Illinois Journal of Mathematics
Volume 56, Number 2, Summer 2012, Pages 307–341
S 0019-2082

DISPERSIVE ESTIMATES FOR MATRIX AND SCALAR SCHRÖDINGER OPERATORS IN DIMENSION FIVE

WILLIAM R. GREEN

ABSTRACT. We investigate the boundedness of the evolution operators e^{itH} and $e^{it\mathcal{H}}$ in the sense of $L^1 \to L^\infty$ for both the scalar Schrödinger operator $H = -\Delta + V$ and the non-selfadjoint matrix Schrödinger operator

$$\mathcal{H} = \begin{bmatrix} -\Delta + \mu - V_1 & -V_2 \\ V_2 & \Delta - \mu + V_1 \end{bmatrix}$$

in dimension five. Here $\mu > 0$ and V_1, V_2 are real-valued decaying potentials. The matrix operator arises when linearizing about a standing wave in certain nonlinear partial differential equations. We apply some natural spectral assumptions on $\mathcal{H}$, including regularity of the edges of the spectrum $\pm\mu$.

1. Introduction

Consider the linear scalar Schrödinger equation with a real-valued decaying potential

$$iu_t - \Delta u + Vu = 0, \quad u(x,0) = f(x) \in \mathcal{S}(\mathbb{R}^5). \tag{1}$$

The scalar Schrödinger operator $H = -\Delta + V$ on $\mathbb{R}^5$ is used to write the solution to (1) as $u(x,t) = e^{itH} f(x)$. Such a function formally solves (1), though it requires some care to interpret the solution operator e^{itH}.

Estimates from $L^2 \to L^2$ for the solution operator are a consequence of e^{itH} being an isometry on L^2 for sufficiently decaying potentials V. Let P_{ac} be projection onto the absolutely continuous spectral subspace of L^2 determined by H. Dispersive, or $L^1 \to L^\infty$ estimates for the evolution operator $e^{itH} P_{ac}(H)$ have been studied under various decay and smoothness condi-

Received September 29, 2010; received in final form January 11, 2012.
The author was partially supported by NSF Grant DMS-09-00865.
2010 *Mathematics Subject Classification.* 35Q41, 42B20.

tions on the potential V. Much work has been done in dimensions $n \leq 3$, see, for instance, the works of Rodnianski and Schlag [46], Schlag [50], Goldberg and Schlag [30], Goldberg [28], [29], Journé, Soffer and Sogge [35], Yajima [66], [65], Erdoğan and Schlag [23], [22] and Weder [62]. Earlier works in the weighted L^2 sense were investigated by Rauch [44], Jensen [33] and Jensen and Kato [34]. The best result in dimension three in print known to the author is that of Goldberg in [28], in which the potential is assumed to lie in $L^p \cap L^q$ with $p < \frac{3}{2} < q$. Very recently, Beceanu and Goldberg proved dispersive estimates hold for scaling-critical potentials, [5], through use of an operator-theoretic Wiener theorem. This result supercedes the $L^p \cap L^q$ result by allowing potentials in the closure of compactly supported functions with respect to the global Kato norm. For a more detailed account of the scalar case, see the survey paper [49].

In higher dimensions, $n > 3$, Goldberg and Visan showed that decay or integrability conditions do not suffice to ensure the dispersive estimates are satisfied. In [31], they construct a compactly supported potential V in $C^\alpha(\mathbb{R}^n)$ for $\alpha < \frac{n-3}{2}$ for which the dispersive estimates cannot hold. Journé, Soffer and Sogge's work applies in higher dimensions, but the assumption of $\hat{V} \in L^1(\mathbb{R}^n)$ necessitates a certain amount of smoothness on the potential. For instance, $V \in H^{\frac{n}{2}+\epsilon}(\mathbb{R}^n)$ ensures that $\hat{V} \in L^1(\mathbb{R}^n)$. Yajima [65], and Yajima and Finco [24] prove dispersive estimates hold if the potential obeys $|V(x)| \lesssim \langle x \rangle^{-(n+2)-}$ and a weighted Fourier transform of the potential obeys a certain integrability condition. This integrability condition is satisfied if more than $\frac{n-1}{2} - \frac{1}{n-2}$ derivatives of V are in L^2. Work on higher dimensional dispersive estimates using techniques of semi-classical analysis has been explored by Vodev [60], [61], Moulin and Vodev [40], and Cardoso, Cuevas and Vodev in [11], [12], [13]. The five-dimensional result in [11] necessitates $V \in C^{1+}(\mathbb{R}^5)$ along with decay of the potential and its derivative. In [20] Erdoğan and the author show that $V \in C^{\frac{n-3}{2}}(\mathbb{R}^n)$ is the sharp smoothness requirement, along with sufficient decay on the potential and its derivatives, for dimensions $n = 5, 7$. This work gives a heuristic argument that shows the smoothness requirement of $V \in C^{\frac{n-3}{2}}(\mathbb{R}^n)$ should hold in all odd dimensions $n > 3$.

It is well known that the presence of a resonance or eigenvalue at zero energy destroys the $|t|^{-n/2}$ decay. This was observed by Rauch to be the generic case in [44]. Rauch, Jensen and Kato [34] and Murata [41] all showed that in dimension three the decay rate will be $|t|^{-1/2}$ in the weighted L^2 setting. The $L^1 \to L^\infty$ case was handled by Erdoğan and Schlag [23] in terms of $|t|^{-1/2}$ and $|t|^{-3/2}$. Yajima [67] and Goldberg [27] independently proved estimates when there are eigenvalues at zero in $\mathbb{R}^3$. In this paper, we avoid the case in which eigenvalues occur by assuming regularity of the edge of the spectrum.

We consider the non-selfadjoint matrix Hamiltonian,

$$\mathcal{H} = \mathcal{H}_0 + V = \begin{bmatrix} -\Delta + \mu & 0 \\ 0 & \Delta - \mu \end{bmatrix} + \begin{bmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{bmatrix}. \tag{2}$$

To the best of the author's knowledge, $L^1 \to L^\infty$ dispersive estimates for this operator have only been studied in dimension three. Notably by Schlag [52], Erdoğan and Schlag [23] and Beceanu [3], [4]. A proof using scattering theory was considered by Marzuola [37]. In [52], it is shown that dispersive estimates hold under the same assumptions on V_i for the matrix case as V in the scalar case, namely regularity of zero and $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 3$. In [23], more decay was required, $\beta > 5$ to handle the cases when zero energy is an eigenvalue or resonance.[1]

The applications of the matrix Schrödinger operator (2) to nonlinear partial differential equations were first studied by Cuccagna [17]. When linearizing about a ground state solution to certain nonlinear Schrödinger equations, one obtains a version of (2) where the potentials depend on the ground state. This is discussed further in Section 2.

In this paper, we prove the following dispersive estimate for the scalar operator.

THEOREM 1.1. *Assume that zero is not an eigenvalue*[2] *of* $H = -\Delta + V$, *where* $V \in C^1(\mathbb{R}^5)$ *with* $|V(x)| \lesssim \langle x\rangle^{-\beta}$ *for some* $\beta > 4$, *and* $|\nabla V(x)| \lesssim \langle x\rangle^{-\alpha}$ *for some* $\alpha > 3$. *Then*

$$\left\| e^{itH} P_{ac} \right\|_{1\to\infty} \lesssim |t|^{-\frac{5}{2}}.$$

This result more than cuts in half the decay requirement needed in [20]. This result relies on the method used in [20] and an improved estimate for the tail of the Born series which is proven in Section 5.

A similar result holds for the matrix Hamiltonian, (2). This requires a few more assumptions on the spectrum of $\mathcal{H}$, which are laid out in the next section. We note that $\mu > 0$ in the definition (2).

THEOREM 1.2. *Assume that* $\pm\mu$ *is not an eigenvalue of* $\mathcal{H}$. *Further for* $i = 1, 2$, $V_i \in C^1(\mathbb{R}^5)$ *with* $|V_i(x)| \lesssim \langle x\rangle^{-\beta}$ *for some* $\beta > 4$, *and* $|\nabla V_i(x)| \lesssim \langle x\rangle^{-\alpha}$ *for some* $\alpha > 3$. *Then*

$$\left\| e^{it\mathcal{H}} P_{c} \right\|_{1\to\infty} \lesssim |t|^{-\frac{5}{2}}.$$

This paper is organized as follows, we first set up the necessary spectral theory to reduce the problem of dispersive estimates to estimating certain oscillatory integrals in Section 2. In Section 3, we use a Born series expansion

[1] During the review period for this article, much work has been done on related problems. Of particular note is the recent work of Erdoğan and the author [21] for matrix operators in two spatial dimensions.

[2] Resonances cannot occur in dimensions $n \geq 5$ as $(-\Delta)^{-2}\langle x\rangle^{-2}$ is a bounded operator on $L^2(\mathbb{R}^n)$ for $n \geq 5$.

and estimate each resulting term, essentially reducing the matrix case to the scalar case as in [20]. In Section 5, we improve the estimates used in [20] to control the tail of the Born series. Finally in Section 6, we apply the scalar Born series tail results to the matrix case.

2. Spectral theory

Consider the matrix Schrödinger operator, given in (2), on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Here $\mu > 0$ and V_1, V_2 are real-valued decaying potentials. It follows from a Weyl's criterion argument that the essential spectrum of $\mathcal{H}$ is $(-\infty, -\mu] \cup [\mu, \infty)$, see [32], [45], for example.

For the spectral theory of the matrix Schrödinger operator, we proceed in the manner of Erdoğan and Schlag [23]. In fact, as most of the proofs presented in [23] are independent of dimension, as such we cite the results without proof. Accordingly, we make the following assumptions:

(A1) $L_- = -\Delta + \mu - V_1 + V_2 \geq 0$,
(A2) $|V_1(x)| + |V_2(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 0$,
(A3) there are no embedded eigenvalues in $(-\infty, -\mu) \cup (\mu, \infty)$.

The first two assumptions are known to apply in the case of the linearized nonlinear Schrödinger equation when the linearization is performed about the positive ground state standing wave. One takes the positive ground state standing wave $\phi(x)$ and assume that for some $\mu > 0$, $\psi(t, x) = e^{it\mu}\phi(x)$ is a standing wave solution of the nonlinear Schrödinger equation

$$i\partial_t \psi + \Delta\psi + |\psi|^{2\gamma}\psi = 0, \tag{3}$$

for some $\gamma > 0$. Assuming that ϕ is the ground state implies that

$$\mu\phi - \Delta\phi = \phi^{2\gamma+1}, \quad \phi > 0.$$

Linearizing about this ground state yields the matrix Schrödinger equation with potentials $V_1 = (\gamma + 1)\phi^{2\gamma}$ and $V_2 = \gamma\phi^{2\gamma}$. It is known that the ground state solutions to (3) exist and are smooth, radial and exponentially decaying. This is proved in [7], [57] by Berestycki and Lions, and Strauss respectively. Uniqueness was proved in [15], [36], [39] by Coffman, Kwong and McLeod and Serrin respectively. Other bound states are investigated by Berestycki and Lions in [8], which expands upon the results in [7].

The orbital (or Lyapunov) stability or instability of the ground states has been a subject of much work. See for example the works of Berestycki and Cazenave [6], Cazenave and Lions [14], Comech and Pelinovsky [16], Grillakis and Shatah [25], [26], Shatah [53], Shatah and Strauss [54], and Weinstein [63], [64]. Much of this work was reviewed in the monographs by Strauss and Sulem and Sulem [58], [59].

The asymptotic stability of ground states is a stronger requirement than orbital stability and has also been well investigated. For small solitons one can see the work of Soffer and Weinstein [55], [56]. For large solitons, see Busalev and Perelman [9], [10], Cuccagna [17], Perelman [42], [43], and Rodnianski, Schlag and Soffer [48], [47].

The Schrödinger equation can show up when linearizing in other nonlinear partial differential equations. A more detailed discussion of the nonlinear Schrödinger case can be found in [22], [52]. For a more thorough review of applications to other nonlinear equations, see the survey paper [51].

We now discuss the needed spectral theory for the matrix Schrödinger operator of (2).

LEMMA 2.1. *Let $\beta > 0$ be arbitrary in* (A2), *then the essential spectrum of $\mathcal{H}$ equals $(-\infty, -\mu] \cup [\mu, \infty)$. Moreover* $\operatorname{spec}(\mathcal{H}) = -\operatorname{spec}(\mathcal{H}) = \overline{\operatorname{spec}(\mathcal{H})} = \operatorname{spec}(\mathcal{H}^*)$, *and* $\operatorname{spec}(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$. *The discrete spectrum of $\mathcal{H}$ consists of eigenvalues $\{z_j\}_{j=1}^N$, $0 \le N \le \infty$, of finite multiplicity. For each $z_j \ne 0$, the algebraic and geometric multiplicity coincide and* $\operatorname{Ran}(\mathcal{H} - z_j)$ *is closed. The zero eigenvalue has finite algebraic multiplicity, that is, the generalized eigenspace $\bigcup_{k=1}^\infty \ker(\mathcal{H}^k)$ has finite dimension. In fact, there is a finite $m \ge 1$ so that $\ker(\mathcal{H}^k) = \ker(\mathcal{H}^{k+1})$ for all $k \ge m$.*

Proof. See Lemma 3 of [23]. □

As in the scalar case, see [30], [20] etc., the proofs will hinge on the limiting absorption principle of Agmon [2]. We now establish such a result for $(\mathcal{H} - z)^{-1}$ for $|z| > \mu$. From the weighted L^2 space $L^{2,\sigma}(\mathbb{R}^n) = \langle x\rangle^{-\sigma} L^2(\mathbb{R}^n)$, we define the space

$$X_\sigma := L^{2,\sigma}\big(\mathbb{R}^5\big) \times L^{2,\sigma}\big(\mathbb{R}^5\big).$$

It is clear that $X_\sigma^* = X_{-\sigma}$. The limiting absorption principle of Agmon is now formulated below.

PROPOSITION 2.2. *Let $\beta > 1$, $\sigma > \frac{1}{2}$ and fix an arbitrary $\lambda_0 > \mu$. Then*

$$\sup_{|\lambda| \ge \lambda_0, \epsilon \ge 0} |\lambda|^{\frac{1}{2}} \big\| \big(\mathcal{H} - (\lambda \pm i\epsilon)\big)^{-1} \big\| < \infty, \tag{4}$$

where the norm is in $X_\sigma \to X_{-\sigma}$.

Proof. See Lemma 4, Proposition 5 and Corollary 6 of [23]. □

To establish dispersive estimates for the matrix case, we need the analogue of projection onto the continuous spectrum. Let P_d be the Riesz projection onto the discrete spectrum of $\mathcal{H}$. Under the assumption that $\pm\mu$ are not eigenvalues of $\mathcal{H}$, we define the following projection

$$P_c = I - P_d.$$

This projection is more complicated in the case where $\pm\mu$ are eigenvalues, see [23]. P_c is now analogous to the projection onto the absolutely continuous spectrum in the scalar case. We project away from the eigenspaces, this requires the assumption that there are no embedded eigenvalues. This assumption seems to hold in the three-dimensional case as evidenced in the numerical studies [18], [38].

