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APPROXIMATION BY POLYNOMIALS AND BLASCHKE PRODUCTS HAVING ALL ZEROS ON A CIRCLE

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ABSTRACT. We show that a nonvanishing analytic function on a sub-disc of the unit disc can be approximated by (a scalar multiple of) a Blaschke product whose zeros lie on a prescribed circle enclosing the sub-disc. We also give a new proof of the analogous classical result for polynomials. A connection is made to universality results for the Riemann zeta function.

1. Introduction

While every analytic function on a disc can be approximated pointwise by a polynomial, it is an interesting problem to determine where the zeros of the polynomial may be chosen. When our attention is focused on bounded analytic functions, there is another class of functions that can be used to approximate: the set of Blaschke products. A finite Blaschke product is a function of the form

$$B(z) = \lambda \prod_{j=1}^{N} \frac{z - a_j}{1 - \overline{a_j} z}, \tag{1.1}$$

where $|a_j| < 1$ for all j and $|\lambda| = 1$. Carathéodory's theorem (see [4, p. 6], for example) shows that if f is an analytic function defined on the open unit disc, $\mathbb{D}$, and f is bounded by 1 in modulus, then there is a sequence $\{B_k\}$ of finite Blaschke products converging to f pointwise on $\mathbb{D}$. Again, it is certainly interesting to ask where the zeros of the approximating Blaschke products may lie.

Looking at more general domains, one natural question is the following: Given a holomorphic function in a Jordan region, when can it be approximated by a polynomial with zeros lying on the boundary? This question was

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answered by G. MacLane [10] in 1949. Curiously, many texts dealing with the study of polynomials or approximation by polynomials do not include reference to this work ([11], [13]), though three different proofs of this result are given by Korevaar [6].

MacLane's work focused on showing that a zero-free holomorphic function can be approximated by a polynomial with zeros on the boundary, when the boundary satisfies certain smoothness conditions. C. Chui [1], [2] looked at the problem of bounded approximation of a zero-free bounded holomorphic function by what he called C-polynomials. Chui showed that every zero-free bounded holomorphic function defined on $\mathbb{D}$ can be boundedly approximated by polynomials with zeros lying on the unit circle. In 1968, Z. Rubinstein showed that given a zero-free holomorphic function with $f(0)=1$, there exists a sequence of C-polynomials mapping 0 to 1 that converges to f uniformly on every compact set in $\mathbb{D}$. In addition, when the function f is bounded, the sequence converges boundedly. One natural approach, that of looking at the zeros of the partial sums of a series, has been further studied by Korevaar and others, see [7], [8], and [9].

Our main result (Corollary 2.3) is that given an analytic function g that has no zeros in a neighborhood of $\{z : |z| \le r\}$, for all $\varepsilon > 0$ and $\delta > 0$ there exists a constant c_B and a finite Blaschke product B with all zeros on the circle $\{z : |z| = r\}$ such that

$$\left|g(z) - c_B B(z)\right| < \varepsilon \quad \text{on } \left\{z : |z| < r - \delta\right\}. \tag{1.2}$$

Our approach also provides a new and relatively simple proof of the fact that a nonvanishing analytic function can be approximated uniformly on compact subsets of a disc by polynomials having zeros on a prescribed circle.

The paper is organized as follows. In Section 2, we consider the special case of discs centered at 0. Then in Section 3.1, we extend the special case to more general sub-discs of the unit disc. In Section 4, we discuss the relationship with universality results for the Riemann zeta function and its connection to random matrix theory.

2. Approximation around 0

Here is a very simple proof that a polynomial p that does not vanish on a neighborhood of the closure of the unit disc $\mathbb{D}$ can be approximated uniformly on compact subsets of $\mathbb{D}$ by polynomials all of whose zeros are on the unit circle:

Note that if the degree of p is m, then the polynomial $p^\star(z) = z^m\overline{p(1/\overline{z})}$ has all zeros inside the unit disc. Let

$$B(z) = p^\star(z)/p(z). \tag{2.1}$$

Then B is analytic in $\mathbb{D}$, continuous on the unit circle, maps $\mathbb{D}$ to itself, the unit circle to itself, and the complement of the closed disc to itself. Therefore,

B is a Blaschke product. Now for $k \in \mathbb{N}$, the set of points in $\overline{\mathbb{D}}$ for which $B(z) = z^{-k}$ lie on the unit circle. Therefore, the polynomial

$$p(z) - z^k p^\star(z) \tag{2.2}$$

has all its zeros on the unit circle and approximates p on compact subsets of $\mathbb{D}$ as $k \to \infty$. This is, more or less, the proof given by Z. Rubinstein [12]. However, it does not seem possible to adapt this proof to the case of Blaschke products. We present an alternate proof of this result in Sections 2.2 and 2.3. Results on Blaschke products appear in Section 3 and as Corollary 2.3, below.

