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TWO-ENDED r-MINIMAL HYPERSURFACES IN EUCLIDEAN SPACE

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ABSTRACT. It is shown that embedded, elliptic r-minimal hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$, $\frac{3}{2}(r+1) \leq n < 2(r+1)$, with two ends, both regular, are catenoids (i.e., rotational hypersurfaces). This extends to this setting previous results by Schoen and Hounie-Leite.

1. Introduction and statement of results

Since its introduction by Aleksandrov, the Tangency Principle, based on the maximum principle for second order elliptic PDEs, has been successfully used to settle many important questions in Differential Geometry. As a remarkable application of this principle, R. Schoen characterized rotational minimal hypersurfaces in $\mathbb{R}^{n+1}$, also known as catenoids.

THEOREM 1.1 ([S]). *If $M \subset \mathbb{R}^{n+1}$ is a complete nonflat minimal hypersurface with two ends, both regular, then M is a catenoid.*

We recall that, roughly speaking, a minimal end is regular if it is asymptotic to the end of a catenoid.

An interesting question is whether this result can be extended to hypersurfaces whose extrinsic geometry satisfies other conditions than minimality. To this effect, recall that if $M \subset \mathbb{R}^{n+1}$ is a hypersurface then one has, at least locally, a unit normal vector field N defining its shape operator $A : TM \to TM$, $A(v) = -D_v N$, where D is the standard derivation on $\mathbb{R}^n$. The n real eigenvalues of the field A of symmetric endomorphisms of TM, say $\kappa_1, \dots, \kappa_n$, are the *principal curvatures* of the immersion. For each $k = 0, 1, \dots, n$, let S_k be the kth elementary symmetric function in the entries of $\kappa = (\kappa_1, \dots, \kappa_n)$, also called the *$k$-curvature* of the immersion.

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Definition 1.1. If $0 \le r \le n-1$ we say that M is *r-minimal* if $S_{r+1} = 0$ identically.

Thus, in our terminology, minimal hypersurfaces are 0-minimal and scalar-flat hypersurfaces are 1-minimal. It is well known [R2] that r-minimal hypersurfaces are critical points, under compactly supported variations, for a natural geometric variational problem, namely, that associated to the functional

$$\mathcal{A}_r(M) = \int_M S_r \, dM. \tag{1.1}$$

Here, dM is the volume element of M. Thus, uniqueness results for r-minimal hypersurfaces pose global constraints on the solutions of this variational problem, besides furnishing generalizations of Theorem 1.1.

One of the difficulties in applying the Tangency Principle to r-minimal hypersurfaces is that the linearized operator associated to the $(r+1)$-curvature, the so-called *Jacobi operator*, is not always elliptic for $r \ge 1$. This question has been completely clarified by Hounie and Leite in a series of important papers [HL1], [HL2]: ellipticity takes place at $p \in M$ if and only if $S_{r+2}(p) \neq 0$ or, equivalently, $\operatorname{rank} A_p \ge r+1$. This motivates the following definition.

Definition 1.2. An r-minimal hypersurface is *elliptic* if its Jacobi operator is elliptic everywhere.

Armed with this concept, Hounie and Leite were able to devise a Tangency Principle for r-minimal hypersurfaces and obtained the following uniqueness result for $r = 1$.

Theorem 1.2 ([HL2]). *If $M^n \subset \mathbb{R}^{n+1}$ is a complete, embedded, elliptic 1-minimal hypersurface with two ends, both regular, then M is a catenoid.*

As in the minimal case, by a catenoid we mean a rotationally invariant r-minimal hypersurface; see Section 3.1 for a description of such objects. Moreover, that the r-minimal end is regular means that, when written as a graph over a hyperplane, this end has the same asymptotic expansion as the end of a catenoid; see Definition 3.1 for a precise discussion. Notice also that, besides ellipticity, another assumption appears here in comparison to Theorem 1.1: M has to be embedded. This is due to the fact that that Tangency Principle in r-minimal case has been established only under this assumption; see Theorem 2.1.

We should remark that, as pointed out by Leite, the proof of the main uniqueness theorem in [HL2] only works for $n = 3$. In a personal communication [L2], it is shown how the argument can be fixed for $n \ge 4$.

The purpose of this paper is to generalize the case $n = 3$ of Theorem 1.2 to a large class of r-minimal hypersurfaces. More precisely, we have the following theorem.

THEOREM 1.3. *Let $M \subset \mathbb{R}^{n+1}$ be a complete, embedded and elliptic r-minimal hypersurface with $\frac{3}{2}(r+1) \leq n < 2(r+1)$. If M has two ends, both regular, then M is a catenoid.*

We believe that the general case can be dealt with by adapting Leite's argument [L2] in the scalar-flat case and we hope to address this question elsewhere.

This paper is organized as follows. In Section 2, we recall several basic facts on r-minimal hypersurfaces, including discussions on the Newton tensors, the Reilly operator L_r and its ellipticity. In Section 3, we recall the classification of rotational r-minimal hypersurfaces (catenoids) and determine their asymptotic expansion; this expansion motivates the definition of regular ends, which is presented there. Also, in this section, we compute the flux of such an end in terms of the coefficient of the leading term of its asymptotic expansion, a basic ingredient in the proof of Theorem 1.3. In fact, this computation is the bulk of the paper as it involves rather delicate estimates and crucially hinges on the assumption $\frac{3}{2}(r+1) \leq n < 2(r+1)$. In order to put this result in its proper perspective, we should mention here that, for $r \geq 1$, the flux of a regular r-minimal end is a second order integral invariant of the homology class of this end. This should be compared with the minimal case, where the flux is first order. Thus, when trying to relate this flux to the leading coefficient in the expansion, we should consider the expansion up to *second* order derivatives, and this is in a sense responsible for the restriction on n and r above. Finally, in Section 4 we combine the previously obtained expression for the flux with the Tangency Principle developed by the above mentioned authors to conclude the proof of Theorem 1.3.

2. Some preliminary facts

As in the Introduction, we will consider a hypersurface $M \subset \mathbb{R}^{n+1}$ with shape operator A. For each $0 \leq k \leq n$, we denote by $S_k = S_k[A]$ its k-*curvature*, so that if A^i_j are the entries of A with respect to some tangential basis then

$$S_k = \frac{1}{k!} \sum_{i_\alpha, j_\alpha = 1}^{n} \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} A^{i_1}_{j_1} \cdots A^{i_k}_{j_k}, \tag{2.1}$$

where $\delta^{i_1 \cdots i_k}_{j_1 \cdots j_k}$ is the generalized Kronecker delta.

If M is a graph, that is, $M = \{(x, u(x)) : x \in \Omega \subset \mathbb{R}^n\}$, with u smooth, then we can use the upward unit normal vector field

$$N(p) = \frac{(-du(x), 1)}{W(x)}, \quad p = \big(x, u(x)\big),$$

to give an orientation to M. Here, $W^2 = 1 + |du|^2$ and $d = D|_{\mathbb{R}^n}$, so that du is the gradient of u. Also, we can naturally push upward the canonical basis

e_i of $\mathbb{R}^n$ in order to have a basis $\mathcal{B}=\{(e_i,u_i(x)),i=1,\dots,n\}$ of T_pM. Here, subscripts will indicate partial differentiation. With respect to this basis, the shape operator is

$$A^i_j=\frac{u_{ij}}{W}-\frac{1}{W^3}\sum_k u_iu_ku_{kj}. \tag{2.2}$$

From (2.1) and (2.2) it is obvious that, if $r\geq 1$, the r-minimality condition locally defines a *fully nonlinear* second order PDE. This should be contrasted to the minimal case, where the corresponding equation is quasi-linear.

Due to this fully nonlinear character it is expected that the linearized, or Jacobi, operator $\mathcal{J}_r$ has its symbol depending on second order data (curvature). Indeed, the principal part of this operator is a divergence type operator with symbol determined by the so-called *Newton tensors* $P_r=P_r[A]$, which are recursively defined by

$$P_0=I,\qquad P_r=S_rI-AP_{r-1}.$$

PROPOSITION 2.1 ([R1]). *If $A=[A^i_j]$ with respect to some tangent basis then, with respect to this basis,*

$$P_r[A]^i_j=\frac{1}{r!}\sum_{i_k,j_k=1}^{n}\delta^{i_1\cdots i_r i}_{j_1\cdots j_r j}A^{i_1}_{j_1}\cdots A^{i_r}_{j_r}. \tag{2.3}$$

DEFINITION 2.1 ([R2]). If $M\subset\mathbb{R}^{n+1}$ is r-minimal we define a second order operator acting on functions by

$$L_ru=\operatorname{div}\big(P_r[A]\nabla u\big),$$

where div and ∇ are the intrinsic divergence and gradient operators on M.

