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# COMPLETE GEOMETRIC UNITARIES IN OPERATOR SPACES

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ABSTRACT. We study the abstract complete geometric notion of unitaries in an operator space characterized in terms of the matrix numerical index, which is a quantity determined by a norm-one element and the matrix numerical radius.

## 1. Introduction

The characterizations of geometric unitaries in Banach spaces have recently been studied in [1], [4], [10], [11]. It is natural to study the properties of unitaries in general operator spaces, since operator spaces are "quantized Banach spaces" which basically means that they are spaces of bounded operators on some Hilbert spaces. Motivated by the work of Huang and Ng [9], we study the abstract complete geometric notion of unitaries in operator spaces.

The notion of matrix numerical index was first introduced in [9] to characterize abstract unital operator spaces. A *matrix numerical range space* $(\mathbf{V}, u)$ will be a pair consisting of an operator space $\mathbf{V}$ and a norm-one element $u \in \mathbf{V}$. We denote by $\mathfrak{S}_1(\mathbf{V})$ the unit sphere of $\mathbf{V}$. Let $(\mathbf{V}, u)$ be a matrix numerical range space. For each $n \in \mathbb{N}$, we define the *n matrix state space* of $u$

$$\mathcal{S}_n(\mathbf{V}; u) := \{\varphi \in \mathrm{CB}(\mathbf{V}, M_n) : \|\varphi\|_{\mathrm{cb}} \leq 1, \varphi(u) = I_n\},$$

and the *matrix numerical radius* of an element $x$ in $M_k(\mathbf{V}) (k \in \mathbb{N})$,

$$\gamma_k^u(x) := \sup\{\|\varphi_k(x)\| : \varphi \in \mathcal{S}_n(\mathbf{V}; u), n \in \mathbb{N}\}$$

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as well as the *matrix numerical index* of $u$, namely

$$n_{\mathrm{cb}}(\mathbf{V};u) := \inf\{\gamma_k^u(x) : x \in \mathfrak{S}_1\big(M_k(\mathbf{V})\big), k \in \mathbb{N}\}.$$

Equivalently, $n_{\mathrm{cb}}(\mathbf{V};u)$ is the greatest constant $t \geq 0$ such that $t\|x\| \leq \gamma_k^u(x)$ for every $x \in M_k(\mathbf{V})$. We call $u$ a *complete geometric unitary* (respectively, *complete strict geometric unitary*) if $n_{\mathrm{cb}}(\mathbf{V};u) > 0$ (respectively, $n_{\mathrm{cb}}(\mathbf{V};u) = 1$). Note that $n_{\mathrm{cb}}(\mathbf{V};u) > 0$ (respectively, $n_{\mathrm{cb}}(\mathbf{V};u) = 1$) if and only if there exist a Hilbert space $\mathbf{H}$ and a completely contractive completely topological injection (respectively, complete isometry) $\Theta : \mathbf{V} \to \mathcal{L}(\mathbf{H})$ such that $\Theta(u) = \mathrm{id}_{\mathbf{H}}$ (see [9, Theorem 2.7]). A matrix numerical range space $(\mathbf{V}, u)$ is called a *unital operator space* if $u$ is a complete strict geometric unitary. Such spaces play a significant role since the birth of operator space theory(see [2, Theorems 1.2.3 and 1.2.9]).

The outline of the paper is as follows. In Section 2, we give a characterization of the unitaries of a unital $C^*$-algebra $\mathbf{A}$ by working in $M_n(\mathbf{A}^*)$ for some $n \in \mathbb{N}$. By this, we introduce some properties of complete geometric unitaries in operator spaces. Next, we use the above results in Section 3 to present that $u \otimes v$ is a complete geometric unitary in the operator space injective tensor product $\mathbf{V} \check{\otimes} \mathbf{W}$, if and only if $u$ and $v$ are complete geometric unitares in $\mathbf{V}$ and $\mathbf{W}$, respectively. We consider complete geometric unitaries in the space $C(\Omega) \check{\otimes} \mathbf{V}$, where $\Omega$ is a compact set. If $F \in C(\Omega) \check{\otimes} \mathbf{V}$ is a complete (respectively, strict) geometric unitary, then $F(t)$ is a complete (respectively, strict) geometric unitary for all $t \in \Omega$. Finally, we devote Section 4 to show that for an isometry $u$ in a unital $C^*$-algebra, $u$ and $u^*$ are complete strict geometric unitaries in $\mathbf{V}$, where $\mathbf{V} := \mathrm{Span}\{e, u, u^*\}$ is a 3-dimensional system.

## 2. Operator spaces characterizations of complete geometric unitaries

We will give a new characterization of unitaries in a unital $C^*$-algebra, which is a generalization of [1, Theorem 2] by C. Akemann and N. Weaver. However, only the final step in the proof of the next theorem is based on an idea of [1, Theorem 2].

THEOREM 2.1. *Let $\mathbf{A}$ be a unital $C^*$-algebra with the identity $e$, and let $u$ be a norm-one element of $\mathbf{A}$. Then the following are equivalent*:

(a) *$u$ is a unitary.*

(b) *For all $n \in \mathbb{N}$, one has*

$$M_n\big(\mathbf{A}^*\big)_{\|\cdot\| \leq 1} \subseteq \left\{\sum_{k=0}^{3} i^k \alpha_k \varphi_k \alpha_k : \varphi_k \in \mathcal{S}_n(\mathbf{A};u), \|\alpha_k\| \leq 1, \alpha_k \in (M_n)_+, \forall k = 0,1,2,3\right\}.$$

(c) *There exists* $n \in \mathbb{N}$ *and* $r > 0$ *such that*

$$M_n(\mathbf{A}^*)_{\|\cdot\| \le r} \subseteq \left\{ \sum_{k=0}^{3} i^k \alpha_k \varphi_k \alpha_k : \varphi_k \in \mathcal{S}_n(\mathbf{A}; u), \|\alpha_k\| \le 1, \alpha_k \in (M_n)_+, \forall k = 0,1,2,3 \right\}.$$

