(\beta_i^k\big)_{1\le i\le n} \quad \text{with } \|\beta_k\|_{q_k^*} \le 1\end{aligned}$$

such that $\alpha = \beta_1 \cdots \beta_k$ i.e. $\alpha_i = \beta_i^1 \cdots \beta_i^k$ for each $1 \le i \le n$.

Then, from the Defant–Voigt theorem that is, $L(X_1,\ldots,X_k;\mathbb{K}) = \Pi_1(X_1,\ldots,X_k;\mathbb{K})$, (see [1, Theorem 3.10], [2, Theorem 3]), we get

$$\left|\sum_{i=1}^n \alpha_i y^* T\big(x_i^1,\ldots,x_i^k\big)\right| = \left|\sum_{i=1}^n y^* T\big(\beta_i^1 x_i^1,\ldots,\beta_i^k x_i^k\big)\right|$$
$$\leq \big\|y^*T\big\| w_1\big(\beta_i^1 x_i^1 \mid 1\leq i\leq n\big)\cdots w_1\big(\beta_i^k x_i^k \mid 1\leq i\leq n\big).$$

But, from Holder's inequality

$$w_1\big(\beta_i^j x_i^j \mid 1\leq i\leq n\big) \leq \|\beta_j\|_{q_j^*} w_{q_j}\big(x_i^j \mid 1\leq i\leq n\big) \leq w_{q_j}\big(x_i^j \mid 1\leq i\leq n\big)$$

for each $1\leq j\leq k$, thus

$$\left|\sum_{i=1}^n \alpha_i y^* T\big(x_i^1,\ldots,x_i^k\big)\right| \leq \|T\| w_{q_1}\big(x_i^1 \mid 1\leq i\leq n\big)\cdots w_{q_k}\big(x_i^k \mid 1\leq i\leq n\big).$$

So

$$\big\|h_T\big(\big(x_i^1\big)_{1\leq i\leq n},\ldots,\big(x_i^k\big)_{1\leq i\leq n}\big)\big\|$$
$$\leq \|T\| w_{q_1}\big(x_i^1 \mid 1\leq i\leq n\big)\cdots w_{q_k}\big(x_i^k \mid 1\leq i\leq n\big),$$

that is, $\|h_T\|\leq\|T\|$. The converse inequality is obvious. □

In the following theorem, we prove the promised predictable connection between the two classes of multilinear operators.

THEOREM 14. *Let k be a natural number, $1<q_1,\ldots,q_k<\infty$ such that $\frac{1}{q_1}+\cdots+\frac{1}{q_k}>k-1$, $1\leq p_j<q_j<\infty$, $1<r_j<\infty$ such that $\frac{1}{p_j}=\frac{1}{q_j}+\frac{1}{r_j}$ for each $1\leq j\leq k$. Then*

$$\mathcal{L}_k\circ(\mathcal{M}_{q_1,p_1},\ldots,\mathcal{M}_{q_k,p_k})\subset\mathcal{M}^k_{(q_1,p_1),\ldots,(q_k,p_k)} \quad \textit{and}$$
$$M^k_{(q_1,p_1),\ldots,(q_k,p_k)}(\cdot)\leq\|\cdot\|_{\mathcal{L}_k\circ(\mathcal{M}_{q_1,p_1},\ldots,\mathcal{M}_{q_k,p_k})}.$$

Proof. Let $U\in\mathcal{L}_k\circ(\mathcal{M}_{q_1,p_1},\ldots,\mathcal{M}_{q_k,p_k})(X_1,\ldots,X_k;Y)$ and consider a factorization of U of the form

$$X_1\times\cdots\times X_k \overset{(A_1,\ldots,A_k)}{\to} Y_1\times\cdots\times Y_k \overset{T}{\to} Z,$$

where A_1 is (q_1,p_1)-mixing, …, A_k is (q_k,p_k)-mixing and T is bounded k-linear.

Let $(x_i^j)_{1\leq i\leq n}\subset X_j$ $(1\leq j\leq k)$ and $\varepsilon>0$. Since A_j is (q_j,p_j)-mixing (linear) $(1\leq j\leq k)$, there exists $\alpha_j=(\alpha_i^j)_{1\leq i\leq n}\subset\mathbb{K}$, $(y_i^j)_{1\leq i\leq n}\subset Y_j$ such that

$$A_j\big(x_i^j\big)=\alpha_i^j y_i^j \quad \text{for each } 1\leq i\leq n,$$
$$\|\alpha_j\|_{r_j} w_{q_j}\big(y_i^j \mid 1\leq i\leq n\big)\leq(1+\varepsilon)M_{q_j,p_j}(A_j)w_{p_j}\big(x_i^j \mid 1\leq i\leq n\big).$$

Then

$$\begin{aligned}U\big(x_i^1,\dots,x_i^k\big) &= \big(T\circ(A_1,\dots,A_k)\big)\big(x_i^1,\dots,x_i^k\big)\\ &= T\big(A_1\big(x_i^1\big),\dots,A_k\big(x_i^k\big)\big)\\ &= \alpha_i^1\cdots\alpha_i^k T\big(y_i^1,\dots,y_i^k\big)\\ &= \alpha_i^1\cdots\alpha_i^k z_i \quad \text{for each } 1\le i\le n,\end{aligned}$$

where

$$z_i = T\big(y_i^1,\dots,y_i^k\big) \quad \text{for each } 1\le i\le n.$$

Since $\frac{1}{[d_k(q_1,\dots,q_k)]^*} = \frac{1}{q_1^*}+\cdots+\frac{1}{q_k^*}$ from Proposition 13 we get

$$\begin{aligned}w_{d_k(q_1,\dots,q_k)}(z_i \mid 1\le i\le n) &= w_{d_k(q_1,\dots,q_k)}\big(T\big(y_i^1,\dots,y_i^k\big)\mid 1\le i\le n\big)\\ &\le \|T\| w_{q_1}\big(y_i^1\mid 1\le i\le n\big)\cdots w_{q_k}\big(y_i^k\mid 1\le i\le n\big).\end{aligned}$$

