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RELATIVE PFAFFIAN CLOSURE FOR DEFINABLY COMPLETE BAIRE STRUCTURES

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Abstract. Speissegger proved that the Pfaffian closure of an o-minimal expansion of the real field is o-minimal. Here we give a first order version of this result: having introduced the notion of definably complete Baire structure, we define the relative Pfaffian closure of an o-minimal structure inside a definably complete Baire structure, and we prove its o-minimality. We derive effective bounds on some topological invariants of sets definable in the Pfaffian closure of an o-minimal expansion of the real field.

1. Introduction

In [Wil99], the author proved that the structure generated by all real Pfaffian functions is o-minimal. Subsequently, Speissegger defined the notion of Pfaffian closure of an o-minimal structure (based on $\mathbb{R}$) and generalized Wilkie's result in [Spe99] by proving that the Pfaffian closure of an o-minimal expansion of the real field is again o-minimal.

Here, we give a version of this result which holds also in a nonarchimedean context. More precisely, having introduced in [FS10] the notion of definably complete Baire structure, we define the relative Pfaffian closure of an o-minimal structure *inside* a definably complete Baire structure, and we prove its o-minimality. As a corollary, we obtain an alternative proof of Speissegger's result, by a modification of the argument in [KM99]. We remark that the first result in this direction is due to Fratarcangeli in [Fra06]. However his definitions and methods are substantially different from ours (he follows [Spe99] whereas we follow [KM99]) and the results he obtains are a special case of our Main Theorem 2.6, as we show in Section 5. There we also compare different notions of Rolle Leaf for expansions of definably complete fields.

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Moreover, our definition of Virtual Rolle multi-Leaf (Definition 3.9) allows us to give in Section 6 effective bounds on a series of topological invariants of sets definable in the Pfaffian closure of an o-minimal expansion of the real field, thus answering a question of Fratarcangeli [Fra06, p. 6].

2. The main result

We recall that Speissegger's results in [Spe99] concern expansions of the real field. Let $\mathbb{R}_0$ be an o-minimal expansion of the real field. Let $U \subseteq \mathbb{R}^n$ be an open subset definable in $\mathbb{R}_0$, and ω be an $\mathbb{R}_0$-definable $\mathcal{C}^1$-form on U which is never 0. An *embedded leaf* with data (U, ω) is a closed connected real submanifold of U of dimension $n-1$ that is orthogonal to ω at every point. A *Rolle Leaf* (RL) is an embedded leaf L which moreover satisfies the condition:

If $\gamma : [0,1] \to U$ is a $\mathcal{C}^1$ curve with end-points in L, then γ is orthogonal to ω in at least one point.

The typical example of a Rolle Leaf is the graph of a $\mathcal{C}^1$ function f such that f is the solution of a first order polynomial differential equation. The Pfaffian closure of an o-minimal expansion of the real field was defined in [Spe99]: the idea is to expand $\mathbb{R}_0$ by all Rolle Leaves with $\mathbb{R}_0$-definable data, and then repeat the procedure inductively.

We now begin to describe the generalized setting in which we will work. We recall that an expansion $\mathbb{K}$ of an orderedfield is a *definably complete Baire structure* if the two following (first-order) conditions hold:

- every definable subset of $\mathbb{K}$ has a supremum in $\mathbb{K} \cup \{\pm\infty\}$.
- $\mathbb{K}$, as a set, is not *definably meager*, i.e. $\mathbb{K}$ is not the union of a definable increasing family of nowhere dense sets.

We refer the reader to [FS10] for the precise definitions and preliminary results about definably complete Baire structures.

We make the same notational choices as in [FS10]. Let $\mathbb{K}_0$ be an o-minimal structure (expanding a field), and $\mathbb{K}$ be an expansion of $\mathbb{K}_0$ that is definably complete and Baire.

Proviso 2.1. In what follows, "definable" will mean "definable in $\mathbb{K}$ with parameters," unless otherwise specified. By "connected" we will mean "definably connected" (in $\mathbb{K}$), by "connected component" we will mean "definably connected component" and by "compact" we will mean "definable, closed and bounded."

Definition 2.2. A $\mathbb{K}$-*manifold* of dimension d (or simply "manifold") is a definable subset M of $\mathbb{K}^n$, such that for every point of $x \in M$ there exist a definable neighbourhood U of x (in $\mathbb{K}^n$), and a definable diffeomorphism $f : U \to \mathbb{K}^d$, such that $U \cap M = f^{-1}(\mathbb{K}^n \times \{0\})$.

DEFINITION 2.3. Let $\omega = a_1\,\mathrm{d}x_1 + \cdots + a_n\,\mathrm{d}x_n$ be a definable $\mathcal{C}^1$ differential form, defined on some definable open subset $U \subseteq \mathbb{K}^n$, such that $\omega \neq 0$ on all U. A *multi-leaf* with data (U,ω) is a $\mathcal{C}^1$ manifold M contained in U and closed in U, of dimension $n-1$, such that M is orthogonal to ω at all of its points (i.e., $T_aM = \ker(\omega(a))$, for every $a \in M$).

We must now face the problem of generalizing the notion of Rolle Leaf to the context of definably complete Baire structures.

We let an *arc* be a definable $\mathcal{C}^1$ map $\gamma : [0,1] \to \mathbb{K}^n$, such that γ' is always nonzero.

The most natural notion of generalized Rolle Leaf would be the following.

DEFINITION 2.4. An *Alternate Rolle Leaf* (ARL) is a connected multi-leaf L with data (U,ω) which moreover satisfies the condition:

If $\gamma : [0,1] \to U$ is an arc in L, then γ is orthogonal to ω in at least one point.

One could think of replacing the use of arcs with the use of connected manifolds of dimension one (see the definition of Rolle Leaf according to Fratarcangeli [Fra06, Definition 1.5]). Unfortunately, it is not clear whether in definably complete Baire structures definable $\mathcal{C}^1$ connected manifolds of dimension one are parameterizable as a finite union of arcs. Also, the drawback of this choice is that it is not possible to express with a first-order formula the fact that a set is definably connected. This fact creates an impediment, as will be clear later (see Section 4), and forces us to modify this definition.

However, for the application we have in mind (see Section 6) we introduce the notion of *Virtual Rolle multi-Leaf* (VRL, see Definition 3.9), which has the advantage of being first order (as ARL is) and at the same time of involving the notion of manifold of dimension one, rather than that of arc.

We are now ready to define the notion of relative Pfaffian closure.

DEFINITION 2.5. Inductive definition: for every $n \in \mathbb{N}$, let $\mathbb{K}_{n+1}$ be the expansion of $\mathbb{K}_n$ to a language L_{n+1} with a new predicate for every VRL with $\mathbb{K}_n$-definable data. Let $L^* = \bigcup_n L_n$ and define the *relative Virtual Pfaffian closure of* $\mathbb{K}_0$ *inside* $\mathbb{K}$, denoted by $\mathcal{VP}(\mathbb{K}_0,\mathbb{K})$, as the L^*-expansion of $\mathbb{K}_0$ where every predicate is interpreted as the corresponding Rolle Leaf.

