

AN EXAMPLE OF WEAKLY AMENABLE AND CHARACTER AMENABLE OPERATOR

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ABSTRACT. A complete characterization of Hilbert space operators that generate weakly amenable algebras remains open, even in the case of compact operator. Farenick, Forrest and Marcoux proposed the question that if T is a compact weakly amenable operator on a Hilbert space \mathfrak{H} , then is T similar to a normal operator? In this paper, we demonstrate an example of compact triangular operator on infinite-dimensional Hilbert space which is a weakly amenable and character amenable operator but is not similar to a normal operator.

1. Introduction

Let \mathfrak{A} be a Banach algebra, and let X be a Banach \mathfrak{A} -bimodule. A *derivation* $D : \mathfrak{A} \rightarrow X$ is a continuous linear map such that $D(ab) = a \cdot D(b) + D(a) \cdot b$, for all $a, b \in \mathfrak{A}$. A derivation $D : \mathfrak{A} \rightarrow X$ is said to be *inner* if there exists $x \in X$ such that $D(a) = a \cdot x - x \cdot a$ for all $a \in \mathfrak{A}$. A Banach \mathfrak{A} -bimodule X is said to be *commutative* if $a \cdot x = x \cdot a$ for each $a \in \mathfrak{A}, x \in X$. For any Banach \mathfrak{A} -bimodule X , its dual X^* is naturally equipped with a Banach \mathfrak{A} -bimodule structure via $[a \cdot f](x) = f(x \cdot a), [f \cdot a](x) = f(a \cdot x), a \in \mathfrak{A}, x \in X, f \in X^*$.

We can now give the definition of amenability, weak amenability and character amenability for a Banach algebra:

DEFINITION 1.1. A Banach algebra \mathfrak{A} is *amenable* if, for each Banach \mathfrak{A} -bimodule X , every derivation $D : \mathfrak{A} \rightarrow X^*$ is inner.

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DEFINITION 1.2. A commutative Banach algebra \mathfrak{A} is *weakly amenable*, if, for each commutative Banach \mathfrak{A} -bimodule X , every derivation $D : \mathfrak{A} \rightarrow X$ is inner.

Let \mathfrak{A} be a Banach algebra and $\sigma(\mathfrak{A})$ the spectrum of \mathfrak{A} , that is, the set of all non-zero multiplicative linear functionals on \mathfrak{A} . If $\varphi \in \sigma(\mathfrak{A}) \cup \{0\}$ and if X is a Banach space, then X can be viewed as left or right Banach \mathfrak{A} -module by the following actions. For $a \in \mathfrak{A}, x \in X$:

$$(1.1) \quad a \cdot x = \varphi(a)x,$$

$$(1.2) \quad x \cdot a = \varphi(a)x.$$

If the left action of \mathfrak{A} on X is given by (1.1), then it is easily verified that the right action of \mathfrak{A} on the dual \mathfrak{A} -module X^* is given by $f \cdot a = \varphi(a)f$, for all $f \in X^*, a \in \mathfrak{A}$. Throughout, by (φ, \mathfrak{A}) -bimodule X , we mean that X is a Banach \mathfrak{A} -bimodule for which the left module action is given by (1.1). (\mathfrak{A}, φ) -bimodule is defined similarly by (1.2). Let $\varphi \in \sigma(\mathfrak{A}) \cup \{0\}$, a Banach algebra \mathfrak{A} is said to be *left φ amenable*, if every derivation D from \mathfrak{A} into the dual \mathfrak{A} -bimodule X^* is inner for all (φ, \mathfrak{A}) -bimodules X ; \mathfrak{A} is said to be *right φ amenable*, if every derivation D from \mathfrak{A} into the dual \mathfrak{A} -bimodule X^* is inner for all (\mathfrak{A}, φ) -bimodules X . \mathfrak{A} is said to be *left character amenable*, if it is left φ amenable for all $\varphi \in \sigma(\mathfrak{A}) \cup \{0\}$; \mathfrak{A} is said to be *right character amenable*, if it is right φ amenable for all $\varphi \in \sigma(\mathfrak{A}) \cup \{0\}$.