Then

$$\begin{aligned}&\|\alpha_1\|_{r_1}\cdots\|\alpha_k\|_{r_k} w_{d_k(q_1,\dots,q_k)}(z_i\mid 1\le i\le n)\\ &\quad\le \|T\|\|\alpha_1\|_{r_1} w_{q_1}\big(y_i^1\mid 1\le i\le n\big)\cdots\|\alpha_k\|_{r_k} w_{q_k}\big(y_i^k\mid 1\le i\le n\big)\\ &\quad\le (1+\varepsilon)^k M_{q_1,p_1}(A_1)\cdots M_{q_k,p_k}(A_k)\\ &\qquad\times\|T\| w_{p_1}\big(x_i^1\mid 1\le i\le n\big)\cdots w_{p_k}\big(x_i^k\mid 1\le i\le n\big).\end{aligned}$$

We deduce that U is $((q_1,p_1),\dots,(q_k,p_k))$-mixing and

$$M^k_{(q_1,p_1),\dots,(q_k,p_k)}(U)\le(1+\varepsilon)^k M_{q_1,p_1}(A_1)\cdots M_{q_k,p_k}(A_k)\|T\|.$$

Since $\varepsilon>0$ is arbitrary, we obtain

$$M^k_{(q_1,p_1),\dots,(q_k,p_k)}(U)\le M_{q_1,p_1}(A_1)\cdots M_{q_k,p_k}(A_k)\|T\|.$$

Taking the infimum over all factorizations of U as above, we obtain $M^k_{(q_1,p_1),\dots,(q_k,p_k)}(U)\le\|U\|_{\mathcal{L}_k\circ(\mathcal{M}_{q_1,p_1},\dots,\mathcal{M}_{q_k,p_k})}$. □

A combination of Theorem 6 and Theorem 14 gives us Corollary 5 in [14], see also Acknowledgement 1.

COROLLARY 15. (i) *Let k be a natural number, $1<q_1,\dots,q_k<\infty$ such that $\frac{1}{q_1}+\cdots+\frac{1}{q_k}>k-1$, $1\le p_j<q_j<\infty$, $1<r_j<\infty$ such that $\frac{1}{p_j}=\frac{1}{q_j}+\frac{1}{r_j}$ for each $1\le j\le k$. Then*

$$\Delta_{r_1,\dots,r_k}\subset\mathcal{M}^k_{(q_1,p_1),\dots,(q_k,p_k)} \quad \textit{and} \quad M^k_{(q_1,p_1),\dots,(q_k,p_k)}(\cdot)\le\Delta_{r_1,\dots,r_k}(\cdot).$$

(ii) *Let k be a natural number, $1\le p<q<k^*$, $k<r<\infty$ such that $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$. Then*

$$\Delta_r^k\subset\mathcal{M}^k_{(q,p)} \quad \textit{and} \quad M^k_{(q,p)}(\cdot)\le\Delta_r^k(\cdot).$$

(iii) *Let k be a natural number, $1<q<\infty$, $1\le p<(kq^*)^*$ i.e. $k<\frac{p^*}{q^*}$ and $1<r<\infty$ such that $\frac{1}{p}=\frac{1}{(kq^*)^*}+\frac{1}{r}$. Then*

$$\Delta_r^k\subset\mathcal{M}^k_{(kq^*)^*,p}\quad\textit{and}\quad M^k_{(kq^*)^*,p}(\cdot)\le\Delta_r^k(\cdot).$$

(iv) *Let k be a natural number, $1<q<\infty$. Then*

$$\Delta_{kq^*}^k\subset\mathcal{M}^k_{(kq^*)^*,1}\quad\textit{and}\quad M^k_{(kq^*)^*,1}(\cdot)\le\Delta_{kq^*}^k(\cdot).$$

Proof. (i) Follows from Theorem 6 and Theorem 14.
(ii) Follows from (i) and Definition 9; see also Example 8(i).
(iii) Follows from (i) and Definition 11; see also Example 8(ii).
(iv) Is a particular case of (iii); see also Definition 12. ☐

We will need further the following result whose simple proof, based on the definition of the infimum, is omitted.

LEMMA 16. *Let k be a natural number, $1<q_1,\dots,q_k<\infty$ such that $\frac{1}{q_1}+\cdots+\frac{1}{q_k}>k-1$, $1\le p_j<q_j<\infty$, $1<r_j<\infty$ such that $\frac{1}{p_j}=\frac{1}{q_j}+\frac{1}{r_j}$ for each $1\le j\le k$.*

(i) *Let $U\in\mathcal{L}_k(X_1,\dots,X_k;Y)$ and $(x_i^1)_{1\le i\le n}\subset X_1,\dots,(x_i^k)_{1\le i\le n}\subset X_k$. Then for all $\varepsilon>0$, there exists $\alpha_1=(\alpha_i^1)_{1\le i\le n}\subset\mathbb{K},\dots,\alpha_k=(\alpha_i^k)_{1\le i\le n}\subset\mathbb{K}$, $(y_i)_{1\le i\le n}\subset Y$ such that*

$$\begin{aligned}&U\big(x_i^1,\dots,x_i^k\big)=\alpha_i^1\cdots\alpha_i^k y_i\quad\textit{for each } 1\le i\le n,\\&\|\alpha_1\|_{r_1}\cdots\|\alpha_k\|_{r_k}w_{d_k(q_1,\dots,q_k)}(y_i\mid 1\le i\le n)\\&\quad\le(1+\varepsilon)\mathfrak{m}^k_{(q_1,p_1),\dots,(q_k,p_k)}\big(U\big(x_i^1,\dots,x_i^k\big)\mid 1\le i\le n\big).\end{aligned}$$