(d) *There exists* $n \in \mathbb{N}$ *such that*

$$\{a \in \mathbf{A} : \varphi(a) = 0, \varphi \in \mathcal{S}_n(\mathbf{A}; u)\} = \{0\}.$$

*Proof.* $(a) \Rightarrow (b)$ Let $\varphi \in M_n(\mathbf{A}^*)$ be a complete contraction. From [5, Theorem 5.3.2] there exist matrix states $\psi_1$ and $\psi_2$ from $\mathbf{A}$ to $M_n$ such that

$$\Phi = \begin{pmatrix} \psi_1 & \varphi \\ \varphi^* & \psi_2 \end{pmatrix} : M_2(\mathbf{A}) \to M_{2n} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \psi_1(a) & \varphi(b) \\ \varphi^*(c) & \psi_2(d) \end{pmatrix}$$

is a matrix state. Let $\Psi := \Phi \circ P$, where $P$ is a completely positive mapping

$$P : \mathbf{A} \to M_2(\mathbf{A}) : a \mapsto \begin{pmatrix} a & a \\ a & a \end{pmatrix}.$$

Then we have the relation

$$\varphi = \sum_{k=0}^{3} \frac{i^k}{4} [1, i^k] \Psi [1, i^k]^*.$$

Set $R_k = \frac{1}{4}[1, i^k]\Psi[1, i^k]^*, k = 0,1,2,3$. It follows from [5, Lemma 5.1.6] that for each $R_k$ there exists a matrix state $\varphi_k \in \mathcal{S}_n(\mathbf{A}; e)$ such that

$$R_k(\cdot) = R_k(e)^{1/2} \varphi_k(\cdot) R_k(e)^{1/2}.$$

This implies that

$$\varphi = \sum_{k}^{3} i^k R_k(e)^{1/2} \varphi_k(\cdot) R_k(e)^{1/2}.$$

Because each $R_k$ is a completely positive completely contractive mapping, we have

$$M_n(\mathbf{A}^*)_{\|\cdot\| \le 1} \subseteq \left\{ \sum_{k=0}^{3} i^k \alpha_k \varphi_k \alpha_k : \varphi_k \in \mathcal{S}_n(\mathbf{A}; e), \|\alpha_k\| \le 1, \alpha_k \in (M_n)_+, \forall k = 0,1,2,3 \right\}.$$

Suppose that $u$ is a unitary and consider the map $T : \mathbf{A} \to \mathbf{A}$ given by $T(a) = ua$. This map is a bijective complete isometry, and hence so is the adjoint map $T^* : \mathbf{A}^* \to \mathbf{A}^*$. More precisely,

$$T_n^*(\varphi)(a) = \varphi(ua) \quad (a \in \mathbf{A})$$

for all $n \in \mathbb{N}$ and $\varphi \in M_n(\mathbf{A}^*) = \mathrm{CB}(\mathbf{A}, M_n)$. It follows that $T_n^*(\mathcal{S}_n(\mathbf{A}; u)) = \mathcal{S}_n(\mathbf{A}; e)$, and so

$$M_n\big(\mathbf{A}^*\big)_{\|\cdot\|\le 1} \subseteq \left\{ \sum_{k=0}^{3} i^k \alpha_k \phi_k \alpha_k : \phi_k \in \mathcal{S}_n(\mathbf{A}; u), \|\alpha_k\| \le 1, \alpha_i \in (M_n)_+, \forall k = 0,1,2,3 \right\}.$$

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are trivial.

(d) $\Rightarrow$ (a). We claim that $u$ is an extreme point of $\mathbf{A}_{\|\cdot\|\le 1}$. If $u$ is not an extreme point, then there exists a nonzero $v \in \mathbf{A}$ such that $u \pm v \in \mathbf{A}_{\|\cdot\|\le 1}$. Thus for each $\varphi \in \mathcal{S}_n(\mathbf{A}; u)$,

$$\begin{aligned} \big\|I_n + \varphi(v) + \varphi(v)^* + \varphi(v)^*\varphi(v)\big\| &= \big\|I_n + \varphi(v)\big\|^2 = \big\|\varphi(u+v)\big\|^2 \le 1, \\ \big\|I_n - \varphi(v) - \varphi(v)^* + \varphi(v)^*\varphi(v)\big\| &= \big\|I_n - \varphi(v)\big\|^2 = \big\|\varphi(u-v)\big\|^2 \le 1. \end{aligned}$$

We conclude that $\|I_n + \varphi(v)^*\varphi(v)\| \le 1$, and thus $\varphi(v) = 0$. This contradicts our hypothesis. Hence $u$ is an extreme point as claimed, and by [7, Theorem 1] we see that $u$ is a partial isometry.

Suppose that $u$ is not a unitary. We can assume that $p = e - u^*u \neq 0$. We will prove that $\varphi(p) = 0$ for any $\varphi \in \mathcal{S}_n(\mathbf{A}; u)$. Fix $\varphi \in \mathcal{S}_n(\mathbf{A}; u)$. Then for each $t \in \mathbb{R}$,

$$\big\|I_n + t\,\mathrm{Re}\,\varphi(p)\big\|^2 = \big\|\mathrm{Re}\,\varphi(u+tp)\big\|^2 \le \|u+tp\|^2 = \big\|uu^* + t^2 p\big\| \le 1 + t^2$$

and

$$\begin{aligned} \big\|I_n - t\,\mathrm{Im}\,\varphi(p)\big\|^2 = \big\|I_n + it\varphi(p)\big\|^2 &\le \big\|\varphi(u+itp)\big\|^2 \\ &\le \|u+itp\|^2 = \big\|uu^* + t^2 p\big\| \le 1 + t^2. \end{aligned}$$

If $r \in \sigma(\mathrm{Re}(\varphi(p)))$, where $\sigma(\mathrm{Re}(\varphi(p)))$ denotes the spectrum of $\mathrm{Re}(\varphi(p))$, then $(1+rt)^2 \le 1 + t^2$ for all real $t$. This implies that $r = 0$ and hence $\mathrm{Re}\,\varphi(p) = 0$. As the same argument as above, we have $\mathrm{Im}\,\varphi(p) = 0$. This contradiction establishes that $\varphi(p) = 0$, as claimed. □

We will gather some facts about matrix numerical index that we shall use in the following results.