First, we consider the case of approximating on a disc centered at 0.

THEOREM 2.1. *Suppose f is analytic and nonvanishing in a neighborhood of $|z| \le r$. Then there exist numbers $|\xi_j| = |\eta_j| = 1$, $A \in \mathbb{C}$, and positive integers $\nu(j)$ such that*

$$f(z) = A \prod_{j=1}^{\infty} \left(1 + \xi_j z^j\right)^{\nu(j)} \left(1 + \eta_j z^j\right) \tag{2.3}$$

for $|z| < \min\{r, 1\}$, with the convergence uniform on $|z| < \min\{r, 1\} - \delta$ for any δ with $0 < \delta < \min\{r, 1\}$.

In particular, f can be approximated on $|z| < \min\{r, 1\} - \delta$ by polynomials having all roots on the unit circle.

THEOREM 2.2. *Under the same conditions as Theorem 2.1, if $R < 1$ then*

$$f(z) = A \prod_{j=1}^{\infty} \left(\frac{1 + \xi_j z^j}{1 + \xi_j R^j z^j} \right)^{\nu(j)} \frac{1 + \eta_j z^j}{1 + \eta_j R^j z^j} \tag{2.4}$$

for $|z| < \min\{r, 1\}$, with the convergence uniform on $|z| < \min\{r, 1\} - \delta$ for any δ with $0 < \delta < \min\{r, 1\}$.

The proofs of Theorem 2.1 and Theorem 2.2 will be presented in Sections 2.2 and 2.3, respectively. In the remaining portion of this section, we show how the result on approximation by Blaschke products follows.

COROLLARY 2.3. *Suppose g is analytic and nonvanishing in a neighborhood of $|z| \le r < 1$. For all $\varepsilon > 0$ and $\delta > 0$ there exists a constant c_B and a Blaschke product B having all zeros on $|z| = r$ such that*

$$\left| g(z) - c_B B(z) \right| < \varepsilon \tag{2.5}$$

for $|z| < r(1 - \delta)$.

Proof. In Theorem 2.2, let $R = r^2$ and set $g(z) = f(z/r)$, so

$$g(rz) = f(z) = a_0 \prod_{j=1}^{\infty} \left(\frac{1 + \xi_j z^j}{1 + \xi_j r^{2j} z^j} \right)^{\nu(j)} \frac{1 + \eta_j z^j}{1 + \eta_j r^{2j} z^j}.$$

By Theorem 2.2, for each $\delta > 0$ we may choose N so that

$$\sup_{|z|<1-\delta}\left|g(rz) - a_0 \prod_{j=1}^{N}\left(\frac{1+\xi_j z^j}{1+\xi_j r^{2j} z^j}\right)^{\nu(j)} \frac{1+\eta_j z^j}{1+\eta_j r^{2j} z^j}\right| < \varepsilon.$$

It remains to rearrange the above product to recognize it as a Blaschke product. Letting $w = rz$ we have

$$\sup_{|w|<r(1-\delta)}\left|g(w) - a_0 \prod_{j=1}^{N}\left(\frac{1+r^{-j}\xi_j w^j}{1+\xi_j r^j w^j}\right)^{\nu(j)} \frac{1+r^{-j}\eta_j w^j}{1+\eta_j r^j w^j}\right| < \varepsilon.$$

Thus,

$$\sup_{|w|<r(1-\delta)}\left|g(w) - a_0 \prod_{j=1}^{N} r^{-2j}\xi_j\eta_j\left(\frac{r^j\overline{\xi_j}+w^j}{1+\xi_j r^j w^j}\right)^{\nu(j)} \frac{r^j\overline{\eta_j}+w^j}{1+\eta_j r^j w^j}\right| < \varepsilon.$$