The following proposition gives a basic property of r-minimal hypersurfaces.

PROPOSITION 2.2 ([R2]). *If $M\subset\mathbb{R}^{n+1}$ is r-minimal and $v\in\mathbb{R}^{n+1}$ then $L_rh_v=0$. Here, $h_v(x)=\langle x,v\rangle$, $x\in M$, is the height function associated to v.*

Since L_0 is the Laplace–Beltrami operator, this proposition generalizes a well-known property of minimal hypersurfaces, namely, height functions are harmonic. It turns out that, as in the minimal case, L_r is the principal part of $\mathcal{J}_r=L_r-(r+2)S_{r+2}$, the Jacobi operator; see [dLdLS] or [Ro] for a proof of this. Thus, $\mathcal{J}_r$ is elliptic precisely where P_r is positive or negative definite. Since when trying to establish a Tangency Principle for r-minimal hypersurfaces, the ellipticity of $\mathcal{J}_r$, and hence of L_r, is a crucial issue, the following result plays a central role in the theory.

PROPOSITION 2.3 ([HL1], [HL2]). *If M is r-minimal then L_r is elliptic at $p\in M$ if and only if $S_{r+2}(p)\neq 0$. Equivalently,* $\operatorname{rank}(A_p)\geq r+1$.

Notice that this justifies Definition 1.2.

In this context, Hounie and Leite [HL1] [HL2] developed a Tangency Principle for r-minimal hypersurfaces, extending a fundamental result established in [S] for minimal hypersurfaces. In order to formulate their results, we need some more notation.

Let $B^{n-1} \subset \mathbb{R}^{n+1}$ be a compact, embedded, boundaryless C^2 submanifold (not necessarily connected) and M an embedded submanifold with $\partial M = B$. Let us consider $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with coordinates $X = (x, x_{n+1})$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose connected boundary satisfies $S_k(\partial\Omega) \geq 0$, $0 \leq k \leq r$, with respect to the inner unit normal. Also, if $\Sigma \subset \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$ are given, we set $\Sigma_{t\pm} = \{X \in \Sigma; \pm x_{n+1} \geq \pm t\}$ and $\Sigma_t^* = \{(x, 2t - x_{n+1}); (x, x_{n+1} \in \Sigma_t)\}$, the *reflection* of Σ_t with respect to $\Pi_t = \{(x,t); x \in \mathbb{R}^n\}$; note that $\Pi_0 = \mathbb{R}^n$. Moreover, if $A, B \subset \mathbb{R}^{n+1}$ we write $A \geq B$ if for any $x \in \mathbb{R}^n$ there holds $x_{n+1} \geq x'_{n+1}$ for any $(x, x_{n+1}) \in A$, $(x, x'_{n+1}) \in B$. Finally, we say that a C^2 submanifold K has *locally bounded slope* (over Π_0) if its tangent planes do no contain the vertical vector $(0,1)$.

THEOREM 2.1. *Let B and Ω as above and assume that*: (i) $B \subset \partial\Omega \times \mathbb{R}$; (ii) B_{0^+} *is a graph with locally bounded slope with* $B^*_{0^+} \geq B_{0^-}$ *and* (iii) M *is r-minimal*, *elliptic and with all of its interior points contained in* $\Omega \times \mathbb{R}$. *Then M_{0^+} is a graph with locally bounded slope satisfying* $M^*_{0^+} \geq M_{0^-}$.

We also consider the flux of certain cycles inside r-minimal hypersurfaces.

DEFINITION 2.2. Given an oriented, smooth $(n-1)$-cycle Σ in an r-minimal immersion M, the *flux* of Σ in the direction of a unit vector $v \in \mathbb{R}^{n+1}$ is defined by

$$\mathrm{Flux}(\Sigma; v) = \int_\Sigma \big\langle P_r[A]\nabla h_v, \xi \big\rangle \, d\Sigma, \tag{2.4}$$

where ξ is the exterior co-normal to Σ and $d\Sigma$ is the volume element of Σ.

REMARK 2.1. In applications, ξ is usually given as part of the description of Σ; see Remark 3.1.

We remark that the flux depends only on the (oriented) homology class of Σ. Indeed, if Σ' is homologous to Σ, then $\Sigma - \Sigma' = \partial\Omega$, where $\Omega \subset M$ is an n-cycle. Since, by Proposition 2.2,

$$\int_\Omega \mathrm{div}\big(P_r[A]\nabla h_v\big) \, dM = \int_\Omega L_r h_v \, dM = 0,$$

integration by parts gives

$$\int_\Sigma \big\langle P_r[A]\nabla h_v, \xi \big\rangle \, d\Sigma = \int_{\Sigma'} \big\langle P_r[A]\nabla h_v, \xi \big\rangle \, d\Sigma',$$

as desired.

Notice that in the minimal case (2.4) reduces to

$$\mathrm{Flux}(\Sigma; v) = \int_\Sigma \langle \nabla h_v, \xi \rangle \, d\Sigma,$$

whose integrand only depends on the metric and not on the shape operator.

3. Regular r-minimal ends

In this section, we compute the flux of a certain class of r-minimal ends, the regular ones, whose behavior at infinity is modeled on the rotational examples (catenoids), so we start by describing this class of r-minimal hypersurfaces; see [HL3].

3.1. A description of catenoids. Consider a smooth curve α in the plane $x_1 x_{n+1}$ which is the graph of a positive function $x_1 = f(x_{n+1})$. If we rotate α around the axis x_{n+1} we obtain a rotationally invariant hypersurface M, parameterized by

$$X(t, \theta) = \big(t, f(t)\theta\big), \quad t = x_{n+1},$$

where $\theta = (\theta_1, \dots, \theta_n)$ is a local parametrization of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and $\mathbb{R}^n$ is viewed as the hyperplane of $\mathbb{R}^{n+1}$ passing through the origin and perpendicular to the axis x_{n+1}. If we take

$$N(t, \theta) = \frac{1}{w}\big(f'(t), -\theta\big), \quad w = \sqrt{1 + f'^2},$$

as the unit normal vector field to M then a computation shows that M is r-minimal if and only if f satisfies the ODE

$$ff'' = \left(\frac{n}{r+1} - 1\right)\big(1 + {f'}^2\big). \tag{3.1}$$

From now on, we simply call *catenoids* those r-minimal hypersurfaces which are rotationally invariant. Equivalently, the catenoids can be described as above in terms of a profile function f satisfying (3.1). More precise information on the global behavior of maximal solutions of (3.1) have been obtained by Hounie and Leite.

PROPOSITION 3.1 ([HL3]). *In the conditions above, if f is a maximal solution of (3.1) determined by $f(0) = \rho_0 > 0$ and $f'(0) = 0$ then f is even, positive, convex and its growth rate at infinity depends on the ratio $\frac{n}{2(r+1)}$ in the following manner:*

1. *If $n \leq 2(r+1)$, then f is defined on $(-\infty, +\infty)$ and has superliner growth with $f(t) = \mathcal{O}(|t|^{\frac{r+1}{2(r+1)-n}})$, as $|t| \to +\infty$.*
2. *If $n > 2(r+1)$, then f blows up in a finite interval $(-L, L)$, with $L \nearrow +\infty$ as $\frac{n}{r+1} \searrow 2$.*

Thus, the ends of the catenoid are asymptotic to parallel planes precisely when $n > 2(r+1)$. Notice also that in the threshold case $n = 2(r+1)$ the profile function approaches a catenary at infinity. We will not consider these cases here. In fact, we start by assuming $n < 2(r+1)$ and work toward a description of the asymptotic behavior of the ends of the catenoid, viewed as a graph over the hyperplane orthogonal to the t-axis, but first we need to determine a nonparametric representation for the ends; see [L] for the computation in case $r = 1$.

We multiply (3.1) by $2f'$ and integrate to obtain

$$\frac{f^q}{1+f'^2} = K$$

for some $K \in \mathbb{R}$, where

$$q = q_{n,r} = 2\left(\frac{n}{r+1} - 1\right). \tag{3.2}$$

If $f(0) = \rho_0 > 0$ and $f'(0) = 0$ as before then $K = \rho_0^q$ and

$$f' = \frac{\sqrt{f^q - \rho_0^q}}{\sqrt{\rho_0^q}}.$$

By Proposition 3.1, f is invertible for $t \geq 0$ and there its inverse $t = u(x_1) = u(x)$ is

$$\frac{u(x)}{\sqrt{\rho_0^q}} = \int_{\rho_0}^{|x|} \frac{dt}{\sqrt{t^q - \rho_0^q}}, \quad |x| \geq \rho_0. \tag{3.3}$$

This gives a graph parametrization for (the upper piece) of the catenoid, namely, $x \mapsto (x, u(x))$.