DEFINITION 1.3. A Banach algebra \mathfrak{A} is said to be *character amenable*, if it is both left character amenable and right character amenable.

The concept of amenable Banach algebras was first introduced by Johnson in [6]. Weak amenability was first defined by Bade, Curtis and Dales in [1], [3]. Character amenability was first defined by Monfared [7]. Ever since its introduction, the concepts of amenability, weak amenability and character amenability have played an important role in research in Banach algebras, operator algebras and harmonic analysis. We only would like to mention the following deep results due to Willis [8] and Farenick, Forrest and Marcoux [4], [5]:

Given a complex separable infinite-dimensional Hilbert space \mathfrak{H} , we write $\mathfrak{B}(\mathfrak{H})$ for the bounded linear operators on \mathfrak{H} . If $T \in \mathfrak{B}(\mathfrak{H})$, denote the norm-closure of $\text{span} \{T^k : k \in \mathbb{N}\}$ by \mathfrak{A}_T , where \mathbb{N} is the set of natural numbers. T is said to be *amenable (weakly amenable or character amenable)* if \mathfrak{A}_T is amenable (respectively, weakly amenable, character amenable).

In [8], Willis showed that:

THEOREM 1.4. *Suppose T is a compact amenable operator, then T is similar to a normal operator.*

In [4], [5] Farenick, Forrest and Marcoux showed that:

THEOREM 1.5. *Suppose T is a triangular operator with respect to an orthonormal basis of \mathfrak{H} , then T is amenable if and only if T is similar to a normal operator whose spectrum has connected complement and empty interior.*

A complete characterisation of Hilbert space operators that generate weakly amenable algebras remains open, even in the case of compact operators. In [4], [5] Farenick, Forrest and Marcoux proposed the following question:

QUESTION 1.6. *If T is a compact weakly amenable operator on \mathfrak{H} , then is T similar to a normal operator?*

It is well known that if \mathfrak{H} is a finite-dimensional Hilbert space, then $T \in \mathfrak{B}(\mathfrak{H})$ is amenable, weakly amenable or character amenable if and only if T is similar to a normal operator. The purpose of this paper is to demonstrate an example of compact triangular operator on an infinite-dimensional Hilbert space which is weakly amenable and character amenable but is not similar to a normal operator.

2. Compact triangular weakly amenable operator

Suppose σ is a compact Hausdorff space. Let $C(\sigma)$ denote the Banach algebra of all continuous functions on σ with the supremum norm $\|f\|_\infty = \sup_{x \in \sigma} |f(x)|$. Throughout this paper, we let $\sigma = \{0, \lambda_1, \lambda_2, \dots\}$, where $\{\lambda_n\}_{n=1}^\infty$ is a sequence of positive real numbers which converge to zero. Let

$$T = \begin{pmatrix} 0 & N^{\frac{1}{2}} \\ 0 & N \end{pmatrix},$$

where N is a normal operator with spectrum σ , then T is an operator on an infinite-dimensional Hilbert space. In this section, we obtain that T is weakly amenable, but is not similar to a normal operator. Especially, if let

$$N = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{pmatrix},$$

then T is a compact triangular operator.

The following lemma is easily verified.

LEMMA 2.1. *Suppose \mathfrak{A} is a commutative Banach algebra which is generated by the idempotent elements in \mathfrak{A} , then \mathfrak{A} is weakly amenable.*

Proof. Let \mathcal{P} denote the set of the idempotent elements in \mathfrak{A} . Assume X is a commutative Banach \mathfrak{A} -bimodule, and $D : \mathfrak{A} \rightarrow X$ is a derivation. For any $p \in \mathcal{P}$, $D(p) = D(p^2) = D(p^3)$ and $D(p^2) = 2pD(p), D(p^3) = 3p^2D(p)$, so $D(p) = 0$. Since $p \in \mathcal{P}$ is arbitrary and \mathfrak{A} is generated by \mathcal{P} , it follows that $D(a) = 0$ for all $a \in \mathfrak{A}$. That is to say, \mathfrak{A} is weak amenable. □

Our main result in this section will be that for any normal operator N with spectrum σ , $T = \begin{pmatrix} 0 & N^{\frac{1}{2}} \\ 0 & N \end{pmatrix}$ is weakly amenable but is not similar to a normal operator.

