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MAHLER'S MEASURES ON FUNCTION SPACES

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ABSTRACT. Recently, I. Pritsker considered a Bergman-space version of Mahler's measure, and obtained many nice properties such as the arithmetic nature, relation with asymptotic zero distribution, etc. (*Illinois J. Math.* **52** (2009) 347–363). In this paper, we define a Fock-space analogue of Mahler's measure, and show a similar version of Lehmer's conjecture. Inspired by this result, we establish an equivalent form of Lehmer's conjecture. Also, this consideration is done on weighted Bergman spaces. However, it is shown that in this case the corresponding form of Lehmer's conjecture fails. In addition, we give an affirmative answer to an approximation question raised by I. Pritsker (*Illinois J. Math.* **52** (2009) 347–363).

1. Introduction

Let $\mathbb{C}[z]$ and $\mathbb{Z}[z]$ denote the set of all polynomials, with complex and integer coefficients, respectively. In [16], Mahler defined a quantity for polynomials, called Mahler's measure today, also see [5], [9], [14]. Precisely, for each polynomial $P \in \mathbb{C}[z]$, set

$$M(P) \triangleq \lim_{r \to 0^+} \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^r d\theta\right)^{\frac{1}{r}},$$

and it turns out that

$$M(P) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P\left(e^{i\theta}\right)\right| d\theta\right).$$

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Then $M(P)$ is called Mahler's measure. This definition can be naturally extended to analytic functions f defined on the unit disk $\mathbb{D}$, whose boundary values make sense a.e. such that $\log|f|$ is integrable on $\partial\mathbb{D}$. Below, P_n always denotes a polynomial with degree n. If we write

$$P_n(z) = a_n \prod_{j=1}^{n} (z - z_j),$$

then applying Jensen's equality shows that

$$M(P_n) = |a_n| \prod_{|z_j|\geq 1} |z_j|.$$

A well-known open problem related to Mahler's measure is known as Lehmer's conjecture, which says that there exists a numerical constant $\zeta_0 > 1$ such that for any $P_n \in \mathbb{Z}[z]$, either $M(P_n) = 1$ or $M(P_n) \geq \zeta_0$. In fact, from a result of Kronecker [15], if $M(P_n) = 1$ for a polynomial $P_n \in \mathbb{Z}[z]$, then either P_n or $-P_n$ is the product of a monomial and cyclotomic polynomials. An irreducible polynomial with integer coefficients is called *cyclotomic* if all the zeros are roots of unity. Therefore, Lehmer's conjecture is equivalent to that there exists a numerical constant $\zeta_0 > 1$ such that for any non-cyclotomic polynomials $P \in \mathbb{Z}[z]$ with $P(0) \neq 0$, $M(P) \geq \zeta_0$. This conjecture has many applications on various fields of mathematics, for example, transcendental number theory, ergodic theory, knot theory, and etc. For details, one may refer to [2], [10], [20], and see [3], [5], [6], [7], [17] for results on Mahler's measure for polynomials in several variables.

Recently, I. Pritsker considered a Bergman-space version of Mahler's measure, and deduced some of its properties. In this paper, those ideas are extended to Fock and other spaces. However, there are significant differences between these two spaces. Let $dA(z)$ denote the area measure over the complex plane, and we may defined a quantity $|P_n|_F$ related to the Gaussian measure $e^{-|z|^2}\frac{dA(z)}{\pi}$. It turns out that there is a numerical constant $c > 1$ such that for any $P_n \in \mathbb{Z}[z]$ with $P_n(0) \neq 0$, $|P_n|_F \geq c$. This, in some sense, shows that a Fock space version of Lehmer's conjecture holds. Furthermore, we give a family of function space versions of Mahler's measure. It is shown that those function space versions of Lehmer's conjecture are equivalent to Mahler's conjecture (see Section 4).

We also consider weighted-Bergman-space versions of Mahler's measure, and define $|P_n|_{\rho^*}$ for each nonzero polynomial, where ρ is related to the weight on these spaces. When $\rho = 1$, it is reduced to the case of Bergman space that was considered by I. Pritsker [18], where he denoted it by $\|P_n\|_0$. It was shown by an example that there exists a sequence of non-cyclotomic polynomials $\{P_n\}$ in $\mathbb{Z}[z]$ with $P_n(0) \neq 0$, satisfying $\lim_{n\to\infty} \|P_n\|_0 = 1$. Under a mild condition, this result is generalized.

In Section 2, we defined a version of Mahler's measure on the Fock space. For polynomials $P(z)$ with integer coefficients and $P(0)\neq 0$, it is shown that this quantity is always larger than a numerical constant $c>1$. In Section 3, we consider a family of heights of polynomials defined by normalized weighted area measures over the unit disk. However, in this case, it is shown that the corresponding Lehmer's conjecture fails. In Section 4 we define a family of function-space versions of Mahler's measure and shows that those versions of Lehmer's conjecture are equivalent to the original conjecture. In Section 5, we give an affirmative answer to an approximation question raised by I. Pritsker [18].

2. Mahler's measure on the Fock space

In this section, we will define a quantity over the Fock space and show that the Fock space version of Mahler's conjecture holds.

Recall that the Fock space $\mathcal{F}$ consists of all entire functions which are square integrable with respect to the Gaussian measure $e^{-|z|^2}\frac{dA(z)}{\pi}$. For each $f\in\mathcal{F}$, its norm is defined by

$$\|f\|=\left(\int_{\mathbb{C}}\big|f(z)\big|^2 e^{-|z|^2}\,\frac{dA(z)}{\pi}\right)^{\frac{1}{2}}.$$

In general, the Fock space $\mathcal{F}_t$ $(0<t<\infty)$ consists of all entire functions f such that

$$\|f\|_t\triangleq\left(\int_{\mathbb{C}}\big|f(z)\big|^t e^{-|z|^2}\,\frac{dA(z)}{\pi}\right)^{\frac{1}{t}}<+\infty.$$