3.2. Regular r-minimal ends. Starting from (3.3), it is possible to obtain the asymptotic expansion for $u = u(x)$ as $|x| \to +\infty$, at least in the case $1 \leq q < 2$ or, equivalently, $\frac{3}{2}(r+1) \leq n < 2(r+1)$; see [HL2] for a similar computation in case $r = 1$.

PROPOSITION 3.2. *Let $u = u(x)$ be defined by (3.3) with $1 \leq q < 2$ and $x_0 \in \mathbb{R}^n$. Then, as $|x| \to \infty$,*

$$\frac{u(x-x_0)}{\sqrt{\rho_0^q}} = A + \frac{2}{2-q}|x|^{1-\frac{q}{2}} - \frac{\langle x_0, x\rangle}{|x|^{1+\frac{q}{2}}} + \frac{\rho_0^q}{2-3q}|x|^{1-\frac{3q}{2}} + \mathcal{O}\big(|x|^{-1-\frac{q}{2}}\big),$$

or, equivalently,

$$\frac{u(x-x_0)}{\sqrt{\rho_0^q}} = A + \frac{2}{2-q}|x|^{2-\frac{n}{r+1}} - \frac{\langle x_0, x\rangle}{|x|^{\frac{n}{r+1}}} + \frac{\rho_0^q}{2-3q}|x|^{4-\frac{3n}{r+1}} + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big).$$

For the proof, we expand the integrand in (3.3) to obtain

$$\begin{aligned}\left(t^q-\rho_0^q\right)^{-\frac{1}{2}} &= t^{-\frac{q}{2}}\left(1-\left(\frac{\rho_0}{t}\right)^q\right)^{-\frac{1}{2}}\\ &= t^{-\frac{q}{2}}\left(1+\frac{1}{2}\left(\frac{\rho_0}{t}\right)^q+\mathcal{O}\left(t^{-2q}\right)\right)\\ &= t^{-\frac{q}{2}}+\frac{1}{2}\rho_0^q t^{-\frac{3q}{2}}+\mathcal{O}\left(t^{-\frac{5q}{2}}\right),\end{aligned}$$

where in the second step we used

$$\frac{1}{\sqrt{1-s}}=1+\frac{1}{2}s+\mathcal{O}\left(s^2\right),$$

with $s=(\frac{\rho_0}{t})^q\to 0$ when $t\to\infty$. In this way, since $q<2$,

$$\begin{aligned}\frac{u(x)}{\sqrt{\rho_0^q}} &= \int_{\rho_0}^{|x|} t^{-\frac{q}{2}}\,dt+\frac{1}{2}\rho_0^q\int_{\rho_0}^{|x|} t^{-\frac{3q}{2}}\,dt+\int_{\rho_0}^{|x|}\mathcal{O}\left(t^{-\frac{5q}{2}}\right)dt\\ &= A+\frac{2}{2-q}|x|^{1-\frac{q}{2}}+\frac{1}{2}\rho_0^q\frac{2}{2-3q}|x|^{1-\frac{3q}{2}}+\mathcal{O}\left(|x|^{1-\frac{5q}{2}}\right),\end{aligned}$$

so that for $x_0\in\mathbb{R}^n$,

$$\frac{u(x-x_0)}{\sqrt{\rho_0^q}} = A+\frac{2}{2-q}|x-x_0|^{1-\frac{q}{2}} \tag{3.4}$$

$$+\frac{1}{2}\rho_0^q\frac{2}{2-3q}|x-x_0|^{1-\frac{3q}{2}}+\mathcal{O}\left(|x|^{1-\frac{5q}{2}}\right). \tag{3.5}$$

We will need the following elementary lemma.

Lemma 3.1. *If $x_0\in\mathbb{R}^n$ and $k\in\mathbb{R}$, then*

$$|x-x_0|^k=|x|^k\left(1-k\frac{\langle x_0,x\rangle}{|x|^2}+\mathcal{O}\left(|x|^{-2}\right)\right).$$

Continuing with the proof, we use the previous lemma with $k=1-\frac{q}{2}$ and $k=1-\frac{3q}{2}$ in (3.5) to obtain

$$\begin{aligned}&\frac{u(x-x_0)}{\sqrt{\rho_0^q}}\\ &\quad= A+\frac{2}{2-q}|x|^{1-\frac{q}{2}}\left(1-\left(1-\frac{q}{2}\right)\frac{\langle x_0,x\rangle}{|x|^2}+\mathcal{O}\left(|x|^{-2}\right)\right)\\ &\qquad+\frac{1}{2}\rho_0^q\frac{2}{2-3q}|x|^{1-\frac{3q}{2}}\left(1-\left(1-\frac{3q}{2}\right)\frac{\langle x_0,x\rangle}{|x|^2}+\mathcal{O}\left(|x|^{-2}\right)\right)\\ &\qquad+\mathcal{O}\left(|x|^{1-\frac{5q}{2}}\right)\\ &\quad= A+\frac{2}{2-q}|x|^{1-\frac{q}{2}}-\frac{\langle x_0,x\rangle}{|x|^{1+\frac{q}{2}}}+\mathcal{O}\left(|x|^{-1-\frac{q}{2}}\right)\end{aligned} \tag{3.6}$$

$$+\frac{\rho_0^q}{2-3q}|x|^{1-\frac{3q}{2}}-\frac{1}{2}\rho_0^q\frac{\langle x_0,x\rangle}{|x|^{1+\frac{3q}{2}}}+\mathcal{O}\big(|x|^{-1-\frac{3q}{2}}\big)$$
$$+\mathcal{O}\big(|x|^{1-\frac{5q}{2}}\big).$$

Observe now that

$$\frac{\langle x_0,x\rangle}{|x|^{1+\frac{3q}{2}}}\in\mathcal{O}\big(|x|^{-\frac{3q}{2}}\big)\subset\mathcal{O}\big(|x|^{-1-\frac{q}{2}}\big)$$

precisely because $-\frac{3q}{2}\le -1-\frac{q}{2}$, which amounts to the assumption $q\ge 1$. By the same reason,

$$\mathcal{O}\big(|x|^{-1-\frac{3q}{2}}\big)\subset\mathcal{O}\big(|x|^{-\frac{3q}{2}}\big)\subset\mathcal{O}\big(|x|^{-1-\frac{q}{2}}\big),\qquad \mathcal{O}\big(|x|^{1-\frac{5q}{2}}\big)\subset\mathcal{O}\big(|x|^{-1-\frac{q}{2}}\big),$$

and leading this information to (3.6) the proof of the proposition follows.

Proposition 3.2 clearly motivates the following definition.

DEFINITION 3.1. An r-minimal end with $\frac{3}{2}(r+1)\le n<2(r+1)$, is *regular* with growth rate $a\neq 0$ if it can be written as the graph of a function $u(x)$ defined on the exterior of a ball in a hyperplane $\Pi\subset\mathbb{R}^{n+1}$ such that, as $|x|\to+\infty$, there holds

$$u(x)=\frac{a}{|x|^{\frac{n}{r+1}-2}}+a_1+\sum_{j=1}^{n}\frac{c_jx_j}{|x|^{\frac{n}{r+1}}}+\frac{a_2}{|x|^{\frac{3n}{r+1}-4}}+\mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big),\tag{3.7}$$

where a,a_1,a_2,c_j are real constants.

3.3. The Newton tensor of a regular end. The next step in computing the flux of a regular end is to determine its Newton tensor.