THEOREM 2.2. *Let $T = \begin{pmatrix} 0 & N^{\frac{1}{2}} \\ 0 & N \end{pmatrix}$, where N is a normal operator with spectrum σ , then T is weakly amenable.*

Proof. By Lemma 2.1, it suffices to show that \mathfrak{A}_T is generated by the idempotent elements in it.

Step 1. $\mathfrak{A}_T = \left\{ \begin{pmatrix} 0 & f(N) \\ 0 & N^{\frac{1}{2}}f(N) \end{pmatrix}; f \in C(\sigma), f(0) = 0 \right\} \triangleq M$, where $f(N)$ is defined by functional calculus of normal operators.

Indeed, for any polynomial $p(z) = \sum_{k=1}^n a_k z^k = z \sum_{k=0}^{n-1} a_{k+1} z^k \triangleq zq(z)$, $p(T)$ has the form

$$\begin{pmatrix} 0 & N^{\frac{1}{2}}q(N) \\ 0 & p(N) \end{pmatrix}.$$

For any $A = \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix} \in \mathfrak{A}_T$, there exists a sequence of polynomials $\{p_n\}$, $p_n(0) = 0$ for all n such that $\|p_n(T) - A\| \rightarrow 0$. i.e. $\|p_n(N) - A_{22}\| \rightarrow 0$ and $\|N^{\frac{1}{2}}q_n(N) - A_{12}\| \rightarrow 0$. Therefore, there exists a function g on σ , such that $\|z^{\frac{1}{2}}q_n - g\|_\infty \rightarrow 0$ and $\|p_n - z^{\frac{1}{2}}g\|_\infty \rightarrow 0$. It follows that $A = \begin{pmatrix} 0 & g(N) \\ 0 & N^{\frac{1}{2}}g(N) \end{pmatrix}$, and $g \in C(\sigma), g(0) = 0$. That is to say, $\mathfrak{A}_T \subseteq M$.

For any $f \in C(\sigma), f(0) = 0$, there exists a sequence of polynomials $\{p_n, p_n(0) = 0\}$ such that $\|p_n - f\|_\infty \rightarrow 0$. Let $p_n = zq_n$, for every n , there exists a polynomial $r_n, r_n(0) = 0$ such that $\|r_n - z^{\frac{1}{2}}q_n\|_\infty < \frac{1}{n}$. Therefore, $\|z^{\frac{1}{2}}r_n - f\|_\infty \leq \|z^{\frac{1}{2}}(r_n - z^{\frac{1}{2}}q_n)\|_\infty + \|p_n - f\|_\infty \rightarrow 0$, and $\|zr_n - z^{\frac{1}{2}}f\|_\infty \rightarrow 0$. It follows that $Tr_n(T) \rightarrow \begin{pmatrix} 0 & f(N) \\ 0 & N^{\frac{1}{2}}f(N) \end{pmatrix}$. That is to say, $M \subseteq \mathfrak{A}_T$.

Step 2. \mathfrak{A}_T is generated by the idempotent elements in it.