3. Born series

Assuming that there are no eigenvalues embedded in the essential spectrum, we begin with the spectral representation

$$
\begin{aligned}
e^{it\mathcal{H}} = \frac{1}{2\pi i}\int_{|\lambda|\geq\mu} e^{it\lambda}\big[\big(\mathcal{H}-(\lambda+i0)\big)^{-1}-\big(\mathcal{H}-(\lambda-i0)\big)^{-1}\big]\,d\lambda \\
+\sum_j e^{it\mathcal{H}}P_{\lambda_j}
\end{aligned}
\tag{5}
$$

with the sum running over the discrete spectrum $\{\lambda_j\}_j$ and P_{λ_j} is the Riesz projection corresponding to λ_j. This representation holds as a consequence of the Hille–Yosida theorem, as shown in [23], [52]

We denote the matrix resolvent operators $R_V^{\pm}(\lambda)=(\mathcal{H}-(\lambda\pm i0))^{-1}$ and $R_0^{\pm}(\lambda)=(\mathcal{H}_0-(\lambda\pm i0))^{-1}$.

We note that P_c is analogous to projection onto the absolutely continuous spectrum in the scalar case. So that, from (5), we wish to examine the operator

$$
e^{it\mathcal{H}}P_c=\frac{1}{2\pi i}\int_{|\lambda|\geq\mu} e^{it\lambda}\big[(R_V^+(\lambda)-R_V^-(\lambda)\big]\,d\lambda.
$$

We write $P_c=P_c^++P_c^-$, where P_c^+ projects onto $[\mu,\infty)$ and P_c^- projects onto $(-\infty,-\mu]$. Now using a finite resolvent expansion and a change of variables $\lambda\to\lambda^2+\mu$,

$$
\begin{aligned}
&\big\langle e^{it\mathcal{H}}P_c^+f,g\big\rangle\\
&\quad=\frac{e^{it\mu}}{\pi i}\int_0^\infty \lambda e^{it\lambda^2}\big\langle\big[R_V^+\big(\lambda^2+\mu\big)-R_V^-\big(\lambda^2+\mu\big)\big]f,g\big\rangle\,d\lambda\\
&\quad=\frac{e^{it\mu}}{\pi i}\Bigg(\sum_{\ell=0}^{2m+1}(-1)^\ell\int_0^\infty \lambda e^{it\lambda^2}\big\langle\big[R_0^+\big(\lambda^2+\mu\big)\big(VR_0^+\big(\lambda^2+\mu\big)\big)^\ell\\
&\qquad -R_0^-\big(\lambda^2+\mu\big)\big(VR_0^-\big(\lambda^2+\mu\big)\big)^\ell\big]f,g\big\rangle\,d\lambda \qquad (6)\\
&\qquad +\int_0^\infty \lambda e^{it\lambda^2}\big\langle\big[\big(R_0^+\big(\lambda^2+\mu\big)V\big)^{m+1}R_V^+\big(\lambda^2+\mu\big)\big(VR_0^+\big(\lambda^2+\mu\big)\big)^{m+1}\\
&\qquad -\big(R_0^+\big(\lambda^2+\mu\big)\big)^{m+1}R_V^-\big(\lambda^2+\mu\big)\big(VR_0^-\big(\lambda^2+\mu\big)\big)^{m+1}\big]f,g\big\rangle\,d\lambda\Bigg). \qquad (7)
\end{aligned}
$$

Here we are calculating the contribution of P_c^+, the projection onto $[\mu,\infty)$, the contribution of P_c^- is done in the same manner.

In odd dimensions $n \geq 3$, $(-\Delta - z)^{-1}$ is an integral operator with kernel

$$(-\Delta - z)^{-1}(x,y) = \frac{i}{4}\left(\frac{z^{\frac{1}{2}}}{2\pi|x-y|}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}\left(z^{\frac{1}{2}}|x-y|\right). \tag{8}$$

Here $H^{(1)}_{\nu}(\cdot)$ is a Hankel function of the first kind of order ν. We use the following explicit representation for the kernel of the limiting resolvent operator $R_0^{\pm}(\lambda^2)$ (see, e.g., [33])

$$R_0^{\pm}(\lambda^2)(x,y) = \mathcal{G}_n\left(\pm\lambda, |x-y|\right),$$

where

$$\mathcal{G}_n(\lambda, r) = C_n \frac{e^{i\lambda r}}{r^{n-2}} \sum_{\ell=0}^{\frac{n-3}{2}} \frac{(n-3-\ell)!}{\ell!(\frac{n-3}{2}-\ell)!}(-2ir\lambda)^{\ell}. \tag{9}$$

We also define

$$\mathcal{G}_1(\lambda, r) = C_1 \frac{e^{i\lambda r}}{\lambda}.$$

LEMMA 3.1. *For $n \geq 3$ and odd, the following recurrence relation holds.*

$$\left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)\mathcal{G}_n(\lambda, r) = \frac{1}{2\pi}\mathcal{G}_{n-2}(\lambda, r).$$

Proof. The proof follows from the recurrence relations of the Hankel functions, found in [1] and the representation of the kernel given in (8). One can also prove this (with a fixed constant instead of 2π) directly using (9). □

For the matrix operator (2), we have

$$R_0^{\pm}\left(\lambda^2 + \mu\right)(x,y) = \begin{pmatrix} \mathcal{G}_n(\lambda, |x-y|) & 0 \\ 0 & \mathcal{J}_n(\lambda, |x-y|) \end{pmatrix}, \tag{10}$$

where $\mathcal{J}_n(\lambda, r) = \mathcal{G}_n(i\sqrt{2\mu + \lambda^2}, r)$. In particular, it is worth noting that $\mathcal{J}_n(\lambda, r)$ is always real-valued and exponentially decaying in r.

COROLLARY 3.2. *For $n \geq 3$, the following recurrence relation holds.*

$$\frac{1}{\lambda}\frac{d}{d\lambda}\mathcal{J}_n(\lambda, r) = \frac{1}{2\pi}\mathcal{J}_{n-2}(\lambda, r).$$

We note that the five-dimensional kernel takes the form:

$$\mathcal{G}_5(\lambda, r) = C_5 e^{i\lambda r}\left(\frac{i\lambda}{r^2} + \frac{1}{r^3}\right).$$

We state the following lemma for the Fourier transforms of certain classes of functions which will be used throughout the paper.

LEMMA 3.3. *Let $\psi_\mu(\lambda)=\sqrt{2\mu+\lambda^2}$, then for $m\in\mathbb{N}_0$*

$$\begin{aligned}
&\sup_{b\geq 0}\bigg\|\int_{\mathbb{R}} e^{-i\lambda u}\lambda\partial_\lambda e^{-b\psi_\mu(\lambda)}\,d\lambda\bigg\|_{\mathcal{M}_u}\\
&\quad=\sup_{\mu\geq 0}\bigg\|\int_{\mathbb{R}} e^{-i\lambda u}\lambda\partial_\lambda e^{-\sqrt{\mu+\lambda^2}}\,d\lambda\bigg\|_{\mathcal{M}_u}<\infty,\\
&\sup_{b\geq 0}\bigg\|\int_{\mathbb{R}} \big(b\psi_\mu(\lambda)\big)^m e^{-b\psi_\mu(\lambda)}e^{-i\lambda u}\,d\lambda\bigg\|_{\mathcal{M}_u}\\
&\quad=\sup_{\mu\geq 0}\bigg\|\int_{\mathbb{R}} \big(\sqrt{\mu+\lambda^2}\big)^m e^{-\sqrt{\mu+\lambda^2}}e^{-i\lambda u}\,d\lambda\bigg\|_{\mathcal{M}_u}<\infty,
\end{aligned}$$

where $\|\cdot\|_{\mathcal{M}}$ stands for the total variation norm of measures.

Proof. The first bound is proven in Section 7 of [52] as well as the $m=0$ case of the second bound. The second bound follows from

$$\begin{aligned}
\partial_\lambda\Big(\big(\sqrt{\mu+\lambda^2}\big)^m e^{-\sqrt{\mu+\lambda^2}}\Big)&=\lambda\big(\sqrt{\mu+\lambda^2}\big)^{m-2}e^{-\sqrt{\mu+\lambda^2}}\big(\sqrt{\mu+\lambda^2}+m\big),\\
\partial_\lambda^2\Big(\big(\sqrt{\mu+\lambda^2}\big)^m e^{-\sqrt{\mu+\lambda^2}}\Big)&=e^{-\sqrt{\mu+\lambda^2}}\big(\sqrt{\mu+\lambda^2}\big)^{m-4}\big[m(m-2)\lambda^2\\
&\quad+\sqrt{\mu+\lambda^2}(2m-1)\lambda^2+\big(\sqrt{\mu+\lambda^2}\big)^2\big(\lambda^2-m\big)\\
&\quad+\big(\sqrt{\mu+\lambda^2}\big)^m\big].
\end{aligned}$$

As for $m\in\mathbb{N}_0$ each of these derivatives are in $L^1(\mathbb{R})$ with norms uniformly bounded in $\mu>0$. It now follows that

$$\sup_{\mu\geq 0}\big(1+u^2\big)\bigg|\int_{\mathbb{R}}\big(\sqrt{\mu+\lambda^2}\big)^m e^{-\sqrt{\mu+\lambda^2}}e^{-i\lambda u}\,d\lambda\bigg|\lesssim 1,$$

and the lemma is proven. □

4. The κth term of the Born series

From (6), to establish Theorem 1.2, we wish to prove estimates of the form

$$\bigg|\int_0^\infty e^{it\lambda^2}\lambda\Im\big[R_0^\pm(\lambda^2+\mu)\big[VR_0^\pm(\lambda^2+\mu)\big]^\kappa\big]e_k\,d\lambda\bigg|\lesssim|t|^{-5/2},$$

here e_k are the unit vectors $e_1=(1,0)^T$ and $e_2=(0,1)^T$. By writing the matrix resolvent $R_0^\pm(\lambda^2+\mu)$ as a diagonal matrix whose entries are the kernels $\mathcal{J}_n(\lambda,r)$ and $\mathcal{G}_n(\lambda,r)$ as in (10), we can reduce down to integral estimates of the same form as considered by Erdoğan and the author in [20].

Let $\mathcal{I}$ and $\mathcal{I}^*$ form a partition of $\{0,1,\dots,\kappa\}$. Noting that $\mathcal{J}_n(\lambda,r)$ is real-valued and even in λ and the imaginary part of $\prod_{j\in\mathcal{I}}\mathcal{G}_n(\lambda,r_j)$ is even, we extend the λ integral to $\mathbb{R}$. We are led to proving estimates of the form

$$\sup_{z_0, z_{\kappa+1}} \left| \int_{\mathbb{R}^{5\kappa+1}} e^{it\lambda^2} \lambda \Im \left[\prod_{j\in\mathcal{I}} \mathcal{G}_5(\lambda, r_j) \prod_{j\in\mathcal{I}^*} \mathcal{J}_5(\lambda, r_j) \prod_{\ell=1}^{\kappa} V(z_\ell) \right] d\vec{z}\, d\lambda \right| \lesssim |t|^{-5/2}, \tag{11}$$

where $r_j = |z_j - z_{j+1}|$ and $d\vec{z} = dz_1 dz_2 \dots dz_\kappa$.

We note that we can immediately ignore the case of $\mathcal{I} = \emptyset$ as $\mathcal{J}_n$ is real-valued. Further if $\mathcal{I}^* = \emptyset$, we can reduce to the scalar case which is handled by Erdoğan and the author in [20] for $n = 5, 7$. As such, we will assume that both $\mathcal{I}$ and $\mathcal{I}^*$ are nonempty. We can now differ from the scalar case by omitting the large λ cut-off function χ_L as $\mathcal{J}_5(\lambda, r)$ is exponentially decaying in λ.

As in the scalar case, we perform $\frac{n-1}{2} = 2$ integration by parts in λ and the left hand side of (11) becomes

$$|t|^{-2} \int_{\mathbb{R}^{5\kappa+1}} e^{it\lambda^2} \lambda \Im \left[\prod_{j\in\mathcal{I}} \mathcal{G}_{5-2\alpha_j}(\lambda, r_j) \prod_{j\in\mathcal{I}^*} \mathcal{J}_{5-2\alpha_j}(\lambda, r_j) \prod_{\ell=1}^{\kappa} V(z_\ell) \right] d\vec{z}\, d\lambda,$$

with $\alpha_0, \dots, \alpha_\kappa \in \mathbb{N}_0$ such that $\sum_{j=0}^{\kappa} \alpha_j = 2$.

4.1. The first term of the Born series. We can assume without loss of generality that $\mathcal{I} = \{1\}$ and $\mathcal{I}^* = \{0\}$ by the assumption that both sets are nonempty. There are now three distinct cases which depend on where the λ derivatives act.

Case 1. Consider the case when both λ derivatives act on the kernel $\mathcal{G}_5(\lambda, r)$. By Lemma 3.1, we have

$$|t|^{-2} \int_{\mathbb{R}^{5+1}} e^{it\lambda^2} \lambda \mathcal{J}_5\big(\lambda, |z_0 - z_1|\big) V(z_1) \mathcal{G}_1\big(\lambda, |z_1 - z_2|\big)\, dz_1\, d\lambda.$$

Recall we have $\psi_\mu(\lambda) = \sqrt{2\mu + \lambda^2}$, We first note the fact that $\|[e^{it(\cdot)^2}]^\vee\|_\infty \lesssim |t|^{-1/2}$, then expanding the kernels as in (9) and ignoring constants, we have an integral of the form $|t|^{-2}$ multiplied by

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{5+1}} e^{it\lambda^2} e^{i\lambda|z_1 - z_2|} e^{-\psi_\mu(\lambda)|z_0 - z_1|} \left(\frac{\psi_\mu(\lambda)}{|z_0 - z_1|^2} + \frac{1}{|z_0 - z_1|^3} \right) V(z_1)\, dz_1\, d\lambda \right| \\
&\quad = \left| \int_{\mathbb{R}^{5+1}} e^{it\lambda^2} e^{i\lambda|z_1 - z_2|} e^{-\psi_\mu(\lambda)|z_0 - z_1|} \left(\frac{1 + \psi_\mu(\lambda)|z_0 - z_1|}{|z_0 - z_1|^3} \right) V(z_1)\, dz_1\, d\lambda \right| \\
&\quad \lesssim |t|^{-1/2} \big\| \mathcal{F}\big((1 + \sqrt{\mu + \lambda^2}) e^{-\sqrt{\mu + \lambda^2}} \big) \big\|_{\mathcal{M}} \int_{\mathbb{R}^5} \frac{|V(z)|}{|z_0 - z_1|^3}\, dz_1 \lesssim |t|^{-1/2},
\end{aligned}$$

where we used Plancherel, Lemma 3.3 and the assumption that $|V(x)| \lesssim \langle x \rangle^{-4-}$ provides more than enough decay to ensure the z_1 integral converges.