PROPOSITION 3.3. *The Newton tensor of a regular r-minimal end as in Definition 3.1 is given by*

$$P_r[A]^i_j=\frac{c_ra^r}{|x|^{n-\frac{q}{2}-1}}\left([\omega_0]^i_j-\frac{n}{r+1}\frac{1}{|x|^2}[\omega_1]^i_j\right)+\mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big),$$

where

$$c_r=\frac{1}{r!}\binom{n}{r}\left(2-\frac{n}{r+1}\right)^r,\qquad [\omega_0]^i_j=\binom{n}{r}\delta^i_j,$$

and

$$[\omega_1]^i_j=\binom{n}{r-1}r\big(\delta^i_j\big(|x|^2-x_i^2\big)-\big(1-\delta^i_j\big)x_ix_j\big).$$

The proof is a somewhat involved computation. We write

$$u(x)=a|x|^{p_{n,r}}+\varphi(x),\quad p_{n,r}=2-\frac{n}{r+1},$$

so that

$$u_i(x)=\frac{p_{n,r}a}{|x|^{\frac{n}{r+1}}}x_i+\varphi_i(x),$$

$$u_{ij}(x) = \frac{p_{n,r}a}{|x|^{\frac{n}{r+1}}}\left(\delta_{ij} - \frac{n}{r+1}\frac{x_i x_j}{|x|^2}\right) + \varphi_{ij},$$

with $\varphi_i \in \mathcal{O}(|x|^{-\frac{n}{r+1}})$ and $\varphi_{ij} \in \mathcal{O}(|x|^{-\frac{n}{r+1}-1})$. We already know from (2.2) that

$$A^i_j = u_{ij} + \mathcal{O}\big(|x|^{-\frac{3n}{r+1}+2}\big),$$

since $W^{-1} = 1 - \frac{1}{2}|du|^2 + \mathcal{O}(|du|^4) = 1 + \mathcal{O}(|x|^{-\frac{2n}{r+1}+2})$. With these preliminaries, we can use (2.3) to compute the Newton tensor. Initially,

$$\begin{aligned}\prod_{\alpha=1}^r A^{i_\alpha}_{j_\alpha} = \prod_{\alpha=1}^r u_{i_\alpha j_\alpha} \\ &+ \sum_{k=1}^r \big(u_{i_1 j_1}\cdots u_{i_{k-1}j_{k-1}}\mathcal{O}\big(|x|^{-\frac{3n}{r+1}+2}\big)u_{i_{k+1}j_{k+1}}\cdots u_{i_r j_r}\big) + \cdots \\ &+ \big(\mathcal{O}\big(|x|^{-\frac{3n}{r+1}+2}\big)\big)^{r-1}\sum_{k=1}^r u_{i_k j_k} + \big(\mathcal{O}\big(|x|^{-\frac{3n}{r+1}+2}\big)\big)^r,\end{aligned}$$

and since $u_{i_k j_k} \in \mathcal{O}(|x|^{-\frac{n}{r+1}})$ and

$$-\frac{3n}{r+1} + 2 = -\frac{n}{r+1} - q,$$

this gives

$$\begin{aligned}\prod_{\alpha=1}^r A^{i_\alpha}_{j_\alpha} = \prod_{\alpha=1}^r u_{i_\alpha j_\alpha} + \big(\mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)\big)^r\mathcal{O}\big(|x|^{-q}\big) + \cdots \\ + \big(\mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)\big)^r\big(\mathcal{O}\big(|x|^{-q}\big)\big)^{r-1} + \big(\mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)\big)^r\big(\mathcal{O}\big(|x|^{-q}\big)\big)^r.\end{aligned}$$

Noticing that

$$\big(\mathcal{O}\big(|x|^{-q}\big)\big)^r \subset \big(\mathcal{O}\big(|x|^{-q}\big)\big)^{r-1} \subset \cdots \subset \mathcal{O}\big(|x|^{-q}\big),$$

we have

$$\begin{aligned}A^{i_1}_{j_1}\cdots A^{i_r}_{j_r} &= u_{i_1 j_1}\cdots u_{i_r j_r} + \big(\mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)\big)^r\mathcal{O}\big(|x|^{-q}\big) \\ &= u_{i_1 j_1}\cdots u_{i_r j_r} + \mathcal{O}\big(|x|^{-n-\frac{q}{2}+1}\big),\end{aligned}$$

since

$$-\frac{nr}{r+1} - q = -n - \frac{q}{2} + 1,$$

and replacing this in (3.9) we get

$$P_r[A]^i_j = \frac{1}{r!}\sum_{i_\alpha, j_\alpha=1}^n \delta^{i_1\cdots i_r i}_{j_1\cdots j_r j}u_{i_1 j_1}\cdots u_{i_r j_r} + \mathcal{O}\big(|x|^{-n-\frac{q}{2}+1}\big). \tag{3.8}$$

We now should compute the product of the second derivatives $u_{i_\alpha j_\alpha}, \alpha = 1, \ldots, n$. To make things easier we write, for each $\alpha = 1, \ldots, n$, $u_{i_\alpha j_\alpha} = \psi_{i_\alpha j_\alpha} + \varphi_{i_\alpha j_\alpha}$, where

$$\psi_{i_\alpha j_\alpha} = \frac{p_{n,r} a}{|x|^{\frac{n}{r+1}}} \left(\delta^{i_\alpha}_{j_\alpha} - \frac{n}{r+1} \frac{x_{i_\alpha} x_{j_\alpha}}{|x|^2} \right),$$

and $\varphi_{i_\alpha j_\alpha}$ is as before. Using the expansion

$$\begin{aligned} \prod_{\alpha=1}^{r} u_{i_\alpha j_\alpha} = \prod_{\alpha=1}^{r} \psi_{i_\alpha j_\alpha} & \\ &+ \sum_{\alpha=1}^{r} \psi_{i_1 j_1} \cdots \psi_{i_{\alpha-1} j_{\alpha-1}} \varphi_{i_\alpha j_\alpha} \psi_{i_{\alpha+1} j_{\alpha+1}} \cdots \psi_{i_r j_r} + \cdots \\ &+ \prod_{\alpha=1}^{r} \varphi_{i_\alpha j_\alpha}, \end{aligned}$$

and moreover that $\psi_{i_\alpha j_\alpha} \in \mathcal{O}(|x|^{-\frac{n}{r+1}})$ and $\varphi_{i_\alpha j_\alpha} \in \mathcal{O}(|x|^{-\frac{n}{r+1}-1}) = \mathcal{O}(|x|^{-1}) \cdot \mathcal{O}(|x|^{-\frac{n}{r+1}})$, we get

$$\begin{aligned} \prod_{\alpha=1}^{r} u_{i_\alpha j_\alpha} &= \prod_{\alpha=1}^{r} \psi_{i_\alpha j_\alpha} + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)^r \mathcal{O}\big(|x|^{-1}\big) + \cdots \\ &\quad + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)^r \mathcal{O}\big(|x|^{-1}\big)^r \\ &= \prod_{\alpha=1}^{r} \psi_{i_\alpha j_\alpha} + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)^r \big[\mathcal{O}\big(|x|^{-1}\big) + \cdots + \mathcal{O}\big(|x|^{-1}\big)^r\big] \\ &= \prod_{\alpha=1}^{r} \psi_{i_\alpha j_\alpha} + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)^r \mathcal{O}\big(|x|^{-1}\big), \end{aligned}$$

where in the last step we used

$$\big(\mathcal{O}\big(|x|^{-1}\big)\big)^r \subset \big(\mathcal{O}\big(|x|^{-1}\big)\big)^{r-1} \subset \cdots \subset \mathcal{O}\big(|x|^{-1}\big).$$

Moreover, since

$$-\frac{nr}{r+1} - 1 = -n + \frac{q}{2},$$

it follows that

$$\begin{aligned} u_{i_1 j_1} \cdots u_{i_r j_r} &= \prod_{\alpha=1}^{r} \frac{p_{n,r} a}{|x|^{\frac{n}{r+1}}} \left(\delta^{i_\alpha}_{j_\alpha} - \frac{n}{r+1} \frac{x_{i_\alpha} x_{j_\alpha}}{|x|^2} \right) + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) \\ &= \frac{p^r_{n,r} a^r}{|x|^{n-\frac{q}{2}-1}} \prod_{\alpha=1}^{r} \left(\delta^{i_\alpha}_{j_\alpha} - \frac{n}{r+1} \frac{x_{i_\alpha} x_{j_\alpha}}{|x|^2} \right) + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big), \end{aligned}$$

and replacing this in (3.8) we obtain

$$
\begin{aligned}
(3.9)\qquad P_r[A]^i_j &= \frac{1}{r!}\frac{p^r_{n,r}a^r}{|x|^{n-\frac{q}{2}-1}}\sum_{i_\alpha,j_\alpha=1}^{n}\delta^{i_1\cdots i_r i}_{j_1\cdots j_r j}\prod_{k=1}^{r}\left(\delta^{i_k}_{j_k}-\frac{n}{r+1}\frac{x_{i_k}x_{j_k}}{|x|^2}\right)\\
&\quad+\mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big)+\mathcal{O}\big(|x|^{-n-\frac{q}{2}+1}\big)\\
&= \frac{1}{r!}\frac{p^r_{n,r}a^r}{|x|^{n-\frac{q}{2}-1}}\sum_{i_\alpha,j_\alpha=1}^{n}\delta^{i_1\cdots i_r i}_{j_1\cdots j_r j}\prod_{k=1}^{r}\left(\delta^{i_k}_{j_k}-\frac{n}{r+1}\frac{x_{i_k}x_{j_k}}{|x|^2}\right)\\
&\quad+\mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big),
\end{aligned}
$$

where here we have used again that $q\geq 1$. The proof of Proposition 3.3 is then completed by using the algebraic lemma below, with $C=n/(r+1)$, to (3.9).