It is easy to verify that if for each n , we let

$$h_n(z) = \begin{cases} \frac{1}{\sqrt{\lambda_n}}, & z = \lambda_n; \\ 0, & z \neq \lambda_n, \end{cases}$$

then $\begin{pmatrix} 0 & h_n(N) \\ 0 & N^{\frac{1}{2}}h_n(N) \end{pmatrix}$ is an idempotent element in \mathfrak{A}_T , and \mathfrak{A}_T is generated by idempotent elements $\left\{ \begin{pmatrix} 0 & h_n(N) \\ 0 & N^{\frac{1}{2}}h_n(N) \end{pmatrix} \right\}_{n=1}^\infty$. The proof is completed. □

PROPOSITION 2.3. *If $T \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ has the form $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$, and T is amenable. Then, there exists an invertible operator $S \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ such that $STS^{-1} = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}$.*

Proof. For each $A \in \mathfrak{A}_T$, A has the form $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with respect to the decomposition $\mathfrak{H}_1 \oplus \mathfrak{H}_2$. Then $\pi : A \longrightarrow \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$ is a representation of \mathfrak{A}_T . Because \mathcal{C}_1 has a Banach \mathfrak{A}_T -bimodule structure defined by $A \cdot a = \pi(A)a, a \cdot A = a\pi(A)$ for all $A \in \mathfrak{A}_T, a \in \mathcal{C}_1$, then $\mathfrak{B}(\mathfrak{H})$ has a dual \mathfrak{A}_T -bimodule structure defined by $A \cdot a = \pi(A)a, a \cdot A = a\pi(A)$ for all $A \in \mathfrak{A}_T, a \in \mathfrak{B}(\mathfrak{H})$.

Define a mapping $D : \mathfrak{A}_T \rightarrow \mathfrak{B}(\mathfrak{H})$ by $D(A) = \begin{pmatrix} 0 & A_{12} \\ 0 & 0 \end{pmatrix}$. Note that

$$D(AB) = \begin{pmatrix} 0 & A_{11}B_{12} + A_{12}B_{22} \\ 0 & 0 \end{pmatrix} = \pi(A)D(B) + D(A)\pi(B).$$

Therefore, D is a derivation. Since \mathfrak{A}_T is amenable, it follows that there exists a $S_0 \in \mathfrak{B}(\mathfrak{H})$ such that $D(A) = \pi(A)S_0 - S_0\pi(A)$ for all $A \in \mathfrak{A}_T$. It follows that $A_{12} = A_{11}S_{12} - S_{12}A_{22}$ for all $A \in \mathfrak{A}_T$.

Let $S = \begin{pmatrix} I & S_{12} \\ 0 & I \end{pmatrix}$, then $STS^{-1} = \begin{pmatrix} T_{11} & T_{12} + S_{12}T_{22} - T_{11}S_{12} \\ 0 & T_{22} \end{pmatrix}$. Since $T_{12} = T_{11}S_{12} - S_{12}T_{22}$, it follows that $STS^{-1} = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}$. □

Finally, we will obtain that $T = \begin{pmatrix} 0 & N^{\frac{1}{2}} \\ 0 & N \end{pmatrix}$ is not similar to a normal operator. Indeed, suppose T is similar to a normal operator, by [4], Theorem 2.7 and the proof of Proposition 2.3, we get that T is amenable and there exists an bounded operator B such that

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & N^{\frac{1}{2}} \\ 0 & N \end{pmatrix} \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}.$$

Therefore, there exists an bounded operator B such that $BN = N^{\frac{1}{2}}$, which is impossible. Hence, T is not similar to a normal operator.

REMARK 2.4. Theorem 2.2 shows that there exists a compact triangular operator with infinite spectrum which is a weakly amenable operator but is not similar to a normal operator. However, we do not know if a compact quasinilpotent operator can be weakly amenable. It would be very interesting to know whether this result is true. Indeed, if any compact quasinilpotent operator can not be weakly amenable, then it is easy to get that a compact operator T with finite spectrum is weakly amenable if and only if T is similar to a normal operator.