CASE 2. Consider when one λ derivative acts on each $\mathcal{G}_5(\lambda, r)$ and $\mathcal{J}_5(\lambda, r)$. By Lemma 3.1 and Corollary 3.2, we have

$$|t|^{-2}\left|\int_{\mathbb{R}^{5+1}} e^{it\lambda^2}\lambda\mathcal{J}_3\big(\lambda,|z_0-z_1|\big)V(z_1)\mathcal{G}_3\big(\lambda,|z_1-z_2|\big)\,dz_1\,d\lambda\right|$$
$$=|t|^{-2}\left|\int_{\mathbb{R}^{5+1}} e^{it\lambda^2}e^{i\lambda|z_1-z_2|}e^{-\sqrt{2\mu+\lambda^2}|z_0-z_1|}\frac{\lambda V(z_1)}{|z_0-z_1||z_1-z_2|}\,dz_1\,d\lambda\right|.$$

Noting that

$$e^{i\lambda|z_1-z_2|}=\frac{i}{\lambda}\nabla_{z_1}e^{i\lambda|z_1-z_2|}\cdot e_{z_2}(z_1),$$

where $e_x(y)$ is the unit vector in direction $x-y$. Now, we can integrate by parts in z_1 once to eliminate the λ power. The derivative can act on the exponential, the potential, either point singularity or the unit vector.

If the derivative acts on the exponential, we gain a term of the form $\sqrt{2\mu+\lambda^2}e_{z_0}(z_1)$, which we can treat as in the first case using Plancherel and Lemma 3.3. We need only bound the integral

$$\int_{\mathbb{R}^5}\frac{|V(z_1)|}{|z_0-z_1|^2|z_1-z_2|}\,dz_1,$$

up to reversal of z_0 and z_2. This integral is bounded if $|V(x)|\lesssim\langle x\rangle^{-2-}$. The same analysis holds if the derivative acts on the point singularity or unit vector as

$$\nabla_{z_1}\cdot e_{z_0}(z_1)=\frac{1-n}{|z_0-z_1|},\qquad \nabla_{z_1}|z_0-z_1|^{-1}=\frac{e_{z_0}(z_1)}{|z_0-z_1|^2}.$$

If the derivative acts on the potential, we need to bound

$$\int_{\mathbb{R}^5}\frac{|\nabla V(z_1)|}{|z_0-z_1||z_1-z_2|}\,dz_1$$

which is bounded if $|\nabla V(x)|\lesssim\langle x\rangle^{-3-}$.

CASE 3. Finally, consider when both λ derivatives act on a $\mathcal{J}_5$. Here we must use that we need only bound the imaginary part of the integrand in (11). By Corollary 3.2, we have

$$\int_{\mathbb{R}^{5+1}} e^{it\lambda^2}\lambda\mathcal{J}_1\big(\lambda,|z_0-z_1|\big)V(z_1)\mathcal{G}_5\big(\lambda,|z_1-z_2|\big)\,dz_1\,d\lambda$$
$$=\int_{\mathbb{R}^{5+1}} e^{it\lambda^2}e^{i\lambda|z_1-z_2|}e^{-\sqrt{2\mu+\lambda^2}|z_0-z_1|}V(z_1)$$
$$\times\left(\frac{\lambda^2}{\sqrt{2\mu+\lambda^2}}\frac{1}{|z_1-z_2|^2}+\frac{\lambda}{\sqrt{2\mu+\lambda^2}}\frac{1}{|z_1-z_2|^3}\right)dz_1\,d\lambda.$$

For the λ^2 term, we note that

$$|z_0 - z_1|^{-1}\lambda\partial_\lambda e^{-\sqrt{2\mu+\lambda^2}|z_0-z_1|} = \frac{\lambda^2}{\sqrt{2\mu+\lambda^2}}e^{-\sqrt{2\mu+\lambda^2}|z_0-z_1|}.$$

Thus Plancherel and Lemma 3.3 will yield the desired bound if

$$\int_{\mathbb{R}^5} \frac{|V(z_1)|}{|z_0-z_1||z_1-z_2|^2}\,dz_1$$

is bounded. This is true when $|V(x)| \lesssim \langle x\rangle^{-2-}$.

For the λ term, we note that everything in the z_1 integral is real except the imaginary exponential, which becomes $\sin(\lambda|z_1-z_2|)$. Writing

$$\sin\big(\lambda|z_1-z_2|\big) = \lambda\int_0^{|z_1-z_2|}\cos(\lambda\alpha)\,d\alpha,$$

we can apply Plancherel and Lemma 3.3 as long as

$$\int_{\mathbb{R}^5} \frac{|V(z_1)|}{|z_0-z_1||z_1-z_2|^2}\,dz_1$$

is bounded. Again if $|V(x)| \lesssim \langle x\rangle^{-2-}$ the above is bounded.

This suffices to show that Theorem 1.2 holds for the first term of the Born series in (6).

4.2. Higher Born series terms. The integrals that arise from the higher Born series terms are of the same form as the $\kappa = 1$ term. The strategies used in the $\kappa = 1$ case can be applied to the higher κ terms. First of all, when a $\mathcal{J}_1(\lambda, r_{j^*})$ occurs, the lowest order λ term can be handled as in Case 3 above, but with a series of iterated integrals of the form

$$\int_{\mathbb{R}^{5\kappa}} \prod_{\ell=1}^{\kappa} V(z_\kappa) \prod_{j\neq j^*} \frac{1}{|z_j - z_{j+1}|^3}.$$

When powers of λ occur, we simply follow the integration by parts scheme for the Born series laid out in [20]. For this, we note that

$$\nabla_z e^{-\sqrt{2\mu+\lambda^2}|x-z|} = \big(-\sqrt{2\mu+\lambda^2}|x-z|\big)e^{-\sqrt{2\mu+\lambda^2}|x-z|}\frac{x-z}{|x-z|^2}. \tag{12}$$

When integrating by parts in the z_j variables, if the derivative acts as in (12), we use the following identity to perform the integration by parts

$$e^{i\lambda|z-y|} = -\frac{i}{\lambda}\big(\nabla_z e^{i\lambda|z-y|}\big)\cdot\frac{y-z}{|y-z|}.$$

The above are needed for the case in which we have $\mathcal{J}_{5-2\alpha_j}(\lambda, |x-z|) \times \mathcal{G}_{5-2\alpha_{j+1}}(\lambda, |z-y|)$. This case is simpler than the scalar case due to the absence of the line singularity E_{xzzy}, which is replaced by the unit vector $\frac{y-z}{|y-z|}$. In [20], the following formula was used to perform the integration by parts

$$e^{i\lambda(|x-z|+|z-y|)} = -\frac{i}{\lambda}\big(\nabla_z e^{i\lambda(|x-z|+|z-y|)}\big)\cdot E_{xzzy},$$

where $E_{xzzy} = \frac{x-z}{|x-z|} - \frac{z-y}{|z-y|}$ is a superposition of two unit vectors. As seen from (12), and Lemma 3.3, integrating by parts leads to positive powers of $\sqrt{2\mu+\lambda^2}$ or acts similar to the scalar case. we simply absorb them into an expression with the decaying exponential whose Fourier transform will be a measure. In effect we can treat any $r_j^{-2}\sqrt{2\mu+\lambda^2}$ as if it were the lower order term r_j^{-3}. It was seen in [20] that the analogous λ term of the free resolvent in the scalar case caused the high energy difficulties and necessitated the smoothness condition on the potential.

From [20], we know how to handle integrals of the form

$$\int_{\mathbb{R}^{5\kappa+1}} e^{it\lambda^2}\lambda^N \frac{1}{r_0^{m_0}}\prod_{j=1}^{\kappa}\frac{V(z_j)}{r_j^{m_j}}\,d\vec{z}\,d\lambda$$

for $N \le \kappa$. This allows us to effectively reduce the matrix Born series terms to those of the scalar Born series.

4.3. Dimensions $n > 5$. In the same way that the Born series terms of the five dimensional matrix case are reduced to the five dimensional scalar case, the integration by parts scheme of [20] will reduce the seven dimensional matrix case to the seven dimensional scalar case. In fact, if one can control the terms of the scalar Born series for higher dimensions as in the scheme suggested in Section 5 of [20], by the method of integration by parts show here one can also control the terms of the matrix Born series.

5. Tail estimates for the scalar case

In this section, we prove a sharper estimate in terms of decay rate on the tail of the Born series given below in (13). Essentially, we estimate the scalar analogue of (7) in the scalar case, and in Section 6 we extend this method to the matrix case. In [20], Erdoğan and the author proved dispersive estimates for the scalar linear Schrödinger equation with optimal smoothness assumptions on the potential V in dimensions five and seven. Their analysis used tail estimates established by Goldberg and Visan in [31]. These estimates required no smoothness on V, but required a decay of $\langle x\rangle^{-\beta}$ for some $\beta > \frac{3n+5}{2}$, far more than the optimally conjectured decay rate of $\beta > 2$. Though in [31], the decay rate was not a main concern since they were constructing a counterexample.

We note that Yajima's result [65] requires that the potential obeys $|V(x)| \lesssim \langle x\rangle^{-(n+2)-}$, and its integrability requirements on the potential do not lead to sharp smoothness requirements. As mentioned in the introduction, a series of work has been done using techniques of semi-classical analysis. In [11], it is shown that dispersive estimates hold if $V \in C^{1+}(\mathbb{R}^5)$ and $|\nabla^j V(x)| \lesssim \langle x\rangle^{-5-}$ for $j=0,1$. In [40], it is shown that low-frequency dispersive estimates require only that $|V(x)| \lesssim \langle x\rangle^{-(n+2)/2-}$, however, their extension to include high frequency requires $\hat{V} \in L^1$ which does not yield optimal smoothness assumptions of the potential. The method laid out here requires slightly more decay, $|V(x)| \lesssim \langle x\rangle^{-4-}$ for both the high energy and low energy portions of the evolution, but requires no smoothness of V.

In this section, we prove dispersive estimates on the tail of the Born series in the scalar case that require only $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 4$ in dimension five. The high energy argument can be extended to arbitrary odd dimensions $n > 3$ assuming only that $|V(x)| \lesssim \langle x\rangle^{-\frac{n+3}{2}-}$. The low energy argument can be duplicated in dimension seven with $\beta > 5$, but cannot be extended in this form to $n > 7$.

In a slight bit of abuse of notation, let $R_0^{\pm}(\lambda^2)$ be the scalar free resolvent operator with kernel given in (9). By iterating the scalar resolvent identity

$$R_V(z) = R_0(z) - R_0(z)VR_V(z)$$

one obtains the Born series representation

$$R_V(z) = \sum_{\kappa=0}^{2m+1} (-1)^\kappa R_0(z)\big[VR_0(z)\big]^\kappa + \big[R_0(z)V\big]^{m+1} R_V(z)\big[VR_0(z)\big]^{m+1}. \tag{13}$$

Here we used that $R_0VR_V = R_VVR_0$. This approach is standard for the scalar case, see for example [20], [28], [30], [31], [46], [50]. We can control the contribution of the terms of the finite Born series in (13) as in [20]. One needs only control the tail of the Born series in (13). That is control the contribution of

$$\big[R_0^{\pm}(\lambda^2)V\big]^{m+1} R_V^{\pm}(\lambda^2)\big[VR_0^{\pm}(\lambda^2)\big]^{m+1}$$

for some $m \in \mathbb{N}_0$ which depends on the dimension. In this section, we prove the following theorem.

THEOREM 5.1. *In dimension five, if zero is regular and $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 4$ and $m \geq 6$,*

$$\sup_{L\geq 1}\left| \int_{\mathbb{R}} e^{it\lambda^2} \lambda \chi_L(\lambda) \int_{\mathbb{R}^{(2m+3)5}} \big(R_0^{\pm}V\big)^{m+1} R_V^{\pm} \big(VR_0^{\pm}\big)^{m+1}\, d\vec{z}\, d\lambda \right| \lesssim |t|^{-\frac{5}{2}}.$$

We state several integral estimates which we use repeatedly in the analysis of the tail of the Born series. We restate Lemma 6.3 of [20].

LEMMA 5.2. *Fix $u_1, u_2 \in \mathbb{R}^n$, and let $0 \le k, \ell < n$, $\beta > 0$, $k + \ell + \beta \ge n$, $k + \ell \neq n$. We have*

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-\beta-} dz}{|z-u_1|^k|z-u_2|^\ell} \lesssim \begin{cases} (\frac{1}{|u_1-u_2|})^{\max(0,k+\ell-n)}, & |u_1-u_2| \le 1, \\ (\frac{1}{|u_1-u_2|})^{\min(k,\ell,k+\ell+\beta-n)}, & |u_1-u_2| > 1. \end{cases}$$

The following consequence of Lemma 5.2 is often useful.

COROLLARY 5.3. *Fix $u_1, u_2 \in \mathbb{R}^n$, and let $0 \le k, \ell < n$, $\beta > 0$, $k+\ell+\beta \ge n$, $k + \ell \neq n$. We have*

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-\beta-}\, dz}{|z-u_1|^k|z-u_2|^\ell} \lesssim \left(\frac{1}{|u_1-u_2|}\right)^{\min(k,\ell,k+\ell+\beta-n)}.$$

We also note Lemma 3.8 of [31].

LEMMA 5.4. *Let μ and σ be such that $\mu < n$ and $n < \sigma + \mu$. Then*

$$\int_{\mathbb{R}^n} \frac{dy}{\langle y\rangle^\sigma |x-y|^\mu} \lesssim \begin{cases} \langle x\rangle^{n-\sigma-\mu}, & \sigma < n, \\ \langle x\rangle^{-\mu}, & \sigma > n. \end{cases}$$

LEMMA 5.5. *Let $0 < \mu, \gamma$ be such that and $n < \gamma + \mu$. Then*

$$\int_{\mathbb{R}^n} \langle x\rangle^{-\gamma} \langle x+y\rangle^{-\mu}\, dx \lesssim \langle y\rangle^{-\min(\gamma,\mu,\gamma+\mu-n)}.$$

Proof. If $|y| < 1$ the result is trivial, assume $|y| > 1$. We divide $\mathbb{R}^n$ into four regions. First, on $|x| < \frac{1}{2}|y|$, we have $|x-y| \approx y$, and this contributes

$$|y|^{-\mu} \int_{|x|<\frac{1}{2}|y|} \langle x\rangle^{-\gamma}\, dx \lesssim |y|^{-\mu} + |y|^{n-\gamma-\mu}$$

by considering the regions $|x| < 1$ and $|x| > 1$ separately. Similarly on $|x+y| < \frac{1}{2}|y|$, this region contributes

$$|y|^{-\gamma} \int_{|x+y|<\frac{1}{2}|y|} \langle x+y\rangle^{-\mu}\, dx \lesssim |y|^{-\gamma} + |y|^{n-\gamma-\mu}.$$

On the complement of the above regions in $|x| < 2|y|$, we have $|x| \approx |x+y| \approx |y|$ and thus this region contributes $|y|^{n-\gamma-\mu}$. Finally, on $|x| > 2|y|$ we have $|x+y| \approx |y|$ and this region contributes

$$\int_{|x|>2|y|} \langle x\rangle^{-\gamma-\mu}\, dx \lesssim |y|^{n-\gamma-\mu}. \qquad \square$$

Finally, we have the following lemma for stationary phase type estimates. This lemma is essentially Lemma 2 in [50].