(ii) *Let $U\in\mathcal{M}^k_{(q_1,p_1),\dots,(q_k,p_k)}(X_1,\dots,X_k;Y)$. Then for all $\varepsilon>0$, all finite systems $(x_i^1)_{1\le i\le n}\subset X_1,\dots,(x_i^k)_{1\le i\le n}\subset X_k$, there exists $\alpha_1=(\alpha_i^1)_{1\le i\le n}\subset\mathbb{K},\dots,\alpha_k=(\alpha_i^k)_{1\le i\le n}\subset\mathbb{K}$, $(y_i)_{1\le i\le n}\subset Y$ such that*

$$\begin{aligned}&U\big(x_i^1,\dots,x_i^k\big)=\alpha_i^1\cdots\alpha_i^k y_i\quad\textit{for each } 1\le i\le n,\\&\|\alpha_1\|_{r_1}\cdots\|\alpha_k\|_{r_k}w_{d_k(q_1,\dots,q_k)}(y_i\mid 1\le i\le n)\\&\quad\le(1+\varepsilon)M^k_{(q_1,p_1),\dots,(q_k,p_k)}(U)w_{p_1}\big(x_i^1\mid 1\le i\le n\big)\cdots w_{p_k}\big(x_i^k\mid 1\le i\le n\big).\end{aligned}$$

In the case when X_1 is (q_1,p_1)-mixing space, …, X_k is (q_k,p_k)-mixing space, we have, obviously,

$$\mathcal{L}_k\circ(\mathcal{M}_{q_1,p_1},\dots,\mathcal{M}_{q_k,p_k})(X_1,\dots,X_k;Y)=\mathcal{L}_k(X_1,\dots,X_k;Y)$$

for each Banach space Y and thus from Theorem 14 and Lemma 16 we get the following corollary.

Corollary 17. *Let k be a natural number, $1 < q_1, \ldots, q_k < \infty$ such that $\frac{1}{q_1} + \cdots + \frac{1}{q_k} > k-1$, $1 \le p_j < q_j < \infty$, $1 < r_j < \infty$ such that $\frac{1}{p_j} = \frac{1}{q_j} + \frac{1}{r_j}$ for each $1 \le j \le k$.*

Let X_1 be (q_1, p_1)-mixing space, ..., X_k be (q_k, p_k)-mixing space, Y a Banach space and $U : X_1 \times \cdots \times X_k \to Y$ a bounded k-linear operator.

Then for all finite systems $(x_i^1)_{1\le i\le n} \subset X_1, \ldots, (x_i^k)_{1\le i\le n} \subset X_k$, all $\varepsilon > 0$, there exists $\alpha_1 = (\alpha_i^1)_{1\le i\le n} \subset \mathbb{K}, \ldots, \alpha_k = (\alpha_i^k)_{1\le i\le n} \subset \mathbb{K}$, $(y_i)_{1\le i\le n} \subset Y$ such that

$$\begin{aligned}
&U\big(x_i^1, \ldots, x_i^k\big) = \alpha_i^1 \cdots \alpha_i^k y_i \quad \text{for each } 1 \le i \le n,\\
&\|\alpha_1\|_{r_1} \cdots \|\alpha_k\|_{r_k} w_{d_k(q_1,\ldots,q_k)}(y_i \mid 1 \le i \le n)\\
&\quad \le (1+\varepsilon) M_{(q_1,p_1)}(X_1) \cdots M_{(q_k,p_k)}(X_k)\\
&\qquad \times \|U\| w_{p_1}\big(x_i^1 \mid 1 \le i \le n\big) \cdots w_{p_k}\big(x_i^k \mid 1 \le i \le n\big).
\end{aligned}$$

In order to state the following corollary, we recall the following coincidence theorems, see [6, Corollary 11.16(a) and (b)], or [16, Corollary 10.18(i) and Corollary 21.5(i)]:

(a) if X has cotype 2, then for all $1 \le p < q \le 2$, all Banach spaces Y we have the equality $\Pi_q(X,Y) = \Pi_p(X,Y)$;
(b) if X has cotype s, with $2 < s < \infty$, then for all $1 \le p < q < s^*$, all Banach spaces Y we have the equality $\Pi_q(X,Y) = \Pi_p(X,Y)$.

In order to unify these coincidence theorems, we introduce a notation. If $2 \le b < \infty$ and $1 \le a < \infty$, we write

$$a \precapprox b^*, \quad \text{if } \begin{cases} a \le 2 & \text{in case } b = 2,\\ a < b^* & \text{in case } b > 2. \end{cases}$$

With this notation the above coincidence theorems can be restated under the form:

Remark 18. Let X be a Banach space of finite cotype, denoted by Cotype(X) and $1 \le p < q \precapprox (\text{Cotype}(X))^*$. Then X is a (q,p)-mixing space, or, equivalent, for all Banach spaces Y we have the equality $\Pi_q(X,Y) = \Pi_p(X,Y)$.

From Remark 18 and Corollary 17, we get the following.