LEMMA 3.2. *Given $C\in\mathbb{R}$ there holds*

$$\sum_{i_\alpha,j_\alpha=1}^{n}\delta^{i_1\cdots i_r i}_{j_1\cdots j_r j}\prod_{k=1}^{r}\left(\delta^{i_k}_{j_k}-C\frac{x_{i_k}x_{j_k}}{|x|^2}\right)=[\omega_0]^i_j-\frac{C}{|x|^2}[\omega_1]^i_j,$$

where

$$[\omega_0]^i_j=\binom{n}{r}\delta^i_j,\qquad [\omega_1]^i_j=\binom{n}{r-1}r\big[\delta^i_j\big(|x|^2-x_i^2\big)-\big(1-\delta^i_j\big)x_ix_j\big].$$

The proof of this lemma is presented in the Appendix.

3.4. The flux of regular ends. Here we finally compute the flux of a regular r-minimal end.

PROPOSITION 3.4. *Let $\Sigma_R=\{(x,u(x)),|x|=R\}$ be the oriented cycle in a regular r-minimal end with growth rate $a\neq 0$ with respect to a hyperplane Π; see Definition 3.1. If $\frac{3}{2}(r+1)\leq n<2(r+1)$, then*

$$(3.10)\qquad \mathrm{Flux}(\Sigma_R;v)=\langle v,\eta\rangle\gamma_r\,\mathrm{Vol}\big(\mathbb{S}^{n-1}\big)a^{r+1},$$

where $\gamma_r=c_r(2-\frac{n}{r+1})\binom{n}{r}$, c_r is as in Proposition 3.3 and η is the positive unit normal to Π.

REMARK 3.1. Since the sign of the growth rate a clearly depends on the choice of orientations we must explain the meaning of this proposition. First, in applications as in Proposition 4.1 below, Σ_R is given an orientation as part of the boundary of an oriented r-minimal domain. This orientation on Σ_R naturally induces an orientation on the sphere $S^{n-1}_R=\{x\in\Pi;|x|=R\}$, so we can give to Π the orientation such that S^{n-1}_R is the oriented boundary of $B^n_R=\{x\in\Pi;|x|=R\}$. Thus, the given orientation on Σ_R uniquely defines an orientation on Π. In this setting, the positive direction in the axis orthogonal to Π, which determines the graph representation of the end, is such that, when added up to the orientation on Π, in this order, gives the (fixed) orientation on

$\mathbb{R}^{n+1}$. It is under this convention that we fix the sign of a and Proposition 3.4 should be interpreted accordingly.

For the proof, we may assume that the end is a graph over the horizontal plane $x_{n+1}=0$, which we identity to $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. Thus, if ξ is the exterior unit co-normal to Σ_R, we have $\xi = \vartheta/|\vartheta|$, where ϑ satisfies

$$\left(\frac{x}{R},0\right) = \vartheta + \left\langle \left(\frac{x}{R},0\right), \frac{1}{W}(-du,1)\right\rangle \frac{1}{W}(-du,1),$$

so that

$$\begin{aligned}
\vartheta &= \left(\frac{x}{R},0\right) + du(x)\left(\frac{x}{R}\right)\left(-\frac{du}{W^2},\frac{1}{W^2}\right)\\
&= \left(\frac{x}{R}, du(x)\left(\frac{x}{R}\right)\right) + du(x)\left(\frac{x}{R}\right)\left((0,-1) + \left(-\frac{du}{W^2},\frac{1}{W^2}\right)\right)\\
&= \left(\frac{x}{R}, du(x)\left(\frac{x}{R}\right)\right) - \frac{1}{W^2}du(x)\left(\frac{x}{R}\right)\big(du,|du|^2\big).
\end{aligned}$$

But

$$\begin{aligned}
\left|\frac{1}{W^2}du(x)\left(\frac{x}{R}\right)\big(du,|du|^2\big)\right| &\le \frac{|du|}{W^2}\sqrt{|du|^2+|du|^4}\\
&= \frac{|du|^2}{W}\\
&= |du|^2\left(1-\frac{1}{2}|du|^2 + \mathcal{O}\big(|du|^4\big)\right)\\
&= \mathcal{O}\big(|x|^{-q}\big),
\end{aligned}$$

so that

$$\vartheta\big(x,u(x)\big) = \left(\frac{x}{R}, du(x)\left(\frac{x}{R}\right)\right) + \mathcal{O}\big(|x|^{-q}\big).$$

If $p=(x,u(x)) \in M$ then T_pM is endowed with the standard basis $\mathcal{B} = \{(e_i, du(x)e_i); i=1,\dots,n\}$. In this basis,

$$\vartheta = \frac{1}{|x|}\begin{pmatrix} x_1\\ \vdots \\ x_n\end{pmatrix} + \mathcal{O}\big(|x|^{-q}\big).$$

Now let $v\in\mathbb{R}^{n+1}$ be a unit vector. In order to compute $\mathrm{Flux}(\Sigma_R; v)$ we must, by (2.4), compute $\langle P_r[A](\nabla h_v), \xi\rangle$. Using that $P_r[A]$ is symmetric and ∇h_v is the tangential component of v, we clearly have

$$\big\langle P_r[A](\nabla h_v), \xi\big\rangle = \frac{1}{|\vartheta|}\big\langle v, P_r[A](\vartheta)\big\rangle.$$

In the basis $\mathcal{B}$, $P_r[A]$ has coefficients given by Proposition 3.3, so that if $1 \le i \le n$, $P_r[A](\vartheta)^i = \sum_j P_r[A]^i_j \vartheta^j$ is given by

$$
\begin{aligned}
P_r[A](\vartheta)^i &= \sum_{j=1}^{n} \left(\frac{c_r a^r}{|x|^{n-\frac{q}{2}-1}} \left([\omega_0]^i_j - \frac{n}{r+1} \frac{1}{|x|^2} [\omega_1]^i_j \right) + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) \right) \vartheta^j \\
&= \frac{c_r a^r}{|x|^{n-\frac{q}{2}-1}} \Bigg(\underbrace{\sum_{j=1}^{n} [\omega_0]^i_j n^j}_{A} - \frac{n}{r+1} \frac{1}{|x|^2} \underbrace{\sum_{j=1}^{n} [\omega_1]^i_j n^j}_{B} \Bigg) \\
&\qquad + \underbrace{\sum_{j=1}^{n} \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) n^j}_{C}.
\end{aligned}
$$

Let us compute A, B and C. We have

$$
\begin{aligned}
A &= \sum_{j=1}^{n} [\omega_0]^i_j \vartheta^j \\
&= \sum_{j=1}^{n} \binom{n}{r} \delta^i_j \left(\frac{x_j}{|x|} + \mathcal{O}\big(|x|^{-q}\big) \right) \\
&= \binom{n}{r} \frac{x_i}{|x|} + \mathcal{O}\big(|x|^{-q}\big), \\
B &= \sum_{j=1}^{n} [\omega_1]^i_j \vartheta^j \\
&= \sum_{j=1}^{n} \binom{n}{r-1} r \big[\delta^i_j \big(|x|^2 - x_i^2\big) - \big(1 - \delta^i_j\big) x_i x_j \big] \left(\frac{x_j}{|x|} + \mathcal{O}\big(|x|^{-q}\big) \right) \\
&= \binom{n}{r-1} r \left(\sum_{j=1}^{n} \delta^i_j \big(|x|^2 - x_i^2\big) \frac{x_j}{|x|} - \sum_{j=1}^{n} \big(1 - \delta^i_j\big) x_i \frac{x_j^2}{|x|} \right) + \mathcal{O}\big(|x|^{-q+2}\big) \\
&= \binom{n}{r-1} r \left(x_i |x| - \frac{x_i^3}{|x|} - x_i |x| + \frac{x_i^3}{|x|} \right) + \mathcal{O}\big(|x|^{-q+2}\big) \\
&= \mathcal{O}\big(|x|^{-q+2}\big),
\end{aligned}
$$

and finally,

$$
\begin{aligned}
C &= \sum_{j=1}^{n} \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) \vartheta^j \\
&= \sum_{j=1}^{n} \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) \left(\frac{x_j}{|x|} + \mathcal{O}\big(|x|^{-q}\big) \right) = \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big).
\end{aligned}
$$

Thus,

$$\begin{aligned} P_r[A](\vartheta)^i &= \frac{c_r a^r}{|x|^{n-\frac{q}{2}-1}}\left(\binom{n}{r}\frac{x_i}{|x|} + \mathcal{O}\big(|x|^{-q}\big) - \frac{n}{r+1}\frac{1}{|x|^2}\mathcal{O}\big(|x|^{-q+2}\big)\right) \\ &\quad + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) \\ &= c_r a^r \binom{n}{r}\frac{x_i}{|x|^{n-\frac{q}{2}}} + \mathcal{O}\big(|x|^{-n-\frac{q}{2}+1}\big) + \mathcal{O}\big(|x|^{-n-\frac{q}{2}+1}\big) \\ &\quad + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) \\ &= \binom{n}{r}\frac{c_r a^r x_i}{|x|^{n-\frac{q}{2}}} + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big), \end{aligned}$$

where we used the assumption $\frac{3}{2}(r+1) \le n$ in the last identity.