Our aim is to prove the following version of Speissegger's theorem.

MAIN THEOREM 2.6. *Let* $\mathbb{K}$ *be a definably complete Baire structure and* $\mathbb{K}_0$ *be an o-minimal reduct of* $\mathbb{K}$*. Then* $\mathcal{VP}(\mathbb{K}_0,\mathbb{K})$ *is o-minimal.*

3. Virtual Rolle multi-Leaves

We now give the precise definition of Virtual Rolle multi-Leaf. The idea is the following: unlike the definition of ARL, where we considered all arcs X, we now consider closed manifolds X of dimension one (not necessarily connected) such that X does not have compact connected components. We want to find a first order condition on X that implies a bound on the number of connected components of X: every component has two "end-points at infinity" (see Definition 3.5 below); hence, if we ask that X has at most $2k$ end-points at infinity, we obtain that X has at most k connected components. It remains to express the requirement that X has no compact components in a first order way: this is done by asking the existence of a definable $\mathcal{C}^1$ function without critical points on X.

Finally, the Rolle condition for a leaf L is expressed by asking that for any X as above that intersects L in a number of points which is greater than the number of connected components of X, there is a point where X is orthogonal to the 1-form defining L.

Definition 3.1. A *weak cell* of dimension d is a $\mathbb{K}_0$-definable set $U \subseteq \mathbb{K}^n$ which is diffeomorphic, via a $\mathbb{K}_0$-definable diffeomorphism ϕ_U, to $\mathbb{K}^d$. For every $0 < t \in \mathbb{K}$, we define $U_t := \phi_U^{-1}(\{x \in \mathbb{K}^d : \|x\| = t\})$.

We consider the diffeomorphism ϕ_U as part of a weak cell: the same subset U of $\mathbb{K}^m$ with two different choices of diffeomorphisms should be considered as two different weak cells. Notice that, for $0 < t \in \mathbb{K}$, U_t is a compact manifold of dimension $d - 1$.

Definition 3.2. Let $U \subseteq \mathbb{K}^n$ be a weak cell. We say that $X \subseteq U$ is a *twine* in U if X is a 1-dimensional $\mathcal{C}^1$ manifold, such that X is closed in U. We say that $X \subseteq U$ is a *good twine* in U if X is a twine in U and moreover there exists a definable $\mathcal{C}^1$ function $\rho : X \to \mathbb{K}$ without critical points.

Remark 3.3. Let $X \subseteq \mathbb{K}^n$ be definable. We denote by $\mathfrak{B}(X)$ the Boolean algebra of definable clopen subsets of X. $\mathfrak{B}(X)$ is finite iff $\mathrm{cc}(X)$ (the number of connected components of X) is finite, and in that case each connected component of X is definable and an atom of $\mathfrak{B}(X)$, and moreover $|\mathfrak{B}(X)| = 2^{\mathrm{cc}(X)}$.

Moreover, for every $m \in \mathbb{N}$, the following are equivalent:

(1) $\mathfrak{B}(X) \leq 2^m$;
(2) $\mathrm{cc}(X) \leq m$;
(3) if $Y_1, \dots, Y_{m+1}$ are disjoint elements of $\mathfrak{B}(X)$, then at least one of them is empty.

Remark 3.4. Let U be a weak cell and X be a twine in U. Let $\emptyset \neq Y \in \mathfrak{B}(X)$. Then, Y is a also twine. If moreover X is good, then Y is also good and not compact. In particular, if X is a good twine and $\mathrm{cc}(X) < \infty$, then no connected component of X is compact (because otherwise the map ρ in Definition 3.2 would have at least one critical point).

Definition 3.5. Let U be a weak cell and X be twine in U. For each $0 < t \in \mathbb{K}$, let $X_t := \{x \in X \cap U_t : X$ is transversal to U_t at $x\}$. We denote by

$$\mathrm{vb}_U(X) := \limsup_{t \to +\infty} |X_t| \in \mathbb{N} \cup \{\infty\},$$

the *virtual boundary* of X.

Notice that X_t is a 0-dimensional manifold, and hence $|X_t| = \mathrm{cc}(X_t)$. Notice also that, unlike the number of connected components, $\mathrm{vb}_U(X)$ can be defined with a first order formula.

Lemma 3.6. *Assume that $\mathbb{K}$ is o-minimal. Let U be a weak cell and $X \subseteq U$ be a good twine in U. If X is a connected, then* $\mathrm{vb}_U(X) = 2$. *More generally,* $\mathrm{vb}_U(X) = 2 \cdot \mathrm{cc}(X)$.

Proof. It suffices to do the case when X is connected. Since $\mathbb{K}$ is o-minimal, X is then the image of some definable $\mathcal{C}^1$ function $\gamma : (0,1) \to U$. The conclusion follows from the o-minimality of $\mathbb{K}$. □

Lemma 3.7. *Let U be a weak cell and X be a good twine in U. If X is nonempty, then* $\mathrm{vb}_U(X) \geq 1$, *and if moreover $\mathbb{K}$ is an expansion of $\mathbb{R}$, then* $\mathrm{vb}_U(X) \geq 2$.

Moreover, if $\mathrm{vb}_U(X)$ *is finite, then* $\mathrm{cc}(X) \leq \mathrm{vb}_U(X)$, *and in particular X has a finite number of connected components, and each component of X is not compact.*

Proof. The case $\mathbb{K}$ expanding $\mathbb{R}$ follows from the fact that each connected component of X (not necessarily definable!) is the image of some $\mathcal{C}^1$ function $f : (0,1) \to \mathbb{R}$, and some standard analysis.

Let $X^0 \subseteq X$ be nonempty, clopen, and definable. Let $I_0 := \{t > 0 : X^0_t \neq \emptyset\}$ (see Definition 3.5). By the Implicit Function theorem, I_0 is an open subset of $\mathbb{K}$. We say that $I \subseteq \mathbb{K}$ is *almost final* in $\mathbb{K}$ if there exists $R > 0$, such that $(R, +\infty) \setminus I$ is nowhere dense in $\mathbb{K}$. □

Claim 1. *I_0 is almost final in $\mathbb{K}$.*

Define $r : U \to \mathbb{K}$, $r(x) := \|\phi_U(x)\|$. Let $R > 0$ be such that there exists $x_0 \in X^0$ with $r(x_0) < R$. Assume, for a contradiction, that there exist $b > a > R$, such that $(a,b) \subseteq (R, +\infty) \setminus I_0$, that is, for every $t \in (a,b)$, X^0 meets U_t only nontransversally.

Let $U_{(a,b)} := r^{-1}((a,b)) = \{x \in U : a < \|\phi_U(x)\| < b\}$ and $Y := X^0 \cap U_{(a,b)}$; notice that Y is open in X^0, and hence open in X. Let $s := r \restriction X_0$; notice that s has only critical points on Y, and therefore s is locally constant on Y. Let $Z := s^{-1}((-\infty, (a+b)/2))$; since s is locally constant on Y, Z is clopen in X^0, and hence in X, and nonempty (because $x_0 \in Z$). By Remark 3.4, Z is not compact; however, Z is closed and bounded in $\mathbb{K}^n$, contradiction, proving the claim.