Corollary 19. *Let k be a natural number, $X_1, \ldots, X_k$ Banach spaces of finite cotype, $1 < q_1, \ldots, q_k < \infty$ such that $\frac{1}{q_1} + \cdots + \frac{1}{q_k} > k-1$ and $q_1 \precapprox (\text{Cotype}(X_1))^*, \ldots, q_k \precapprox (\text{Cotype}(X_k))^*$. Let $1 \le p_1 < q_1, \ldots, 1 \le p_k < q_k$ and $1 < r_1, \ldots, r_k < \infty$ be such that $\frac{1}{p_1} = \frac{1}{q_1} + \frac{1}{r_1}, \ldots, \frac{1}{p_k} = \frac{1}{q_k} + \frac{1}{r_k}$.*

Let Y be a Banach space and $U : X_1 \times \cdots \times X_k \to Y$ a bounded k-linear operator. Then for all finite systems $(x_i^1)_{1\le i\le n} \subset X_1, \ldots, (x_i^k)_{1\le i\le n} \subset X_k$, all

$\varepsilon > 0$, *there exists* $\alpha_1 = (\alpha_i^1)_{1\le i\le n} \subset \mathbb{K}, \dots, \alpha_k = (\alpha_i^k)_{1\le i\le n} \subset \mathbb{K}$, $(y_i)_{1\le i\le n} \subset Y$ *such that*

$$\begin{aligned}
&U(x_i^1,\dots,x_i^k) = \alpha_i^1 \cdots \alpha_i^k y_i \quad \textit{for each } 1 \le i \le n,\\
&\|\alpha_1\|_{r_1} \cdots \|\alpha_k\|_{r_k} w_{d_k(q_1,\dots,q_k)}(y_i \mid 1 \le i \le n)\\
&\quad \le (1+\varepsilon) M_{(q_1,p_1)}(X_1) \cdots M_{(q_k,p_k)}(X_k)\\
&\qquad \times \|U\| w_{p_1}(x_i^1 \mid 1 \le i \le n) \cdots w_{p_k}(x_i^k \mid 1 \le i \le n).
\end{aligned}$$

As a concrete illustration of Corollary 19, we get the following "spliting result" in the case of bilinear operators.

Corollary 20. *Suppose that X and Y both have cotype 2 and let $1 \le p < q < 2 < r < \infty$ be such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then for each Banach space Z, each bounded bilinear operator $U : X \times Y \to Z$, each $\varepsilon > 0$ and each $(x_i)_{1\le i\le n} \subset X$, $(y_i)_{1\le i\le n} \subset Y$, there exists $\alpha = (\alpha_i)_{1\le i\le n} \subset \mathbb{K}$, $\beta = (\beta_i)_{1\le i\le n} \subset \mathbb{K}$, $(z_i)_{1\le i\le n} \subset Z$ such that*

$$\begin{aligned}
&U(x_i, y_i) = \alpha_i \beta_i z_i \quad \textit{for each } 1 \le i \le n,\\
&\|\alpha\|_r \|\beta\|_r w_{(\frac{q^*}{2})^*}(z_i \mid 1 \le i \le n)\\
&\quad \le (1+\varepsilon) M_{(q,p)}(X) M_{(q,p)}(Y) \|U\| w_p(x_i \mid 1 \le i \le n) w_p(y_i \mid 1 \le i \le n).
\end{aligned}$$

The following theorem is the first version of a possible multilinear variant of Pietsch's composition formula (PIV).

Theorem 21. *Let k be a natural number, $1 < q_1, \dots, q_k < \infty$ such that $\frac{1}{q_1} + \dots + \frac{1}{q_k} > k-1$, $1 \le b_j < p_j < q_j < \infty$ for each $1 \le j \le k$. Then*

$$\mathcal{M}^k_{(q_1,p_1),\dots,(q_k,p_k)} \circ (\mathcal{M}_{p_1,b_1}, \dots, \mathcal{M}_{p_k,b_k}) \subset \mathcal{M}^k_{(q_1,b_1),\dots,(q_k,b_k)}.$$

Proof. Define $1 < r_j < \infty$ such that $\frac{1}{p_j} = \frac{1}{q_j} + \frac{1}{r_j}$ for each $1 \le j \le k$, $1 < c_j < \infty$ by $\frac{1}{b_j} = \frac{1}{p_j} + \frac{1}{c_j}$ for each $1 \le j \le k$. Let us consider the diagram

$$X_1 \times \dots \times X_k \overset{(A_1,\dots,A_k)}{\to} Y_1 \times \dots \times Y_k \overset{T}{\to} Z,$$

where A_1 is (p_1,b_1)-mixing, …, A_k is (p_k,b_k)-mixing and T is $((q_1,p_1),\dots,(q_k,p_k))$-mixing.

Let $\varepsilon > 0$ and take $(x_i^j)_{1\le i\le n} \subset X_j$ $(1 \le j \le k)$. Since A_j is (p_j,b_j)-mixing (linear) $(1 \le j \le k)$, there exists $\alpha_j = (\alpha_i^j)_{1\le i\le n} \subset \mathbb{K}$, $(y_i^j)_{1\le i\le n} \subset Y_j$ such that

$$\begin{gathered}
A_j(x_i^j) = \alpha_i^j y_i^j \quad \text{for each } 1 \le i \le n,\\
\|\alpha_j\|_{c_j} w_{p_j}(y_i^j \mid 1 \le i \le n) \le (1+\varepsilon) M_{p_j,b_j}(A_j) w_{b_j}(x_i^j \mid 1 \le i \le n).
\end{gathered}$$