LEMMA 5.6. *Let* $\phi'(0)=0$ *and* $1\le\phi''\le C$. *Then,*

$$\Big|\int_{-\infty}^{\infty} e^{it\phi(\lambda)}a(\lambda)\,d\lambda\Big| \lesssim \int_{|\lambda|<|t|^{-\frac12}} \big|a(\lambda)\big|\,d\lambda \\ + |t|^{-1}\int_{|\lambda|>|t|^{-\frac12}} \bigg(\frac{|a(\lambda)|}{|\lambda^2|}+\frac{|a'(\lambda)|}{|\lambda|}\bigg)\,d\lambda.$$

Proof. Let $\eta\in C_c^\infty$ be such that $\eta(x)=1$ if $|x|<1$ and $\eta(x)=0$ if $|x|>2$. Let $\eta_2(x)=\eta(x/2|t|^{-1/2})$. Writing $1=\eta_2+(1-\eta_2)$, we rewrite the integral as follows

$$\Big|\int_{-\infty}^{\infty} e^{it\phi(\lambda)}a(\lambda)\,d\lambda\Big| \lesssim \Big|\int_{-\infty}^{\infty} e^{it\phi(\lambda)}a(\lambda)\eta_2(\lambda)\,d\lambda\Big| \\ + \Big|\int_{-\infty}^{\infty} e^{it\phi(\lambda)}a(\lambda)\big(1-\eta_2(\lambda)\big)\,d\lambda\Big|.$$

The first term is bounded as in the claim since $\operatorname{supp}(\eta_2)=[-|t|^{-\frac12},|t|^{-\frac12}]$. For the second term, we integrate by parts once in λ to bound with

$$|t|^{-1}\Big|\int_{-\infty}^{\infty} e^{it\phi(\lambda)}\bigg(\frac{a(\lambda)(1-\eta_2(\lambda))}{\phi'(\lambda)}\bigg)'\,d\lambda\Big|.$$

By Taylor's theorem,

$$\phi'(\lambda)=\phi'(0)+\lambda\phi''(c)=\lambda\phi''(c)$$

for some c between 0 and λ. By assumptions, we have that ϕ'' is bounded above and below, we have

$$\big|\phi'(\lambda)\big|\approx|\lambda|.$$

This completes the proof of the desired bound. □

5.1. Five dimensional high energy. For the high energy portion of the evolution, we examine

$$\Big\langle e^{itH}\chi\Big(\frac{\sqrt{H}}{L}\Big)\Big[1-\chi\Big(\frac{\sqrt{H}}{\lambda_0}\Big)\Big]P_{ac}f,g\Big\rangle \\ =\int_0^\infty e^{it\lambda^2}\lambda\chi\Big(\frac{\lambda}{L}\Big)\Big[1-\chi\Big(\frac{\lambda}{\lambda_0}\Big)\Big] \\ \times\big\langle\big[R_V\big(\lambda^2+i0\big)-R_V\big(\lambda^2-i0\big)\big]f,g\big\rangle\frac{d\lambda}{\pi i}. \tag{14}$$

Here $\lambda_0>0$ is a small constant which is determined by the analysis in Section 5.2.

We first establish some properties of iterated resolvents.

LEMMA 5.7. *If $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 4$ in dimension five, for any $\sigma > \frac{1}{2}$,*

$$\Big\| \int_{\mathbb{R}^5} R_0^{\pm}(\lambda^2)(x,z)V(z)R_0^{\pm}(\lambda^2)(z,y)\,dz \Big\|_{L_x^{2,-\sigma}} \lesssim \langle\lambda\rangle^2,$$

$$\Big\| \int_{\mathbb{R}^5} \Big(1+\frac{1}{\lambda}\Big)\frac{d}{d\lambda}\big[R_0^{\pm}(\lambda^2)(x,z)V(z)R_0^{\pm}(\lambda^2)(z,y)\big]\,dz \Big\|_{L_x^{2,-\sigma-1}} \lesssim \langle\lambda\rangle^2,$$

$$\Big\| \int_{\mathbb{R}^5} \frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}\big[R_0^{\pm}(\lambda^2)(x,z)V(z)R_0^{\pm}(\lambda^2)(z,y)\big]\,dz \Big\|_{L_x^{2,-\sigma-2}} \lesssim \langle\lambda\rangle,$$

uniformly for $y \in \mathbb{R}^5$.

Proof. Using the explicit expansion for the kernel of $R_0^{\pm}$ from (9), we apply Lemma 5.2. We can see that one integration is enough to establish local L^2 behavior. Considering the slowest decaying terms that result from the integration, one can establish the weighted L^2 behavior. □

Our analysis depends on estimates on the limiting absorption principle of Agmon, [2], and some estimates we establish for certain functions.

LEMMA 5.8 (The limiting absorption principle). *In dimension n, for all $\lambda > \lambda_0$,*

$$\Big\| \Big(\frac{d}{d\lambda}\Big)^j R_0^{\pm}(\lambda^2) \Big\|_{L^{2,\frac{1}{2}+j+} \to L^{2,-\frac{1}{2}-j-}} \lesssim \begin{cases} \lambda^{-1}, & j=0,\\ 1, & j\geq 1,\end{cases}$$

$$\Big\| \Big(\frac{d}{d\lambda}\Big)^j R_V^{\pm}(\lambda^2) \Big\|_{L^{2,\frac{1}{2}+j+} \to L^{2,-\frac{1}{2}-j-}} \lesssim \begin{cases} \lambda^{-1}, & j=0,\\ 1, & j\geq 1,\end{cases}$$

where $0 \leq j \leq \frac{n+1}{2}$.

This result is due to Agmon, [2]. For another proof, one can see [19].

We define the following kernels

$$G_{\pm,x}(\lambda^2)(\cdot) := e^{\mp i\lambda|x|}R_0(\lambda^2 \pm i0)(\cdot,x). \tag{15}$$

Such kernels have been used first by Yajima, see [66], and in the three-dimensional case by Goldberg and Schlag [30]. These kernels inserted in (14) will allow us to differentiate more than $\frac{n-1}{2}$ times in λ without leading to growth in x. Further, this modulation merely shifts the phase when using stationary phase methods. As such, we examine

$$\begin{aligned} &\int_0^\infty e^{it\lambda^2}e^{\pm i\lambda(|x|+|y|)}\chi(\lambda/L)\big[1-\chi(\lambda/\lambda_0)\big] \\ &\qquad \times \lambda\big\langle VR_V^{\pm}(\lambda^2)V\big(R_0^{\pm}(\lambda^2)V\big)^m G_{\pm,x}(\lambda^2), \big(R_0^{\mp}(\lambda^2)V\big)^m G_{\pm,x}^*(\lambda^2)\big\rangle\,d\lambda. \end{aligned} \tag{16}$$

We now define the following.

$$J_y^{\pm}(\lambda,\cdot) := \int_{\mathbb{R}^5} R_0^{\pm}(\lambda^2)(\cdot,z)V(z)G_{\pm,y}(\lambda^2)(z)\,dz. \tag{17}$$

We establish estimates on three λ derivatives of $J_y^\pm(\lambda)$.

LEMMA 5.9. *If* $|V(x)| \lesssim \langle x\rangle^{-\beta}$ *for some* $\beta > 4$, *the following estimates hold for* λ *derivatives of* $J_y^\pm$. *For* $0 \leq j \leq 2$,

$$\left|\left(\frac{d}{d\lambda}\right)^j J_y^\pm(\lambda, x)\right| \lesssim \langle\lambda\rangle^2 \begin{cases} |x-y|^{-1}, & |x-y|<1, \\ |x-y|^{j-2}, & |x-y|>1, \end{cases}$$

and

$$\left|\frac{d^3}{d\lambda^3} J_y^\pm(\lambda, x)\right| \lesssim \langle\lambda\rangle^2 \begin{cases} |x-y|^{-1}, & |x-y|<1, \\ \lambda\langle x\rangle, & |x-y|>1. \end{cases}$$

Proof. We note the following inequality.

$$\left|\left(\frac{d}{d\lambda}\right)^j \left[e^{i\lambda\phi}\left(a\lambda^2+b\lambda+c\right)\right]\right| \lesssim \lambda^2\left[a\phi^j\right] + \lambda\left[a\phi^{j-1}+b\phi^j\right] + \left[a\phi^{j-2}+b\phi^{j-1}+c\phi^j\right],$$

where we take $\phi^\ell = 0$ if $\ell < 0$. Taking $\phi = |x-z|+|z-y|-|y|$ and a, b, c the coefficients of the λ powers that arise in $J_y^\pm$. The proof now follows Lemma 5.2, the fact that $||z-y|-|y|| \lesssim \langle z\rangle$.

For $j = 3$, on must use that $|\phi| \lesssim \langle x\rangle + \langle z\rangle$ at most once to avoid growth in $|x-z|$. □

COROLLARY 5.10. *If* $|V(x)| \lesssim \langle x\rangle^{-\beta}$ *for some* $\beta > 4$, *and* $0 \leq j \leq 3$,

$$\left\|\left(\frac{d}{d\lambda}\right)^j J_y^\pm(\lambda, \cdot)\right\|_{L^{2,-\sigma}} \lesssim \langle\lambda\rangle^2$$

for $\sigma > j + \frac{1}{2}$.

We can improve Corollary 5.10 to push forward decay in y by increasing the degree of the polynomial weight.

PROPOSITION 5.11. *If* $|V(x)| \lesssim \langle x\rangle^{-\beta-}$ *for some* $\beta > 4$, *then*

$$\left\|J_y^\pm(\lambda, z)\right\|_{L_z^{2,-\frac{3}{2}-}} \lesssim \frac{\langle\lambda\rangle^2}{\langle y\rangle^2},$$

$$\left\|\frac{d}{d\lambda} J_y^\pm(\lambda, z)\right\|_{L_z^{2,-\frac{5}{2}-}} \lesssim \frac{\langle\lambda\rangle^2}{\langle y\rangle^2}.$$

Proof. The statement for $j = 0$ arises from the following calculations. First, consider the contribution of the λ^2 term of (17) to the weighted L^2 norm.

$$\left[\int_{\mathbb{R}^5}\left|\int_{\mathbb{R}^5} \frac{\langle z\rangle^{-4-}}{|x-z|^2|z-y|^2}\,dz\right|^2 \langle x\rangle^{-3-}\right]^{\frac{1}{2}}.$$

We note that

$$\frac{1}{|x-z|^2|z-y|^2} \lesssim \frac{1}{|x-y|^2}\left[\frac{1}{|x-z|^2} + \frac{1}{|z-y|^2}\right].$$

Using this fact and Lemma 5.4, we bound with

$$\begin{aligned}
&\bigg[\int_{\mathbb{R}^5}\bigg|\int_{\mathbb{R}^5}\frac{\langle z\rangle^{-4-}}{|x-z|^2|z-y|^2}\,dz\bigg|^2\langle x\rangle^{-3-}\,dx\bigg]^{\frac12}\\
&\quad\lesssim\bigg[\int_{\mathbb{R}^5}\bigg|\int_{\mathbb{R}^5}\langle z\rangle^{-4-}\bigg(\frac{1}{|x-z|^2}+\frac{1}{|z-y|^2}\bigg)\,dz\bigg|^2\frac{\langle x\rangle^{-3-}}{|x-y|^4}\,dx\bigg]^{\frac12}\\
&\quad\lesssim\bigg[\int_{\mathbb{R}^5}\big(\langle x\rangle^{-2}+\langle y\rangle^{-2}\big)\frac{\langle x\rangle^{-3-}}{|x-y|^4}\,dx\bigg]^{\frac12}\lesssim\frac{1}{\langle y\rangle^2}.
\end{aligned}$$

We handle the λ^0 term of (17) similarly, we note

$$\frac{1}{|x-z|^3|z-y|^3}\lesssim\frac{1}{|x-y|^2}\bigg[\frac{1}{|x-z|^4}+\frac{1}{|z-y|^4}\bigg]. \tag{18}$$

Thus, this term contributes the following to the weighted L^2 norm.

$$\begin{aligned}
&\bigg[\int_{\mathbb{R}^5}\bigg|\int_{\mathbb{R}^5}\langle z\rangle^{-4-}\bigg(\frac{1}{|x-z|^4}+\frac{1}{|z-y|^4}\bigg)\,dz\bigg|^2\frac{\langle x\rangle^{-3-}}{|x-y|^4}\,dx\bigg]^{\frac12}\\
&\quad\lesssim\bigg[\int_{\mathbb{R}^5}\frac{\langle x\rangle^{-3-}}{|x-y|^4}\big(\langle x\rangle^{-6}+\langle y\rangle^{-6}\big)\,dx\bigg]^{\frac12}\lesssim\frac{1}{\langle y\rangle^2}.
\end{aligned}$$

We note that the λ coefficient of $J_y^{\pm}$ is bounded by the sum its λ^2 and λ^0 coefficients.

For the term with a λ derivative, we take a bit more care. We note that in Lemma 5.9, we bound $|\phi|\lesssim|x-z|+\langle z\rangle$, for this estimate, we wish to retain the $|x-z|$ decay, so instead we use the bound $|\phi|\lesssim\langle x\rangle+\langle z\rangle$. Again, we need only concern ourselves with the λ^2 and λ^0 terms. We bound these terms as follows,

$$\frac{\lambda^2}{|x-z|^2|z-y|^2}\big(\langle x\rangle+\langle z\rangle\big), \tag{19}$$

$$\frac{1}{|x-z|^3|z-y|^3}\big(\langle x\rangle+\langle z\rangle\big). \tag{20}$$

We note that each bound is a sum of two terms. However, if we take the term with $\langle x\rangle$, it reduces down to the case with no derivatives since this term merely cancels out the extra weight $\sigma>\frac52$. Let us first consider the λ^2 term. We need to bound the weighted L^2 norm, that is for the $\langle z\rangle$ term of (19), we use

$$\begin{aligned}
&\bigg[\int_{\mathbb{R}^5}\bigg|\int_{\mathbb{R}^5}\frac{\langle z\rangle^{-3-}}{|x-z|^2|z-y|^2}\,dz\bigg|^2\langle x\rangle^{-5-}\,dx\bigg]^{\frac12}\\
&\quad\lesssim\bigg[\int_{\mathbb{R}^5}\frac{\langle x\rangle^{-5-}}{|x-y|^4}\,dx\bigg]^{\frac12}\lesssim\frac{1}{\langle y\rangle^2}.
\end{aligned}$$

Where we used Lemma 5.2 in the second to last line with $\max(0,k+\ell-n)\le \min(k,\ell,k+\ell+\beta-n)=2$ as $k=\ell=2$ and $\beta=3$, and we used Lemma 5.4 in the last line.