Assume, for a contradiction, that $m := \mathrm{vb}_U(X) < \mathrm{cc}(X)$. Let $X^1, \dots, X^{m+1} \in \mathfrak{B}(X)$ be disjoint and nonempty, and, for each $j \le m+1$, let $I_j := \{t > 0 : X^j_t \neq \emptyset\}$. Then, by Claim 1, each I_j is almost final in $\mathbb{K}$; thus $\bigcap_j I_j$ is almost final in $\mathbb{K}$, and hence $\mathrm{vb}_U(X) \ge m+1$, contradiction.

The remainder follows from Remark 3.4.

Remark 3.8. Let $X \subset \mathbb{K}^n$ be a weak cell of dimension 1. Then, X is a good twine in itself, and $\mathrm{vb}_X(X) = 2$.

Definition 3.9. A *Virtual Rolle multi-Leaf* (VRL) is a multi-leaf L with data (U, ω) which satisfies the following condition: for every $n \in \mathbb{N}$, for every $V \subseteq U \times \mathbb{K}^n$ weak cell and every X good twine in V, if $|X \cap (L \times \mathbb{K}^n)| > \mathrm{vb}_V(X)$, then X is orthogonal to $\pi^*(\omega)$ in at least one point, where $\pi^*(\omega)$ is the pullback of ω via the canonical projection $\pi : U \times \mathbb{K}^n \to U$.

With the notation of the above definition, if $\mathrm{vb}_V(X)$ is infinite, then the premise is false, and therefore the condition is automatically satisfied (for the given X). Therefore, to verify whether L is a VRL, we need to check only the good twines X such that $\mathrm{vb}_V(X)$ is finite. Moreover, by Lemma 3.7, such a good twine X satisfies $\mathrm{cc}(X) \le \mathrm{vb}_V(X)$. Therefore, if $X_1, \dots, X_m$ are the components of X, and $|X \cap L| > \mathrm{vb}_V(X)$, then for at least one i we have $|X_i \cap L| > 1$.

4. o-minimality of $\mathcal{VP}(\mathbb{K}_0, \mathbb{K})$

In this section, we prove Theorem 2.6. We will assume the reader to have familiarity with [FS09]. In particular, recall Definitions 3.1, 3.3 and 3.7. However, we will use a condition which, albeit stronger than the DAC^N condition ([FS09, Definition 3.7]), is easier to state and to work with.

Definition 4.1. A partial function $f : \mathbb{K}^n \hookrightarrow \mathbb{K}$ is an *admissible partial function* if it is definable, its domain is open in $\mathbb{K}^n$, it is continuous, and its graph is closed in $\mathbb{K}^{n+1}$. Suppose that $\mathcal{S}$ is a weak structure over $\mathbb{K}$. We say that $\mathcal{S}$ is *determined by its $\mathcal{C}^N$ admissible partial functions* (DPC^N) for all N, if, for each $n \in \mathbb{N}$ and $A \in \mathcal{S}_n$, there exists $m \ge n$, such that, for every $N \in \mathbb{N}$, there exists a set $B_N \subseteq \mathbb{K}^m$, which is a finite union of sets of the form $V(f_{N,i})$, where each $f_{N,i} : \mathbb{K}^m \hookrightarrow \mathbb{K}$ is an admissible $\mathcal{C}^N$ partial function, and $A = \Pi^m_n(B_N)$.

Notice that if $\mathcal{S}$ is a semi-closed o-minimal weak structure which is DPC^N for all N, then it is DAC^N for all N. Therefore, the following theorem is an immediate consequence of [FS09, Theorem 3.8]

Theorem 4.2 (Generalized theorem of the complement). *Suppose that $\mathcal{S}$ is a semi-closed o-minimal weak structure over $\mathbb{K}$, which is DPC^N for all N. Then the Charbonnel closure $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ is an o-minimal weak structure over $\mathbb{K}$,* *which is closed under complementation. Hence, $\widetilde{\mathcal{S}}$ is an o-minimal first-order structure.*

Hence, all we need to do is to prove that $\mathcal{VP}(\mathbb{K}_0,\mathbb{K})$ is the Charbonnel closure of some weak structure $\mathcal{S}$ which satisfies the hypotheses of the above theorem.

For the rest of this section, a Rolle Leaf will be a Virtual Rolle multi-Leaf (in particular, we are *not* requiring that a Rolle Leaf is definably connected).

Definition 4.3. Let $\mathrm{Rolle}(\mathbb{K}_0,\mathbb{K}) = \{(\mathrm{Rolle}(\mathbb{K}_0,\mathbb{K}))_n | n \in \mathbb{N}\}$ be such that $(\mathrm{Rolle}(\mathbb{K}_0,\mathbb{K}))_n$ consists of all the finite unions of sets $A \cap L_1 \cap \cdots \cap L_k$, which we call *basic Rolle sets*, where $A \subseteq \mathbb{K}^n$ is $\mathbb{K}_0$-definable, and each L_i is a Rolle Leaf with data (U_i,ω_i) in $\mathbb{K}_0$.

We will show the following proposition.

Proposition 4.4. $\mathrm{Rolle}(\mathbb{K}_0,\mathbb{K})$ *is a semi-closed o-minimal weak structure, satisfying* DPC^N *for all* N.

Since $\mathrm{Rolle}(\mathbb{K}_0,\mathbb{K})$ generates $\mathbb{K}_1$ in Definition 2.5, this, together with Theorem 4.2, shows that $\mathbb{K}_1$ is o-minimal; by applying inductively the same result to each $\mathbb{K}_n$, we obtain a proof of Theorem 2.6.

We will prove Proposition 4.4 via a series of lemmas.

Remark 4.5. Every basic Rolle set is the projection of another basic Rolle set, such that all the open sets U_i in the data are the same open set U (the proof is as in [KM99, ¶3.4]).

Lemma 4.6. $\mathrm{Rolle}(\mathbb{K}_0,\mathbb{K})$ *is a weak structure.*

Proof. As in [KM99, Lemma 3]. Notice that $\mathrm{Rolle}(\mathbb{K}_0,\mathbb{K})$ is closed under cartesian products by definition of VRL. □

Proposition 4.7. *Let $\Omega = (\omega_1,\dots,\omega_q)$ be a tuple of $\mathbb{K}_0$-definable nonsingular 1-forms defined on some common open subset U of $\mathbb{K}^n$, and let A be a $\mathbb{K}_0$-definable subset of U. Then, there is a natural number N such that, whenever L_i is a VRL of $\omega_i = 0$ for each $i = 1,\dots,q$, then $A \cap L_1 \cap \cdots \cap L_q$ is the union of at most N connected manifolds. Moreover, N can be chosen independent of the parameters used in defining Ω, U, and A.*

The proof of the above proposition is in Section 4.1.