Remark 2.2. Let $(\mathbf{V}, u)$ be a matrix numerical range space.

(a) Let $\mathbf{W}$ be an operator space and $\Psi : \mathbf{V} \to \mathbf{W}$ is complete isometry. Then we have $n_{\mathrm{cb}}(\mathbf{V}; u) \le n_{\mathrm{cb}}(\mathbf{W}; \Psi(u))$.
(b) We denote by $Q_u$ the canonical complete contraction from $\mathbf{V}$ to $\mathbf{V}_u$, where $N_u := \{v \in \mathbf{V} : \gamma_1^u(v) = 0\}$. Then $(\mathbf{V}_u, Q_u(u))$ is a unital operator space. If $n_{\mathrm{cb}}(\mathbf{V}; u) > 0$, then $n_{\mathrm{cb}}(\mathbf{V}; u) = \|Q_u^{-1}\|_{\mathrm{cb}}^{-1}$ (see [9, Lemma 2.4]).
(c) If $n_{\mathrm{cb}}(\mathbf{V}; u) > 0$, then $u$ is an extreme point of the closed unit ball of $\mathbf{V}$.
(d) $n_{\mathrm{cb}}(\mathbf{V}; u) = n_{\mathrm{cb}}(\mathbf{V}^{**}; u)$.

THEOREM 2.3. *Let* $(\mathbf{V},u)$ *be a matrix numerical range space. Then* $u$ *is a complete geometric unitary if and only if there exists* $0<r\le 1$ *such that*

$$M_n(\mathbf{V}^*)_{\|\cdot\|<r} \subseteq \left\{\sum_{k=0}^{3} i^k\alpha_k\varphi_k\alpha_k : \varphi_k\in\mathcal{S}_n(\mathbf{V},u), \|\alpha_k\|\le 1, \alpha_k\in(M_n)_+, k=0,1,2,3\right\}$$

*for all* $n\in\mathbb{N}$.

*Proof.* Suppose that $u$ is a complete geometric unitary, we will show that

$$M_n(\mathbf{V}^*)_{\|\cdot\|<n_{\mathrm{cb}}(\mathbf{V};u)} \subseteq \left\{\sum_{k=0}^{3} i^k\alpha_k\varphi_k\alpha_k : \varphi_k\in\mathcal{S}_n(\mathbf{V},u), \|\alpha_k\|\le 1, \alpha_k\in(M_n)_+, k=0,1,2,3\right\}$$

for all $n\in\mathbb{N}$.

We first do the case $n_{\mathrm{cb}}(\mathbf{V},u)=1$. Then there exist a Hilbert space $\mathbf{H}$ and a complete isometry $\Theta:\mathbf{V}\to\mathcal{L}(\mathbf{H})$ such that $\Theta(u)=\mathrm{id}_{\mathbf{H}}$ by [9, Theorem 2.7]. Therefore, $(\Theta^*)_n : M_n(\mathcal{L}(\mathbf{H})^*)\to M_n(\mathbf{V}^*)$ is a surjective quotient mapping and

$$(\Theta^*)_n\big(\mathcal{S}_n(\mathcal{L}(\mathbf{H}),I_{\mathbf{H}})\big)=\mathcal{S}_n(\mathbf{V};u)$$

by Arveson–Wittstock–Hahn–Banach theorem [5, Theorem 4.1.5]) for any $n\in\mathbb{N}$. Thus for each $\psi\in M_n(\mathbf{V}^*)_{\|\cdot\|<1}$, there exists $\phi\in\mathrm{CB}(\mathcal{L}(\mathbf{H}),M_n)_{\|\cdot\|\le 1}$ such that $\Theta_n^*(\phi)=\psi$. It follows from Theorem 2.1 that we can write

$$\phi=\sum_{k=0}^{3} i^k\alpha_k\phi_k\alpha_k,$$

where $\alpha_k\in(M_n)_+$, $\|\alpha_k\|\le 1$ and $\phi_k\in\mathcal{S}_n(\mathcal{L}(\mathbf{H});I_{\mathbf{H}})$ $(k=0,1,2,3)$. We have proved that if $n_{\mathrm{cb}}(\mathbf{V};u)=1$ then for each $n\in\mathbb{N}$,

$$M_n(\mathbf{V}^*)_{\|\cdot\|<1} \subseteq \left\{\sum_{k=0}^{3} i^k\alpha_k\varphi_k\alpha_k : \varphi_k\in\mathcal{S}_n(\mathbf{V};u), \|\alpha_k\|\le 1, \alpha_k\in(M_n)_+, \forall k=0,1,2,3\right\}.$$

To deal with the general case when $n_{\mathrm{cb}}(\mathbf{V};u)>0$, we consider the map $(Q_u^*)_n : \mathrm{CB}(\mathbf{V}_u,M_n)\to\mathrm{CB}(\mathbf{V},M_n)$ given by $Q_u$. It is clear that $(Q_u^*)_n$ is a completely contractive complete isomorphism and

$$(Q_u^*)_n\big(\mathcal{S}_n(\mathbf{V}_u;Q_u(u))\big)\subseteq\mathcal{S}_n(\mathbf{V};u).$$

In fact, we get that

$$n_{\mathrm{cb}}(\mathbf{V};u)\|\varphi\|_{\mathrm{cb}}\le\big\|(Q_u^*)_n(\varphi)\big\|_{\mathrm{cb}}\quad\big(\varphi\in M_n(\mathbf{V}_u^*)\big).$$

On the other hand, for any $\varphi\in\mathcal{S}_n(\mathbf{V};u)$,

$$\|\varphi_k(x)\|\le\gamma_k^u(x)=\|(Q_u)_k(x)\|_k\quad\big(x\in M_k(\mathbf{V})\big).$$

Hence, there exists $\psi \in \mathrm{CB}(\mathbf{V}_u, M_n)$ with $\varphi = \psi \circ Q_u$ and $\|\psi\|_{\mathrm{cb}} \leq 1$. This shows that $(Q_u^*)_n$ is a surjection (and hence a bijection) from $\mathcal{S}_n(\mathbf{V}_u; Q_u(u))$ to $\mathcal{S}_n(\mathbf{V}; u)$. Since $n_{\mathrm{cb}}(\mathbf{V}_u, Q_u(u)) = 1$ by Remark 2.2(b), we can apply the above argument to obtain the general case.