Turning our attention to (20), the λ^0 term, as in the case of no derivatives and using (18), we need to bound

$$\begin{aligned}&\bigg[\int_{\mathbb{R}^5}\bigg|\int_{\mathbb{R}^5}\langle z\rangle^{-3-}\Big(\frac{1}{|z-y|^3}+\frac{1}{|x-z|^3}\Big)\,dz\bigg|^2\frac{\langle x\rangle^{-5-}}{|x-y|^4}\bigg]^{\frac12}\\ &\quad\lesssim\bigg[\int_{\mathbb{R}^5}\big(\langle y\rangle^{-2}+\langle x\rangle^{-2}\big)\frac{\langle x\rangle^{-5-}}{|x-y|^4}\bigg]^{\frac12}\lesssim\frac{1}{\langle y\rangle^2}.\end{aligned}$$

Where we used Lemma 5.4 throughout this calculation. ☐

We rewrite (16) as

$$\begin{aligned}I^{\pm}_{x,y}(t)&=\int_0^\infty e^{it\lambda^2\pm i\lambda(|x|+|y|)}\chi_L(\lambda)\big(1-\chi_0(\lambda)\big)\\ &\qquad\times\lambda\big\langle VR_V^{\pm}V\big(R_0^{\pm}V\big)^{m-1}J_y^{\pm}(\lambda,\cdot),\big(R_0^{\pm}V\big)^{m-1}J_x^{\pm}(\lambda,\cdot)\big\rangle\,d\lambda\\ &=\int_0^\infty e^{it\lambda^2\pm i\lambda(|x|+|y|)}a^{\pm}_{x,y}(\lambda)\,d\lambda.\end{aligned}$$

For $0<t<1$, we note that by Corollary 5.10 and Lemma 5.8,

$$\begin{aligned}\big|a^{\pm}_{x,y}(\lambda)\big|&\lesssim\lambda\|J_x\|_{L^{2,-\frac12-}}\big\|R_0^{\pm}V\big\|^{2m-2}_{L^{2,-\frac12-}\to L^{2,-\frac12-}}\|R_VV\|_{L^{2,-\frac12-}\to L^{2,-\frac12-}}\\ &\qquad\times\|V\|_{L^{2,-\frac12-}\to L^{2,\frac12+}}\big\|J_y^{\pm}\big\|_{L^{2,-\frac12-}}\\ &\lesssim\lambda\langle\lambda\rangle^4\big(\lambda^{-1}\big)^{2m-2}\lambda^{-1}\lesssim\langle\lambda\rangle^{6-2m}.\end{aligned}$$

We see taking $m=4$ suffices and that

$$\big|I^{\pm}_{x,y}(t)\big|\lesssim\int_0^\infty\langle\lambda\rangle^{-2}\,d\lambda\lesssim 1.$$

When $t>1$, we note that the phase has critical point $\lambda_1=\mp\frac{|x|+|y|}{2t}$. We have that $a^{\pm}_{x,y}$ has three derivatives in λ that satisfy the following bounds

$$\begin{aligned}\big|a^{\pm}_{x,y}(\lambda)\big|&\lesssim\langle\lambda\rangle^{-2}\big(\langle x\rangle\langle y\rangle\big)^{-2}&&\text{for all }\lambda>1,\\ \bigg|\Big(\frac{d}{d\lambda}\Big)^j a^{\pm}_{x,y}(\lambda)\bigg|&\lesssim\langle\lambda\rangle^{-2}&&\text{for }j=1,2,3\text{ for all }\lambda>1.\end{aligned}$$

In particular, this justifies taking $L=\infty$ in (16).

We note that for $I^{+}_{x,y}$, the critical point of the phase is outside of the support of $a^{\pm}_{x,y}$. Three integration by parts in λ yield the bound $|I^{+}_{x,y}(t)|\lesssim|t|^{-3}$.

We can similarly integrate by parts and bound $|I^-_{x,y}(t)| \lesssim |t|^{-3}$ away from the critical point of the phase. Further if $\lambda_1 \ll \lambda_0$, we can again inegrate by parts three times. Finally, if $\lambda_1 \gtrsim \lambda_0$ we can also have the bound $\max(|x|, |y|) \gtrsim |t|$. From Lemma 5.6 and Proposition 5.11, we see that stationary phase contributes $|t|^{-1/2}(\langle x\rangle\langle y\rangle)^{-2} \lesssim |t|^{-5/2}$, as desired.

5.2. Five dimensional low energy. We now control the low-energy portion of the evolution,

$$\begin{aligned}(21)\qquad &\langle e^{itH}\chi(\sqrt{H}/\lambda_0)P_{ac}f, g\rangle \\ &\quad = \int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda/\lambda_0)\langle [R_V(\lambda^2+i0) - R_V(\lambda^2-i0)f, g\rangle \frac{d\lambda}{\pi i}.\end{aligned}$$

We use the resolvent identity

$$R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)V\big(I + R_0^\pm(\lambda^2)V\big)^{-1}R_0^\pm(\lambda^2). \tag{22}$$

The free resolvents in dimension five have explicit expansion

$$R_0^\pm(\lambda^2)(x,y) = C_5 e^{\pm i\lambda|x-y|}\frac{1 \mp i\lambda|x-y|}{|x-y|^3}. \tag{23}$$

We note that in particular $R_0^\pm(\lambda^2)(x,y)$ is not locally in $L^2(\mathbb{R}^5)$, so it cannot be Hilbert–Schmidt.

We follow the approach of Goldberg and Schlag in [30], in particular we establish invertibility of $S^\pm(\lambda) = (I + R_0^\pm(\lambda^2)V)$ as a perturbation from zero energy. We then expand in a Neumann series in certain Hilbert–Schmidt norms.

We note the following Proposition from [31].

Proposition 5.12. *Suppose $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > \frac{n+1}{2}$ and also that zero energy is neither an eigenvalue nor a resonance of $H = -\Delta + V$. Then*

$$\sup_{\lambda\geq 0}\big\|\big[S^\pm\big]^{-1}(\lambda)\big\|_{L^{2,-\sigma}\to L^{2,-\sigma}} < \infty$$

for all $\sigma \in (\frac{1}{2}, \beta - \frac{1}{2})$.

We rewrite $R_0^\pm(\lambda^2) = R_0^\pm(0) + B_5^\pm(\lambda)$. Then, we can write

$$\big[I + R_0^\pm(\lambda^2)\big]^{-1} = S_0^{-1}\big[I + B_5^\pm(\lambda)VS_0^{-1}\big]^{-1}.$$

The integral kernel has form

$$B_5^\pm(\lambda) = C_5\bigg(e^{i\lambda|x-y|}\frac{1 \mp i\lambda|x-y|}{|x-y|^3} - \frac{1}{|x-y|^3}\bigg), \tag{24}$$

which satisfies the size estimate

$$\big|B_5^\pm(\lambda^2)\big| \lesssim \frac{\lambda}{|x-y|^2}. \tag{25}$$

This follows as $|e^{i\theta}-1|\lesssim|\theta|$.

PROPOSITION 5.13. *If $\sigma,\alpha>\frac{1}{2}$, and $\sigma+\alpha>3$, then*

$$\sup_{\lambda\geq 0}\lambda^{-1}\big\|B_5^{\pm}(\lambda)\big\|_{\mathrm{HS}(\sigma,-\alpha)}\leq C_{\sigma,\alpha}.$$

Proof. We note that

$$\begin{aligned}\big\|B_5^{\pm}(\lambda)\big\|^2_{\mathrm{HS}(\sigma,-\alpha)}&=\lambda^2\int\int_{\mathbb{R}^{10}}\frac{\langle x\rangle^{-2\sigma}\langle y\rangle^{-2\alpha}}{|x-y|^4}\,dx\,dy\\&\lesssim\lambda^2\int_{\mathbb{R}^5}\langle y\rangle^{-2\alpha-\min(4,2\sigma-1)}\,dy\lesssim\lambda^2\end{aligned}$$

by Lemma 5.4. □

COROLLARY 5.14. *If $\sigma,\alpha>\frac{1}{2}$, and $\sigma+\alpha>3$, then*

$$\lim_{\lambda\to 0}\big\|B_5^{\pm}(\lambda)\big\|_{\mathrm{HS}(\sigma,-\alpha)}=0.$$

COROLLARY 5.15. *If $|V(x)|\lesssim\langle x\rangle^{-\beta}$ for some $\beta>4$, then*

$$\lim_{\lambda\to 0}\big\|B_5^{\pm}(\lambda)VS_0^{-1}\big\|_{\mathrm{HS}(\sigma,\sigma)}=0$$

for all $\sigma\in(-\frac{7}{2},-\frac{1}{2})$.

Proof. We know $VS_0^{-1}:L^{2,\sigma}\to L^{2,\sigma+4+}$ for $-\frac{7}{2}<\sigma<-\frac{1}{2}$ from Proposition 5.12. Corollary 5.14 implies that $\|B_5^{\pm}\|_{\mathrm{HS}(\sigma+4+,\sigma)}\to 0$ as $\lambda\to 0$. □

For derivatives of $B_5^{\pm}(\lambda)(x,y)$, we note that $(B_5^{\pm})'(\lambda)(x,y)=Ce^{\pm i\lambda|x-y|}\times\frac{\lambda}{|x-y|}$. So that $\frac{1}{\lambda}\frac{d}{d\lambda}B_5^{\pm}(\lambda)(x,y)=CR_3(\lambda^2)(x,y)$ where $R_3(\lambda^2)(x,y)$ is the three-dimensional free resolvent. Further, we have that $\frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}B_5^{\pm}(\lambda^2)(x,y)=Ce^{\pm i\lambda|x-y|}$.

CLAIM. $\|\frac{1}{\lambda}\frac{d}{d\lambda}B_5^{\pm}(\lambda)\|_{\mathrm{HS}(\sigma,-\alpha)}\leq C$ *if $\sigma,\alpha>\frac{3}{2}$ and $\sigma+\alpha>4$.*

CLAIM. $\|\frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}B_5^{\pm}(\lambda)\|_{\mathrm{HS}(\sigma,-\alpha)}\leq C$ *if $\sigma,\alpha>\frac{5}{2}$.*

We prove the existence of $\lambda_0>0$ such that for $0<\lambda<\lambda_0$ we can expand

$$\tilde{B}_5^{\pm}(\lambda)=\big[I+B_5^{\pm}(\lambda)VS_0^{-1}\big]^{-1}$$

as a Neumann series in the norms $\|\cdot\|_{\mathrm{HS}(\sigma,\sigma)}$ for $-\frac{7}{2}<\sigma<-\frac{1}{2}$. The symmetry $\tilde{B}_5^{-}(\lambda)=\tilde{B}_5^{+}(-\lambda)$ is still valid.

Define $\chi_0(\lambda)=\chi(\frac{\lambda}{\lambda_0})$, and note that (21) and (22) lead us to bounding

$$\begin{aligned}(26)\qquad\sup_{x,y\in\mathbb{R}^5}\bigg|\int_0^{\infty}\lambda\chi_0(\lambda)\Bigg[\Bigg(&\sum_{j=0}^{3}R_0^{+}(\lambda^2)\big[-VR_0^{+}(\lambda^2)\big]^j\\&-R_0^{-}(\lambda^2)\big[-(\lambda^2)VR_0^{-}(\lambda^2)\big]^j\Bigg)\end{aligned}$$

$$
\begin{aligned}
&- \big[R_0^+(\lambda^2)VR_0^+(\lambda^2)VS_0^{-1}\tilde{B}_5^+R_0^+(\lambda^2)VR_0^+(\lambda^2)\\
&- R_0^-(\lambda^2)VR_0^-(\lambda^2)VS_0^{-1}\tilde{B}_5^-R_0^-(\lambda^2)VR_0^-(\lambda^2)\big]\Bigg](x,y)\,d\lambda\Bigg|. \qquad (27)
\end{aligned}
$$

We note that (26) is the low-energy part of the Schrödinger evolution which is known to disperse if $|\nabla^j V(x)| \lesssim \langle x\rangle^{-3-}$ for $j=0,1$. This was proved in [20] and is nearly optimal with respect to decay of the potential. We bound the tail as follows.

$$
\begin{aligned}
\sup_{x,y\in\mathbb{R}^5}\bigg|&\int_{-\infty}^{\infty} e^{it\lambda^2}\lambda\int\int_{\mathbb{R}^{10}} A\big(\lambda,|x-x_1|\big)VS_0^{-1}\big(\chi_0\tilde{B}_5^+\big)(\lambda)(x_1,x_2)\\
&\times A\big(\lambda,|x_2-y|\big)\,dx_1\,dx_2\,d\lambda\bigg|,
\end{aligned}
$$

where $A=\int R_0^\pm VR_0^\pm$, integrated in the appropriate variable, which is seen to be in certain weighted L^2 spaces by Lemma 5.7. Following the standard approach first laid out in [46] by Rodnianski and Schlag, we integrate by parts in λ twice and bound.

$$
\begin{aligned}
\sup_{x,y\in\mathbb{R}^5}\bigg|\frac{1}{t^2}&\int_{-\infty}^{\infty} e^{it\lambda^2}\int\int_{\mathbb{R}^{10}} \frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}A(\lambda,|x-x_1|)\\
&\times VS_0^{-1}\big(\chi_0\tilde{B}_5^+\big)(\lambda)(x_1,x_2)A\big(\lambda,|x_2-y|\big)\,dx_1\,dx_2\,d\lambda\bigg|. \qquad (28)
\end{aligned}
$$

There are several different cases to consider, depending on where the λ derivatives act. We first consider when derivatives don't act on the cut-off function $\chi_0(\lambda)$. We estimate the resolvent and its derivatives as a mapping from $L^{2,\sigma}(\mathbb{R}^n)$ to $L^{2,-\alpha}(\mathbb{R}^n)$. We recall that the Hilbert–Schmidt norm is defined by

$$
\|R\|^2_{\mathrm{HS}(\sigma,-\alpha)}=\int\int_{\mathbb{R}^{2n}}\langle x\rangle^{-2\sigma}\big|R(x,y)\big|^2\langle y\rangle^{-2\alpha}\,dx\,dy
$$

Consider the term from (28) which arises when all derivative act on $\tilde{B}_5^+$, the other cases are similarly managed. Using Parseval and the fact that $\|(e^{it(\cdot)^2})^\vee\|_{L^\infty}\lesssim |t|^{-\frac12}$, we need only bound

$$
\begin{aligned}
\sup_{x,y\in\mathbb{R}^5}|t|^{-\frac52}&\int_{-\infty}^{\infty}\bigg|\int\int_{\mathbb{R}^{10}} A_x(x_1)VS_0^{-1}\bigg(\chi_0\frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}\widetilde{B_5^+}\bigg)^\vee(\xi)(x_1,x_2)\\
&\times A_y(x_2)\,dx_1\,dx_2\bigg|\,du, \qquad (29)
\end{aligned}
$$

with $\xi:=u+|x-x_1|+|y-x_2|$. We note that we only use the fact that the kernels $A(\lambda,\cdot)$ are in weighted L^2 spaces. Their λ dependence is not important to our mapping argument, see Remark 5.19 below. We now denote them as $A_x(x_1)$ and $A_y(x_2)$ since their properties with respect to x,y are what we are most interested in here. Now, using Fubini, we interchange the order of integration to bound (29) with

$$\sup_{x,y\in\mathbb{R}^5} |t|^{-\frac{5}{2}} \int\int_{\mathbb{R}^{10}} \int_{\mathbb{R}} \big|A_x(x_1)V\big| \Big| S_0^{-1}\Big(\chi_0 \frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}\widetilde{B_5^+}\Big)^{\vee}(\xi)(x_1,x_2)\Big|$$
$$\times \big|A_y(x_1)\big|\, du\, dx_1\, dx_2.$$
$$\leq \sup_{x,y\in\mathbb{R}^5} |t|^{-\frac{5}{2}} \|A_x V\|_{L^{2,3+}} \Big\| \int \Big| S_0^{-1}\Big(\chi_0 \frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}\widetilde{B_5^+}\Big)^{\vee}(u)\Big|\, du\Big\|_{\mathrm{HS}(-1-,-3-)}$$
$$\times \|A_y\|_{L^{2,-1-}}.$$

The weighted L^2 bounds follow from Lemma 5.7. We need only control the size of

$$\Big\| \int \Big| S_0^{-1}\Big(\chi_0 \frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}\widetilde{B_5^+}\Big)^{\vee}(u)\Big|\, du\Big\|_{\mathrm{HS}(-1-,-3-)}.$$

Using Minkowski allows us to bring the Hilbert–Schmidt norm inside the integral. By Proposition 5.12, S_0^{-1} is a bounded operator on $L^{2,-3-}(\mathbb{R}^5)$ and the fact that composition of a bounded operator with a Hilbert–Schmidt operator is Hilbert–Schmidt, we can reduce to showing the existence of a value $\lambda_0 > 0$ such that

$$\int_{\mathbb{R}} \Big\| \Big(\chi_0 \frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}\widetilde{B_5^+}\Big)^{\vee}(u)\Big\|_{\mathrm{HS}(-1-,-3-)} du < \infty.$$

To this end, we establish the following estimates.