Fix $f\in\bigcap_{t>0}\mathcal{F}_t$, and it is easy to see that $\|f\|_t$ is increasing. Therefore, $\lim_{t\to 0^+}\|f\|_t$ always exists. In particular, if $f=P_n$ is a polynomial, then it turns out

$$\lim_{t\to 0^+}\|P_n\|_t=\exp\left(\int_{\mathbb{C}}\log\big|P_n(z)\big|e^{-|z|^2}\,\frac{dA(z)}{\pi}\right).$$

Now we define

$$|P_n|_F=\exp\left(\int_{\mathbb{C}}\log\big|P_n(z)\big|e^{-|z|^2}\,\frac{dA(z)}{\pi}\right)\geq\big|P_n(0)\big|, \tag{1}$$

and put

$$S(x)=\int_1^\infty\frac{e^{-tx^2}}{t}\,dt,\quad x>0.$$

It turns out that the function S has some relation with the Gamma function Γ, defined by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt,\quad \operatorname{Re}z>0,$$

which can be analytically extended to the whole complex plane except for $0, -1, -2, \ldots$, see [1], [8], [21], for example. In fact, the following identity is known:

$$\int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_1^\infty \frac{e^{-t}}{t}\,dt = -\Gamma'(1) = \gamma, \tag{2}$$

where γ is the gamma constant, that is,

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right) = 0.57721\ldots.$$

By (2), $S(1) = \int_0^1 \frac{1-e^{-t}}{t}\,dt - \gamma$. Therefore, expanding e^{-t} shows that

$$\begin{aligned} S(1) &= \int_0^1 \frac{1-\sum_{n=0}^\infty \frac{(-1)^n}{n!}t^n}{t}\,dt - \gamma \\ &= \sum_{n=1}^\infty (-1)^{n-1}\frac{1}{n!n} - \gamma. \end{aligned}$$

The constant $S(1)$ has a close relation with $|P_n|_F$, whose explicit form is given as follows.

THEOREM 1. *If $P_n \in \mathbb{C}[z]$ and $z_1, \ldots, z_n$ are its zeros, then*

$$\begin{aligned} |P_n|_F = M(P_n)&\exp\left(\frac{1}{2}\sum_{|z_j|\ge 1} S\big(|z_j|\big)\right) \\ &\times \exp\left(\frac{1}{2}\sum_{|z_j|<1}\left(-\int_{|z_j|^2}^1 \frac{1-e^{-r}}{r}\,dr + S(1)\right)\right). \end{aligned}$$

By (2), the above formula for $|P_n|_F$ can also be written as

$$|P_n|_F = M(P_n)\exp\left(\frac{1}{2}\sum_{|z_j|\ge 1} S\big(|z_j|\big)\right)\exp\left(\frac{1}{2}\sum_{|z_j|<1}\left(\int_0^{|z_j|^2} \frac{1-e^{-r}}{r}\,dr - \gamma\right)\right).$$

This theorem has the following consequence.

COROLLARY 2. *For any nonconstant polynomial $P_n \in \mathbb{Z}[z]$ with $P_n(0) \neq 0$, we have $|P_n|_F \ge \exp(\frac{S(1)}{2}) = 1.115\ldots$. Moreover, the smallest value $\exp(\frac{S(1)}{2})$ is attained exactly by the followings: $z+1$, $z-1$, $-z+1$ and $-z-1$.*

This corollary can be viewed as an affirmative answer to the Fock space version of Lehmer's conjecture. Also, notice that the constant $\exp(\frac{S(1)}{2})$ is less than the smallest known Mahler measure $M(L) = 1.17628\ldots$, where $L(z) = z^{10}+z^9-z^7-z^6-z^5-z^3-z^4+z+1$ [13].

Proof. For a polynomial P_n with

$$(3)\quad \begin{aligned} P_n(z) &= \sum_{i=0}^{n} a_i z^i = a_n \prod_{i=1}^{n}(z-z_i),\\ M(P_n) &= |a_n| \prod_{|z_j|\geq 1} |z_j| = |a_0| \prod_{|z_j|\leq 1} \frac{1}{|z_j|}. \end{aligned}$$

There are two cases under consideration.

Case I. There is an i_0 such that $|z_{i_0}| < 1$. In this case, notice that the function $g(x) = \frac{e^{\frac{x^2-1}{2}}}{x}$ is always larger than 1 for $0 < x < 1$ [18]. Then by Theorem 1 and (3), we have

$$|P_n|_F \geq |a_0| \prod_{|z_j|<1} \left(\frac{\exp(\frac{|z_j|^2-1}{2})}{|z_j|} \exp\left(\frac{S(1)}{2}\right)\right) \geq \exp\left(\frac{S(1)}{2}\right).$$

Case II. There is no z_j such that $|z_j| < 1$. From the identity

$$a_0 = (-1)^n a_n \prod_{i=1}^{n} z_n,$$

it is easy to see that either all $|z_j| = 1$ or $|a_0| \geq 2$. Then again by Theorem 1 and (3),

$$|P_n|_F = M(P_n) \exp\left(\frac{1}{2} \sum_{|z_j|\geq 1} S\big(|z_j|\big)\right) \geq \min\left\{\exp\left(\frac{S(1)}{2}\right), 2\right\} = \exp\left(\frac{S(1)}{2}\right).$$

The remaining follows from a close look at the above discussion. The proof is complete. □

Remark 1. Furthermore, the constant $\exp(\frac{S(1)}{2})$ is not a limit point of

$$\big\{|P_n|_F; P_n \in \mathbb{Z}[z] \text{ and } P_n \text{ is a nonconstant polynomial satisfying } P_n(0) \neq 0\big\}.$$

To see this, it is enough to show that for any P_n having at least two zeros, $|P_n|_F$ is away from $\exp(\frac{S(1)}{2})$. Fix a constant a ($a > 1$) and write

$$P_n(z) = \sum_{i=0}^{n} a_i z^i = a_n \prod_{i=1}^{n}(z-z_i).$$

Combining Theorem 1 with (3) shows that

$$|P_n|_F \geq |a_0| \prod_{|z_j|<1} \left(\frac{\exp(\frac{|z_j|^2-1}{2})}{|z_j|} \exp\left(\frac{S(1)}{2}\right)\right) \prod_{|z_j|\geq 1} \exp\left(\frac{S(|z_j|)}{2}\right).$$

We may assume that $|a_0| = 1$; otherwise $|P_n|_F \geq |a_0| \geq 2 > \exp(\frac{S(1)}{2})$. There are two cases under consideration.