Conversely, for each $\psi \in M_n(\mathbf{V}^*)_{\|\cdot\|<1}$ $(n \in \mathbb{N})$, we can find $\alpha_k \in (M_n)_+$, $\|\alpha_k\| \leq 1$ and $\varphi_k \in \mathcal{S}_n(\mathbf{V}; u)$ $(k = 0, 1, 2, 3)$ such that

$$r\psi = \sum_{k=0}^{3} i^k \alpha_k \varphi_k \alpha_k.$$

This means that for every $x$ in $M_k(\mathbf{V})$,

$$\gamma_k^u(x) \geq r\|x\|/4$$

and so $n_{\mathrm{cb}}(\mathbf{V}; u) \geq r/4$. □

We obtain directly the following interesting corollary from Theorem 2.1 and Theorem 2.3.

Corollary 2.4. *Let* $\mathbf{A}$ *be a unital* $C^*$*-algebra, and let* $u$ *be a norm-one element of* $\mathbf{A}$*. Then* $u$ *is complete geometric unitary if and only if it is a unitary.*

Proposition 2.5. *Let* $\mathbf{V}$ *be a finite dimensional operator space with* $u \in \mathfrak{S}_1(\mathbf{V})$*. Then* $u$ *is a complete geometric unitary if and only if there exists* $n \in \mathbb{N}$ *such that*

$$\{v \in \mathbf{V} : \varphi(v) = 0, \varphi \in \mathcal{S}_n(\mathbf{V}; u)\} = \{0\}.$$

*Proof.* If $u$ is a complete geometric unitary, then from Theorem 2.3, for each $n \in \mathbb{N}$,

$$\{v \in \mathbf{V} : \varphi(v) = 0, \varphi \in \mathcal{S}_n(\mathbf{V}; u)\} = \{0\}.$$

Conversely, by the hypothesis $\gamma_n^u$ induces a norm on $M_n(\mathbf{V})$. Thus $N_u = \{0\}$ and the canonical complete contraction $Q_u$ is the identity mapping. By the Inverse Mapping theorem, $Q_u^{-1}$ is bounded. If $\mathbf{V}$ is an $m$-dimensional operator space, then $\|Q_u^{-1}\|_{\mathrm{cb}} \leq m\|Q_u^{-1}\|$ by [5, Corollary 2.2.4]. It follows from Remark 2.2(b) that

$$n_{\mathrm{cb}}(\mathbf{V}; u) = \|Q_u^{-1}\|_{\mathrm{cb}}^{-1} > 0.$$ □

Proposition 2.6. *Let* $\{\mathbf{V}_\lambda : \lambda \in \Lambda\}$ *be a family of operator spaces. If* $u = (u_\lambda)$ *is a complete geometric unitary in the* $l^\infty$ *direct sum* $[\bigoplus_{\lambda \in \Lambda} \mathbf{V}_\lambda]_\infty$*, then each* $u_\lambda$ *is a complete geometric unitary. In this case,*

$$n_{\mathrm{cb}}\Big(\Big[\bigoplus_{\lambda \in \Lambda} \mathbf{V}_\lambda\Big]_\infty; (u_\lambda)\Big) = \inf\{n_{\mathrm{cb}}(\mathbf{V}_\lambda; u_\lambda) : \lambda \in \Lambda\}.$$

*Proof.* If $u=(u_\lambda)$ is a complete geometric unitary, then from Remark 2.2(c), $u=(u_\lambda)$ is an extreme point. Thus, each $u_\lambda$ is a norm-one element. We will prove that

$$n_{\mathrm{cb}}\left(\left[\bigoplus_{\lambda\in\Lambda}\mathbf{V}_\lambda\right]_\infty;u\right)=\inf\bigl\{n_{\mathrm{cb}}(\mathbf{V}_\lambda;u_\lambda):\lambda\in\Lambda\bigr\}.$$

This completes the proof.

Given fixed $\lambda_0\in\Lambda$, one has clearly that

$$\left[\bigoplus_{\lambda\in\Lambda}\mathbf{V}_\lambda\right]_\infty=\mathbf{V}_{\lambda_0}\oplus_\infty\left[\bigoplus_{\lambda\neq\lambda_0}\mathbf{V}_\lambda\right]_\infty.$$

Set $\mathbf{U}:=[\bigoplus_{\lambda\neq\lambda_0}\mathbf{V}_\lambda]_\infty$ and $e=(u_\lambda)_{\lambda\neq\lambda_0}$. Then $e$ is a norm-one element on $\mathbf{U}$. Fixing a linear functional $f\in\mathcal{S}_1(\mathbf{V};u_{\lambda_0})$, we define an operator

$$\Psi:\mathbf{V}_{\lambda_0}\to\mathbf{V}_{\lambda_0}\oplus_\infty\mathbf{U}\quad\text{by }\Psi(v):=\bigl(v,f(v)e\bigr)\ (v\in\mathbf{V}_{\lambda_0}).$$