Lemma 5.16. *The inverse Fourier transform of $\chi_0 B_5^+$ in λ satisfies*

$$\int_{-\infty}^{\infty} \big\| \big[\chi_0 B_5^+\big]^{\vee}(u)\big\|_{\mathrm{HS}(\sigma,-\alpha)}\, du < C\lambda_0\big(1+\lambda_0^{\frac{1}{2}+}\big)$$

if $\sigma,\alpha > 1$ and $\sigma+\alpha > \frac{7}{2}$.

Proof. We note that by construction of B_5^+, we can use Taylor's theorem to cancel out the non-locally L^2 behavior. Specifically,

$$\big|\big[B_5^+\chi_0\big]^{\vee}(\xi)\big| \lesssim \frac{\lambda_0^2}{|x-y|^2}.$$

We use this on the region $|\xi| < \frac{2}{\lambda_0}$, and the fact that $|x-y|^{-2}$ has finite Hilbert–Schmidt norm under the above hypotheses. One can see with two applications of Lemma 5.4 the contribution of this region to the ξ integral is bounded by $C\lambda_0$.

On the region $|\xi| > \frac{2}{\lambda_0}$, we must take some additional care. Consider the region on which $|\xi| < \frac{1}{2}|x-y|$, it then follows that $|x-y|^{-1} \lesssim \lambda_0$, and

$||x-y|+\xi| \approx |x-y| \gtrsim |\xi|$. Here we use

$$\begin{aligned}\big|\big[B_5^+\chi_0\big]^\vee(\xi)\big| &\lesssim \lambda_0^2 \sum_{j=0}^{1} \frac{\lambda_0^{-j}}{|x-y|^{2+j}} \left(\frac{d}{d\xi}\right)^j \big[\chi^\vee\big(\lambda_0(\xi+|x-y|)\big)+\chi^\vee(\lambda_0\xi)\big] \\ &\lesssim \frac{\lambda_0^2}{|x-y|^2}\langle\lambda_0\xi\rangle^{-10} \lesssim \frac{\lambda_0^2}{|x-y|^2}\frac{1}{\lambda_0^{10}|\xi|^{10}}.\end{aligned}$$

Again, the Hilbert–Schmidt norm is finite, and this region contributes $\lambda_0^{\frac{n-3}{2}}$ to the integral.

On the region where $|\xi| > 2|x-y|$, we have that $||x-y|+\xi| \approx |\xi|$, so using Taylor's theorem on the inverse Fourier transform, we have

$$\big|\big[B_5^+\chi_0\big]^\vee(\xi)\big| \lesssim \frac{\lambda_0^2}{|x-y|^2}\frac{d}{d\xi}\big[\chi^\vee(\lambda_0\xi)+\chi^\vee(\lambda_0 c)\big].$$

Here $c \in B(\xi, \frac{1}{2}|\xi|)$. So the inverse Fourier transforms of the cut-off are of comparable in size. The bound then follows as in the previous region.

Finally, one must take care in the annular region $\frac{1}{2}|x-y| < |\xi| < 2|x-y|$. We do not use the bound that arises from the use of Taylor's theorem, but instead bound as in the region $|\xi| < \frac{1}{2}|x-y|$,

$$\big|\big[B_5^+\chi_0\big]^\vee(\xi)\big| \lesssim \frac{\lambda_0^2}{|x-y|^2}\big[\big\langle\lambda_0\big(\xi+|x-y|\big)\big\rangle^{-10}+\langle\lambda_0\xi\rangle^{-10}\big].$$

The second term with $\langle\lambda_0\xi\rangle^{-10}$ can be handled as in the previous two regions. The first term requires more care as we can have $\xi+|x-y| = 0$ on this region. Let us now consider the contribution of this region to the square of the Hilbert–Schmidt norm. We wish to bound

$$(30)\quad \lambda_0^2\int_{|\xi|\gtrsim\frac{1}{\lambda_0}}\left[\int\int_{|x-y|\approx|\xi|}\frac{\langle x\rangle^{-2\sigma}\langle\lambda_0(\xi+|x-y|)\rangle^{-20}\langle y\rangle^{-2\alpha}}{|x-y|^4}\,dx\,dy\right]^{\frac{1}{2}}d\xi.$$

By a switch to polar coordinates and scaling the λ_0 out of the integrals, we have

$$\begin{aligned}(30) &= \lambda_0^{\frac{1}{2}}\int_{|s|\gtrsim1}\left[\int_{S^4}\int_{r\approx|s|}\int_{\mathbb{R}^5}\langle x\rangle^{-2\sigma}\langle s+r\rangle^{-20}\left\langle x+\frac{r\theta}{\lambda_0}\right\rangle^{-2\alpha}dx\,dr\,d\theta\right]^{\frac{1}{2}}d\xi \\ &\lesssim \lambda_0^{\frac{1}{2}}\int_{|s|\gtrsim1}\left[\int_{S^4}\int_{r\approx|s|}\langle s+r\rangle^{-20}\left(\frac{r}{\lambda_0}\right)^{-q}dr\,d\theta\right]d\xi \\ &\lesssim \lambda_0^{\frac{1+q}{2}}\int_{|s|\gtrsim1}|s|^{-\frac{q}{2}}\,ds \lesssim \lambda_0^{\frac{1+q}{2}}.\end{aligned}$$

As in Lemma 5.17, we use Lemma 5.5 to estimate the x integral with $q = \min(2\alpha, 2\sigma, 2\alpha+2\sigma-5)$. By assumption, we have $q > 2$. ☐

Lemma 5.17. *The inverse Fourier transform of* $\chi_0 \frac{1}{\lambda}\frac{d}{d\lambda}B_5^+$ *in* λ *satisfies*

$$\int_{-\infty}^{\infty} \left\| \left[\chi_0 \frac{1}{\lambda}\frac{d}{d\lambda}B_5^+ \right]^\vee (u) \right\|_{\mathrm{HS}(\sigma,-\alpha)} du < C < \infty$$

uniformly as $\lambda_0 \to 0$ *if* $\sigma, \alpha > 2$, *and* $\sigma + \alpha > \frac{9}{2}$.

Proof. First note that by Lemma 3.1,

$$\frac{1}{\lambda}\frac{d}{d\lambda}B_5^+ = CR_3^+(\lambda^2) = C\frac{e^{i\lambda|x-y|}}{|x-y|}.$$

It now follows that, up to a constant multiplier,

$$\left[\chi_0 \frac{1}{\lambda}\frac{d}{d\lambda}B_5^+\right]^\vee (u) = \left[\chi_0 \frac{e^{i\lambda|x-y|}}{|x-y|}\right]^\vee (u) = \frac{\lambda_0 \chi^\vee(\lambda_0(u+|x-y|))}{|x-y|}.$$

By scaling of the inverse Fourier transform and the fact that $\chi \in \mathcal{S}(\mathbb{R})$, in this case we have

$$\left| \left[\chi_0 \frac{1}{\lambda}\frac{d}{d\lambda}B_5^+ \right]^\vee (u) \right| \lesssim \frac{\lambda_0}{|x-y|}\big\langle \lambda_0(u+|x-y|)\big\rangle^{-10}. \tag{31}$$

We first consider when $|u| \le \frac{2}{\lambda_0}$. Ignoring the $\langle \lambda_0(u+|x-y|)\rangle^{-10}$ in (31), it is clear, by Lemma 5.4, that $\sigma, \alpha > 2$, $\sigma + \alpha > \frac{9}{2}$ is sufficient to establish

$$\left\| \left[\chi_0 \frac{1}{\lambda}\frac{d}{d\lambda}B_5^+ \right]^\vee (u) \right\|_{\mathrm{HS}(\sigma,-\alpha)} \lesssim \lambda_0.$$

Thus,

$$\int_{|u|\le \frac{2}{\lambda_0}} \left\| \left[\chi_0 \frac{1}{\lambda}\frac{d}{d\lambda}B_5^+ \right]^\vee (u) \right\|_{\mathrm{HS}(\sigma,-\alpha)} du \lesssim \int_{|u|\le\frac{2}{\lambda_0}} \lambda_0 \, du \lesssim 1.$$

Now if $|u| \ge \frac{2}{\lambda_0}$, we have

$$\left| \left[\chi_0 \frac{1}{\lambda}\frac{d}{d\lambda}B_5^+ \right]^\vee (u) \right| \lesssim \frac{\lambda_0}{|x-y|}\big\langle \lambda_0(u+|x-y|)\big\rangle^{-10}.$$

We can further bound by

$$\left| \left[\chi_0 \frac{1}{\lambda}\frac{d}{d\lambda}B_5^+ \right]^\vee (u) \right| \lesssim \begin{cases} \dfrac{1}{\lambda_0^8 |u|^{10}|x-y|}, & |x-y| \le \dfrac{1}{2}|u|, \\ \dfrac{1}{\lambda_0^8 |u|^{10}|x-y|}, & |x-y| \ge 2|u|, \\ \dfrac{\lambda_0}{|u|}\langle \lambda_0(u+|x-y|)\rangle^{-10}, & \dfrac{1}{2}|u| < |x-y| < 2|u|. \end{cases}$$

The first two regions, the estimates follow from the triangle inequality. The integral of the Hilbert–Schmidt norm is bounded as before,

$$\int_{|u|\ge\frac{2}{\lambda_0}} \frac{1}{\lambda_0^8|u|^{10}} \big\| |x-y|^{-1} \big\|_{\mathrm{HS}(\sigma,-\alpha)} du \lesssim \frac{1}{\lambda_0^8}\int_{|u|\ge\frac{2}{\lambda_0}} |u|^{-10} du \lesssim \lambda_0.$$

We now need only bound the function on the annular region. We change the y integral to polar, $y = x + r\theta$ where $\theta \in S^4$, the four-dimensional sphere. We now need to bound

$$\lambda_0 \int_{|u| \gtrsim \frac{1}{\lambda_0}} \left[\int_{S^4} \int_{|r| \approx |u|} \int_{\mathbb{R}^5} \langle x \rangle^{-2\sigma} \big\langle \lambda_0 (r+u) \big\rangle^{-20} \times \langle x + r\theta \rangle^{-2\alpha} r^4 \, dx \, dr \, d\theta \right]^{\frac{1}{2}} \frac{du}{|u|}. \tag{32}$$

Rescaling, and defining $F(x,r,s,\theta) := \langle x \rangle^{-2\sigma} \langle r+s \rangle^{-20} \langle x + \frac{r\theta}{\lambda_0} \rangle^{-2\alpha}$, we have

$$\begin{aligned}
(32) &\lesssim \lambda_0^{-\frac{3}{2}} \int_{|s| \gtrsim 1} \left[\int_{S^4} \int_{r \approx |s|} \int_{\mathbb{R}^5} F(x,r,s,\theta) r^4 \, dx \, dr \, d\theta \right]^{\frac{1}{2}} \frac{ds}{|s|} \\
&\lesssim \lambda_0^{\frac{q-3}{2}} \int_{|s| \gtrsim 1} \frac{1}{|s|} \left[\int_{r \approx |s|} r^{4-q} \langle r+s \rangle^{-20} \, dr \right]^{\frac{1}{2}} ds \\
&\lesssim \lambda_0^{\frac{q-3}{2}} \int_{|s| \gtrsim 1} |s|^{1-q} \, ds \lesssim \lambda_0^{\frac{q-3}{2}}.
\end{aligned}$$

Where we used Lemma 5.5 with $q = \min(2\alpha, 2\sigma, 2\alpha + 2\sigma - 5)$ and the fact that $q - 1 > 1$. □

LEMMA 5.18. *The inverse Fourier transform of $\chi_0 \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+$ in λ satisfies*

$$\int_{-\infty}^{\infty} \left\| \left[\chi_0 \frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+ \right]^{\vee} (u) \right\|_{\mathrm{HS}(3+,-3-)} du < C < \infty$$

uniformly as $\lambda_0 \to 0$.

Proof. This proof is identical in form to that of Lemma 15 in [30]. We note that by Lemma 3.1,

$$\frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+(\lambda) = \big(R_3^+ \big)' \big(\lambda^2 \big).$$

Where R_3 is the three-dimensional free resolvent. The need for larger Hilbert–Schmidt weights is a consequence of the ambient space being $\mathbb{R}^5$ instead of $\mathbb{R}^3$. □

We now present two cases that arise in the analysis of (28), that in which no λ derivatives act on $\tilde{B}_5^+$ and that in which all λ derivatives act on $\tilde{B}_5^+$. The intermediate case of one derivative acting on $\tilde{B}_5^+$ is handled similarly.

No derivatives act on $\tilde{B}_5^+$:

If no λ derivatives act on $\tilde{B}_5^+(\lambda)$ in (28), they must act on the leading and trailing $A(\lambda, \cdot)$ terms. From Lemma 5.7, we see that we must establish

$$V S_0^{-1} \big[\chi_0 \tilde{B}_5^+ \big]^{\vee} \in L^1 \big(L^{2,\gamma-3-}, L^{2,\gamma+} \big)$$

for $\frac{1}{2} \le \gamma \le \frac{5}{2}$. From Proposition 5.12, we need only establish that

$$\big[\chi_0 \tilde{B}_5^+\big]^\vee \in L^1\big(L^{2,\gamma-3-}, L^{2,\gamma-4-}\big).$$

We define $\tilde{B}_5^+$ as a convergent Neumann series

$$\tilde{B}_5^+(\lambda) = \big[I + B_5^+(\lambda) V S_0^{-1}\big]^{-1} = \sum_{n=0}^{\infty} \big(-B_5^+(\lambda) V S_0^{-1}\big)^n. \tag{33}$$

We define $\chi_1(\lambda) = \chi(\frac{\lambda}{2\lambda_0})$ so that $\chi_1^n \chi_0 = \chi_0$ for any $n \ge 0$. We use this and (33) to define $\chi_0 B_5^+$ as a Neumann series.

$$\begin{aligned} \chi_0 \tilde{B}_5^+(\lambda) &= \chi_0 \big[I + B_5^+(\lambda) V S_0^{-1}\big]^{-1} \\ &= \chi_0(\lambda) \sum_{n=0}^{\infty} \big(-\chi_1(\lambda) B_5^+(\lambda) V S_0^{-1}\big)^n. \end{aligned} \tag{34}$$

Upon applying the inverse Fourier transform to (34), we note that as in the scalar case multiplication of operator-valued functions yields convolution of their inverse Fourier transforms. We can bound the L^1 norm of the repeated convolutions by the product of the L^1 norms of each piece provided the range of each operator is contained in the domain of the operator following it. So that we have

$$\begin{aligned} &\big\| \big[\chi_0 \tilde{B}_5^+\big]^\vee \big\|_{L^1(L^{2,\gamma-3-}, L^{2,\gamma-4-})} \\ &\quad \le \big\| \chi_0^\vee I \big\|_{L^1(L^{2,\gamma-3-}, L^{2,\gamma-4-})} \\ &\qquad + \sum_{n=1}^{\infty} \big\| \big[\chi_0 B_5^+\big]^\vee V S_0^{-1} \big\|_{L^1(\mathrm{HS}(\gamma-3-,\gamma-4-))}. \end{aligned} \tag{35}$$

In view of Proposition 5.12 and Lemma 5.16 we see that the sum converges for λ_0 chosen small enough.