Case I. There is an i_0 such that $|z_{i_0}|<1$. We may even assume that z_{i_0} is the only z_j satisfying $|z_j|<1$, since otherwise we would have

$$|P_n|_F \ge \prod_{|z_j|<1}\left(\frac{\exp(\frac{|z_j|^2-1}{2})}{|z_j|}\exp\left(\frac{S(1)}{2}\right)\right) \ge \exp\bigl(S(1)\bigr).$$

Now either $|z_{i_0}|>\frac{1}{a}$ or $|z_{i_0}|\le\frac{1}{a}$. If $|z_{i_0}|>\frac{1}{a}$, then by the identity

$$1=|a_0|=\left|(-1)^n a_n\prod_{i=1}^{n} z_n\right|,$$

we have $|z_j|\le a$ for all $|z_j|\ge 1$. Therefore,

$$\begin{aligned}|P_n|_F &\ge \prod_{|z_j|<1}\left(\frac{\exp(\frac{|z_j|^2-1}{2})}{|z_j|}\exp\left(\frac{S(1)}{2}\right)\right)\prod_{|z_j|\ge 1}\exp\left(\frac{S(|z_j|)}{2}\right)\\ &\ge \exp\left(\frac{S(1)}{2}\right)\exp\left(\frac{S(a)}{2}\right).\end{aligned}$$

Otherwise, we have $|z_{i_0}|\le\frac{1}{a}$, forcing

$$|P_n|_F \ge \frac{\exp(\frac{|\frac{1}{a}|^2-1}{2})}{|\frac{1}{a}|}\exp\left(\frac{S(1)}{2}\right).$$

Case II. There is no z_j such that $|z_j|<1$. By the proof of Corollary 2, all $|z_j|=1$. Therefore $|P_n|_F\ge\min\{2,\exp(S(1))\}$.

Now put

$$b=\min\left\{\frac{\exp(\frac{|\frac{1}{a}|^2-1}{2})}{|\frac{1}{a}|},\exp\left(\frac{S(a)}{2}\right)\right\}>1,$$

and we have

$$|P_n|_F\ge\min\left\{2,b\exp\left(\frac{S(1)}{2}\right)\right\}>\exp\left(\frac{S(1)}{2}\right)$$

for any P_n ($P_n\in\mathbb{Z}[z]$) having at least two zeros. Therefore, $\exp(\frac{S(1)}{2})$ is not a limit point of

$$\bigl\{|P_n|_F; P_n\in\mathbb{Z}[z] \text{ and } P_n \text{ is a nonconstant polynomial satisfying } P_n(0)\ne 0\bigr\}.$$

To close this section, we give the proof of Theorem 1.

Proof of Theorem 1. Assume that P_n is a nonzero polynomial and $z_1,\dots,$ z_n are its zeros. By (1), we have

$$\log|P_n|_F = 2\int_0^\infty e^{-r^2}r\left(\frac{1}{2\pi}\int_0^{2\pi}\log\bigl|P_n\bigl(re^{i\theta}\bigr)\bigr|\,d\theta\right)dr.$$

Applying Jensen's formula, we get

$$\begin{aligned}\log|P_n|_F &= \int_0^\infty \Big(\log|a_n| + \sum_{|z_j|\geq r} \log|z_j| + \sum_{|z_j|<r} \log r\Big) e^{-r^2}\, dr^2 \\ &= \log|a_n| + \sum_{j=1}^n \Big(\log|z_j| + \frac{1}{2}\int_{|z_j|^2}^\infty \big(\log r - \log|z_j|^2\big) e^{-r}\, dr\Big).\end{aligned}$$

Then write

$$\begin{aligned}I\big(|z_j|\big) &= \log|z_j| + \frac{1}{2}\int_{|z_j|^2}^\infty \big(\log r - \log|z_j|^2\big) e^{-r}\, dr \\ &= \log|z_j| + \frac{1}{2}\int_{|z_j|^2}^\infty \frac{e^{-r}}{r}\, dr \\ &= \log|z_j| + \frac{1}{2} S\big(|z_j|\big).\end{aligned} \tag{4}$$

When $|z_j| < 1$,

$$\begin{aligned}I\big(|z_j|\big) &= \frac{1}{2}\log|z_j|^2 + \frac{1}{2}\Big(\int_{|z_j|^2}^1 \frac{e^{-r}}{r}\, dr + \int_1^\infty \frac{e^{-r}}{r}\, dr\Big) \\ &= \frac{1}{2}\Big(-\int_{|z_j|^2}^1 \frac{1-e^{-r}}{r}\, dr + \int_1^\infty \frac{e^{-r}}{r}\, dr\Big) \\ &= \frac{1}{2}\Big(-\int_{|z_j|^2}^1 \frac{1-e^{-r}}{r}\, dr + S(1)\Big).\end{aligned} \tag{5}$$

Combining (3) and (4) with (5), we get

$$\begin{aligned}|P_n|_F &= |a_n| \prod_{|z_j|\geq 1} \Big(|z_j| \exp\Big(\frac{S(|z_j|)}{2}\Big)\Big) \\ &\quad \times \exp\Big(\frac{1}{2}\sum_{|z_j|<1}\Big(-\int_{|z_j|^2}^1 \frac{1-e^{-r}}{r}\, dr + S(1)\Big)\Big) \\ &= M(P_n) \exp\Big(\frac{1}{2}\sum_{|z_j|\geq 1} S\big(|z_j|\big)\Big) \\ &\quad \times \exp\Big(\frac{1}{2}\sum_{|z_j|<1}\Big(-\int_{|z_j|^2}^1 \frac{1-e^{-r}}{r}\, dr + S(1)\Big)\Big).\end{aligned}$$

The proof is complete. □

REMARK 2. Recently, I. Pritsker studied an areal analog $\|P_n\|_0$ for Mahler measure, see [18], [19]. Given a polynomial $P_n \in \mathbb{C}[z]$ with zeros z_j, it turns out that

$$\|P_n\|_0 = M(P_n)\exp\biggl(\sum_{|z_j|\le 1}\frac{|z_j|^2-1}{2}\biggr),$$

and then it is easy to see that $\|P_n\|_0 \le |P_n|_F$. From this, one may check that almost all results for $\|P_n\|_0$ in Sections 2 and 3 in [18] remain valid for $|P_n|_F$.