Then

$$\begin{aligned}
&(T \circ (A_1,\dots,A_k))(x_i^1,\dots,x_i^k)\\
&\quad = T(A_1(x_i^1),\dots,A_k(x_i^k)) = \alpha_i^1 \cdots \alpha_i^k T(y_i^1,\dots,y_i^k) \quad \text{for each } 1 \le i \le n.
\end{aligned}$$

Since T is $((q_1,p_1),\dots,(q_k,p_k))$-mixing, from Lemma 16(ii) there exists $\lambda_1 = (\lambda_i^1)_{1\le i\le n}\subset\mathbb{K},\dots,\lambda_k=(\lambda_i^k)_{1\le i\le n}\subset\mathbb{K}$, $(z_i)_{1\le i\le n}\subset Z$ such that

$$\begin{aligned}
&T\big(y_i^1,\dots,y_i^k\big)=\lambda_i^1\cdots\lambda_i^k z_i \quad \text{for each } 1\le i\le n,\\
&\|\lambda_1\|_{r_1}\cdots\|\lambda_k\|_{r_k}w_{d_k(q_1,\dots,q_k)}(z_i\mid 1\le i\le n)\\
&\quad\le(1+\varepsilon)M^k_{(q_1,p_1),\dots,(q_k,p_k)}(T)w_{p_1}\big(y_i^1\mid 1\le i\le n\big)\cdots w_{p_k}\big(y_i^k\mid 1\le i\le n\big).
\end{aligned}$$

Then

$$\big(T\circ(A_1,\dots,A_k)\big)\big(x_i^1,\dots,x_i^k\big)=\alpha_i^1\lambda_i^1\cdots\alpha_i^k\lambda_i^k z_i \quad \text{for each } 1\le i\le n.$$

Define also $\frac{1}{c_j}+\frac{1}{r_j}=\frac{1}{s_j}$ for each $1\le j\le k$. Then, from Holder's inequality

$$\|\alpha_j\lambda_j\|_{s_j}\le\|\alpha_j\|_{c_j}\|\lambda_j\|_{r_j}\quad \text{for each } 1\le j\le k.$$

Moreover, we have

$$\begin{aligned}
&\|\alpha_1\lambda_1\|_{s_1}\cdots\|\alpha_k\lambda_k\|_{c_k}w_{d_k(q_1,\dots,q_k)}(z_i\mid 1\le i\le n)\\
&\quad\le\|\alpha_1\|_{c_1}\cdots\|\alpha_k\|_{c_k}\|\lambda_1\|_{r_1}\cdots\|\lambda_k\|_{r_k}w_{d_k(q_1,\dots,q_k)}(z_i\mid 1\le i\le n)\\
&\quad\le(1+\varepsilon)M^k_{(q_1,p_1),\dots,(q_k,p_k)}(T)\\
&\qquad\times\|\alpha_1\|_{c_1}w_{p_1}\big(y_i^1\mid 1\le i\le n\big)\cdots\|\alpha_k\|_{c_k}w_{p_k}\big(y_i^k\mid 1\le i\le n\big)\\
&\quad\le(1+\varepsilon)^{k+1}M^k_{(q_1,p_1),\dots,(q_k,p_k)}(T)\\
&\qquad\times M_{p_1,b_1}(A_1)\cdots M_{p_k,b_k}(A_k)w_{b_1}\big(x_i^1\mid 1\le i\le n\big)\cdots w_{b_k}\big(x_i^k\mid 1\le i\le n\big).
\end{aligned}$$

We deduce that $T\circ(A_1,\dots,A_k)$ is $((q_1,b_1),\dots,(q_k,b_k))$-mixing and

$$\begin{aligned}
&M^k_{(q_1,b_1),\dots,(q_k,b_k)}\big(T\circ(A_1,\dots,A_k)\big)\\
&\quad\le(1+\varepsilon)^{k+1}M_{p_1,b_1}(A_1)\cdots M_{p_k,b_k}(A_k)M^k_{(q_1,p_1),\dots,(q_k,p_k)}(T)
\end{aligned}$$

which conclude the proof, since $\varepsilon>0$ is arbitrary. □

From Theorem 21 and Definitions 9, 11, 12 we get

Corollary 22. (i) *Let k be a natural number, $1\le p<q<k^*$, $k<r<\infty$ such that $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$ and $1\le b_1,\dots,b_k<p$. Then*

$$\mathcal{M}^k_{(q,p)}\circ(\mathcal{M}_{p,b_1},\dots,\mathcal{M}_{p,b_k})\subset\mathcal{M}^k_{(q,b_1),\dots,(q,b_k)}.$$

(ii) *Let k be a natural number, $1<q<\infty$, $1\le p<(kq^*)^*$ that is, $k<\frac{p^*}{q^*}$ and $1<r<\infty$ such that $\frac{1}{p}=\frac{1}{(kq^*)^*}+\frac{1}{r}$ and $1\le b_1,\dots,b_k<p$. Then*

$$\mathcal{M}^k_{((kq^*)^*,p)}\circ(\mathcal{M}_{p,b_1},\dots,\mathcal{M}_{p,b_k})\subset\mathcal{M}^k_{((kq^*)^*,b_1),\dots,((kq^*)^*,b_k)}.$$

In the proof of a second possible version of a multilinear Pietsch multiplication formula, (PIV), we need the following well-known result. For the sake of completeness we include its proof.