3. Compact triangular character amenable operator

Let Q be the Volterra operator on infinite-dimensional Hilbert space, then by [7], Corollary 2.7 and [2], Corollary 5.11, Q is a compact quasinilpotent operator which is character amenable. However, the lattice of invariant subspaces of Q is a continuous nest, i.e. Q is not a triangular operator. In this section, we will prove that $T = \begin{pmatrix} 0 & N^{\frac{1}{2}} \\ 0 & N \end{pmatrix}$ is a character amenable operator,

for any normal operator N with spectrum σ . Hence, there exists a compact triangular operator which is a character amenable operator but is not similar to a normal operator.

In [7], Monfared obtained a necessary and sufficient condition for a Banach algebra to be character amenable.

LEMMA 3.1. *A Banach algebra \mathfrak{A} is character amenable if and only if $\ker \phi$ has a bounded approximate identity for every $\phi \in \sigma(\mathfrak{A}) \cup \{0\}$.*

PROPOSITION 3.2. *Let $T = \begin{pmatrix} 0 & N^{\frac{1}{2}} \\ 0 & N \end{pmatrix}$, then for any polynomial p , $\lambda_n \in \mathbb{C}$,*

$$(\lambda_n I - T)p(\lambda_n I - T) = \begin{pmatrix} \lambda_n p(\lambda_n I) & -q(N)N^{\frac{1}{2}} \\ 0 & (\lambda_n I - N)p(\lambda_n I - N) \end{pmatrix},$$

where $\{q\}$ is a polynomial which satisfy the equation $\lambda_n p(\lambda_n) - (\lambda_n - z)p(\lambda_n - z) = zq(z)$.

Proof. Note that if T has the form $T = \begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}$, with $\{N_i\}$ a collection of commuting operators, then $T^k = \begin{pmatrix} N_1^k & A_k N_2 \\ 0 & N_3^k \end{pmatrix}$, where $N_1^k - N_3^k = (N_1 - N_3)A_k$ for all $k \in \mathbb{N}$.

It is easy to check that for any $k > 1$ and $T = \begin{pmatrix} 0 & N^{\frac{1}{2}} \\ 0 & N \end{pmatrix}$,

$$(\lambda_n I - T)^k = \begin{pmatrix} \lambda_n^k I & -N^{k-1}N^{\frac{1}{2}} \\ 0 & (\lambda_n - N)^k \end{pmatrix},$$

and for any polynomial p

$$(\lambda_n I - T)p(\lambda_n I - T) = \begin{pmatrix} \lambda_n p(\lambda_n I) & -q(N)N^{\frac{1}{2}} \\ 0 & (\lambda_n I - N)p(\lambda_n I - N) \end{pmatrix},$$

where $\{q\}$ is a polynomials which satisfy the equation $\lambda_n p(\lambda_n) - (\lambda_n - z)p(\lambda_n - z) = zq(z)$. □

Using Lemma 3.1, and Proposition 3.2, we will prove T is a character amenable operator.

THEOREM 3.3. *Let $T = \begin{pmatrix} 0 & N^{\frac{1}{2}} \\ 0 & N \end{pmatrix}$, then T is character amenable.*

Proof. By Lemma 3.1, it suffices to show that \mathfrak{A}_T and $\mathfrak{A}_{\lambda_n I - T}$ have bounded approximate identities.

It is verity that \mathfrak{A}_T has a bounded approximate identity. Indeed, for any polynomial $p(z) = \sum_{k=1}^n a_k z^k = z \sum_{k=0}^{n-1} a_{k+1} z^k \triangleq zq(z)$, $p(T)$ and $p(T)T$ has the form

$$p(T) = \begin{pmatrix} 0 & N^{\frac{1}{2}}q(N) \\ 0 & p(N) \end{pmatrix}, \quad p(T)T = \begin{pmatrix} 0 & N^{\frac{1}{2}}p(N) \\ 0 & p(N)N \end{pmatrix}.$$

Let $\{p_n\}$ be the sequence of polynomials such that $\{p_n(N)\}$ is the bounded approximate identities of \mathfrak{A}_N , then $\{p_n(T)\}$ is the bounded approximate identities of \mathfrak{A}_T .