3. Counterparts of Mahler's measure defined over the unit disk

In this section, we consider a family of heights of polynomials defined by normalized weighted area measures over the unit disk. However, it is shown that the corresponding Lehmer's conjecture fails.

Recently, on the Bergman space, I. Pritsker defined a quantity $\|P_n\|_0$ for polynomials P_n. Precisely,

$$\|P_n\|_0 = \exp\biggl(\int_{\mathbb{D}} \log\bigl|P_n(z)\bigr|\frac{dA(z)}{\pi}\biggr),$$

and it turns out that

$$\|P_n\|_0 = M(P_n)\exp\biggl(-\frac{1}{2}\sum_{|z_j|\le 1}\int_{|z_j|^2}^1 dr\biggr).$$

In [19], it was shown that there is a sequence of non-cyclotomic polynomials $P_n \in \mathbb{Z}[z]$ with $P_n(0)\ne 0$, satisfying

$$\lim_{n\to\infty}\|P_n\|_0 = 1.$$

In this section, we will show that this also happens in the case of weighted Bergman spaces, see Corollary 4.

Now we assume that τ *is a nonnegative continuous function over* $[0,1)$ *satisfying* $\lim_{r\to 1^-}\tau(r)=1$. Define

$$|P_n|_\tau = M(P_n)\exp\biggl(-\frac{1}{2}\sum_{|z_j|<1}\int_{|z_j|^2}^1 \tau(r)\,dr\biggr), \tag{6}$$

where z_j are all zeros of P_n with $|z_j|<1$. We also *assume that for any polynomial* $P_n\in\mathbb{Z}[z]$ *with* $P_n(0)\ne 0$, $|P_n|_\tau \ge 1$. In many cases, these assumptions hold.

Example 1. It is natural to extend the quantity $\|P_n\|_0$ to the case of weighted Bergman space. Let ρ be a continuous positive function on $[0,1)$, and denote $L_a^2(\mathbb{D},\rho(|z|^2)\,dA(z))$ the weighted Bergman space consisting of all holomorphic functions over $\mathbb{D}$, which are square integrable with respect to $\rho(|z|^2)\,dA(z)$. For each polynomial $P_n \in \mathbb{C}[z]$, its norm is defined by

$$\|P_n\|_\rho \triangleq \left(\int_{\mathbb{D}} \bigl|P_n(z)\bigr|^2 \rho\bigl(|z|^2\bigr)\frac{dA(z)}{\pi}\right)^{\frac{1}{2}}.$$

We also assume that $\int_0^1 \rho(r)\,dr = 1$, which is the normalizing condition; that is, $\|1\|_\rho = 1$.

We define

$$|P_n|_{\rho^*} = \exp\left(\int_{\mathbb{D}} \log\bigl|P_n(z)\bigr|\rho\bigl(|z|^2\bigr)\frac{dA(z)}{\pi}\right).$$

Then *for each* $P_n \in \mathbb{Z}[z]$ *with* $P_n(0) \neq 0$, $|P_n|_{\rho^*} \geq 1$. The reasoning is as follows. Since $|P_n(z)|^t$ is subharmonic for each $t > 0$, it follows that

$$\int_{\mathbb{D}} \bigl|P_n(z)\bigr|^t \rho\bigl(|z|^2\bigr)\frac{dA(z)}{\pi} \geq \bigl|P_n(0)\bigr| \geq 1,$$

and hence

$$|P_n|_{\rho^*} = \lim_{t\to 0^+}\left(\int_{\mathbb{D}} \bigl|P_n(z)\bigr|^t \rho\bigl(|z|^2\bigr)\frac{dA(z)}{\pi}\right)^{\frac{1}{t}} \geq 1,$$

as desired.

Now define $F(r) = \int_0^r \rho(t)\,dt$, and it is clear that $F(0) = 0$ and $F(1) = 1$. Set $\tau(r) = \frac{F(r)}{r}, 0 < r < 1; \tau(0) = \rho(0)$, and put

$$G(x) = \exp\left(-\frac{1}{2}\int_{x^2}^1 \tau(r)\,dr\right).$$

We will see that

$$|P_n|_{\rho^*} = M(P_n)\prod_{|z_j|<1} G\bigl(|z_j|\bigr),$$

where z_j are zeros of P_n with $|z_j| < 1$. Then $|P_n|_{\rho^*}$ is a special case of $|P_n|_\tau$ since $\tau(1) = 1$.

The reasoning is as follows. For a polynomial P_n with

$$P_n(z) = \sum_{i=0}^n a_i z^i = a_n \prod_{i=1}^n (z - z_i),$$

we have

$$\begin{aligned}
\log|P_n|_{\rho^*} &= \frac{1}{\pi}\int_0^1 \rho\bigl(r^2\bigr) r\,dr \int_0^{2\pi} \log\bigl|P_n\bigl(re^{i\theta}\bigr)\bigr|\,d\theta \\
&= 2\int_0^1 \rho\bigl(r^2\bigr) r\,dr \frac{1}{2\pi}\int_0^{2\pi} \log\bigl|P_n\bigl(re^{i\theta}\bigr)\bigr|\,d\theta \\
&= \int_0^1 \left(\log|a_n| + \sum_{|z_j|\geq r}\log|z_j| + \sum_{|z_j|<r}\log r\right)\rho\bigl(r^2\bigr)2r\,dr
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\log|a_n| + \sum_{|z_j|\ge 1} \log|z_j|\Big) \\
&\quad + 2\sum_{|z_j|<r}\Big(\int_0^{|z_j|} \log|z_j|\rho\big(r^2\big)r\,dr + \int_{|z_j|}^1 \rho\big(r^2\big)r\log r\,dr\Big) \\
&\equiv \Big(\log|a_n| + \sum_{|z_j|\ge 1} \log|z_j|\Big) + \mathcal{I}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathcal{I} &= \sum_{|z_j|<r}\Big(\int_0^{|z_j|^2} \log|z_j|\rho(r)\,dr + \frac{1}{2}\int_{|z_j|^2}^1 \rho(r)\log r\,dr\Big) \\
&= \sum_{|z_j|<r}\Big(\log|z_j|F\big(|z_j|^2\big) + \frac{1}{2}\int_{|z_j|^2}^1 \log r\,dF(r)\Big) \\
&= -\sum_{|z_j|<r}\Big(\frac{1}{2}\int_{|z_j|^2}^1 \frac{F(r)}{r}\,dr\Big),
\end{aligned}
$$