On the other hand, the last component of $P_r[A](\vartheta)$ is

$$\begin{aligned} du(x)\big(P_r[A](\vartheta)\big) &= \left(\frac{p_{n,r}a}{|x|^{\frac{n}{r+1}}}x + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)\right)\left(\binom{n}{r}\frac{c_r a^r x}{|x|^{n-\frac{q}{2}}} + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big)\right) \\ &= p_{n,r}\binom{n}{r}\frac{c_r a^{r+1}}{|x|^{n-1}} + \mathcal{O}\big(|x|^{-\frac{q}{2}}\big)\mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) \\ &\quad + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)\mathcal{O}\big(|x|^{-n+\frac{q}{2}+1}\big) + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big)\mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) \\ &= p_{n,r}\binom{n}{r}\frac{c_r a^{r+1}}{|x|^{n-1}} + \mathcal{O}\big(|x|^{-n}\big) + \mathcal{O}\big(|x|^{-n}\big) + \mathcal{O}\big(|x|^{-n-1}\big) \\ &= p_{n,r}\binom{n}{r}\frac{c_r a^{r+1}}{|x|^{n-1}} + \mathcal{O}\big(|x|^{-n}\big), \end{aligned}$$

and we finally have

$$\begin{aligned} \big\langle v, P_r[A](\vartheta)\big\rangle &= \binom{n}{r}\frac{c_r a^r}{|x|^{n-\frac{q}{2}}}\sum_{i=1}^{n} v_i x_i + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big) \\ &\quad + p_{n,r}\binom{n}{r}\frac{c_r a^{r+1}}{|x|^{n-1}} v_{n+1} + \mathcal{O}\big(|x|^{-n}\big) \\ &= \binom{n}{r}\frac{c_r a^r}{|x|^{n-\frac{q}{2}}}\sum_{i=1}^{n} v_i x_i + p_{n,r}\binom{n}{r}\frac{c_r a^{r+1}}{|x|^{n-1}} v_{n+1} \\ &\quad + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big), \end{aligned}$$

and replacing this into (2.4) we get

$$\begin{aligned} \mathrm{Flux}(\Sigma_R, v) = \int_{\Sigma_R} \frac{1}{|\vartheta|}\Bigg\{ & c_r a^r \binom{n}{r}\frac{1}{|x|^{n-\frac{q}{2}}}\sum_{i=1}^{n} v_i x_i \\ & + c_r a^{r+1} p_{n,r}\binom{n}{r}\frac{1}{|x|^{n-1}} v_{n+1} + \mathcal{O}\big(|x|^{-n+\frac{q}{2}}\big)\Bigg\}\, d\Sigma_R \end{aligned}$$

$$= c_r a^r \binom{n}{r} \int_{\Sigma_R} \frac{1}{|\vartheta|} \sum_{i=1}^{n} \frac{v_i x_i}{R^{n-\frac{q}{2}}}\, d\Sigma_R$$
$$+ c_r a^{r+1} p_{n,r} \binom{n}{r} \int_{\Sigma_R} \frac{1}{|\vartheta|} \frac{v_{n+1}}{R^{n-1}}\, d\Sigma_R$$
$$+ \int_{\Sigma_R} \frac{1}{|\vartheta|} \mathcal{O}\big(R^{-n+\frac{q}{2}}\big)\, d\Sigma_R.$$

The computation is completed observing that, since the flux depends only on the homology class of the cycle, we can take $R \to +\infty$, so that $|\vartheta| \to 1$ and $d\Sigma_R$ asymptotically approaches $d\omega_R = R^{n-1}\, d\omega$, the volume element of $S_R^{n+1} \subset \mathbb{R}^n$. Here, $d\omega$ is the volume element of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Now, the first integral asymptotes

$$R^{\frac{q}{2}-1} \int_{\mathbb{S}^{n-1}} \sum_{i=1}^{n} v_i x_i\, d\omega,$$

and since $q < 2$ it vanishes due to the symmetry of integrand. On the other hand, the third integral asymptotes

$$\int_{\mathbb{S}^{n-1}} \mathcal{O}\big(R^{\frac{q}{2}-1}\big)\, d\omega,$$

and it vanishes again because $q < 2$. Thus,

$$\begin{aligned}\mathrm{Flux}(\Sigma_R, \mathrm{v}) &= c_r a^{r+1} p_{n,r} \binom{n}{r} \lim_{R\to\infty} \int_{\Sigma_R} \frac{1}{|\vartheta|} \frac{v_{n+1}}{R^{n-1}}\, d\Sigma_R \\ &= c_r a^{r+1} p_{n,r} \binom{n}{r} \int_{\mathbb{S}^{n-1}} d\omega,\end{aligned}$$

and this concludes the proof of the Proposition.

4. The proof of Theorem 1.3

In this section, we finally prove Theorem 1.3. We will need an auxiliary proposition which says that, in the conditions of the theorem, the ends of the hypersurface are balanced.

PROPOSITION 4.1. *If $M \hookrightarrow \mathbb{R}^{n+1}$ is a complete, embedded and oriented r-minimal hypersurface with $\frac{3}{2}(r+1) \le n < 2(r+1)$ with two ends, both regular, then the ends are parallel with the same growth rate.*

Before starting the proof, we explain the meaning of the balancing conclusion. First, there exist planes Π_1 and Π_2 above which the ends are expressed as graphs. Take R large enough so that the intersection of the cylinders of radius R over the hyperplanes contains the compact piece of M, say M_R, complementary to the ends. Note that $\partial M_R = \Sigma_R^1 \cup \Sigma_R^2$, a union of two cycles, which we endow with the boundary orientation. By Remark 3.1, this completely determines the asymptotic expansions of the ends, so that in particular the growth rates, say a and b, are determined. We want to show that $a = b$.

If $v \in \mathbb{R}^{n+1}$ is a unit vector then Proposition 2.2 and integration by parts gives

$$\operatorname{Flux}\bigl(\Sigma_R^1, v\bigr) + \operatorname{Flux}\bigl(\Sigma_R^2, v\bigr) = 0$$

so that by (3.10),

$$a^{r+1}\langle v, \eta_1\rangle + b^{r+1}\langle v, \eta_2\rangle = 0, \tag{4.1}$$

where η_i is the positively oriented unit normal to $\Pi_i, i = 1, 2$.

Attached to each end we have the corresponding coordinate system relatively to which the expansions are written, say (x, x_{n+1}) and (y, y_{n+1}). We denote the corresponding graphing functions by $u = u(x)$ and $v = v(y)$, respectively. This determines two orthogonal decompositions of $\mathbb{R}^{n+1}$, namely,

$$(x, x_{n+1}) \in \Pi_1 \oplus [\eta_1], \qquad (y, y_{n+1}) \in \Pi_2 \oplus [\eta_2],$$

and we can consider an orientation preserving orthogonal map $\mathcal{T} : \Pi_1 \oplus [\eta_1] \to \Pi_2 \oplus [\eta_2]$ with $\mathcal{T}\eta_1 = \eta_2$. Using this in (4.1), we get

$$\bigl\langle a^{r+1} v + b^{r+1}\mathcal{T}^T v, \eta_1\bigr\rangle = 0,$$

that is, $\mathcal{T}^T$ maps Π_1 over itself, so that $\Pi_1 = \Pi_2$ and we should have $\eta_2 = \eta_1$ or $\eta_2 = -\eta_1$. In both cases, we already conclude that the ends are parallel.