It is easily verified that $\Psi$ is a complete isometry such that $\Psi(u_{\lambda_0})=u$. Thus, for each $\lambda_0\in\Lambda$

$$n_{\mathrm{cb}}\left(\left[\bigoplus_{\lambda\in\Lambda}\mathbf{V}_\lambda\right]_\infty;u\right)\le n_{\mathrm{cb}}(\mathbf{V}_{\lambda_0};u_{\lambda_0}).$$

On the other hand, let $k\in\mathbb{N}$ and $v=(v_\lambda)\in M_k([\bigoplus_{\lambda\in\Lambda}\mathbf{V}_\lambda]_\infty)$. Then for every $\varepsilon>0$, there exists $\lambda_1\in\Lambda$ such that

$$\|v_{\lambda_1}\|_k>\|v\|_k-\varepsilon.$$

Set $\mathbf{W}:=[\bigoplus_{\lambda\neq\lambda_1}\mathbf{V}_\lambda]_\infty$ and $e=(u_\lambda)_{\lambda\neq\lambda_1}$. For all $m,n\in\mathbb{N}$, $\varphi\in\mathcal{S}_m(\mathbf{V}_{\lambda_1};u_{\lambda_1})$ and $\phi\in\mathcal{S}_n(\mathbf{W};e)$, we consider the operator $\theta\in\mathrm{CB}(\mathbf{V}_{\lambda_1}\oplus^\infty\mathbf{W},M_{m+n})$ defined by

$$\theta\bigl((x,y)\bigr):=\varphi(x)\oplus\phi(y)\quad(x\in\mathbf{V}_{\lambda_1},y\in\mathbf{W}).$$

We clearly have

$$\theta\in\mathcal{S}_{m+n}(\mathbf{V}_{\lambda_1}\oplus_\infty\mathbf{W};u)\quad\text{and}\quad\bigl\|\varphi_k(v_{\lambda_1})\bigr\|\le\bigl\|\theta_k(v)\bigr\|.$$

It follows that

$$\bigl(\|v\|_k-\varepsilon\bigr)n_{\mathrm{cb}}(\mathbf{V}_{\lambda_1};u_{\lambda_1})<\|v_{\lambda_1}\|_k n_{\mathrm{cb}}(\mathbf{V}_{\lambda_1};u_{\lambda_1})\le\gamma_k^{u_{\lambda_1}}(v_{\lambda_1})\le\gamma_k^{(u_\lambda)}(v).$$

Consequently,

$$n_{\mathrm{cb}}\left(\left[\bigoplus_{\lambda\in\Lambda}\mathbf{V}_\lambda\right]_\infty;u\right)\ge\inf\bigl\{n_{\mathrm{cb}}(\mathbf{V}_\lambda;u_\lambda):\lambda\in\Lambda\bigr\}.$$

□

Let $\Omega$ be a topological space and $\mathbf{V}$ an operator space. We let $B(\Omega,\mathbf{V})$ be the space of all bounded Borel measurable mappings from $\Omega$ into $\mathbf{V}$. Then $B(\Omega,\mathbf{V})$ as a subspace of $l^\infty(\Omega,\mathbf{V})$ is an operator space.

PROPOSITION 2.7. *Let $\Omega$ be a topological space and $\mathbf{V}$ an operator space. If $F \in B(\Omega, \mathbf{V})$ is a complete geometric unitary, then $F(t)$ is a complete geometric unitary for each $t \in \Omega$. In this case,*

$$n_{\mathrm{cb}}\big(B(\Omega,\mathbf{V});F\big) = \inf\big\{n_{\mathrm{cb}}\big(\mathbf{V};F(t)\big) : t \in \Omega\big\}.$$

*Proof.* Since $F \in B(\Omega, \mathbf{V})$ is a complete geometric unitary, we see that $F$ is an extreme point of the closed unit ball of $B(\Omega, \mathbf{V})$. It follows that each $F(t)$ is a norm-one element in $\mathbf{V}$. Otherwise, there exist $t_0 \in \Omega$ and $\varepsilon > 0$ such that $\|F(t_0)\| \leq 1 - \varepsilon < 1$. Set $x := F(t_0)$ and $E_{t_0} := \{t \in \Omega : F(t) = x\}$. Then $E_{t_0}$ is a Borel set and $\|F \pm \varepsilon\chi_{E_{t_0}} x \in \| \leq 1$. We have $F = 1/2(F + \varepsilon\chi_{E_{t_0}} x) + 1/2(F - \varepsilon\chi E_{t_0} x)$. This leads to a contradiction that $F$ is an extreme point.

For each $s \in \Omega$, set $u := F(s)$ and $E_s := \{t \in \Omega : F(t) = u\}$. Fix $f \in \mathcal{S}_1(\mathbf{V}; u)$ and consider the mapping $\Theta : \mathbf{V} \to B(\Omega, \mathbf{V})$ defined by

$$\Theta(v) = \chi_{E_s} v + f(v)\chi_{E_s^c} F.$$

Then $\Theta$ is a complete isometry and $\Theta(u) = F$. It follows that

$$n_{\mathrm{cb}}\big(B(\Omega,\mathbf{V});F\big) \leq n\big(\mathbf{V},F(s)\big),$$

and so

$$n_{\mathrm{cb}}\big(B(\Omega,\mathbf{V});F\big) \leq \inf\big\{n\big(\mathbf{V};F(t)\big) : t \in \Omega\big\}.$$

Conversely, we can regard $B(\Omega, \mathbf{V})$ as a subspace of $l_\infty(\Omega, \mathbf{V})$. By Proposition 2.6, the reverse inequality holds. □

## 3. Complete geometric unitaries in the injective tensor product

DEFINITION 3.1. For any two operator spaces $\mathbf{V}$ and $\mathbf{W}$ we define the *injective matrix norm* $\|\cdot\|$ on $\mathbf{V} \otimes \mathbf{W}$ by setting

$$\|u\|_\vee = \sup\big\{\big\|(f \otimes g)_n(u)\big\| : f \in M_p\big(\mathbf{V}^*\big), g \in M_q\big(\mathbf{W}^*\big), \|f\|, \|g\| \leq 1\big\}$$

for each matrix $u \in M_n(\mathbf{V} \otimes \mathbf{W})$. We define the *operator space injective tensor product* $\mathbf{V} \check{\otimes} \mathbf{W}$ to be the completion of the operator space $(\mathbf{V} \otimes \mathbf{W}, \|\cdot\|_\vee)$.