then

$$
\begin{aligned}
\log|P_n|_{\rho^*} = &\Big(\log|a_n| + \sum_{|z_j|\ge 1} \log|z_j|\Big) \\
&- \sum_{|z_j|<r}\Big(\frac{1}{2}\int_{|z_j|^2}^1 \frac{F(r)}{r}\,dr\Big).
\end{aligned}
$$

Therefore, $|P_n|_{\rho^*} = M(P_n)\prod_{|z_j|<1} G(|z_j|)$, as desired.

We have the following proposition.

PROPOSITION 3. *Suppose that τ is a nonnegative continuous function over $[0,1)$ satisfying $\lim_{r\to 1^-}\tau(r)=1$, and $|P_n|_\tau$ is defined by (6). Then there is a sequence of non-cyclotomic polynomials $\{P_n\}$ in $\mathbb{Z}[z]$ with $P_n(0)\neq 0$, satisfying*

$$\lim_{n\to\infty} |P_n|_\tau = 1.$$

COROLLARY 4. *Let $|P_n|_{\rho^*}$ be defined as in Example 1. Then there is a sequence of non-cyclotomic polynomials $\{P_n\}$ in $\mathbb{Z}[z]$ with $P_n(0)\neq 0$, satisfying*

$$\lim_{n\to\infty} |P_n|_{\rho^*} = 1.$$

As done in Example 1, write

$$G(x) = \exp\Big(-\frac{1}{2}\int_{x^2}^1 \tau(r)\,dr\Big),$$

and then by definition we rewrite

$$|P_n|_\tau = M(P_n) \prod_{|z_j|<1} G\big(|z_j|\big), \tag{7}$$

where z_j are zeros of P_n with $|z_j| < 1$. Since $\tau(r) \to 1$ $(r \to 1^-)$, there is a positive infinitesimals $o(1)$ such that $\tau(r) \geq 1 - o(1)$. For example, we may choose $o(1)$ to be $|1 - \tau(r)|$.

Now we are ready to give the proof of Proposition 3.

Proof of Proposition 3. Now put $P_n(z) = z^n + z + 1$, and it is known that each P_n $(n > 2)$ is not cyclotomic [10, p. 78, Exercise 3.12]. To prove Proposition 3, it suffices to show that $\lim_{n\to\infty} |P_n|_\tau = 1$. First, observe that $\{M(P_n)\}$ is bounded since $|P_n| \leq 3$ on the unit circle. In fact,

$$\lim_{n\to\infty} M(P_n) = 1.38135\ldots,$$

see [4], [20]. Also, we will see that for this sequence $\{P_n\}$,

$$\lim_{n\to\infty} \min\big\{|z| < 1 : P_n(z) = 0\big\} = 1.$$

To see this, suppose conversely that there is a subsequence $\{P_{n_k}\}$ and $\{w_k\} \subseteq \mathbb{D}$ such that $P_{n_k}(w_k) = 0$ and

$$\lim_{k\to\infty} |w_k| = a \quad \text{with } 0 \leq a < 1.$$

However, we have $|P_{n_k}(w_k)| \geq 1 - |w_k| - |w_k|^{n_k}$, forcing

$$\limsup_{k\to\infty} \big|P_{n_k}(w_k)\big| \geq 1 - a.$$

This is a contradiction to the assumption $P_{n_k}(w_k) = 0$.

For each fixed P_n, let $\{z_j\}$ be its zeros satisfying $|z_j| < 1$, and write $|z_j| = 1 - \varepsilon_j$. Since

$$\lim_{n\to\infty} \min\big\{|z| < 1 : P_n(z) = 0\big\} = 1,$$

these ε_j tends to zero uniformly as n tends to infinity. We assume that ε_j are small enough as required. By the definition of G and our convention below Corollary 4, there is a positive infinitesimal $o(1)$ such that

$$\frac{G(|z_j|)}{|z_j|} \leq \frac{\exp(-\frac{1}{2}\int_{|z_j|^2}^1 (1 - o(1))\, dr)}{1 - \varepsilon_j} \tag{8}$$

as n tends to ∞. Here and below, $o(1)$ always denotes a positive infinitesimal, which may differs in the context. Roughly speaking, we assume that it is enough small.

Noticing that by Taylor's expansion of e^{-x}, for any enough small x $(x > 0)$,

$$e^{-x} \leq 1 - x + \frac{x^2}{2},$$

and that

$$
\begin{aligned}
-\frac{1}{2}\int_{|z_j|^2}^{1}\bigl(1-o(1)\bigr)\,dr &= -\frac{1}{2}\int_{(1-\varepsilon_j)^2}^{1}\bigl(1-o(1)\bigr)\,dr \\
&= -\left(\varepsilon_j-\frac{\varepsilon_j^2}{2}\right)\bigl(1-o(1)\bigr) \\
&\le \bigl(-1+o(1)\bigr)\varepsilon_j,
\end{aligned}
$$

we get

$$
\exp\left(-\frac{1}{2}\int_{|z_j|^2}^{1}\bigl(1-o(1)\bigr)\right)dr \le 1-\varepsilon_j+o(1)\varepsilon_j, \tag{9}
$$

which, combined with (8), shows that

$$
\frac{G(z_j)}{|z_j|} \le \frac{1-\varepsilon_j+o(1)\varepsilon_j}{1-\varepsilon_j} \le 1+o(1)\varepsilon_j. \tag{10}
$$