We check the mappings for (35) with $\gamma = \frac{5}{2}$. Here we need to establish the sum holds in $L^1(\mathrm{HS}(-\frac{1}{2}-, -\frac{3}{2}-))$.

$$\begin{aligned} \big[\chi_0 B_5^+\big]^\vee V S_0^{-1} &\in L^1\big(\mathrm{HS}(-1/2-, -1-)\big), \\ \big[\chi_0 B_5^+\big]^\vee V S_0^{-1} &\in L^1\big(\mathrm{HS}(-1-, -1-)\big), \\ \big[\chi_0 B_5^+\big]^\vee V S_0^{-1} &\in L^1\big(\mathrm{HS}(-1-, -3/2-)\big). \end{aligned}$$

The other values of γ that arise are treated similarly.

All derivatives act on $\tilde{B}_5^+$:

If all the λ derivatives in (28) act on $\tilde{B}_5^+(\lambda)$, we have that both the leading at the trailing $A(\lambda, \cdot)$ terms are in $L^{2,-\frac{1}{2}-}(\mathbb{R}^5)$. We need to show that

$$\bigg(\frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}\big[V S_0^{-1} \chi_0 \tilde{B}_5^{\pm}(\lambda)\big]\bigg)^\vee \in L^1\big(L^{2,-\frac{1}{2}-}, L^{2,\frac{1}{2}+}\big).$$

From Proposition 5.12, we need only show

$$\left(\frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}\big[\chi_0\tilde{B}_5^{\pm}(\lambda)\big]\right)^{\vee}\in L^1\big(L^{2,-\sigma-},L^{2,\frac{1}{2}+}\big)$$

for $\sigma\in(\frac{1}{2},\frac{7}{2})$.

We defined $\tilde{B}_5^{+}$ as a convergent Neumann series in (33), we now consider the action of derivatives on this series. If both derivatives act on $\tilde{B}_5^{+}$, and for the time being we assume the derivatives do not act on the cut-off χ_0, we have the following Neumann series to consider.

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}(-1)^n\Bigg[\big(\chi_1 B_5^{+}VS_0^{-1}\big)^m \tag{36}$$

$$\times\left(\chi_0\frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}B_5^{+}VS_0^{-1}\right)\big(\chi_1 B^{+}VS_0^{-1}\big)^{n-(m+1)}$$

$$+\sum_{j=0}^{m-1}\big(\chi_1 B_5^{+}VS_0^{-1}\big)^j\left(\chi_0\frac{d}{d\lambda}B_5^{+}VS_0^{-1}\right)\big(\chi_1 B_5^{+}VS_0^{-1}\big)^{m-(j+1)}$$

$$\times\left(\chi_1\frac{1}{\lambda}\frac{d}{d\lambda}B_5^{+}VS_0^{-1}\right)\big(\chi_1 B_5^{+}VS_0^{-1}\big)^{n-(m+1)} \tag{37}$$

$$+\big(\chi_1 B_5^{+}VS_0^{-1}\big)^m\left(\chi_0\frac{1}{\lambda}\frac{d}{d\lambda}B_5^{+}VS_0^{-1}\right)$$

$$\times\sum_{j=0}^{n-m-2}\big(\chi_1 B_5^{+}VS_0^{-1}\big)^j \tag{38}$$

$$\times\left(\chi_1\frac{d}{d\lambda}B_5^{+}VS_0^{-1}\right)\big(\chi_1 B_5^{+}VS_0^{-1}\big)^{n-(j-m-1)}\Bigg].$$

Three subcases arise from the Neumann series above. We first present the subcase of (36), then the analysis for (37). We note that the analysis of (38) is similar to that of (37).

As in the case when no derivatives act on $\tilde{B}_5^{+}$, we evaluate the Neumann series in the L^1 norm, but this time it will take values in different Hilbert–Schmidt spaces. We note that the terms of the series in (36) are controlled by Lemmas 5.16 and 5.18.

$$\big(\chi_1 B_5^{+}\big)^{\vee}VS_0^{-1}\in L^1\big(\mathrm{HS}(-1/2-,-1-)\big),$$
$$\left(\chi_0\frac{d}{d\lambda}\frac{1}{\lambda}\frac{d}{d\lambda}B_5^{+}\right)^{\vee}VS_0^{-1}\in L^1\big(\mathrm{HS}(-1-,-3-)\big),$$
$$\big(\chi_1 B_5^{+}\big)^{\vee}VS_0^{-1}\in L^1\big(\mathrm{HS}(-3-,-3-)\big).$$

Finally, one uses $VS_0^{-1}: L^{2,-3-}\to L^{2,\frac{1}{2}+}$. Again, the series for (36) converges for λ_0 small enough.

For the series in (37), we apply the same process. This time we use Lemmas 5.16 and 5.17 to control the various terms.

$$\begin{aligned}
\big(\chi_1 B_5^+\big)^\vee V S_0^{-1} V S_0^{-1} &\in L^1\big(\mathrm{HS}(-1/2-,-1-)\big),\\
\left(\chi_0 \frac{1}{\lambda}\frac{d}{d\lambda} B_5^+\right)^\vee V S_0^{-1} &\in L^1\big(\mathrm{HS}(-1-,-2-)\big),\\
\big(\chi_1 B_5^+\big)^\vee V S_0^{-1} &\in L^1\big(\mathrm{HS}(-2-,-3/2-)\big),\\
\left(\chi_1 \frac{d}{d\lambda} B_5^+\right)^\vee &\in L^1\big(\mathrm{HS}(-3/2-,-3/2-)\big).
\end{aligned}$$

Now, one uses that $VS_0^{-1}: L^{2,-\frac{3}{2}-} \to L^{2,\frac{1}{2}+}$. Again, the series for (37) converges for λ_0 small enough.

Remark 5.19. In our analysis we have not accounted for the powers of λ that arise from terms of the leading and trailing resolvents, $A(\lambda,\cdot)$. Each contributes a sum of terms $\lambda^2+\lambda+1$. Our analysis concentrated only on when the zero order λ term arose. To handle higher λ powers, one notes that the estimates considered in Lemmas 5.16, 5.17, and 5.18 gain positive powers of λ_0 for each power of λ that occurs due to scaling considerations.

Similar estimates hold when derivatives act on the cut-off χ_0, since $(\frac{1}{\lambda}\frac{d}{d\lambda})^k\chi \in \mathcal{S}(\mathbb{R})$ is supported on the annulus $|\lambda|\approx 1$. By scaling considerations, each application of a derivative or $\frac{1}{\lambda}$ multiplies the bounds in Lemmas 5.16, 5.17 and 5.18 by λ_0^{-1}.

5.3. Higher dimensional tail estimates. The low energy argument used in dimension five in this paper cannot be extended to dimensions $n>7$ as the kernel $R_n^\pm(\lambda^2)-R_0^\pm(0)$ is not locally $L^2(\mathbb{R}^n)$. Thus, one cannot estimate these kernels as Hilbert–Schmidt operators on weighted L^2 spaces. The $n=7$ low energy argument is essentially the same as the five dimensional argument, though one needs to take one more λ derivative and work with larger Hilbert–Schmidt weights due to the ambient space being $\mathbb{R}^7$. By the arguments here and in Sections 4 and 6, one can prove the analogue of Theorems 1.1 and 1.2 in dimension seven if zero energy is regular and the potential and its first two derivatives decay like $\langle x\rangle^{-8-}$.

For dimensions $n>7$, we note that one can adapt the tail argument of Goldberg and Visan in Section 4 of [31]. This relies on the observation that one can bound the kernels $\frac{d^j}{d\lambda^j}R_0^\pm(\lambda^2+\mu)$ with fractional integral operators. While this will not lead to sharp results with respect to the decay rate of the potentials, the method we use here no longer applies when $n>7$.

6. Tail estimates for the matrix case

It now remains to transfer the tail estimates of the scalar case in Section 5 to the matrix case, (7). As this process is nearly identical to the scalar case in Section 5, as such we only provide a sketch, omitting most of the details.

First, for high energy evolution, we again need the modulated kernels. With a slight abuse of notation, we define

$$G_{\pm,x}(\lambda^2)(x_1) := \begin{bmatrix} e^{\mp i\lambda|x|} & 0 \\ 0 & 1 \end{bmatrix} R_0^{\pm}(\lambda^2+\mu)(x_1,x). \tag{39}$$

Here we modulated the scalar kernel to the same end as the scalar case, we need not modulate the exponentially decaying portion of the kernel. Let $\chi(\lambda)$ be a cut-off away from $\pm\mu$, as appropriate. Again, let $e_1=(1,0)^T$ and $e_2=(0,1)^T$, so that upon removing f,g from (7) we wish to bound

$$\begin{aligned} &\bigg| \int_0^\infty e^{it\lambda^2} e^{\pm i\lambda(|x|+|y|)} \chi(\lambda)\lambda \big\langle V R_V^{\pm}(\lambda^2) V \big(R_0^{\pm}(\lambda^2)V\big)^m G_{\pm,y}(\lambda^2)e_1, \\ &\qquad \big(R_0^{\mp}(\lambda^2)V^*\big)^m G_{\pm,x}^*(\lambda^2)e_1 \big\rangle \, d\lambda \bigg|, \end{aligned} \tag{40}$$

as well as

$$\begin{aligned} &\bigg| \int_0^\infty e^{it\lambda^2} e^{\pm i\lambda|x|} \chi(\lambda)\lambda \big\langle V R_V^{\pm}(\lambda^2) V \big(R_0^{\pm}(\lambda^2)V\big)^m G_{\pm,y}(\lambda^2)e_2, \\ &\qquad \big(R_0^{\mp}(\lambda^2)V^*\big)^m G_{\pm,x}^*(\lambda^2)e_1 \big\rangle \, d\lambda \bigg|, \\ &\bigg| \int_0^\infty e^{it\lambda^2} e^{\pm i\lambda|y|} \chi(\lambda)\lambda \big\langle V R_V^{\pm}(\lambda^2) V \big(R_0^{\pm}(\lambda^2)V\big)^m G_{\pm,y}(\lambda^2)e_1, \\ &\qquad \big(R_0^{\mp}(\lambda^2)V^*\big)^m G_{\pm,x}^*(\lambda^2)e_2 \big\rangle \, d\lambda \bigg|, \\ &\bigg| \int_0^\infty \chi(\lambda)\lambda \big\langle V R_V^{\pm}(\lambda^2) V \big(R_0^{\pm}(\lambda^2)V\big)^m G_{\pm,y}(\lambda^2)e_2, \\ &\qquad \big(R_0^{\mp}(\lambda^2)V^*\big)^m G_{\pm,x}^*(\lambda^2)e_2 \big\rangle \, d\lambda \bigg|, \end{aligned}$$

by $|t|^{-5/2}$ uniformly in $x,y\in\mathbb{R}^5$. Similar to the scalar case, one can easily check that

$$\begin{aligned} &\sup_{y\in\mathbb{R}^5} \bigg\| \frac{d^j}{d\lambda^j} R_0^{\pm} V G_{\pm,y} e_k \bigg\|_{L^{2,-\sigma}} \lesssim \frac{\langle\lambda\rangle^2}{\langle y\rangle^2} \quad \text{if } \sigma > \frac{3}{2}+j, \text{ and } 0\le j\le 1, \\ &\sup_{y\in\mathbb{R}^5} \bigg\| \frac{d^j}{d\lambda^j} R_0^{\pm} V G_{\pm,y} e_k \bigg\|_{L^{2,-\sigma}} \lesssim \langle\lambda\rangle^2 \quad \text{if } \sigma > \frac{1}{2}+j, \text{ and } 0\le j\le 3, \end{aligned} \tag{41}$$

for $k=1,2$. As in the scalar case, we rewrite the high energy integral of (7) as

$$I_M(x,y,t) := \int_0^\infty e^{it\lambda^2} e^{\pm i\lambda(|x|+|y|)} a^{\pm}_{x,y}(\lambda)\, d\lambda.$$

Here we use the subscript "M" to denote the matrix tail integral. Now, with the limiting absorption principle, Proposition 2.2 along with the estimates of (41) we can conclude the following bounds on three derivatives of $a^{\pm}_{x,y}(\lambda)$.

$$\begin{aligned}\left|\frac{d^j}{d\lambda^j} a^{\pm}_{x.y}(\lambda)\right| &\lesssim \frac{\langle\lambda\rangle^{-2}}{\langle x\rangle^2\langle y\rangle^2} \quad \text{for } j=0,1, \text{ and } \lambda\geq 1,\\ \left|\frac{d^j}{d\lambda^j} a^{\pm}_{x.y}(\lambda)\right| &\lesssim \langle\lambda\rangle^{-2} \qquad\quad \text{for } j=2,3, \text{ and } \lambda\geq 1.\end{aligned} \tag{42}$$

This ensures that the integral in (40) converges. We need to take m sufficiently large to ensure ample iterations of the limiting absorption principle to provide the λ decay and $|V(x)|\lesssim \langle x\rangle^{-4-}$ as in the scalar case.

For $|t|>1$, I_M^+ can be estimated by integrating by parts three times as the critical point of the phase is outside of the support of $a^+_{x,y}(\lambda)$. This yields a bound of $|I_M^+|\lesssim |t|^{-3}$. For I_M^-, one must take care as the critical point of the phase $\lambda_1 = \frac{|x|+|y|}{2t}$ can be in the support of $a^-_{x,y}(\lambda)$. Away from the critical point, one can integrate by parts three times which yields a bound of $|t|^{-3}$. If λ_1 is in the support of $a^-_{x,y}$, then the bound $\max(|x|,|y|)\gtrsim |t|$ and stationary phase contributes $|t|^{-1/2}(\langle x\rangle\langle y\rangle)^{-2}\lesssim |t|^{-5/2}$ as desired. For $|t|<1$, one sees from (42) the $|I_M^{\pm}|\lesssim 1$ holds.

For the integrals following (40) with one e_1 and one e_2, the same analysis holds with the critical point being $\frac{|x|}{2t}$ or $\frac{|y|}{2t}$ respectively. For the case of two e_2s, one notes that the critical point is now at $\lambda=0$ which is outside of the support of the integrand. Thus, integrating by parts three times for $|t|$ large and no times for $|t|$ small establishes the desired bounds.