Proposition 4.8. *Let U be a $\mathbb{K}_0$-definable open subset of $\mathbb{K}^n$, and ω be a $\mathbb{K}_0$-definable 1-form on U, such that $\omega \neq 0$ on all U. Let L be a multi-leaf with data (U,ω). Let C be a definable connected $\mathcal{C}^1$ manifold of dimension at most $n-1$ contained in U, such that C is orthogonal to ω at all of its points. Then, either C is contained in L, or C is disjoint from L.*

Proof. [Fra06, Lemma 5.4]. □

We need the following results on o-minimal structures. By "cell" we will always mean "cell in the sense of $\mathbb{K}_0$."

LEMMA 4.9. *Let n and N be natural numbers greater than* 0. *Define* $\pi := \Pi_n^{n+1}$.

(1) *For every $Y \subseteq \mathbb{K}^n$ closed and $\mathbb{K}_0$-definable, there exists $f : \mathbb{K}^n \to [0,1]$ $\mathbb{K}_0$-definable and $\mathcal{C}^N$, such that $Y = V(f)$. Moreover, if Z is a closed and $\mathbb{K}_0$-definable subset of $\mathbb{K}^n$ disjoint from Y, then we can require that $Z = V(1-f)$.*

(2) *For every $\mathbb{K}_0$-definable $X \subseteq \mathbb{K}^n$ there exists $Y \subseteq \mathbb{K}^{n+1}$, also $\mathbb{K}_0$-definable, such that Y is closed and $X = \pi(Y)$. If moreover X is a $\mathcal{C}^N$ cell, then Y can be chosen to be a* (*closed*) *$\mathcal{C}^N$ cell of the same dimension as X.*

(3) *Let $X \subseteq \mathbb{K}^n$ be $\mathbb{K}_0$-definable. Then, there exists $f : \mathbb{K}^{n+1} \to \mathbb{K}$ $\mathbb{K}_0$-definable and $\mathcal{C}^N$, such that $X = \pi(V(f))$.*

(4) *Let $Y \subseteq \mathbb{K}^n$ be a* closed *$\mathcal{C}^N$ cell. Then, there exists a $\mathbb{K}_0$-definable $\mathcal{C}^N$ retraction*: $\mathbb{K}^n \to Y$.

Sketch of proof. (1) The proof in [vdDM96, Corollary C.12] works in any o-minimal structure expanding a field, and not only in $\mathbb{R}$.

(2) is clear (see [vdD98]) and (3) follows immediately from (1) and (2).

(4) It is easy to see that there exists a $\mathbb{K}_0$-definable open neighbourhood U of Y and a $\mathbb{K}_0$-definable $\mathcal{C}^N$ retraction $r_0 : U \to Y$. Let V be another $\mathbb{K}_0$-definable open neighbourhood of Y, such that $\overline{V} \subseteq U$. By (1), there exists $h : \mathbb{K}^n \to [0,1]$ $\mathbb{K}_0$-definable and $\mathcal{C}^N$, such that $Y = h^{-1}(1)$ and $\mathbb{K}^n \setminus V = h^{-1}(0)$. Let $\phi : Y \to \mathbb{K}^d$ be a $\mathcal{C}^N$ $\mathbb{K}_0$-definable diffeomorphism. Define

$$r(x) := \begin{cases} \phi^{-1}\big(h(x)\cdot\phi\big(r_0(x)\big)\big) & \text{if } x \in U; \\ \phi^{-1}(0) & \text{if } x \notin \overline{V}. \end{cases}$$ □

LEMMA 4.10. $\mathrm{Rolle}(\mathbb{K}_0, \mathbb{K})$ *is semi-closed.*

Proof. We use:

(1) union commutes with projection;

(2) the class of projections (from various $\mathbb{K}^n$) of closed sets in $\mathrm{Rolle}(\mathbb{K}_0, \mathbb{K})$ is closed under intersections.

It suffices to prove that any Rolle Leaf $L \subseteq \mathbb{K}^n$ is the projection of a closed set in $\mathrm{Rolle}(\mathbb{K}_0, \mathbb{K})$. Let (U, ω) be the data (definable in $\mathbb{K}_0$) of L. Consider $f : U \to \mathbb{K}$, $f(x) := 1/d(x, \mathbb{K}^n \setminus U)$ (where d is the distance function). Notice that f is an admissible partial function with domain U. Let $\pi := \Pi_n^{n+1}$, $\tilde{\omega} := \pi^*(\omega)$ be the pullback of ω to $U \times \mathbb{K}$, and $\tilde{L} := \pi^{-1}(L)$; note that $\tilde{L}$ is a Rolle Leaf, with data $(U \times \mathbb{K}, \tilde{\omega})$. Define $C := L \cap F$, where F is the graph of f. Then, C is closed in $\mathbb{K}^{n+1}$, and $\pi(C) = L$. □

LEMMA 4.11. $\mathrm{Rolle}(\mathbb{K}_0, \mathbb{K})$ *is an o-minimal weak structure.*

Proof. The conclusion can be easily obtained from Proposition 4.7, reasoning as in [Spe99, Corollary 2.7]. □

Hence, we can conclude that $\mathrm{Rolle}(\mathbb{K}_0, \mathbb{K})$ is a semi-closed o-minimal weak structure. The last step is proving that $\mathrm{Rolle}(\mathbb{K}_0, \mathbb{K})$ satisfies DPC^N for all N, and hence is an o-minimal *structure*.

The following observation explains why it is easier to work with admissible partial functions instead of correspondences.

REMARK 4.12. (1) Let $f : \mathbb{K}^n \hookrightarrow \mathbb{K}$ be a definable continuous partial function, with domain an open set U. Then, f is admissible iff, for every $x \in \operatorname{bd} U$,

$$\lim_{\substack{y \to x,\\ y \in U}} \|f(y)\| = +\infty.$$

(2) Let $f, g : \mathbb{K}^n \hookrightarrow \mathbb{K}$ be two admissible partial functions. Then $h := f^2 + g^2$ is an admissible partial function, such that $\mathrm{dom}(h) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and $V(h) = V(f) \cap V(g)$.

The following lemma explains the reason why we are able to use admissible partial functions instead of admissible correspondences.

LEMMA 4.13. *Let U be an open $\mathcal{C}^N$ cell in $\mathbb{K}^n$, $\theta := \Pi^n_{n-1}$, and $U' := \theta(U)$ be the basis of U. Let $\omega := a_1 \,\mathrm{d}x_1 + \cdots + a_n \,\mathrm{d}x_n$ be a $\mathcal{C}^N$ 1-form on U, such that $a_n \equiv 1$, and let F be a Rolle Leaf with data (U, ω). Then, F is the graph of a $\mathcal{C}^{N+1}$ partial function $f : W \to \mathbb{K}$, with open domain $W \subseteq U'$. Moreover, for every $N \in \mathbb{N}$, there exists an admissible $\mathcal{C}^N$ partial function $g_N : \mathbb{K}^{n+1} \hookrightarrow \mathbb{K}$ in* $\mathrm{Rolle}(\mathbb{K}_0, \mathbb{K})$, *such that $f = \pi(V(g_N))$, where $\pi := \Pi^{n+1}_n$.*

Proof. Let us prove that F is the graph of a (single-valued) partial function f. The fact that f is a $\mathcal{C}^{N+1}$ partial function with open domain is then clear. If not, there exist $\bar{x} \in V$ and $y_1 < y_2 \in \mathbb{K}$, such that, for $i = 1, 2$, $p_i := (\bar{x}, y_i) \in F$. Let J be the "vertical" segment with endpoints p_1 and p_2. By the Rolle condition, there exists $q \in J$ such that J is orthogonal to ω at q. Since J is vertical, this means that $\omega(q)$ is "horizontal," contradicting the fact that $a_n \equiv 1$.