Returning to (4.1), we get $(a^{r+1} \pm b^{r+1})\langle v, \eta_1\rangle = 0$, and taking $v = \eta_1$ we conclude that $a^{r+1} \pm b^{r+1} = 0$. In case r is odd we necessarily have $a^{r+1} - b^{r+1} = 0$ so that $|a| = |b|$. Moreover, $\eta_2 = -\eta_1$ so that $y_{n+1} = -x_{n+1}$ and $|y| = |x|$, which means that $\mathcal{T}$ restricted to Π_1 changes orientation. If we compare the two ends in the same coordinate system, say (x, x_{n+1}), then $v(y)$ becomes $-v(x)$ and b becomes $-b$. Thus, if $b = -a$ the ends are asymptotic to each other. Now, since M is embedded, we may displace a hyperplane parallel to Π_1 starting from infinity in the direction opposed to the ends. This hyperplane will eventually touch M at a point where all the principal curvatures have the same sign. But the ellipticity of M implies that at least $r + 1$ of the curvature are non-null and this contradicts r-minimality. Thus, in this case, $a = b$ as desired.

The case where r is even is more immediate. First, if $a^{r+1} - b^{r+1} = 0$ we promptly conclude that $a = b$ and $\eta_1 = -\eta_2$, as desired. If, on the other hand, $a^{r+1} + b^{r+1} = 0$ we have $a = -b$ computed in the same coordinated system, since $\eta_1 = \eta_2$ in this case. This obviously means that if we compute the expression of v in the system determined by $-\eta_2$ the growth rates coincide and the proposition is proved in any case.

We now are ready to prove the main result in this work, namely, Theorem 1.3. Given Proposition 4.1, the proof is an adaptation of the argument given in [S] (see also [HL2]) so we merely give a sketch emphasizing the points where their arguments should be modified. By Proposition 4.1, if we write both ends relative to the same coordinate system then we have, after possibly vertically translating Π_1, the asymptotic expansions are

$$u(x) = \frac{a}{|x|^{\frac{n}{r+1}-2}} + a_1 + \mathcal{O}\big(|x|^{-\frac{n}{r+1}+1}\big),$$

$$v(x) = -\frac{a}{|x|^{\frac{n}{r+1}-2}} - a_1 + \mathcal{O}\big(|x|^{-\frac{n}{r+1}+1}\big).$$

In these coordinates we consider the hyperplane $\Pi_t = \{(x,t); x \in \mathbb{R}^n$, with $t > 0$ fixed. Choose $R > 0$ so that $|x| > R$ implies $u(x) + v(x) < 2t$, which is possible due to the above expansions. In the notation in the paragraph preceding Theorem 2.1, this means that $B^*_{t^+} \geq B_{t^-}$, where $B = M \cap \partial\mathcal{C}$ and $\mathcal{C}$ is the infinite cylinder having $B^n_R \subset \Pi_1$ as basis. By Theorem 2.1, $(M \cap \mathcal{C})^*_{t^+} \geq (M \cap \mathcal{C})_{t^-}$ and setting $t \to 0$ we conclude that $M^*_{0^+} \geq M_{0^-}$. Changing x_{n+1} by $-x_{n+1}$ and repeating the argument we have $M^*_{0^-} \geq M_{0^+}$, so that in fact $M^*_{0^+} = M_{0^-}$, that is, M is symmetric with respect to the reflection leaving Π_1 invariant.

In order to proceed, we need to determine the symmetry axis of M. We already know that the expansion of M at infinity is

$$u(x) = \frac{a}{|x|^{\frac{n}{r+1}-2}} + a_1 + \frac{\langle c, x\rangle}{|x|^{\frac{n}{r+1}}} + \frac{a_2}{|x|^{\frac{3n}{r+1}-4}} + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big), \tag{4.2}$$

so that if we take $x = y - \beta$ and use Lemma 3.1 we get

$$\begin{aligned} \frac{a}{|x|^{\frac{n}{r+1}-2}} &= \frac{a}{|y-\beta|^{\frac{n}{r+1}-2}} \\ &= \frac{a}{|y|^{\frac{n}{r+1}-2}}\left(1 - p_{n,r}\frac{\langle \beta, y\rangle}{|y|^2} + \mathcal{O}\big(|y|^{-2}\big)\right) \\ &= \frac{a}{|y|^{\frac{n}{r+1}-2}} - ap_{n,r}\frac{\langle \beta, y\rangle}{|y|^{\frac{n}{r+1}}} + \mathcal{O}\big(|y|^{-\frac{n}{r+1}}\big). \end{aligned} \tag{4.3}$$

Moreover, an easy computation shows that

$$\frac{\langle c, x\rangle}{|x|^{\frac{n}{r+1}}} = \frac{\langle c, y\rangle}{|y|^{\frac{n}{r+1}}} + \mathcal{O}\big(|y|^{-\frac{n}{r+1}}\big),$$

$$\frac{a_2}{|x|^{\frac{3n}{r+1}-4}} = \frac{a_2}{|y|^{\frac{3n}{r+1}-4}} + \mathcal{O}\big(|y|^{3-\frac{3n}{r+1}}\big),$$

and $\mathcal{O}(|x|^{-\frac{n}{r+1}}) = \mathcal{O}(|y|^{-\frac{n}{r+1}})$, so that

$$\begin{aligned} u(y-\beta) &= \frac{a}{|y|^{\frac{n}{r+1}-2}} - ap_{n,r}\frac{\langle \beta, y\rangle}{|y|^{\frac{n}{r+1}}} + \frac{\langle c, y\rangle}{|y|^{\frac{n}{r+1}}} \\ &\quad + \mathcal{O}\big(|y|^{-\frac{n}{r+1}}\big) + \frac{a_2}{|y|^{\frac{3n}{r+1}-4}} + \mathcal{O}\big(|y|^{3-\frac{3n}{r+1}}\big) \\ &= \frac{a}{|y|^{\frac{n}{r+1}-2}} + \frac{\langle ap_{n,r}\beta + c, y\rangle}{|y|^{\frac{n}{r+1}}} + \frac{a_2}{|y|^{\frac{3n}{r+1}-4}} + +\mathcal{O}\big(|y|^{-\frac{n}{r+1}}\big), \end{aligned}$$

since $\frac{3}{2} \leq \frac{n}{r+1}$. Choosing $\beta = -c/ap_{n,r}$ and changing y by x we get

$$u(x) = \frac{a}{|x|^{\frac{n}{r+1}-2}} + a_1 + \frac{a_2}{|x|^{\frac{3n}{r+1}-4}} + \mathcal{O}\big(|x|^{-\frac{n}{r+1}}\big), \tag{4.4}$$

so that the new x_{n+1}-axis is a good candidate for the symmetry axis.

We now look at the symmetry of M with respect to the hyperplane Π_0 given by $x_1 = 0$. Let $B = M \cap \{|x_{n+1}| = \Lambda\} = B^+ \cup B^-$, where $B^\pm = M \cap \{x_{n+1} = \pm\Lambda\}$ and $\Lambda > 0$ is large enough. We consider the parallel hyperplanes Π_t given by $x_1 = t$, $t > 0$. From (4.4), we have

$$\frac{\partial u}{\partial x_1} = ap_{n,r}\frac{x_1}{|x|^{\frac{n}{r+1}}} + a_2\left(4 - \frac{3n}{r+1}\right)\frac{x_1}{|x|^{\frac{3n}{r+1}-2}} + \mathcal{O}\big(|x|^{-\frac{n}{r+1}-1}\big),$$

which is positive for $x_1 \geq t > 0$ and $|x|$ large enough. Thus, B_{t+} is a graph over Π_t with bounded slope. Moreover, since $B^\pm$ asymptotes a sphere as $R \to +\infty$, we see easily that $B^*_{t+} \geq B_{t-}$. Again by Theorem 2.1, $(M \cap \{|x_{n+1}| \leq \Lambda\})^*_{t+} \geq (M \cap \{|x_{n+1}| \leq \Lambda\})_{t-}$ and letting $t \to 0$ we get $(M \cap \{|x_{n+1}| \leq \Lambda\})^*_{0+} \geq (M \cap \{|x_{n+1}| \leq \Lambda\})_{0-}$. Changing x_1 by $-x_1$ and repeating the argument, $(M \cap \{|x_{n+1}| \leq \Lambda\})^*_{0-} \geq (M \cap \{|x_{n+1}| \leq \Lambda\})_{0+}$, so that in fact $(M \cap \{|x_{n+1}| \leq \Lambda\})^*_{0+} = (M \cap \{|x_{n+1}| \leq \Lambda\})_{0-}$, that is, M is invariant under the reflection relative to Π_0. The full symmetry of M now follows from the observation that the right-hand side of (4.4) is, up to the error term, rotationally invariant. This means that the above argument can be used after replacing Π_0 by any hyperplane containing the x_{n+1}-axis. The theorem is proved.