Letting $\alpha_j = r^j\overline{\xi_j}$ and $\beta_j = r^j\overline{\eta_j}$, we have

$$\sup_{|w|<r(1-\delta)}\left|g(w) - \left(a_0 \prod_{j=1}^{N} r^{-2j}\xi_j\eta_j\right)C(w)\right| < \varepsilon, \tag{2.6}$$

where $C(w) = \prod_{j=1}^{N}(\frac{\alpha_j+w^j}{1+\overline{\alpha_j}w^j})^{\nu(j)}(\frac{\beta_j+w^j}{1+\overline{\beta_j}w^j})$. Since each factor of C is a Möbius transformation composed with w^j, each factor is a Blaschke product and therefore C is a Blaschke product as well. □

Before giving the proof of Theorems 2.1 and 2.2, we describe the construction in a context that avoids the issue of convergence.

2.1. Formal power series as infinite products.

Corollary 2.4. *Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a formal power series with $a_n \in \mathbb{C}$ and $a_0 \neq 0$. We can write*

$$f(z) = a_0 \prod_{j=1}^{\infty}\left(1+\xi_j z^j\right)^{\nu(j)}\left(1+\eta_j z^j\right), \tag{2.7}$$

where $|\xi_j| = |\eta_j| = 1$ and $\nu(j)$ is a nonnegative integer.

We will use Lemma 2.5 below to define the product representations inductively, and then we provide a proof of Proposition 2.4. We thank the referee for helping us give a clearer proof of the lemma.

Lemma 2.5. *Let $R_0 > 0$. Every $w \in \mathbb{C}$ can be written as $w = m\xi + \eta$ where $|\xi| = |\eta| = R_0$ and m is a positive integer. There are four such representations if $|w| > R_0$, two representations if $0 < |w| \leq R_0$, and infinitely many if $w = 0$.*

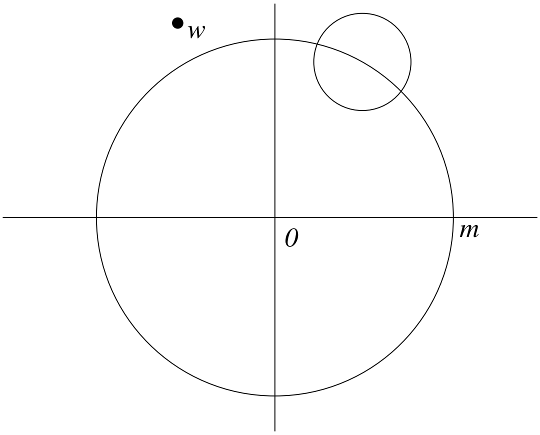

FIGURE 1. In the figure, w is a complex number with $|w|>1$, and the larger circle has radius $m=\text{floor}(|w|)$. The small circle has radius 1 and center on the large circle. As we move the small circle around the large circle, keeping the center on the large circle, it hits the point w twice. The other two representations can be seen by replacing m by $m+1$. The construction must be modified slightly if $|w|$ is an integer.

Proof. We need only consider the case $R_0=1$. Note that the problem is equivalent to looking for $\eta\in\partial\mathbb{D}$ such that $|w-\eta|\in\mathbb{N}$. If $|w|>1$, then the result follows from the fact that

$$\text{dist}(w,\partial\mathbb{D})\le|w-\eta|\le 2+\text{dist}(w,\partial\mathbb{D}), \tag{2.8}$$

with each inequality being sharp for some $\eta\in\partial\mathbb{D}$. If $|w|\le 1$, then we use the fact that

$$\text{dist}(w,\partial\mathbb{D})\le|w-\eta|\le 2-\text{dist}(w,\partial\mathbb{D}). \tag{2.9}$$

That there are four possible representations for $|w|>1$ and two for $0<|w|\le 1$ can be seen from Figure 1. □

Proof of Proposition 2.4. We may assume $a_0=1$. By Lemma 2.5 with $R_0=1$, we can choose ξ_1, η_1, and $\nu(1)$ so that $a_1=\nu(1)\xi_1+\eta_1$.