Hence, F is a closed subset of U, and satisfies all conditions for being the graph of an admissible $\mathcal{C}^N$ partial function $f : \mathbb{K}^{n-1} \hookrightarrow \mathbb{K}$, except that F might not be closed in $\mathbb{K}^n$ (but only in U). If $U = \mathbb{K}^n$, we can easily conclude as in [KM99, Lemma 6]. Otherwise, we have more work to do.

Let $\phi' : \mathbb{K}^{n-1} \xrightarrow{\sim} U'$ and $\phi : \mathbb{K}^n \xrightarrow{\sim} U$ be $\mathbb{K}_0$-definable $\mathcal{C}^N$ diffeomorphisms, such that $\phi' \circ \theta = \theta \circ \phi$. Let $\tilde{F} := \phi^{-1}(F)$, and $\tilde{\omega} := \phi^*(\omega)$. Then, $\tilde{F}$ is a Rolle Leaf, with data $(\mathbb{K}^n, \tilde{\omega})$. Moreover, $\tilde{F}$ is the graph of a $\mathcal{C}^N$ admissible partial function $\tilde{f} : \mathbb{K}^{n-1} \hookrightarrow \mathbb{K}$ (in fact, $\tilde{F}$ is closed in $\mathbb{K}^n$).

Define $\tilde{g}(x_1, \ldots, x_n) := \tilde{f}(x_1, \ldots, x_{n-1}) - x_n$, $\tilde{g} : \mathbb{K}^n \hookrightarrow \mathbb{K}$. By [FS09, Lemma 4.5], $\tilde{g}$ is an admissible partial function; notice that $\tilde{F} = V(\tilde{g})$. We would like to pullback $\tilde{g}$ via ϕ; the problem is that $\tilde{g} \circ \phi^{-1}$ might not have a closed graph, because ϕ^{-1} is not defined on all $\mathbb{K}^n$.

Let $D \subseteq \mathbb{K}^{n+1}$ be a *closed* $\mathcal{C}^N$ cell, such that $\pi(D) = U$, let $f_{N,1} : \mathbb{K}^{n+1} \to \mathbb{K}$ be a $\mathcal{C}^N$ and $\mathbb{K}_0$-definable (total!) function, such that $D = V(f_{N,1})$, and $r : \mathbb{K}^{n+1} \to D$ be a $\mathbb{K}_0$-definable $\mathcal{C}^N$ retraction (D, $f_{N,1}$ and r exist by Lemma 4.9). Let $f_{N,2} := \tilde{g} \circ \phi^{-1} \circ \pi \circ r$. Notice that $\phi^{-1} \circ \pi \circ r$ is a total $\mathcal{C}^N$ function, and therefore, by [FS09, Lemma 4.4], $f_{N,2}$ is admissible; $f_{N,1}$ is also obviously admissible. Let $h_N := f_{N,1}^2 + f_{N,2}^2 : \mathbb{K}^{n+1} \hookrightarrow \mathbb{K}$; clearly, g_N is an admissible $\mathcal{C}^N$ partial function in $\widetilde{\mathcal{S}}$, and $F = \pi(V(h_N))$. $\square$

Notice that the following lemma does not imply Lemma 4.10, because the DPC^N condition does not imply that a weak structure is semi-closed.

LEMMA 4.14. Rolle$(\mathbb{K}_0, \mathbb{K})$ *satisfies* DPC^N *for all* N.

Proof. Proceeding as in the proof of Lemma 4.10, one can see that it is enough to prove:

(*) If $U \subseteq \mathbb{K}^n$ is open and definable in $\mathbb{K}_0$, ω is a $\mathcal{C}^1$ form, also definable in $\mathbb{K}_0$, and L is a Rolle Leaf with data (U, ω), then, for every $N \geq 1$, there is a set $S \subseteq \mathbb{K}^{2n}$, such that S is a finite union of sets of the form $V(f_{N,i})$, where each $f_{N,i} : \mathbb{K}^{2n} \hookrightarrow \mathbb{K}$ is a $\mathcal{C}^N$ admissible partial function in $\widetilde{\text{Rolle}(\mathbb{K}_0, \mathbb{K})}$, and $L = \pi(S)$, where $\pi := \Pi^{2n}_n$.

Note that the above claim is the DPC^N hypothesis for L, with $m = 2n$. We prove (*) by induction on n. If $n = 0$ or $n = 1$, the conclusion is clear. So, we can assume $n \geq 2$, and that (*) is true for any $n' < n$.

Fix N. Do a decomposition of U into $\mathcal{C}^N$ cells E_i, such that on each open cell ω is a $\mathcal{C}^N$ form. Let $E := E_i$. It suffices to prove (*) for $L \cap E$. We proceed by further induction on $\dim(E)$.

CASE 1. Assume that $\dim(E) < n$. Let X be the set of points $x \in E$ such that ω is orthogonal to E in x. Notice that X is $\mathbb{K}_0$-definable; decompose E into $\mathcal{C}^1$ cells compatibly with X. Let E' be a cell in this second decomposition. It suffices to prove (*) for $L \cap E'$.

CASE 1a. If $E' \subseteq X$, then, by definition, E' is orthogonal to ω at all of its points. By Proposition 4.8, E' is either disjoint or contained in L. Hence, $L \cap E'$ is either empty or a cell, and thus, by Lemma 4.9, it satisfies (*).

CASE 1b. Assume that $E' \subseteq E \setminus X$. Assume moreover that $d := \dim(E') = \dim(E)$. After a permutation of coordinates, we can assume that E' is the graph of a $\mathcal{C}^N$ function $f : V \to \mathbb{K}^{n-d}$, where $V \subseteq \mathbb{K}^d$ is an open cell. Let $\sigma : E' \to V$ be the canonical projection, and $\tau : V \to E'$, $x \mapsto (x, f(x))$ be its inverse. Notice that σ and τ are $\mathcal{C}^N$ diffeomorphisms; let $\omega \restriction E'$ be the projection of ω onto the tangent space TE' (that is, the pullback of ω to E' via the inclusion map), $\tilde{\omega} := \tau^*(\omega \restriction E')$ be the its pullback to V, and $\tilde{L} := \sigma(L \cap E')$. Since $E' \subseteq E \setminus X$ and $\dim(E') = \dim(E)$, ω is never orthogonal to E'; thus, $\omega \restriction E'$ is never 0, and $\tilde{\omega}$ is never 0. Moreover, $\tilde{L}$ is a Rolle Leaf with data $(V, \tilde{\omega})$. Thus, by inductive hypothesis, there exist finitely many admissible $\mathcal{C}^N$ partial functions $g_{N,i} : \mathbb{K}^{2d} \hookrightarrow \mathbb{K}$ in $\widetilde{\mathcal{S}}$, such that $\tilde{L} = \bigcup_i \Pi^{2m}_m(V(g_{N,i}))$. Let $h_N : \mathbb{K}^{n+1} \to \mathbb{K}$ be a $\mathcal{C}^N$ $\mathbb{K}_0$-definable function, such that $E' = \Pi^{n+1}_n(V(h))$. For each i, define

$$f_{N,i}(x_1, \dots, x_d, y_{d+1}, \dots, y_n, z_{n+1}, \dots, z_{n+d}) \\ := h_N(x_1, \dots, x_d, y_{d+1}, \dots, y_n)^2 + g_{N,i}(x_1, \dots, x_d, z_{n+1}, \dots, z_{n+d})^2.$$