Since there exists a constant $M>0$ such that $M(P_n)\le M\ (n\ge 1)$, we have

$$
\prod_{|z_j|<1}\frac{1}{|z_j|}\le M;
$$

that is,

$$
\prod_{|z_j|<1}\frac{1}{1-\varepsilon_j}\le M.
$$

Therefore,

$$
-\sum_{|z_j|<1}\log(1-\varepsilon_j)\le \log M. \tag{11}
$$

Now fix $\delta>0$. When n is sufficiently large, for each j with $|z_j|<1$, ε_j are small enough to satisfying

$$
\varepsilon_j \le -2\log(1-\varepsilon_j).
$$

This, combined with (11), immediately gives that

$$
\begin{aligned}
\sum_{|z_j|<1}\log\bigl(1+o(1)\varepsilon_j\bigr) \le \sum_{|z_j|<1}\log(1+\delta\varepsilon_j) &\le \sum_{|z_j|<1}\delta\varepsilon_j \\
&\le -2\delta\sum_{|z_j|<1}\log(1-\varepsilon_j)\le 2\delta\log M.
\end{aligned}
$$

Then by (10),

$$
\prod_{|z_j|<1}\frac{G(|z_j|)}{|z_j|}\le \prod_{|z_j|<1}\bigl(1+o(1)\varepsilon_j\bigr)\le M^{2\delta}.
$$

On the other hand, by (7) and (3), $|P_n|_\tau = \prod_{|z_j|<1} \frac{G(z_j)}{|z_j|}$. Thus

$$\limsup_{n\to\infty} |P_n|_\tau \le M^{2\delta},$$

and by the arbitrariness of δ, we have

$$\limsup_{n\to\infty} |P_n|_\tau \le 1.$$

By our assumption, $|P_n|_\tau \ge 1$ for any $P_n \in \mathbb{Z}[z]$ satisfying $P_n(0) \ne 0$. Therefore,

$$\liminf_{n\to\infty} |P_n|_\tau = 1,$$

and hence

$$\lim_{n\to\infty} |P_n|_\tau = 1.$$

The proof is complete. □

4. An equivalent version for Lehmer's conjecture

In this section, we will establish an equivalent version for Lehmer's conjecture.

We adopt the notation of Section 3. Define

$$|P_n|_\tau = M(P_n) \exp\left(-\frac{1}{2} \sum_{|z_j|<1} \int_{|z_j|^2}^1 \tau(r)\, dr\right),$$

where z_j are all zeros of P_n with $|z_j| < 1$. However, in this section, *we assume that τ is a continuous function on $[0,1)$ satisfying $0 \le \tau \le 1$ and*

$$\limsup_{r\to 1^-} \tau(r) < 1.$$

Clearly, $|P_n|_\tau \le M(P_n)$.

Recently, on the Bergman space, I. Pritsker defined a quantity $\|P_n\|_0$ for polynomials P_n, which turns out to be

$$\|P_n\|_0 = M(P_n) \exp\left(-\frac{1}{2} \sum_{|z_j|\le 1} \int_{|z_j|^2}^1 dr\right).$$

Clearly, $\|P_n\|_0 \le |P_n|_\tau$, and then for each $P_n \in \mathbb{Z}[z]$ with $P_n(0) \ne 0$, $|P_n|_\tau \ge \|P_n\|_0 \ge 1$.

First, let us see an example.

EXAMPLE 2. In Section 2, for each nonzero polynomial $P_n \in \mathbb{C}[z]$, we have defined $|P_n|_F$, which turns out to be

$$|P_n|_F = M(P_n) \exp\left(\frac{1}{2} \sum_{|z_j|\ge 1} S\big(|z_j|\big)\right) \times \exp\left(\frac{1}{2} \sum_{|z_j|<1} \left(-\int_{|z_j|^2}^1 \frac{1-e^{-r}}{r}\, dr + S(1)\right)\right).$$

The above quantity $|P_n|_F$ induces another one, say

$$|P_n|'_F \triangleq M(P_n)\exp\left(-\frac{1}{2}\sum_{|z_j|<1}\int_{|z_j|^2}^{1}\frac{1-e^{-r}}{r}\,dr\right).$$

Since $0<1-e^{-r}\le r$ for $0<r\le 1$, we have $0\le \frac{1-e^{-r}}{r}\le 1$. Also, notice that $\frac{1-e^{-r}}{r}|_{r=1}<1$. Therefore, $|P_n|'_F$ is a quantity as desired.

Here, we will establish a statement that is equivalent to Lehmer's conjecture, which says that there is a constant ζ_0 ($\zeta_0>1$) such that for each $P_n\in\mathbb{Z}[z]$, either $M(P_n)=1$ or $M(P_n)\ge\zeta_0$. As mentioned in the Introduction, Lehmer's conjecture is equivalent to saying that there is a constant ζ_0 ($\zeta_0>1$) such that for each non-cyclotomic polynomial P_n in $\mathbb{Z}[z]$, $M(P_n)\ge\zeta_0$.

The following theorem gives an equivalent version for Lehmer's conjecture, which can be also regarded as the counterpart of Proposition 3.

THEOREM 5. *Suppose τ is a continuous function on $[0,1)$ satisfying $0\le\tau\le 1$ and*

$$\limsup_{r\to 1^-}\tau(r)<1.$$

Then the followings are equivalent:

(i) *Lehmer's conjecture holds;*

(ii) *There exists a constant c ($c>1$) such that for any non-cylotomic $P_n\in\mathbb{Z}[z]$ with $P_n(0)\ne 0$, $|P_n|_\tau\ge c$.*

To prove this theorem, we need to establish a lemma. Now define

$$h(x)=\frac{\exp(-\frac{1}{2}\int_{x^2}^{1}\tau(r)\,dr)}{x},\quad x>0. \tag{12}$$

By a simple calculation, it is easy to see that $h'(x)<0$ for $0<x<1$, and hence $h(x)\ge 1$ $(0<x\le 1)$.