Set

$$P_1(z)=(1+\xi_1 z)^{\nu(1)}(1+\eta_1 z) \tag{2.10}$$

and note that $P_1(z)=1+a_1z+O(z^2)$. Therefore,

$$\frac{f(z)}{P_1(z)}=\sum_{j=0}^{\infty}b_jz^j, \tag{2.11}$$

where $b_0 = 1$ and $b_1 = 0$. Now choose ξ_2, η_2, and $\nu(2)$ so that $b_2 = \nu(2)\xi_2 + \eta_2$. Setting

$$P_2(z) = \left(1 + \xi_2 z^2\right)^{\nu(2)} \left(1 + \eta_2 z^2\right) = 1 + b_2 z^2 + O\left(z^3\right) \tag{2.12}$$

we have

$$P_1(z) P_2(z) = 1 + a_1 z + b_2 z^2 + O\left(z^3\right) = 1 + a_1 z + a_2 z^2 + O\left(z^3\right). \tag{2.13}$$

Proceeding inductively, we obtain $f(z) = \prod_j P_j(z)$. □

COROLLARY 2.6. *Suppose*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.14}$$

is a formal power series with $a_n \in \mathbb{C}$ *and* $a_0 \neq 0$, *and let* $R < 1$. *We can write*

$$f(z) = a_0 \prod_{j=1}^{\infty} \left(\frac{1 + \xi_j z^j}{1 + \xi_j R^j z^j} \right)^{\nu(j)} \frac{1 + \eta_j z^j}{1 + \eta_j R^j z^j}, \tag{2.15}$$

where $|\xi_j| = |\eta_j| = 1$ *and* $\nu(j)$ *is a nonnegative integer.*

Proof of Proposition 2.6. Note that

$$\frac{1 + \xi_j z^j}{1 + \xi_j R^j z^j} = 1 + \left(1 - R^j\right) \xi_j z^j + O\left(z^{2j}\right). \tag{2.16}$$

So everything goes through in the previous proof, with the modification that we apply Lemma 2.5 with $R_0 = 1 - R^j$. □

COROLLARY 2.7. *Suppose* g *is analytic in a neighborhood of* 0 *with* $g(0) \neq 0$, *and suppose* $0 < r < 1$. *Then for all* $J > 0$ *there exists a constant* c_J *and a Blaschke product* B_J *having all zeros on* $|z| = r$ *such that*

$$g(z) - c_J B_J(z) = O\left(z^J\right), \tag{2.17}$$

as $z \to 0$.

Proof. The proof of Corollary 2.7 is the same as the proof of Corollary 2.3. □

The proofs given above are just formal calculations, and it is not clear what convergence properties the infinite products might have. Even if f represents an analytic function in a neighborhood of the origin, the products can only converge where f does not vanish. The convergence of the products will depend on the growth of the numbers $\nu(j)$, and the above constructions do not appear to shed light on this. In the next section, we organize the proof in a different way that may appear more cumbersome, but it gives information about the analytic properties of the infinite product.

2.2. Proof of Theorem 2.1. We will need one technical lemma before turning to the proof of our main result.

LEMMA 2.8. *Suppose $\nu(j) \geq 0$ for $j \geq 0$. If there exist $\kappa > 1$ and $C > 0$ so that*

$$(n+1)\nu(n+1) \leq n^2 + C\kappa^n + \sum_{\substack{j|(n+1) \\ 1 \leq j \leq n}} j\nu(j) \tag{2.18}$$

for all $n \geq 1$, then there exists C' so that $n\nu(n) \leq C'\kappa^n$ for all $n \geq 1$.

Proof. First, choose N so that

$$\frac{n}{\kappa^{(n+1)/2}} < \frac{2}{3} \tag{2.19}$$

if $n \geq N$. Then choose C' so that

(1) $C' > 3C$,
(2) $n^2 < \frac{1}{3}C'\kappa^n$ for all n, and
(3) $n\nu(n) < C'\kappa^n$ for $n \leq N$.

Note that (2) uses only the fact that $\kappa > 1$, and (3) uses only that $\kappa > 0$ and N is finite.