Notice that each $f_{N,i}$ is an admissible partial function in $\widetilde{\mathcal{S}}$, and that $L \cap E' = \Pi^{n+d}_n(\bigcup_i V(f_{N,i}))$.

CASE 1c. If $\dim(E') < \dim(E)$, we can apply the induction on $\dim(E)$, and conclude that $L \cap E'$ satisfies (*).

CASE 2. If $E := E_i$ is an open cell, then $L \cap E$ is a Rolle Leaf with data (E, ω); hence, by substituting U with E, w.l.o.g. we can assume that U is an open cell.

Let $\omega := a_1 \,\mathrm{d}x_1 + \cdots + a_n \,\mathrm{d}x_n$, and $V_j := \{\bar{x} \in U : a_j(\bar{x}) \neq 0\}$, $j = 1, \dots, n$. Note that V_j is open and definable in $\mathbb{K}_0$. Decompose again U into $\mathcal{C}^N$ cells, in a way compatible with each V_j. For the nonopen cells, apply the induction on $\dim(E)$. If E' is an open cell, then it is contained in some V_j.

We claim that, w.l.o.g., $E' \subseteq V_n$. In fact, assume that E' is contained in V_j. Permute the coordinates, exchanging x_j with x_n, and rename the V_l accordingly. Notice that in the new system of coordinates $E' \subseteq V_n$, but E' might no longer be a cell; decompose further E' into cells. For the nonopen ones, proceed as in Case 1. For the open ones, the claim is true.

Thus, by substituting U with E' and permuting the coordinates, we are reduced to the case $a_n(\bar{x})$ is never 0 on U, and therefore we can assume that a_n is the constant function 1. Hence, we can apply Lemma 4.13, and we are done. ☐

REMARK 4.15. In the proof of [KM99, Lemma 6] there is a gap, in that the function $f_{N,2}$ in the proof of Lemma 4.13 might not be a total function: this is *the* reason why we had to work with admissible partial functions instead of total functions. For instance, let $f : \mathbb{R} \hookrightarrow \mathbb{R}$ be the partial function $f(x) := 1/x$, defined on $\mathbb{R}_+$, and $F \subset \mathbb{R}^2$ be the graph of f. Let $\omega(x,y) := y^2 \,\mathrm{d}x + \mathrm{d}y$ be a 1-form defined on $\mathbb{R}^2$. Then, F is a $\mathcal{C}^\infty$ Rolle Leaf of $\omega = 0$. In fact, f solves the differential equation $f' = -f^2$, and therefore we can apply [Spe99, Example 1.3].

4.1. Proof of Proposition 4.7.

We will assume familiarity with [Fra06].

Some important but easy observations are the following ones:

- [Fra06, Lemma 5.9] does not require that the manifolds L_i are connected, and therefore can be applied to L_i multi-leaves.
- [Fra06, Proposition 5.10] does not use neither the conditions that the L_i are connected nor the Rolle condition, and remains true for L_i multi-leaves.
- [Fra06, Proposition 5.7] can be used in the following form:

PROPOSITION 4.16. *Let U and V be definable open subsets of $\mathbb{K}^n$, and let $\sigma : V \to U$ be a definable diffeomorphism. Let ω be a definable 1-form on U, and L be a multi-leaf with data (U, ω). Then, $\sigma^{-1}(L)$ is a multi-leaf with data $(V, \sigma^*(\omega))$.*

If L is a VRL and σ is $\mathbb{K}_0$-definable, then $\sigma^{-1}(L)$ is a VRL.

Hence, the Rolle condition is used directly only at the end of the proof, on [Fra06, p. 39]. We will show how to use the Virtual Rolle condition instead.

The proof will proceed by induction on q. If $q = 0$, the conclusion follows from o-minimality of $\mathbb{K}_0$; hence, we can assume $q \geq 1$.

For the inductive step, we assume that we have already proved the conclusion for $q - 1$: that is, we assume that we have proved the result for every $(q-1)$-tuple Ω' of $\mathbb{K}_0$-definable nonsingular 1-forms defined on some open set U' of $\mathbb{K}^{n'}$, for every $\mathbb{K}_0$-definable set $A' \subseteq U'$, and for every corresponding $(q-1)$-tuple of VRL with data (U', Ω').

Fix U, Ω, $L_1, \ldots, L_q$, and A as in the assumption of the theorem. Let $d := \dim(A)$. We prove the conclusion by a further induction on d.

As in the proof of [Fra06, Theorem 1.7], we can reduce to the case that A is a $\mathbb{K}_0$-definable $\mathcal{C}^1$-cell of dimension $d \geq q$, contained in U, and Ω is transverse to A; that is, for every $a \in A$, the projections of (the vector fields associated to) $\omega_1, \ldots, \omega_q$ on $T_a(A)$ are linearly independent. Notice that "Ω transverse to A" is equivalent to "the projection on $T(A)$ of the q-form $\omega_1 \wedge \cdots \wedge \omega_q$ is never null."

If $d > q$, we can conclude by induction on d as in [Fra06, p. 39, "CASE $d > q$"]; as we noticed before, the Rolle condition is not used in [Fra06, Proposition 5.10], and therefore we can use it in our situation.

Hence, it remains to treat the case $d = q$.

If $d = q$, we treat first as a way of exemplification the case $d = q = 1$. Then, A is a good twine in itself, thus, by the Rolle condition, and the fact that ω_1 is transverse to A, $|A \cap L_1| \leq \mathrm{vb}_A(A) = 2$, and we are done.

In general, if $d = q$, define $L' := A \cap L_1 \ldots L_{q-1}$ (or $L' := A$ if $q = 1$). Notice that L' is a twine in A. Let $\omega' := \omega'_1 \wedge \cdots \wedge \omega'_{q-1}$, where each ω'_i is the projection of ω_i onto (the tangent space of) A. Notice that ω' is a nonsingular $(q-1)$-form on A. If we identify ω' with the corresponding vector field on A, then ω' is always tangent to L'. Notice also that $A \cap L_1 \cap \cdots \cap L_q$ is a 0-dimensional manifold, and therefore $\mathrm{cc}(A \cap L_1 \cap \cdots \cap L_q) = |L' \cap L_q|$.