LEMMA 6. *Given a constant $c_0>1$, there exist constants s and c with $0<s<1<c$ and satisfying the following: for each finite sequence $\{x_i\}_{i=1}^n$ with each $x_i\in[s,1)$, if $\prod_{i=1}^n\frac{1}{x_i}\ge c_0>1$, then*

$$\prod_{i=1}^n h(x_i)\ge c.$$

Proof. By our previous assumption,

$$\limsup_{r\to 1^-}\tau(r)<1.$$

So there are constants $a,s\in(0,1)$ such that $\tau(r)\le a$ for $s^2\le r<1$. Also, we assume that $1-s$ is small enough such that the followings hold for each y satisfying $0<y\le 1-s$:

$$e^{-y}\ge 1-y,\qquad 2y\ge-\log(1-y)\quad\text{and}\quad\log(1+y)\ge\frac{y}{2}. \tag{13}$$

Therefore, we have

$$\exp\left(-\frac{1}{2}\int_{x^2}^{1}\tau(r)\,dr\right) \geq \exp\left(-\frac{1}{2}a\big(1-x^2\big)\right) \geq 1-\frac{a}{2}\big(1-x^2\big), \tag{14}$$

where $s \leq x < 1$.

For a given sequence $\{x_i\}_{i=1}^{n}$ which satisfies $\prod_{i=1}^{n}\frac{1}{x_i} \geq c_0 > 1$ and $s \leq x_i < 1$ $(1 \leq i \leq n)$, write $\varepsilon_i = 1 - x_i$. Then by (12) and (14), we have

$$\begin{aligned}
\prod_{i=1}^{n} h(x_i) &\geq \prod_{i=1}^{n}\frac{1-\frac{a}{2}(1-x_i^2)}{x_i} \\
&\geq \prod_{i=1}^{n}\frac{1-a(1-x_i)}{x_i} \\
&= \prod_{i=1}^{n}\frac{1-a\varepsilon_i}{1-\varepsilon_i} \\
&\geq \prod_{i=1}^{n}\big(1+a'\varepsilon_i\big),
\end{aligned} \tag{15}$$

where $a' = 1 - a > 0$. Since $s \leq x_i < 1$, i.e. $0 < \varepsilon_i \leq 1 - s$, then by (13)

$$2\varepsilon_i \geq -\log(1-\varepsilon_i)$$

and

$$\log\big(1+a'\varepsilon_i\big) \geq \frac{a'}{2}\varepsilon_i.$$

Therefore,

$$\sum_{i=1}^{n}\log\big(1+a'\varepsilon_i\big) \geq \frac{a'}{2}\sum_{i=1}^{n}\varepsilon_i \geq \frac{a'}{4}\sum_{i=1}^{n}-\log(1-\varepsilon_i),$$

which, combined with the inequality

$$\prod_{i=1}^{n}\frac{1}{1-\varepsilon_i} = \prod_{i=1}^{n}\frac{1}{x_i} \geq c_0,$$

gives that

$$\prod_{i=1}^{n}\big(1+a'\varepsilon_i\big) \geq \left(\prod_{i=1}^{n}\frac{1}{1-\varepsilon_i}\right)^{\frac{a'}{4}} \geq c_0^{\frac{a'}{4}} > 1.$$

Then letting $c = c_0^{\frac{a'}{4}}$, by (15) we get $\prod_{i=1}^{n} h(x_i) \geq c$, as desired. □

Now we are ready to prove Theorem 5.

Proof of Theorem 5. Since for each polynomial P_n, we have $|P_n|_\tau \le M(P_n)$, (ii) $\Rightarrow$ (i) is straightforward. It remains to consider (i) $\Rightarrow$ (ii). For this, suppose $P_n(z) = \sum_{i=0}^n a_i z^i$ is a non-cyclotomic polynomial with integer coefficients. Also, we assume that both a_0 and a_n are nonzero, and z_j are all zeros of P_n with $|z_j| < 1$. By (3) and (6),

$$|P_n|_\tau = |a_0| \prod_{|z_j|<1} h\big(|z_j|\big).$$

Next, we will show that $|P_n|_\tau$ has a lower bound $c > 1$. There are two cases under consideration.

Case I: $|a_0| \ge 2$. By the comments after Theorem 5, for each $x \in (0,1)$, $h(x) \ge 1$, and then

$$|P_n|_\tau = |a_0| \prod_{|z_j|<1} h\big(|z_j|\big) \ge 2.$$

Case II: $|a_0| = 1$. In this case, $|P_n|_\tau = \prod_{|z_j|<1} h(|z_j|)$.

Now let s be as in Lemma 6. If there is a z_j such that $|z_j| \le s$, then $|P_n|_\tau \ge h(s) > 1$. Otherwise, all $|z_j|$ with $|z_j| < 1$ must satisfy $|z_j| > s$. By Lehmer's conjecture, there is a constant $\zeta_0 > 1$ such that $M(P_n) = \prod_{|z_j|<1} \frac{1}{|z_j|} \ge \zeta_0$. Then by Lemma 6, there is a numerical constant $c' > 1$ such that

$$\prod_{|z_j|<1} h\big(|z_j|\big) \ge c'.$$

That is, $|P_n|_\tau \ge c'$.

Put $c = \min\{c', h(s), 2\} > 1$, and we have $|P_n|_\tau \ge c$. The proof is complete. $\square$

5. Approximation by polynomials in $\mathbb{Z}[z]$ in Mahler's measure

Recently, I. Pritsker considered several questions of approximation by polynomials in $\mathbb{Z}[z]$. In the Hardy space $H^p(\mathbb{D})$ $(0 < p < \infty)$, define

$$\|f\|_{H^p} \triangleq \left(\int_0^{2\pi} \big|f\big(e^{i\theta}\big)\big|^p \frac{d\theta}{2\pi}\right)^{\frac{1}{p}}, \quad f \in H^p(\mathbb{D}).$$

It was shown in [18] that for each $f \in H^p$, if there is a sequence of polynomials $\{P_n\}$ in $\mathbb{Z}[z]$ such that

$$\lim_{n\to\infty} \|f - P_n\|_{H^p} = 0,$$

then $f \in \mathbb{Z}[z]$. I. Pritsker asked whether this conclusion also holds for $p = 0$, i.e. for approximation of functions in Mahler's measure. In this section, we give an affirmative answer to this question under a mild condition. In fact, we show that this is true for Nevanlinna functions.