Proposition 23. *Let k be a natural number, $1 < p_1, \dots, p_k < \infty$ and n be a natural number. Then for each $\lambda = (\lambda_i)_{1\le i\le n} \in l^n_{v_k(p_1,\dots,p_k)}$ there exists $\lambda^1 = (\lambda^1_i)_{1\le i\le n} \in l^n_{p_1}, \dots, \lambda^k = (\lambda^k_i)_{1\le i\le n} \in l^n_{p_k}$ such that*

$$\lambda = \lambda^1 \cdots \lambda^k \quad \textit{i.e. } \lambda_i = \lambda^1_i \cdots \lambda^k_i \quad \textit{for each } 1 \le i \le n,$$
$$\|\lambda\|_{v_k(p_1,\dots,p_k)} = \left\|\lambda^1\right\|_{p_1} \cdots \left\|\lambda^k\right\|_{p_k}.$$

Proof. Write $v_k = v_k(p_1, \dots, p_k)$ and take

$$\lambda^1_i = \begin{cases} 0, & \text{if } \lambda_i = 0, \\ (\operatorname{sgn} \lambda_i)|\lambda_i|^{\frac{v_k}{p_1}}, & \text{if } \lambda_i \ne 0, \end{cases}$$
$$\lambda^2_i = \begin{cases} 0, & \text{if } \lambda_i = 0, \\ |\lambda_i|^{\frac{v_k}{p_2}}, & \text{if } \lambda_i \ne 0, \end{cases} \qquad \dots,$$
$$\lambda^k_i = \begin{cases} 0, & \text{if } \lambda_i = 0, \\ |\lambda_i|^{\frac{v_k}{p_k}}, & \text{if } \lambda_i \ne 0, \end{cases} \qquad \text{for each } 1 \le i \le n. \qquad \square$$

The following theorem is a second possible version of a multilinear Pietsch multiplication formula (PIV).

Theorem 24. *Let k be a natural number, $1 \le p_j < q_j < b_j < \infty$ for each $1 \le j \le k$ and $\frac{1}{b_1} + \dots + \frac{1}{b_k} > k - 1$. Then*

$$\mathcal{M}_{d_k(b_1,\dots,b_k), d_k(q_1,\dots,q_k)} \circ \mathcal{M}^k_{(q_1,p_1),\dots,(q_k,p_k)} \subset \mathcal{M}^k_{(b_1,p_1),\dots,(b_k,p_k)}.$$

Proof. Define $1 < r_j < \infty$ such that $\frac{1}{p_j} = \frac{1}{q_j} + \frac{1}{r_j}$ for each $1 \le j \le k$, $\frac{1}{q_j} = \frac{1}{b_j} + \frac{1}{c_j}$ and $\frac{1}{p_j} = \frac{1}{b_j} + \frac{1}{s_j}$ for each $1 \le j \le k$ and where $\frac{1}{s_j} = \frac{1}{r_j} + \frac{1}{c_j}$ for each $1 \le j \le k$. Let us consider the diagram

$$X_1 \times \dots \times X_k \xrightarrow{A} Y \xrightarrow{T} Z,$$

where A is $((q_1, p_1), \dots, (q_k, p_k))$-mixing and T is $(d_k(b_1, \dots, b_k), d_k(q_1, \dots, q_k))$-mixing.

Take $(x^j_i)_{1\le i\le n} \subset X_j$ $(1 \le j \le k)$ and $\varepsilon > 0$. From Lemma 16(ii), there exists $\alpha^1 = (\alpha^1_i)_{1\le i\le n} \subset \mathbb{K}, \dots, \alpha^k = (\alpha^k_i)_{1\le i\le n} \subset \mathbb{K}$, $(y_i)_{1\le i\le n} \subset Y$ such that

$$A\big(x^1_i, \dots, x^k_i\big) = \alpha^1_i \cdots \alpha^k_i y_i \quad \text{for each } 1 \le i \le n,$$
$$\left\|\alpha^1\right\|_{r_1} \cdots \left\|\alpha^k\right\|_{r_k} w_{d_k(q_1,\dots,q_k)}(y_i \mid 1 \le i \le n)$$
$$\le (1+\varepsilon) M^k_{(q_1,p_1),\dots,(q_k,p_k)}(A) w_{p_1}\big(x^1_i \mid 1 \le i \le n\big) \cdots w_{p_k}\big(x^k_i \mid 1 \le i \le n\big).$$

Then, from the hypotheses we get

$$\frac{1}{d_k(q_1, \dots, q_k)} = \frac{1}{d_k(b_1, \dots, b_k)} + \frac{1}{v_k(c_1, \dots, c_k)}$$

and, by the fact that T is $(d_k(b_1,\dots,b_k), d_k(q_1,\dots,q_k))$-mixing (linear), there exists $\lambda=(\lambda_i)_{1\le i\le n}\subset\mathbb{K}$, $(z_i)_{1\le i\le n}\subset Z$ such that

$$\begin{aligned}&T(y_i)=\lambda_i z_i \quad \text{for each } 1\le i\le n,\\ &\|\lambda\|_{v_k(c_1,\dots,c_k)} w_{d_k(b_1,\dots,b_k)}(z_i \mid 1\le i\le n)\\ &\quad\le (1+\varepsilon) M_{d_k(b_1,\dots,b_k),d_k(q_1,\dots,q_k)}(T) w_{d_k(q_1,\dots,q_k)}(y_i \mid 1\le i\le n).\end{aligned}$$

Since $\frac{1}{v_k(c_1,\dots,c_k)}=\frac{1}{c_1}+\cdots+\frac{1}{c_k}$ from Proposition 23, there exists $\lambda^1=(\lambda_i^1)_{1\le i\le n}\in l_{c_1}^n,\dots,\lambda^k=(\lambda_i^k)_{1\le i\le n}\in l_{c_k}^n$ such that

$$\lambda=\lambda^1\cdots\lambda^k \quad \text{i.e.} \quad \lambda_i=\lambda_i^1\cdots\lambda_i^k \quad \text{for each } 1\le i\le n,$$
$$\|\lambda\|_{v_k(c_1,\dots,c_k)}=\left\|\lambda^1\right\|_{c_1}\cdots\left\|\lambda^k\right\|_{c_k}.$$