THEOREM 3.2. *Let $(\mathbf{V}, u)$ and $(\mathbf{W}, v)$ be matrix numerical range spaces. Then $u \otimes v$ is a complete (respectively, strict) geometric unitary if and only if $u$ and $v$ are complete (respectively, strict) geometric unitaries.*

*Proof.* We suppose that $u \otimes v$ is a complete (respectively, strict) geometric unitary. Since $\mathbf{V}$ and $\mathbf{W}$ can be regarded as a subspace of $\mathbf{V} \check{\otimes} \mathbf{W}$ through $x \mapsto x \otimes v$ and $y \mapsto u \otimes y$, it follows that $u$ and $v$ are complete (respectively, strict) geometric unitaries.

Conversely, if $u$ and $v$ are complete strict geometric unitaries. We assume that $\phi : \mathbf{V} \to \mathcal{L}(\mathbf{H})$ and $\psi : W \to \mathcal{L}(\mathbf{K})$ are complete isometries such that $\phi(u) = \mathrm{id}_{\mathbf{H}}$ and $\psi(v) = \mathrm{id}_{\mathbf{K}}$. It follows from [5, Proposition 8.1.6] that $\phi \otimes \psi : \mathbf{V} \check{\otimes} \mathbf{W} \to \mathcal{L}(\mathbf{H} \otimes \mathbf{K})$ is a complete isometry with $\phi \otimes \psi(u \otimes v) = \mathrm{id}_{\mathbf{H}\otimes\mathbf{K}}$. We conclude that $u \otimes v$ is a complete strict geometric unitary.

For the general case, we suppose that $u$ and $v$ are complete geometric unitaries. We can regard $\mathbf{V}\check{\otimes}\mathbf{W}$ as a subspace of $\mathrm{CB}(\mathbf{V}^*,\mathbf{W})$ by [5, Proposition 8.1.2]. Fix $\Phi\in M_n(\mathbf{V}\check{\otimes}\mathbf{W})$ with $\|\Phi\|=1$. Given $\varepsilon>0$, there exist $m\in\mathbb{N}$, $\psi\in M_m(\mathbf{V}^*)_{\|\cdot\|<1}$ such that $\|\Phi_m(\psi)\|>1-\varepsilon$. Since $u$ is a complete geometric unitary, from Theorem 2.3 we can find $\alpha_m\in(M_m)_+$, $\|\alpha_k\|\le 1$ and $\varphi_k\in\mathcal{S}_m(\mathbf{V};u)$, $(k=0,1,2,3)$ such that

$$n_{\mathrm{cb}}(\mathbf{V};u)\psi=\sum_{k=0}^{3}i^k\alpha_k\varphi_k\alpha_k.$$

Thus, there exists $\varphi\in\mathcal{S}_n(\mathbf{V};u)$ such that

$$\big\|\Phi_n(\varphi)\big\|>n_{\mathrm{cb}}(\mathbf{V};u)(1-\varepsilon)/4.$$

Hence, by the definition of $n_{\mathrm{cb}}(\mathbf{W};v)$, there exist $p\in\mathbb{N}$, $\phi\in\mathcal{S}_p(\mathbf{W};v)$ such that

$$\big\|\phi_{mn}\big(\Phi_n(\varphi)\big)\big\|>n_{\mathrm{cb}}(\mathbf{W};v)\big\|\Phi_n(\varphi)\big\|-\varepsilon.$$

We define an operator $\theta\in\mathrm{CB}(\mathbf{V}\check{\otimes}\mathbf{W},M_{np})$ by

$$\theta(\Psi)=\phi_n\big(\Psi_n(\varphi)\big)\quad\text{for each }\Psi\in\mathbf{V}\check{\otimes}\mathbf{W}.$$

Now $\theta\in\mathcal{S}_{np}(\mathbf{V}\check{\otimes}\mathbf{W};u\otimes v)$, $\theta_m(\Phi)=\phi_{mn}(\Phi_n(\varphi))$ and

$$\big\|\theta_m(\Phi)\big\|>n_{\mathrm{cb}}(\mathbf{W};v)\big\|\Phi_n(\varphi)\big\|-\varepsilon>n_{\mathrm{cb}}(\mathbf{V};u)n_{\mathrm{cb}}(\mathbf{W};v)/4-2\varepsilon.$$

The desired inequality $n_{\mathrm{cb}}(\mathbf{V}\check{\otimes}\mathbf{W};u\otimes v)\ge n_{\mathrm{cb}}(\mathbf{V};u)n_{\mathrm{cb}}(\mathbf{W};v)/4$ follows. □

Let $\Omega$ be a compact Hausdorff space and $\mathbf{V}$ an operator space. It is well known that for each $f\in(C(\Omega)\check{\otimes}\mathbf{V})^*$, there is a unique weakly regular set function $m:\sum\to\mathbf{V}^*$ so that $f(F)=\int_K F\,dm$ for each $F\in C(\Omega)\check{\otimes}\mathbf{V}=C(\Omega,\mathbf{V})$ (see [3, Theorem 2.2]). Then we can regard $B(\Omega,\mathbf{V})$ as a subspace of $(C(\Omega)\check{\otimes}\mathbf{V})^{**}$ by the way

$$G(f)=\int_K G\,dm,\quad G\in B(\Omega,\mathbf{V}).$$

THEOREM 3.3. *Let* $\mathbf{V}$ *be an operator space and* $\Omega$ *a compact Hausdorff space. If* $F\in C(\Omega)\check{\otimes}\mathbf{V}$ *is a complete* (*respectively, strict*) *geometric unitary, then for all* $t\in\Omega$, $F(t)$ *is a complete* (*respectively, strict*) *geometric unitary. In this case,*

$$n_{\mathrm{cb}}\big(C(\Omega)\check{\otimes}\mathbf{V};F\big)=\inf\big\{n_{\mathrm{cb}}\big(\mathbf{V};F(t)\big):t\in\Omega\big\}.$$