We have to further decompose A in order to transform L' into a *good* twine. Fix a map $p : A \to \mathbb{K}$, such that p is $\mathbb{K}_0$-definable, is $\mathcal{C}^1$, and has no critical points on A. For every $x \in A$, let $c(x)$ be the gradient vector of p at x (by definition, $c(x)$ is tangent to A).

Define A_{crit} to be the set of points in A such that ω' is orthogonal to c, and $A_{\mathrm{reg}} := A \setminus A_1$. After a further cell decomposition, w.l.o.g. we can assume that either $A = A_{\mathrm{reg}}$, or $A = A_{\mathrm{crit}}$.

If $A = A_{\mathrm{reg}}$, let ρ be the restriction of p to L'. Notice that, by definition of A_{reg}, ρ is a definable $\mathcal{C}^1$ function without critical points, and hence L' is a good twine in A. Fix a $\mathbb{K}_0$-definable diffeomorphism ϕ_A between A and $\mathbb{K}^d$, and define A_t accordingly. By induction on q, there is $N \in \mathbb{N}$ such that $L' \cap A_t \cap \{x \in A : \omega' \text{ is not orthogonal to } A_t\}$ has at most N connected components, where N does not depend on t. Hence, by definition, $\mathrm{vb}_A(L') \leq N$. Thus, since L_q is a VRL and Ω is transverse to A, $|L' \cap L_q| \leq N$, and we are done.

If instead $A = A_{\mathrm{crit}}$, for every $t \in \mathbb{K}$ let $B(t) := \{x \in A : p(x) = t\}$: each $B(t)$ is a $\mathbb{K}_0$-definable set of dimension $d - 1$. By induction on q, L' has a uniformly bounded number of connected components $M_1, \dots, M_r$. Moreover, p is constant on each M_i, and therefore for each $i \leq r$ there exists $t_i \in \mathbb{K}$ such that $M_i \subseteq B(t_i)$. Thus, $M_i \cap L_q \subseteq L_1 \cap \dots \cap L_q \cap B(t_i)$, and therefore

$$A \cap L_1 \cap \dots \cap L_q = \bigcup_{i=1}^{r} B(t_i) \cap L_1 \cap \dots \cap L_q.$$

By induction on d, there exists a uniform (independent from t) bound r' for $\mathrm{cc}(B(t_i) \cap L_1 \cap \dots \cap L_q)$, and therefore $\mathrm{cc}(A \cap L_1 \cap \dots \cap L_q) \leq rr'$.

5. Variants of the rolle property

In this section, we compare different notions of Rolle Leaves: the original definition of Rolle Leaf (RL), due to Speissegger, which makes sense only for expansions of the real field was given at the beginning of Section 2. Alternate Rolle Leaves (ARL) and Virtual Rolle multi-Leaves (VRL) were defined in Definitions 2.4 and 3.9, respectively.

Definition 5.1. A *Rolle Leaf according to Fratarcangeli* (FRL) is a connected multi-leaf L with data (U, ω), which moreover satisfies the condition: for every $m \in \mathbb{N}$, if $X \subset U \times \mathbb{K}^m$ is a definable connected $\mathcal{C}^1$ submanifold of $U \times \mathbb{K}^m$ of dimension one, and X intersects L in at least two points, then X is orthogonal to ω in at least one point (compare with [Fra06, Definition 1.5]).

Proposition 5.2. *Let $\mathbb{K}$ be an expansion of the real field. Then every RL is a VRL.*

In particular, we recover Speissegger's theorem as a special case of ours.

Proof of Proposition 5.2. Let $L \subset \mathbb{R}^n$ be a RL with data (U, ω). Let $V \subseteq U$ be a weak cell and X be a good twine in V. Assume that $|X \cap L| > \mathrm{vb}_V(X) =: m$ (the case when $V \subseteq U \times \mathbb{R}^k$ can be treated similarly). We must show that X is orthogonal to ω in at least one point; assume, for a contradiction, that this is not the case. Let X_i be a connected component of X (notice that X_i is not necessarily definable). Since X is a good twine, X_i is not compact; moreover, X has at most m connected components. Hence, X_i intersects L

in at least two points, for some connected component X_i. Thus, since L is a RL and X_i is arc-connected, X_i is orthogonal to ω in at least one point, contradiction. □

PROPOSITION 5.3. *Let $\mathbb{K}$ be definably complete. Then every FRL is a VRL.*

In particular, we recover Fratarcangeli's theorem as a special case of ours.

Proof of Proposition 5.3. Let L be a FRL with data (U, ω). Let $V \subseteq U$ be a weak cell and $X \subseteq V$ be a good twine in V, such that $|X \cap L| > \mathrm{vb}_V(X) =: m$ (for simplicity, we are dealing with the case $n = 0$ in Definition 3.9). By Lemma 3.7, X has at most m connected components; therefore, there exists Y component of X such that $|Y \cap X| \geq 2$. Thus, since L is a FRL, Y it orthogonal to ω at some point. □

REMARK 5.4. Notice that a priori it is not clear whether every RL is an FRL: consider an expansion of the real field and let L be a connected multi-leaf with data (U, ω); let X be 1-dimensional *definably connected but not topologically connected* manifold. Then it could happen that L meets two distinct connected components of X without X being anywhere orthogonal to ω.

There is the following question left. Let $\mathbb{K}$ be definably complete and Baire. Let $F : \mathbb{K}^n \to \mathbb{K}$ be a definable $\mathcal{C}^\infty$ function satisfying $\frac{\partial F}{\partial x_i} = g_i(x, F(x))$, for some $\mathcal{C}^\infty$ $\mathbb{K}_0$-definable function $g_i : \mathbb{K}^n \to \mathbb{K}$. Let $\mathbb{K}_0(F)$ be the expansion of $\mathbb{K}_0$ by F. Is $\mathbb{K}_0(F)$ o-minimal? Let C be the graph of F, and ω be the 1-form on $U := \mathbb{K}^{n+1}$ $g_1 \, \mathrm{d}x_1 + \cdots + g_n \, \mathrm{d}x_n - \mathrm{d}y$. Notice that C is a connected leaf with data (U, ω). The question has positive answer if C is a VRL. We don't know if this is true, but, since being a VRL is a first-order condition, we can add either the condition "C is VRL" to the axioms of $\mathbb{K}_0(F)$, or we can add the condition "every graph of a Pfaffian function is a VRL" to the axioms of $\mathbb{K}$. In both ways, we obtain an axiomatization of $\mathbb{K}_0(F)$ that ensures o-minimality.

6. Effective bounds

In this section, we apply our results to derive uniform and effective bounds on some topological invariants (e.g., the number of connected components) of sets definable in the Pfaffian closure of an o-minimal expansion of the real field.