We now need to establish dispersive bounds for λ near the edge of the essential spectrum, $\pm\mu$. We need to show that

$$\begin{aligned}&\big\langle e^{it\mathcal{H}}\big(1-\chi(\mathcal{H})\big)P_c^+ f, g\big\rangle\\ &\quad= \frac{e^{it\mu}}{\pi i}\int_0^\infty e^{it\lambda^2}\lambda(1-\chi)\big(\lambda^2+\mu\big)\\ &\qquad\times \big\langle \big[R_V^+\big(\lambda^2+\mu\big) - R_V^-\big(\lambda^2+\mu\big)\big] f, g\big\rangle\, d\lambda\end{aligned} \tag{43}$$

is bounded by $|t|^{-5/2}\|f\|_1\|g\|_1$. We employ the resolvent identity

$$\begin{aligned}&R_V^{\pm}\big(\lambda^2+\mu\big)\\ &\quad= R_0^{\pm}\big(\lambda^2+\mu\big) - R_0^{\pm}\big(\lambda^2+\mu\big)\big[I+R_0^{\pm}\big(\lambda^2+\mu\big)V\big]^{-1}R_0^{\pm}\big(\lambda^2+\mu\big)\end{aligned} \tag{44}$$

and write $R_0^\pm(\lambda^2+\mu)=R_0^\pm(\mu)+B^\pm(\lambda)$. Then we have

$$\big[I+R_0^\pm\big(\lambda^2+\mu\big)\big]^{-1}=S_0^{-1}\big[I+B^\pm(\lambda)VS_0^{-1}\big]^{-1},$$

where here $S_0=I+R_0^\pm(\mu)$. By the definition of the matrix resolvent, (10), up to a constant multiple

$$R_0^\pm(\mu)(x,y)=\begin{bmatrix}\frac{1}{|x-y|^3} & 0\\ 0 & \frac{e^{-\sqrt{2\mu}|x-y|}}{|x-y|^3}(\sqrt{2\mu}|x-y|+1)\end{bmatrix}.$$

As in the scalar case, invertibility of S_0 follows from the Fredholm alternative and the assumption that $\pm\mu$ is regular. From the fact that

$$\lim_{\lambda\to 0}\big\|B^\pm(\lambda)VS_0^{-1}\big\|_{\mathrm{HS}(\sigma,\sigma)}=0$$

for all $\sigma\in(-\frac{7}{2},-\frac{1}{2})$ where $\mathrm{HS}(\sigma,\sigma)$ is the Hilbert–Schmidt norm of $L^{2,\sigma}\times L^{2,\sigma}\to L^{2,\sigma}\times L^{2,\sigma}$. This allows us to expand

$$\widetilde{B^\pm}(\lambda)=\big[I+B^\pm(\lambda)VS_0^{-1}\big]^{-1}$$

in a Neumann series in Hilbert–Schmidt norms. The proof of the dispersive bound now follows as in the low energy of the scalar case given in Section 5. One needs to make small adjustments for the exponentially decaying resolvent using the fact that

$$\int_{-\infty}^{\infty}e^{i\tau\lambda}e^{-a\sqrt{2\mu+\lambda^2}}\big(-a\sqrt{2\mu+\lambda^2}+1\big)\,d\lambda:=\nu_a(d\tau)$$

is a measure with mass $\sup_{a>0}\|\nu_a\|<\infty$. See Lemma 3.3.

Acknowledgment. The author would like to thank Burak Erdoğan for many helpful discussions concerning this project.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, DC, 1964. MR 0167642

[2] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup Pisa Cl. Sci. (4) **2** (1975), no. 2, 151–218. MR 0397194

[3] M. Beceanu, *A critical centre-stable manifold for the cubic focusing Schroedinger equation in three dimensions*, Ph.D. thesis, Univ. Chicago, 2009, available at `arXiv:0909.4029`. MR 2713222

[4] M. Beceanu, *New estimates for time-independent Schrödinger equation*, Duke Math. J. **159** (2011), no. 3, 417–477. MR 2831875

[5] M. Beceanu and M. Golberg, *Schrödinger dispersive estimates for a scaling-critical class of potentials*, Comm. Math. Phys. **314** (2012), 471–481. MR 2958960

[6] H. Berestycki and T. Cazenave, *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein–Gordon non linéaires*, C. R. Acad. Sci. Paris. **293** (1981), 489–492. MR 0646873

[7] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Ration. Mech. Anal. **82** (1983), no. 4, 313–345. MR 0695535

[8] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. II. Existence of infinitely many solutions*, Arch. Ration. Mech. Anal. **82** (1983), no. 4, 347–375. MR 0695536

[9] V. S. Buslaev and G. S. Perelman, *Scattering for the nonlinear Schrödinger equation: States that are close to a soliton* (russian), Algebra i Analiz **4** (1992), no. 6, 63–102; translation in St. Petersburg Math. J. **4** (1993), no. 6, 1111–1142. MR 1199635

[10] V. S. Buslaev and G. S. Perelman, *On the stability of solitary waves for nonlinear Schrödinger equations. Nonlinear evolution equations*, Amer. Math. Soc. Transl. Ser. 2, vol. 164, Amer. Math. Soc., Providence, RI, 1995, pp. 75–98. MR 1334139

[11] F. Cardoso, C. Cuevas and G. Vodev, *Dispersive estimates for the Schrödinger equation in dimensions four and five*, Asymptot. Anal. **62** (2009), no. 3–4, 125–145. MR 2521760

[12] F. Cardoso, C. Cuevas and G. Vodev, *Dispersive estimates for the Schrödinger equation with potentials of critical regularity*, Cubo **11** (2009), no. 5, 57–70. MR 2568252

[13] F. Cardoso, C. Cuevas and G. Vodev, *High frequency dispersive estimates for the Schrödinger equation in high dimensions*, Asymptot. Anal. **71** (2011), no. 4, 207–225. MR 2815001

[14] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561. MR 0677997

[15] C. V. Coffman, *Uniqueness of positive solutions of $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions*, Arch. Ration. Mech. Anal. **46** (1972), 81–95. MR 0333489

[16] A. Comech and D. Pelinovsky, *Purely nonlinear instability of standing waves with minimal energy*, Comm. Pure Appl. Math. **56** (2003), no. 11, 1565–1607. MR 1995870

[17] S. Cuccagna, *Stablization of solutions to nonlinear Schrödinger equations*, Comm. Pure Appl. Math. **54** (2001), no. 9, 1110–1145. MR 1835384

[18] L. Demanet and W. Schlag, *Numerical verification of a gap condition for linearized NLS*, Nonlinearity **19** (2006), 829–852. MR 2214946

[19] M. B. Erdoğan, M. Goldberg and W. Schlag, *Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions*, Forum Math. **21** (2009), no. 4, 687–722. MR 2541480

[20] M. B. Erdoğan and W. R. Green, *Dispersive estimates for the Schrödinger equation with $C^{(n-3)/2}$ potentials in odd dimensions*, Internat. Math. Res. Notices **13** (2010), 2532–2565. MR 2669658

[21] M. B. Erdoğan and W. R. Green, *Dispersive estimates for matrix Schrdinger operators in dimension two.* Discrete Contin. Dyn. Syst. **33** (2013), no. 10, 4473-4495. MR 3049087

[22] M. B. Erdoğan and W. Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: I*, Dyn. Partial Differ. Equ. **1** (2004), 359–379. MR 2127577

[23] M. B. Erdoğan and W. Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or eigenvalue at zero energy in dimension three: II*, J. Anal. Math. **99** (2006), 199–248. MR 2279551

[24] D. Finco and K. Yajima, *The L^p boundedness of wave operators for Schrödinger operators with threshold singularities II. Even dimensional case*, J. Math. Sci. Univ. Tokyo **13** (2006), no. 3, 277–346. MR 2284406

[25] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. **74** (1987), no. 1, 160–197. MR 0901236

[26] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal. **94** (1990), 308–348. MR 1081647

[27] M. Goldberg, *A Dispersive bound for three-dimensional Schrödinger operators with zero energy eigenvalues*, Comm. Partial Differential Equations **35** (2010), 1610–1634. MR 2754057

[28] M. Goldberg, *Dispersive bound for the three-dimensional Schrödinger equation with almost critical potentials*, Geom. Funct. Anal. **16** (2006), no. 3, 517–536. MR 2238943

[29] M. Goldberg, *Dispersive estimates for the three-dimensional Schrödinger equation with rough potentials*, Amer. J. Math. **128** (2006) 731–750. MR 2230923

[30] M. Goldberg and W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*, Comm. Math. Phys. **251** (2004), no. 1, 157–178. MR 2096737

[31] M. Goldberg and M. Visan, *A counterexample to dispersive estimates*, Comm. Math. Phys. **266** (2006), no. 1, 211–238. MR 2231971

[32] P. D. Hislop and I. M. Sigal, *Introduction to spectral theory. With applications to Schrödinger operators*, Applied Mathematical Sciences, vol. 113, Springer, New York, 1996. MR 1361167

[33] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions results in $L^2(\mathbb{R}^m)$, $m \geq 5$*, Duke Math. J. **47** (1980), no. 1, 57–80. MR 0563367

[34] A. Jensen and T. Kato, *Spectral properties of Schrödinger and time-decay of wave functions*, Duke Math. J. **46** (1979), no. 3, 583–611. MR 0544248

[35] J.-L. Journé, A. Soffer and C.D. Sogge, *Decay estimates for Schrödinger operators*, Comm. Pure Appl. Math. **44** (1991), 573–611. MR 1105875

[36] M. K. Kwong, *Uniqueness of positive solutions of $\Delta u + u^p = 0$ in $\mathbb{R}^n$*, Arch. Ration. Mech. Anal. **65** (1989), 243–266. MR 0969899

[37] J. Marzuola, *Dispersive estimates using scattering theory for matrix Hamiltonian equations*, Discrete Contin. Dyn. Syst. **30** (2011), no. 4, 995–1036. MR 2850934

[38] J. Marzuola and G. Simpson, *Spectral analysis for matrix Hamiltonian operators*, Nonlinearity **24** (2011), 389–429. MR 2754405

[39] K. McLeod and J. Serrin, *Nonlinear Schrödinger equation. Uniqueness of positive solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$*, Arch. Ration. Mech. Anal. **99** (1987), 115–145. MR 0886933

[40] S. Moulin and G. Vodev, *Low-frequency dispersive estimates for the Schrödinger group in higher dimensions*, Asymptot. Anal. **55** (2007), no. 1–2, 49–71. MR 2373806

[41] M. Murata, *Asymptotic expansions in time for solutions of Schrödinger-type equations*, J. Funct. Anal. **49** (1982), no. 1, 10–56. MR 0680855

[42] G. Perelman, *On the formation of singularities in solutions of the critical nonlinear Schrödinger equation*, Ann. Henri Poincaré **2** (2001), no. 4, 605–673. MR 1852922

[43] G. Perelman, *Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations*, Comm. Partial Differential Equations **29** (2004), no. 7–8, 1051–1095. MR 2097576

[44] J. Rauch, *Local decay of scattering solutions to Schrödinger's equation*, Comm. Math. Phys. **61** (1978), no. 2, 149–168. MR 0495958

[45] M. Reed and B. Simon, *Methods of modern mathematical physics. I: Functional analysis, IV: Analysis of operators*, Academic Press, New York, NY, 1972. MR 0493419

[46] I. Rodnianski and W. Schlag, *Time decay for solutions of the Schrödinger equations with rough and time-dependent potentials*, Invent. Math. **155** (2004), no. 3, 451–513. MR 2038194

[47] I. Rodnianski, W. Schlag and A. Soffer, *Asymptotic stability of N-soliton states of NLS*, preprint, 2004.

[48] I. Rodnianski, W. Schlag and A. Soffer, *Dispersive analysis of charge transfer models*, Comm. Pure Appl. Math. **58** (2005), no. 2, 149–216. MR 2094850

[49] W. Schlag, *Dispersive estimates for Schrödinger operators: A survey*, Mathematical aspects of nonlinear dispersive equations, Ann. of Math. Stud., vol. 163, Princeton Univ. Press, Princeton, NJ, 2007, pp. 255-285. MR 2333215

[50] W. Schlag, *Dispersive estimates for Schrödinger operators in dimension two*, Comm. Math. Phys. **257** (2005), no. 1, 87–117. MR 2163570

[51] W. Schlag, *Spectral theory and nonlinear partial differential equations: A survey*, Discrete Contin. Dyn. Syst. **15** (2006), no. 3, 703–723. MR 2220744

[52] W. Schlag, *Stable manifolds for an orbitally unstable nonlinear Schrödinger equation*, Ann. of Math. (2) **169** (2009), no. 1, 139–227. MR 2480603

[53] J. Shatah, *Stable standing waves of nonlinear Klein–Gordon equations*, Comm. Math. Phys. **91** (1983), no. 3, 313–327. MR 0723756

[54] J. Shatah and W. Strauss, *Instability of nonlinear bound states*, Comm. Math. Phys. **100** (1985), no. 2, 173–190. MR 0804458

[55] A. Soffer and M. Weinstein, *Multichannel nonlinear scattering for nonintegrable equations*, Comm. Math. Phys. **133** (1990), 119–146. MR 1071238

[56] A. Soffer and M. Weinstein, *Multichannel nonlinear scattering. II. The case of anisotropic potentials and data*, J. Differential Equations **98** (1992), 376–390. MR 1170476

[57] W. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162. MR 0454365

[58] W. A. Strauss, *Nonlinear wave equations*, CBMS Regional Conference Series in Mathematics, vol. 73, Amer. Math. Soc., Providence, RI, 1989. MR 1032250

[59] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, Applied Mathematical Sciences, vol. 139, Springer, New York, 1999. MR 1696311

[60] G. Vodev, *Dispersive estimates of solutions to the Schrödinger equation*, Ann. Henri Poincaré **6** (2005), no. 6, 1179–1196. MR 2189381

[61] G. Vodev, *Dispersive estimates of solutions to the Schrödinger equation in dimensions* $n \geq 4$, Asymptot. Anal. **49** (2006), no. 1–2, 61–86. MR 2260557

[62] R. Weder, $L^p - L^{p'}$ *estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential*, J. Funct. Anal. **170** (2000), no. 1, 37–68. MR 1736195

[63] M. I. Weinstein, *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal. **16** (1985), no. 3, 472–491. MR 0783974

[64] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), no. 1, 51–67. MR 0820338

[65] K. Yajima, *The* L^p *Boundedness of wave operators for Schrödinger operators with threshold singularities. I. The odd dimensional case*, J. Math. Sci. Univ. Tokyo **13** (2006), 43–94. MR 2223681

[66] K. Yajima, L^p*-boundedness of wave operators for two-dimensional Schrödinger operators*, Comm. Math. Phys. **208** (1999), no. 1, 125–152. MR 1729881

[67] K. Yajima, *Dispersive estimate for Schrödinger equations with threshold resonance and eigenvalue*, Comm. Math. Phys. **259** (2005), no. 2, 475–509. MR 2172692

William R. Green, Department of Mathematics, Eastern Illinois University, Charleston, IL 61920, USA

Current address: Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute, IN 47803, USA

E-mail address: green@rose-hulman.edu