*Proof.* Since $C(\Omega)\check{\otimes}\mathbf{V}$ is a subspace of $B(\Omega,\mathbf{V})$, and since moreover $B(\Omega,\mathbf{V})$ is a subspace of $(C(\Omega)\check{\otimes}\mathbf{V})^{**}$, it follows from Remark 2.2(a) and (d) that

$$n_{\mathrm{cb}}\big(C(\Omega)\check{\otimes}\mathbf{V};F\big)=n_{\mathrm{cb}}\big(B(\Omega,\mathbf{V});F\big)=n_{\mathrm{cb}}\big(\big(C(\Omega)\check{\otimes}\mathbf{V}\big)^{**};F\big).$$

Thus, $F$ is a complete (respectively, strict) geometric unitary in $B(\Omega,\mathbf{V})$. By Proposition 2.7, the conclusion follows. □

Definition 3.4. A matrix numerical range space $(\mathbf{V}, u)$ is called a *minimal unital operator space* if there exist a compact Hausdorff space $\Omega$ and a complete isometry $\Phi : \mathbf{V} \to C(\Omega)$ satisfying $\Phi(u) = I$.

Proposition 3.5. *Let* $(\mathbf{V}, u)$ *and* $(\mathbf{W}, v)$ *be matrix numerical range spaces. If* $(\mathbf{V}, u)$ *is a minimal unital operator space, then*

$$n_{\mathrm{cb}}(\mathbf{V} \check{\otimes} \mathbf{W}; u \otimes v) = n_{\mathrm{cb}}(\mathbf{W}; v).$$

*Proof.* If $(\mathbf{V}, u)$ is a minimal unital operator space, then there exist a compact Hausdorff space $\Omega$ and a complete isometry $\Phi : \mathbf{V} \to C(\Omega)$ satisfying $\Phi(u) = I$. Since $\mathbf{W}$ can be regarded as a subspace of $\mathbf{V} \check{\otimes} \mathbf{W}$ through $w \mapsto u \otimes w$ and $\mathbf{V} \check{\otimes} \mathbf{W}$ can be regarded as a subspace of $C(\Omega) \check{\otimes} \mathbf{W}$ through $x \otimes w \mapsto \Phi(x) \otimes w$, we have

$$n_{\mathrm{cb}}(\mathbf{W}; v) \geq n_{\mathrm{cb}}(\mathbf{V} \check{\otimes} \mathbf{W}; u \otimes v) \geq n_{\mathrm{cb}}\big(C(\Omega) \check{\otimes} \mathbf{W}; I \otimes v\big).$$

It suffices to show that $n_{\mathrm{cb}}(C(\Omega) \check{\otimes} \mathbf{W}; I \otimes v) \geq n_{\mathrm{cb}}(\mathbf{W}; v)$. Fix $k \in \mathbb{N}$ and

$$g \in M_k\big(C(\Omega) \check{\otimes} \mathbf{W}\big) = C\big(\Omega, M_k(\mathbf{W})\big).$$

We may find $t_0 \in \Omega$ such that $\|g\| = \|g(t_0)\|$. For any $n \in \mathbb{N}$ and $\varphi \in \mathcal{S}_n(\mathbf{W}; v)$, we consider the operator $\phi \in \mathrm{CB}(C(\Omega) \check{\otimes} \mathbf{W}, M_n)$ given by

$$\phi(f) := \varphi\big(f(t_0)\big), \quad f \in C(\Omega) \check{\otimes} \mathbf{W}.$$

It is easy to check that

$$\phi \in \mathcal{S}_n\big(C(\Omega) \check{\otimes} \mathbf{W}; I \otimes v\big) \quad \text{and} \quad \phi_k(g) = \varphi_k\big(g(t_0)\big).$$

Hence,

$$\gamma_k^{I \otimes u}(g) \geq \gamma_k^u\big(g(t_0)\big) \geq n_{\mathrm{cb}}(\mathbf{W}; v),$$

and so $n_{\mathrm{cb}}(C(\Omega) \check{\otimes} \mathbf{W}; I \otimes u) \geq n_{\mathrm{cb}}(\mathbf{W}; v)$. □

## 4. Complete geometric unitaries in 3-dimensional operator systems

Let $\mathbf{A}$ be a unital $C^*$-algebra with $u \in \mathfrak{S}_1(\mathbf{A})$. Then $u$ is a complete geometric unitary if and only if $u$ is a unitary by Corollary 2.4. Thus, an isometry $u$ in $\mathbf{A}$ may be not a complete geometric unitary. We define a 3-dimensional operator system by setting $\mathbf{V} := \mathrm{span}\{e, u, u^*\}$. It is interesting to ask whether $u$ in $\mathbf{V}$ is a complete geometric unitary. To answer this question, we need to introduce some more definitions and results.

A (concrete) *operator system* is a self-adjoint unital subspace of a unital $C^*$-algebra. Let $\mathbf{V}$ be an operator system with the identity $e$. Consider the set $\mathrm{CP}_k(\mathbf{V})$ of all completely positive linear maps $\psi : \mathbf{V} \to M_k$. If $\psi, \phi \in \mathrm{CP}_k(\mathbf{V})$, then the notation $\psi \leq_{\mathrm{cb}} \phi$ means that $\phi - \psi \in \mathrm{CP}_k(\mathbf{V})$. A *matrix state* on $\mathbf{V}$ is an element $\phi \in \mathrm{CP}_k(\mathbf{V})$ such that $\phi(e) = I_k$. A matrix state $\phi \in \mathrm{CP}_k(\mathbf{V})$ is said to be *pure* if for every $\psi \in \mathrm{CP}_k(\mathbf{V})$ satisfying $\psi \leq_{\mathrm{cb}} \phi$ there is a $t \in [0,1] \subseteq \mathbb{R}$ such that $\psi = t\phi$. If $\phi : \mathbf{V} \to M_k$ is a linear mapping such that $\phi(e) = I_k$, then $\phi$ is completely positive if and only if it is a complete contraction (see [5, Corollary 5.1.2]). Thus in the case of an operator system, $\mathcal{S}_k(\mathbf{V}; e)$ is the set of all matrix states $\phi : \mathbf{V} \to M_k$.