Let T_0 be a recursively axiomatized (not necessarily complete) o-minimal theory (if T_0 is not recursively axiomatized, then the effective results of these section are still valid with respect to an oracle for T_0). Let $\mathbb{R}_0$ be an o-minimal expansion of the real field, which is a model of T_0 and let $\mathcal{P}(\mathbb{R}_0)$ be the Pfaffian closure of $\mathbb{R}_0$ (in the sense of [Spe99]).

DEFINITION 6.1. Let $X \subseteq \mathbb{R}^n$ be definable in $\mathcal{P}(\mathbb{R}_0)$. We call the *topological complexity of* X (denoted by t.c.(X)) the least $N \in \mathbb{N}$, such that there exist:

(1) a simplicial complex Z composed of fewer than N simplices, each of dimension at most N;

(2) and a $\mathcal{P}(\mathbb{R}_0)$-definable homeomorphism $f : X \approx |Z|$.

Note that, since $\mathcal{P}(\mathbb{R}_0)$ is o-minimal, the topological complexity is a well defined natural number [vdD98, Theorem 8.1.7 and ¶9.2.1].

Let X be defined by a formula φ, where some of the variables are evaluated as a suitable tuple of parameters. This definition will involve a finite number of Rolle Leaves $L_1, \dots, L_k$. As one can see from the inductive definition of Pfaffian closure, every leaf L_i will have data (U_i, ω_i) definable (by a formula ϕ_i, where some of the variables are evaluated as a suitable tuple of parameters) in terms of a finite number of Rolle Leaves $L_{i,1}, \dots, L_{i,n_i}$ of lower complexity (i.e., appearing at some earlier stage of the inductive construction). Hence, to the set X (or better, to its definition φ) we can associate a finite sequence $F_1 = L_1, \dots, F_k = L_k, F_{k+1} = L_{1,1}, \dots, F_{k+1+n_1} = L_{1,n_1}, \dots$ of Rolle Leaves, which are involved in its definition. The aim of the following definition is to code the set X by this sequence of leaves (cf. [Fra06], [GV04]).

DEFINITION 6.2. Let $\mathcal{L}(P)$ be the language of $\mathbb{R}_0$ to which we adjoin a countable set of new predicates $\{P_1, \dots, P_n, \dots\}$. A *format* of a definable set X is the following finite sequence of $\mathcal{L}(P)$-formulae (without parameters): $(\varphi, \mathbf{P}, \Phi)$, where

- for a suitable choice of parameters $\bar{a}$, the set X is defined by $\varphi(\cdot, \bar{a})$;
- $\mathbf{P} = (P_1, \dots, P_m)$ and every P_i represents a Rolle Leaf F_i involved in this definition of X;
- $\Phi = (\phi_1, \dots, \phi_m)$, where ϕ_i is an $\mathcal{L}(P_{i+1}, \dots, P_m)$-formula, and, for a suitable choice of parameters $\bar{a}_i$, the formula $\phi_i(\cdot, \bar{a}_i)$ defines the graph of ω_i on U_i, where (U_i, ω_i) is the data of the leaf F_i.

Note that in the definition only formulas without parameters appear. In particular, if ω is a definable form on a definable open set U, then every Rolle Leaf with data (U, ω) corresponds to the same format.

EXAMPLE 6.3. Let $X = L_1 \cup L_2$, where L_i are Rolle Leaves with data (U_i, ω_i). Let L_3 be a Rolle Leaf with $\mathbb{R}_0$-definable data (U_3, ω_3). Suppose (U_1, ω_1) are $\langle \mathbb{R}_0, L_3 \rangle$-definable and (U_2, ω_2) are $\mathbb{R}_0$-definable. Let the graphs of $\omega_1, \omega_2, \omega_3$ be defined by formulas $\phi_1(\bar{a}_1, \bar{x}, \bar{y}), \phi_2(\bar{a}_2, \bar{x}, \bar{y}), \phi_3(\bar{a}_3, \bar{x}, \bar{y})$ respectively, where $\bar{a}_1, \bar{a}_2, \bar{a}_3$ are tuples of parameters. Then a format for X is given by the $L_0 \cup \{P_1, P_2, P_3\}$-formulas $(\varphi, \mathbf{P}, \Phi)$, where $\varphi = P_1 \vee P_2; \mathbf{P} = (P_1, P_2, P_3); \Phi = (\phi_1, \phi_2, \phi_3)$.

The next definition requires the notion of Rolle Leaf to be first order. This is the reason why we introduced Virtual Rolle multi-Leaves: the property of being a VRL is type-definable, that is, it can be expressed by a countable (recursive) conjunction of first order formulae.

Definition 6.4. Let X be a definable set and $\theta = (\varphi, \mathbf{P}, \Phi)$ be a format for X. Let T_θ be the first order theory (in the language of $\mathbb{R}_0$ adjoined with the predicates $P_1, \dots, P_m$) with the following *recursive* (but not necessarily complete) axiomatization:

- Axioms of T_0;
- Axioms of Definably Complete Baire Structure;
- ϕ_i defines the graph of a nonsingular $\mathcal{C}^1$ 1-form ω_i on some definable open subset U_i;
- P_i is a VRL with data (U_i, ω_i).

We now show the existence of a bound on the topological complexity of X, which depends (recursively) only on a format for X.

Theorem 6.5. *There is a recursive function η which, given a set X definable in $\mathcal{P}(\mathbb{R}_0)$ and a format θ for X, returns a natural number $\eta(\theta)$ which is an upper bound on the topological complexity of X.*

Proof. Let X be a definable set and $\theta = (\varphi, \mathbf{P}, \Phi)$ be a format for X. Note that, by Theorem 2.6, the theory T_θ is o-minimal. In particular, there is a natural number N such that t.c.$(X) < N$. Moreover, T_θ is recursively enumerable, hence we can recursively enumerate all the formulas which (for every choice of the parameters) are provable in this theory. Take the first formula in this enumeration which defines a homeomorphism between the set defined by φ and some simplicial complex Z. Define $\eta(\theta)$ as the number of complexes which form Z. □

Corollary 6.6. *There are recursive bounds on the following topological invariants of sets definable in $\mathcal{P}(\mathbb{R}_0)$: number of connected components, sum of the Betti numbers, number of generators of the fundamental group.*

Proof. Let X be a set definable in $\mathcal{P}(\mathbb{R}_0)$ and N be the recursive bound on t.c.(X) given by the above theorem. Let F be a simplicial complex with at most N simplices, each of them of dimension at most N, such that $|F|$ is homeomorphic to X (F exists by definition of t.c.(X)). Clearly, the number of connected components of $|F|$ (and hence of X) is at most N.

If F were a *closed* complex, by classical algebraic topology theory, N would be also a bound for the other mentioned topological invariants. Otherwise, let F' be the barycentric subdivision of F. By [EW08, Lemma 7.1], there exists a closed simplicial complex C which is also a sub-complex of F', such that $|F'|$ (and hence X) is homotopic to $|C|$. Since C is a closed complex, the number m of simplices of C gives an upper bound to the sum of the Betti numbers of $|C|$ (and hence of X), and to the number of generators of $\pi_1(X, x_0)$ (for any $x_0 \in X$), and m is bounded by a recursive function of N. □

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