Appendix: The proof of Lemma 3.2

The purpose of this Appendix it to prove Lemma 3.2. We set

$$\omega^i_j = \sum_{i_\alpha, j_\alpha = 1}^{n} \delta^{i_1 \cdots i_r i}_{j_1 \cdots j_r j} \prod_{k=1}^{r}\left(\delta^{i_k}_{j_k} - C\frac{x_{i_k}x_{j_k}}{|x|^2}\right),$$

so that

$$\omega^i_j = \sum_{l=0}^{r}(-1)^l \frac{C^l}{|x|^{2l}}[\omega_l]^i_j,$$

where

$$\begin{aligned}[\omega_l]^i_j = &\sum_{i_\alpha, j_\alpha = 1}^{n} \delta^{i_1 \cdots i_r i}_{j_1 \cdots j_r j} \\ &\times \sum_{1 \leq k_1 < \cdots < k_l \leq r} \delta^{h}_{j_1} i_1 \cdots x_{i_{k_1}} x_{j_{k_1}} \cdots x_{i_{k_l}} x_{j_{k_l}} \cdots \delta^{i_r}_{j_r} \\ = &\binom{n}{r-l} \sum_{i_\alpha, j_\alpha = 1}^{n} \sum_{1 \leq k_1 < \cdots < k_l \leq r} \delta^{i_{k_1} \cdots i_{k_l} i}_{j_{k_1} \cdots j_{k_l} j} x_{i_{k_1}} x_{j_{k_1}} \cdots x_{i_{k_l}} x_{j_{k_l}}.\end{aligned} \tag{A.1}$$

We have

$$
\begin{aligned}
[\omega_0]^i_j &= \sum_{i_\alpha, j_\alpha=1}^{n} \delta^{i_1\cdots i_r i}_{j_1\cdots j_r j}\delta^{i_1}_{j_1}\cdots\delta^{i_r}_{j_r} \\
&= \sum_{i_\alpha=1}^{n} \delta^{i_1\cdots i_r i}_{i_1\cdots i_r j} \\
&= \binom{n}{r}\delta^i_j
\end{aligned}
$$

and

$$
\begin{aligned}
[\omega_1]^i_j &= \sum_{i_\alpha, j_\alpha=1}^{n} \delta^{i_1\cdots i_r i}_{j_1\cdots j_r j}\sum_{k=1}^{r}\delta^{i_1}_{j_1}\cdots x_{i_k}x_{j_k}\cdots\delta^{i_r}_{j_r} \\
&= \sum_{k=1}^{r}\sum_{i_\alpha, j_\alpha=1}^{n} \delta^{i_1\cdots i_k\cdots i_r i}_{i_1\cdots j_k\cdots i_r j}x_{i_k}x_{j_k} \\
&= \binom{n}{r-1}\sum_{k=1}^{r}\sum_{i_k, j_k=1}^{n} \delta^{i_k i}_{j_k j}x_{i_k}x_{j_k}.
\end{aligned}
$$

In case $i=j$, $\delta^{i_k i}_{j_k i}\neq 0$ only if $i_k=j_k$ with $i_k\neq i$. Thus,

$$
\begin{aligned}
[\omega_1]^i_i &= \binom{n}{r-1}\sum_{k=1}^{r}\sum_{i_k=1; i_k\neq i}^{n} \delta^{i_k i}_{i_k i}x_{i_k}^2 \\
&= \binom{n}{r-1} r\big(x_1^2+\cdots+x_{i-1}^2+x_{i+1}^2+\cdots+x_n^2\big) \\
&= \binom{n}{r-1} r\big(|x|^2-x_i^2\big).
\end{aligned}
$$

On the other hand, if $i\neq j$ then $\delta^{i_k i}_{j_k j}\neq 0$ only if $i_k=j$ and $j_k=i$, and we have

$$
\begin{aligned}
[\omega_1]^i_j &= \binom{n}{r-1}\sum_{k=1}^{r}\delta^{ji}_{ij}x_ix_j \\
&= -\binom{n}{r-1} r x_i x_j,
\end{aligned}
$$

so that in general,

$$
[\omega_1]^i_j = \binom{n}{r-1} r\big[\delta^i_j\big(|x|^2-x_i^2\big)-\big(1-\delta^i_j\big)x_ix_j\big].
$$

The proof of the lemma will be completed if we show that $\omega_l = 0$ for $l \geq 2$. Again we consider two cases. If $i = j$ then $[\omega_l]_i^i$ is proportional to

$$\sum_{i_\alpha, j_\alpha = 1}^{n} \sum_{1 \leq k_1 < \cdots < k_l \leq r; k_m \neq i} \delta^{i_{k_1} \cdots i_{k_l}}_{j_{k_1} \cdots j_{k_l}} x_{i_{k_1}} x_{j_{k_1}} \cdots x_{i_{k_l}} x_{j_{k_l}},$$

and since $\delta^{i_{k_1} \cdots i_{k_l}}_{j_{k_1} \cdots j_{k_l}}$ is skew-symmetric in the indexes $j_{k_1}, \ldots, j_{k_l}$ and the product $x_{i_{k_1}} x_{j_{k_1}} \cdots x_{i_{k_l}} x_{j_{k_l}}$ is symmetric in these same indexes, this sum clearly vanishes, as desired. Now fix i and j such that $i \neq j$. Let us look initially at the case $l = 2$. Thus $\delta^{i_{k_1} i_{k_2} i}_{j_{k_1} j_{k_2} j} \neq 0$ only if $j = i_{k_1}$ or $j = i_{k_2}$. Let us assume, for example, that $j = i_{k_1}$ and set $i_{k_2} = k$ for simplicity. Then the corresponding sum is clearly a multiple of

$$x_i x_j \sum_{k \neq i,j} x_k^2 \big(\delta^{jki}_{ikj} + \delta^{jki}_{kij}\big) = x_i x_j \sum_{k \neq i,j} x_k^2 \big(\delta^{jki}_{ikj} - \delta^{jki}_{ikj}\big) = 0,$$

and this proves that $\omega_2 = 0$.

A similar cancelation takes place if $l \geq 3$ and $i \neq j$. We illustrate the argument by considering only the case $l = 3$, since in general the difficulty is mostly notational. Again, $\delta^{i_{k_1} i_{k_2} i_{k_3} i}_{j_{k_1} j_{k_2} j_{k_3} j} \neq 0$ only if j equals one of the indexes i_{k_1}, i_{k_2} or i_{k_3}. The sum (A.1) splits accordingly and we may assume, without loss of generality, that $j = i_{k_1}$. Setting $i_{k_2} = k$ and $i_{k_3} = m$ the corresponding sum is

$$x_i x_j \sum_{k,m \neq i,j} x_k^2 x_m^2 \big(\delta^{jkmi}_{ikmj} + \delta^{jkmi}_{imkj} + \delta^{jkmi}_{kimj} + \delta^{jkmi}_{kmij} + \delta^{jkmi}_{mkij} + \delta^{jkmi}_{mikj}\big) = 0.$$

The proof of Lemma 3.2 is complete.

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References

[dLdLS] de Lima, L. L., de Lira, J. H. and Silva, J., *New r-minimal hypersurfaces via perturbative methods*, J. Geom. Anal. **21** (2011), no. 4, 1132–1156. MR 2836594

[HL1] Hounie, J. and Leite, M. L., *The maximun principle for hypersurfaces with vanishing curvature functions*, J. Differential Geom. **41** (1995), 247–258. MR 1331967

[HL2] Hounie, J. and Leite, M. L., *Two-ended hypersurfaces with zero scalar curvature*, Indiana Univ. Math. J. **47** (1998), 867–882. MR 1736975

[HL3] Hounie, J. and Leite, M. L., *Uniqueness and non-existence theorems for hypersurfaces with $H_r = 0$*, Ann. Global Anal. Geom. **17** (1999), 397–407. MR 1715154

[L] Leite, M. L., *Rotational hypersurfaces on space forms with constante scalar survature*, Manuscripta Math. **67** (1990), 289–304. MR 1046990

[L2] Leite, M.L., *Errata to "Two-ended hypersurfaces with zero scalar curvature"*, by J. Hounie and M. L. Leite, Indiana Univ. Math. J. **48** (1999) (private communication).

[R1] Reilly, R, *On the Hessian of a function and the curvatures of its graph*, Michigan Math. J., **20** (1974), no. 4, 373–383. MR 0334045

[R2] Reilly, R, *Variarional properties of functions of the mean curvatures for hypersurfaces in space forms,* J. Differential Geom. **8** (1973), 465–477. MR 0341351

[Ro] Rosenberg, H., *Hypersurfaces of constant curvature in space forms*, Bull. Sci. Math., **117** (1993), no. 2, 211–239. MR 1216008

[S] Schoen, R., *Uniqueness, symmetry and embeddedness of minimal surfaces,* J. Differential Geom. **18** (1983), 791–809. MR 0730928

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