DEFINITION 4.1. A *matrix convex* set in a vector space $\mathbf{V}$ is a collection $K = (K_n)$ of subsets $K_n \subseteq M_n(\mathbf{V})$ such that

$$\sum_i \gamma_i^* x_i \gamma_i \in K_n$$

for all $x_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ for $i = 1, \ldots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = I_n$.

Given a collection $S = (S_n)$ of subsets $S_n \subseteq M_n(\mathbf{V})$ for some local convex vector space $\mathbf{V}$, we define the *closed matrix convex hull* $\overline{co}(S)$ to be the smallest closed matrix convex set containing $S$. We can describe $\overline{co}(S)$ as the closure of all elements $x \in M_n(\mathbf{V})$ of the form

$$\sum_i \gamma_i^* x_i \gamma_i \in K_n$$

for all $x_i \in S_{n_i}$ and $\gamma_i \in M_{n_i,n}$ for $i = 1, \ldots, k$ satisfying $\sum_i \gamma_i^* \gamma_i = I_n$.

DEFINITION 4.2. We define a *compact matrix convex set* to be a matrix convex subset $K = (K_n)$ of a locally convex vector space $\mathbf{V}$ such that each $K_n$ is compact in the product topology in $M_n(\mathbf{V})$.

DEFINITION 4.3. Suppose that $K = (K_n)$ is a matrix convex set in a vector space $\mathbf{V}$. A *matrix extreme point* of $K$ is an element $v \in K_n$ for some $n \in \mathbb{N}$ with the property: whenever

$$v = \sum_{i=1}^k \gamma_i^* v_i \gamma_i,$$

with $v_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ $(i = 1, \ldots, k)$ such that each $\gamma_i$ is a right-invertible complex matrix satisfying $\sum_{i=1}^k \gamma_i^* \gamma = I_n$, we have each $n_i = n$ and $v = u_i v_i u_i$ for some unitary $u_i \in M_n$.

We denote by $\partial K_n$ the (possibly empty) set of matrix extreme points in $K_n$.

If $\mathbf{V}$ is a 3-dimensional operator system, then it has a basis in a particularly useful form: $\mathbf{V} = \operatorname{span}\{e, u, u^*\}$, where $u$ is called the "generator" of $\mathbf{V}$. We can easily get the following proposition from [6, Theorem 3.1 and Theorem 3.2] as well as [6, Proposition 5.2].

LEMMA 4.4. *Assume that* $\mathbf{V} := \operatorname{span}\{e, u, u^*\}$ *is a* 3*-dimensional system in a unital* $C^*$*-algebra and that* $\|u\| = 1$ *and* $\mathbb{T} \subseteq \sigma(u)$, *where* $\sigma(u)$ *denotes the spectrum of* $u$. *Then* $u$ *is a complete strict geometric unitary.*

*Proof.* Let $\mathcal{S} := (\mathcal{S}_n(\mathbf{V}; e))_{n\in\mathbb{N}}$. Then $\mathcal{S}$ is a compact matrix convex set in the point-norm topology (see [6] for details). Thus, [6, Theorem 3.2] tells us that $\mathcal{S}$ is the closure of the matrix convex hull of the matrix extreme points of $\mathcal{S}$. Then for any $k\in\mathbb{N}$ and $x\in M_k(\mathbf{V})$,

$$\begin{aligned}\|x\|_k &= \sup\{\|\varphi_k(x)\| : \varphi\in\mathcal{S}_n(\mathbf{V}; e), n\in\mathbb{N}\}\\ &= \sup\{\|\phi_k(x)\| : \phi\in\partial\mathcal{S}_n(\mathbf{V}; e), n\in\mathbb{N}\}.\end{aligned}$$

Moreover, [6, Theorem 3.1] together with [6, Proposition 5.2] tells us that

$$\begin{aligned}\|x\|_k &= \sup\{\|\phi_k(x)\| : \phi\in\partial\mathcal{S}_n(\mathbf{V}; e), n\in\mathbb{N}\}\\ &= \sup\{\|\phi_k(x)\| : \phi\in\mathcal{S}_1(\mathbf{V}; e) \text{ and } |\phi(u)| = 1\}.\end{aligned}$$

This completes the proof. □

THEOREM 4.5. *Assume that* $\mathbf{V} := \operatorname{span}\{e, u, u^*\}$ *is a* 3*-dimensional system in a unital* $C^*$*-algebra, where* $u$ *is an isometry. Then* $u$ *and* $u^*$ *are complete strict geometric unitaries.*

*Proof.* We can assume that $\mathbf{V}\subseteq\mathcal{L}(\mathbf{H})$ on a Hilbert space $\mathbf{H}$. If $u$ is a unilateral shift operator on $\mathbf{H}$, then $\operatorname{sp}(u) = \operatorname{sp}(u^*) = \mathbb{D}$ (the closed unit disc) by [8, the solution of Problem 82]. It follows from Lemma 4.4 that $n_{\mathrm{cb}}(\mathbf{V}; u) = n_{\mathrm{cb}}(\mathbf{V}; u^*) = 1$. If $u$ is an isometry, we recall from [8, Problem 149] that every isometry is either a unitary, or a direct sum of one or more copies of the unilateral shift, or a direct sum of a unitary operator and some copies of the unilateral shift. Using Proposition 2.6, we obtain the required result. □

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