An analytic function f over the unit disk f is called in *the Nevanlinna class* if the subharmonic function $\log^+|f|$ has a harmonic majorant. This condition is equivalent to that f has bounded characteristic, that is, there are two bounded analytic functions g and h such that $f=\frac{g}{h}$, see [11, pp. 69–75]. For a Nevanlinna function f, its boundary values make sense and then we may define

$$M(f)\triangleq\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(e^{i\theta}\right)\right|d\theta\right)<\infty.$$

Then we have the following result.

THEOREM 7. *Suppose f is a Nevanlinna function over $\mathbb{D}$. If $P_n\in\mathbb{Z}[z], n\in\mathbb{Z}_+$, satisfying*

$$\lim_{n\to\infty}M(f-P_n)=0,$$

then $f\in\mathbb{Z}[z]$.

Before continuing, let us make an observation. For each analytic function h over $\mathbb{D}$, $\log|h|$ is subharmonic. From this and the inner–outer decomposition of $H^2(\mathbb{D})$-functions, it is easy to give the following (see [11], [12], for example).

LEMMA 8. *Given a nonzero function $h\in H^2(\mathbb{D})$, there exists an $r\in(0,1)$ such that*

$$\sup_{r\le t\le 1}\left|\int_0^{2\pi}\log\left|h\left(te^{i\theta}\right)\right|\frac{d\theta}{2\pi}\right|<\infty.$$

Also, we have

$$\int_0^{2\pi}\log\left|h\left(te^{i\theta}\right)\right|\frac{d\theta}{2\pi}\le\int_0^{2\pi}\log\left|h\left(e^{i\theta}\right)\right|\frac{d\theta}{2\pi},\quad 0<t<1.$$

Then we are ready to prove Theorem 7.

Proof of Theorem 7. Suppose that f is a Nevanlinna function over $\mathbb{D}$ and there are polynomials $P_n\in\mathbb{Z}[z], n\in\mathbb{Z}_+$, satisfying

$$\lim_{n\to\infty}M(f-P_n)=0.$$

Write

$$f(z)=\sum_{k=0}^{\infty}a_kz^k,\quad z\in\mathbb{D}.$$

Assume conversely that $f(z)\notin\mathbb{Z}[z]$, and then there are two cases under consideration.

(1) All a_n are integers and there exist infinitely many n such that $a_n\neq 0$;
(2) There exists a minimal k_0 such that $a_{k_0}\neq\mathbb{Z}$.

Then we claim that in both cases, there exists a constant a such that

$$\liminf_{r\to 1^-}\int_0^{2\pi}\log|f-P_n|\left(re^{i\theta}\right)d\theta\ge a. \tag{16}$$

In fact, in case (1), let $j=j(n)$ be the largest integer such that $\frac{f(z)-P_n(z)}{z^j}$ is analytic in $\mathbb{D}$. Since $\frac{f(z)-P_n(z)}{z^j}|_{z=0}$ is a nonzero integer, it follows that

$$\begin{aligned}\liminf_{r\to1^-}\int_0^{2\pi}\log|f-P_n|\big(re^{i\theta}\big)\,d\theta &= \liminf_{r\to1^-}\int_0^{2\pi}\log\left|\frac{f-P_n}{z^j}\right|\big(re^{i\theta}\big)\,d\theta\\ &\geq \log\left|\frac{f(z)-P_n(z)}{z^j}\right|\Bigg|_{z=0}\geq 0.\end{aligned}$$

The discussion for case (2) is similar.

Since f is a Nevanlinna function, write $f=\frac{g}{h}$, where both g and h are bounded analytic function over $\mathbb{D}$. By Lemma 8, assume that there is an $r_0\in(0,1)$ and $M>0$ satisfying

$$\sup_{r_0\leq t\leq 1}\left|\int_0^{2\pi}\log\big|h\big(te^{i\theta}\big)\big|\frac{d\theta}{2\pi}\right|=M<\infty. \tag{17}$$

Since

$$\begin{aligned}&\int_0^{2\pi}\log|f-P_n|\big(re^{i\theta}\big)\,d\theta\\ &\quad=\int_0^{2\pi}\log\left|\frac{g}{h}-P_n\right|\big(re^{i\theta}\big)\,d\theta\\ &\quad=-\int_0^{2\pi}\log\big|h\big(re^{i\theta}\big)\big|\,d\theta+\int_0^{2\pi}\log|g-hP_n|\big(re^{i\theta}\big)\,d\theta,\end{aligned}$$

then by (17),

$$\begin{aligned}&\int_0^{2\pi}\log|g-hP_n|\big(re^{i\theta}\big)\,d\theta\\ &\quad=\int_0^{2\pi}\log|f-P_n|\big(re^{i\theta}\big)\,d\theta+\int_0^{2\pi}\log\big|h\big(re^{i\theta}\big)\big|\,d\theta\\ &\quad\geq\int_0^{2\pi}\log|f-P_n|\big(re^{i\theta}\big)\,d\theta-M.\end{aligned}\tag{18}$$

Combining (17) and (18) with Lemma 8 implies that

$$\begin{aligned}\int_0^{2\pi}\log|f-P_n|\big(e^{i\theta}\big)\,d\theta &= -\int_0^{2\pi}\log\big|h\big(e^{i\theta}\big)\big|\,d\theta+\int_0^{2\pi}\log|g-hP_n|\big(e^{i\theta}\big)\,d\theta\\ &\geq -M+\int_0^{2\pi}\log|g-hP_n|\big(re^{i\theta}\big)\,d\theta\\ &\geq -2M+\int_0^{2\pi}\log|f-P_n|\big(re^{i\theta}\big)\,d\theta.\end{aligned}$$

Then by (16), we get

$$\int_0^{2\pi} \log |f - P_n|\big(e^{i\theta}\big)\, d\theta \geq -2M + a,$$

and hence

$$M(f - P_n) \geq \exp(-2M + a),$$

which is a contradiction to $\lim_{n\to\infty} M(f - P_n) = 0$. Therefore, $f \in \mathbb{Z}[z]$. The proof is complete. □

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