Then

$$(T\circ A)\big(x_i^1,\dots,x_i^k\big)=\alpha_i^1\lambda_i^1\cdots\alpha_i^k\lambda_i^k z_i \quad \text{for each } 1\le i\le n.$$

From $\frac{1}{s_j}=\frac{1}{r_j}+\frac{1}{c_j}$ by Holder's inequality, we have

$$\left\|\alpha^j\lambda^j\right\|_{s_j}\le\left\|\alpha^j\right\|_{r_j}\left\|\lambda^j\right\|_{c_j} \quad \text{for each } 1\le j\le k.$$

We deduce

$$\begin{aligned}&\left\|\alpha^1\lambda^1\right\|_{s_1}\cdots\left\|\alpha^k\lambda^k\right\|_{s_k} w_{d_k(b_1,\dots,b_k)}(z_i \mid 1\le i\le n)\\ &\quad\le\left\|\alpha^1\right\|_{r_1}\cdots\left\|\alpha^k\right\|_{r_k}\left\|\lambda^1\right\|_{c_1}\cdots\left\|\lambda^k\right\|_{c_k} w_{d_k(b_1,\dots,b_k)}(z_i \mid 1\le i\le n)\\ &\quad=\left\|\alpha^1\right\|_{r_1}\cdots\left\|\alpha^k\right\|_{r_k}\|\lambda\|_{v_k(c_1,\dots,c_k)} w_{d_k(b_1,\dots,b_k)}(z_i \mid 1\le i\le n)\\ &\quad\le(1+\varepsilon) M_{d_k(b_1,\dots,b_k),d_k(q_1,\dots,q_k)}(T)\\ &\qquad\times\left\|\alpha^1\right\|_{r_1}\cdots\left\|\alpha^k\right\|_{r_k} w_{d_k(q_1,\dots,q_k)}(y_i \mid 1\le i\le n)\\ &\quad\le(1+\varepsilon)^2 M_{d_k(b_1,\dots,b_k),d_k(q_1,\dots,q_k)}(T)\\ &\qquad\times M^k_{(q_1,p_1),\dots,(q_k,p_k)}(A) w_{p_1}\big(x_i^1 \mid 1\le i\le n\big)\cdots w_{p_k}\big(x_i^k \mid 1\le i\le n\big).\end{aligned}$$

This means that $T\circ A$ is $((b_1,p_1),\dots,(b_k,p_k))$-mixing and

$$\begin{aligned}&M^k_{(b_1,p_1),\dots,(b_k,p_k)}(T\circ A)\\ &\quad\le(1+\varepsilon)^2 M^k_{(q_1,p_1),\dots,(q_k,p_k)}(T) M_{d_k(b_1,\dots,b_k),d_k(q_1,\dots,q_k)}(A).\end{aligned}$$

This concludes the proof, since $\varepsilon>0$ is arbitrary. □

From Theorem 24 and Definition 9, we get

Corollary 25. *Let k be a natural number, $1\le p<q<b<k^*$. Then*

$$\mathcal{M}_{(\frac{b^*}{k})^*,(\frac{q^*}{k})^*}\circ\mathcal{M}^k_{(q,p)}\subset\mathcal{M}^k_{(b,p)}.$$

The following result is one of possible multilinear variants of Pietsch's basic formula (PI), see also [14, Theorem 2.4, Theorem 3.3, Theorem 4.3] for another possible multilinear variants.

THEOREM 26. *Let k be a natural number, $1 < q_1, \dots, q_k < \infty$ such that $\frac{1}{q_1} + \cdots + \frac{1}{q_k} > k-1$, $1 \le p_j < q_j < \infty$, $1 < r_j < \infty$ such that $\frac{1}{p_j} = \frac{1}{q_j} + \frac{1}{r_j}$ for each $1 \le j \le k$ and $1 < b < \infty$ such that $d_k(q_1, \dots, q_k) \le b$. Then*

$$\Pi_{b,d_k(q_1,\dots,q_k)} \circ \mathcal{M}^k_{(q_1,p_1),\dots,(q_k,p_k)} \subset \Pi^k_{v_{k+1}(r_1,\dots,r_k,b);p_1,\dots,p_k}.$$

Proof. Let us consider the diagram

$$X_1 \times \cdots \times X_k \xrightarrow{A} Y \xrightarrow{T} Z$$

in which A is $((q_1,p_1),\dots,(q_k,p_k))$-mixing and T is $(b, d_k(q_1,\dots,q_k))$-summing.

Let $\varepsilon > 0$ and take $(x_i^j)_{1\le i \le n} \subset X_j$ ($1 \le j \le k$). From Lemma 16(ii), there exists $\alpha^1 = (\alpha_i^1)_{1\le i\le n} \subset \mathbb{K}, \dots, \alpha^k = (\alpha_i^k)_{1 \le i \le n} \subset \mathbb{K}$, $(y_i)_{1\le i\le n} \subset Y$ such that

$$A\big(x_i^1,\dots,x_i^k\big) = \alpha_i^1 \cdots \alpha_i^k y_i \quad \text{for each } 1 \le i \le n,$$
$$\big\|\alpha^1\big\|_{r_1} \cdots \big\|\alpha^k\big\|_{r_k} w_{d_k(q_1,\dots,q_k)}(y_i \mid 1 \le i \le n)$$
$$\le (1+\varepsilon) M^k_{(q_1,p_1),\dots,(q_k,p_k)}(A) w_{p_1}\big(x_i^1 \mid 1 \le i \le n\big) \cdots w_{p_k}\big(x_i^k \mid 1 \le i \le n\big).$$