Hence, we only to need prove that $\mathfrak{A}_{\lambda_n I - T}$ has a bounded approximate identity for any n .

For each fixed n , assume that f_n is a smooth function defined on $[\lambda_n - \lambda_1, \lambda_n]$ and satisfies that

$$f_n(z) = \begin{cases} 0, & z = 0; \\ 1, & z \in [\lambda_n - \lambda_1, \lambda_n - \lambda_{n-1}] \cup [\lambda_n - \lambda_{n+1}, \lambda_n]. \end{cases}$$

There exists a sequence of polynomials $\{p_k, p_k(0) = 0\}$ such that p_k and p'_k (the derivative of p_k) converge to f_n and f'_n uniformly on $[\lambda_n - \lambda_1, \lambda_n]$, respectively. Since $\lambda_n I - N$ is a normal operator with spectrum $\sigma(\lambda_n I - N) = \{\lambda_n, \lambda_n - \lambda_1, \lambda_n - \lambda_2, \dots\}$, and for any $f \in C(\sigma(\lambda_n I - N))$, $f(0) = 0$, we have $f f_n = f_n f = f$, it follows that $f_n(\lambda_n I - N)$ is an identity of $\mathfrak{A}_{\lambda_n I - N}$ and $f_n(\lambda_n I) = \lambda_n I$. Hence, $\|(\lambda_n I - N)p_k(\lambda_n I - N) - (\lambda_n I - N)\| \rightarrow 0$ and $\|p_k(\lambda_n I) - I\| \rightarrow 0$, when $k \rightarrow \infty$.

By Proposition 3.2,

$$(\lambda_n I - T)p_k(\lambda_n I - T) = \begin{pmatrix} \lambda_n p_k(\lambda_n I) & -q_k(N)N^{\frac{1}{2}} \\ 0 & (\lambda_n I - N)p_k(\lambda_n I - N) \end{pmatrix},$$

where $\{q_k\}$ is a sequence of polynomials which satisfy the equation $\lambda_n p_k(\lambda_n) - (\lambda_n - z)p_k(\lambda_n - z) = zq_k(z)$. Note that

$$q_k(z) - p_k(\lambda_n - z) = \frac{\lambda_n p_k(\lambda_n) - \lambda_n p_k(\lambda_n - z)}{z} = \lambda_n p'_k(\xi_{k,n,z})$$

for some $\xi_{k,n,z} \in [\lambda_n - \lambda_1, \lambda_n]$ and for all $z \in \sigma$. Hence, $\{q_k\}$ is bounded on σ .

Since $\|Nq_k(N) - N\| = \|\lambda_n p_k(\lambda_n I) - (\lambda_n I - N)p_k(\lambda_n I - N) - N\| \rightarrow 0$, it follows that $\{q_k(N)\}$ is a bounded approximate identity for \mathfrak{A}_N . Hence, $\|N^{\frac{1}{2}}q_k(N) - N^{\frac{1}{2}}\| \rightarrow 0$. Therefore, $\{p_k(\lambda_n I - T)\}$ is a bounded approximate identity for $\mathfrak{A}_{\lambda_n I - T}$. □

REMARK 3.4. Theorem 3.3 shows that there exists a compact triangular operator with infinite spectrum which is a character amenable operator but is not similar to a normal operator. Moreover, by Lemma 3.1 we can describe character amenable operator with finite spectrum: If $T \in \mathfrak{B}(\mathfrak{H})$ with finite spectrum $\sigma(T) = \{\delta_1, \delta_2, \dots, \delta_n\}$, then T is similar to

$$\begin{pmatrix} \delta_1 I + Q_1 & & & \\ & \delta_2 I + Q_2 & & \\ & & \ddots & \\ & & & \delta_n I + Q_n \end{pmatrix},$$

where Q_k is a quasinilpotent operator for $1 \leq k \leq n$. By Lemma 3.1, T is character amenable if and only if \mathfrak{A}_{Q_k} has a bounded approximate identity for $1 \leq k \leq n$.

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