Then from

$$(T\circ A)\big(x_i^1,\dots,x_i^k\big) = T\big(A\big(x_i^1,\dots,x_i^k\big)\big) = \alpha_i^1 \cdots \alpha_i^k T(y_i)$$

the equality, $\frac{1}{v_{k+1}(r_1,\dots,r_k,b)} = \frac{1}{r_1} + \cdots + \frac{1}{r_k} + \frac{1}{b}$ and Holder's inequality we get

$$l_{v_{k+1}(r_1,\dots,r_k,b)}\big((T\circ A)\big(x_i^1,\dots,x_i^k\big) \mid 1 \le i \le n\big)$$
$$\le \big\|\alpha^1\big\|_{r_1} \cdots \big\|\alpha^k\big\|_{r_k} l_b\big(T(y_i) \mid 1 \le i \le n\big).$$

Since T is $(b, d_k(q_1,\dots,q_k))$-summing

$$l_b\big(T(y_i) \mid 1 \le i \le n\big) \le \pi_{b,d_k(q_1,\dots,q_k)}(T) w_{d_k(q_1,\dots,q_k)}(y_i \mid 1 \le i \le n).$$

We deduce

$$l_{v_{k+1}(r_1,\dots,r_k,b)}\big((T\circ A)\big(x_i^1,\dots,x_i^k\big) \mid 1 \le i \le n\big)$$
$$\le \pi_{b,d_k(q_1,\dots,q_k)}(T)\big\|\alpha^1\big\|_{r_1} \cdots \big\|\alpha^k\big\|_{r_k} w_{d_k(q_1,\dots,q_k)}(y_i \mid 1 \le i \le n)$$
$$\le (1+\varepsilon) M^k_{(q_1,p_1),\dots,(q_k,p_k)}(A)$$
$$\times \pi_{b,d_k(q_1,\dots,q_k)}(T) w_{p_1}\big(x_i^1 \mid 1 \le i \le n\big) \cdots w_{p_k}\big(x_i^k \mid 1 \le i \le n\big).$$

This means that $T\circ A$ is $(v_{k+1}(r_1,\dots,r_k,b); p_1,\dots,p_k)$-summing and

$$\pi^k_{v_{k+1}(r_1,\dots,r_k,b);p_1,\dots,p_k}(T\circ A) \le (1+\varepsilon) M^k_{(q_1,p_1),\dots,(q_k,p_k)}(A) \pi_{b,d_k(q_1,\dots,q_k)}(T).$$

Since $\varepsilon > 0$ is arbitrary, we get the statement. □

COROLLARY 27. (i) *Let k be a natural number, $1 \le p < q < k^*$ and $k < r < \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then*

$$\Pi_{(\frac{q^*}{k})^*} \circ \mathcal{M}^k_{(q,p)} \subset \Pi^k_{(\frac{p^*}{k})^*;p} \quad \text{and} \quad \Pi_{(\frac{q^*}{k})^*} \circ \Delta^k_r \subset \Pi^k_{(\frac{p^*}{k})^*;p}.$$

(ii) *Let k be a natural number, $1<q<\infty$, $1\le p<(kq^*)^*$ i.e. $k<\frac{p^*}{q^*}$ and $1<r<\infty$ such that $\frac{1}{p}=\frac{1}{(kq^*)^*}+\frac{1}{r}$. Then*

$$\Pi_q\circ\mathcal{M}^k_{((kq^*)^*,p)}\subset\Pi^k_{s_k;p}\quad\textit{and}\quad\Pi_q\circ\Delta^k_r\subset\Pi^k_{s_k;p},\quad\textit{where }\frac{1}{s_k}=\frac{1}{k^*q^*}+\frac{1}{(\frac{p^*}{k})^*}.$$

(iii) *Let k be a natural number, $1<q<\infty$. Then*

$$\Pi_q\circ\mathcal{M}^k_{((kq^*)^*,1)}\subset\Pi^k_{s_k;1}\quad\textit{and}\quad\Pi_q\circ\Delta^k_{kq^*}\subset\Pi^k_{s_k;1},\quad\textit{where }\frac{1}{s_k}=1+\frac{1}{k^*q^*}.$$

Proof. (i) We have

$$\frac{1}{v_{k+1}(r,\dots,r,(\frac{q^*}{k})^*)}=\frac{k}{r}+\frac{1}{(\frac{q^*}{k})^*}=\frac{k}{r}+1-\frac{k}{q^*}=1-\frac{k}{p^*}=\frac{1}{(\frac{p^*}{k})^*}.$$

From Theorem 26, we get the first part of the statement. The second follows from the first part and Corollary 15(ii).

(ii) We have

$$\begin{aligned}\frac{1}{v_{k+1}(r,\dots,r,(kq^*)^*)}&=\frac{k}{r}+\frac{1}{(kq^*)^*}\\&=k\left(\frac{1}{p}-1+\frac{1}{kq^*}\right)+1-\frac{1}{kq^*}\\&=1-\frac{k}{p^*}+\frac{1}{q^*}\cdot\left(1-\frac{1}{k}\right)\\&=\frac{1}{k^*q^*}+\frac{1}{(\frac{p^*}{k})^*}.\end{aligned}$$

From Theorem 26, we get the first part of the statement. The second follows from the first part and Corollary 15(iii).

(iii) Follows from (ii) in which we take $r=kq^*$. ☐

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