Now we prove the desired estimate by induction. Suppose $n\nu(n) \leq C'\kappa^n$ for $n \leq M$, where $M > N$, and suppose $n = M+1$. Using the first two conditions on C', the induction hypothesis, and the fact that all proper divisors of $n+1$ are at most $(n+1)/2$, we have

$$\begin{aligned}(n+1)\nu(n+1) &< \frac{1}{3}C'\kappa^n + \frac{1}{3}C'\kappa^n + \sum_{j \leq (n+1)/2} j\nu(j) \\ &\leq \frac{2}{3}C'\kappa^n + \sum_{j \leq (n+1)/2} C'\kappa^j \\ &\leq \frac{2}{3}C'\kappa^n + \frac{n+1}{2}C'\kappa^{(n+1)/2} \\ &= \frac{2}{3}C'\kappa^n + C'\kappa^{n+1}\frac{n+1}{2\kappa^{(n+1)/2}} \\ &\leq C'\kappa^{n+1}.\end{aligned} \tag{2.20}$$

The last inequality follows from $n > N$ and the choice of N. That completes the proof of Lemma 2.8. □

Proof of Theorem 2.1. Let

$$\frac{f'}{f}(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{2.21}$$

Since f'/f is analytic in a disc slightly larger than $|z|<r$, there is a $C>0$ and $\kappa>1/r$ so that

$$|a_n|<C\kappa^n \tag{2.22}$$

for all n. If $r\geq 1$ we set $\kappa=1+\delta$ for some $\delta>0$. In particular, $\kappa>1$.

Let

$$g(z)=g_J(z)=\prod_{j=1}^{J}\left(1+\xi_j z^j\right)^{\nu(j)}\left(1+\eta_j z^j\right), \tag{2.23}$$

where $|\xi_j|=|\eta_j|=1$ and $\nu(j)$ is a non-negative integer. We will choose those parameters so that the first J terms in the Taylor series for g'/g match those of f'/f.

We have

$$\begin{aligned}\frac{g'}{g}(z) &= \sum_{1\leq j\leq J}\left(j\nu(j)\xi_j z^{j-1}\frac{1}{1+\xi_j z^j}+j\eta_j z^{j-1}\frac{1}{1+\eta_j z^j}\right)\\ &= -z^{-1}\sum_{1\leq j\leq J} j\sum_{m=0}^{\infty}\big(\nu(j)(-1)^{m+1}\xi_j^{m+1}z^{j(m+1)}\\ &\qquad +(-1)^{m+1}\eta_j^{m+1}z^{j(m+1)}\big)\\ &= -\sum_{k=0}^{\infty}z^k\sum_{\substack{1\leq j\leq J\\ j|(k+1)}}(-1)^{\frac{k+1}{j}}j\big(\nu(j)\xi_j^{\frac{k+1}{j}}+\eta_j^{\frac{k+1}{j}}\big)\\ &= \sum_{k=0}^{\infty}b_k z^k,\end{aligned} \tag{2.24}$$

say. On the third line we change the summation index $k=j(m+1)-1$, and the notation $n|m$, read "n divides m," means that m/n is an integer.

Now we show how to choose the parameters in g to match the Taylor series coefficients of the logarithmic derivatives.

By Lemma 2.5, we can choose a non-negative integer $\nu(1)$ and complex numbers $|\xi_1|=|\eta_1|=1$ so that $-a_0=\nu(1)\xi_1+\eta_1$. Thus, $b_0=a_0$, and we have matched the first Taylor series coefficients of f'/f and g'/g.

Since the only positive integer that divides $0+1=1$ is 1, we will continue to have $b_0=a_0$ no matter what we later choose for ξ_j, η_j, and $\nu(j)$ for $j\geq 2$. Likewise, once we have chosen ξ_j, η_j, and $\nu(j)$ for $j\leq K$ so that $a_j=b_j$ for $j\leq K-1$, we will continue to have $a_j=b_j$ for $j\leq K-1$ because if $j|(k+1)$ then $j\leq k+1$.

To choose ξ_j, η_j, and $\nu(j)$ for $j\geq 2$, we have to deal with the fact that the $j=1$ terms make a contribution to all of those coefficients. Similarly, the $j=2$ terms contribute to all of the later even-index coefficients, and so on.

Breaking the sum defining b_K into two parts, we find

$$
\begin{aligned}
b_K &= -\sum_{\substack{1\le j\le K\\ j|(K+1)}} (-1)^{\frac{K+1}{j}} j\big(\nu(j)\xi_j^{\frac{K+1}{j}} + \eta_j^{\frac{K+1}{j}}\big) \\
&\quad - \sum_{\substack{K+1\le j\le J\\ j|(K+1)}} (-1)^{\frac{K+1}{j}} j\big(\nu(j)\xi_j^{\frac{K+1}{j}} + \eta_j^{\frac{K+1}{j}}\big) \\
&= -\sum_{\substack{1\le j\le K\\ j|(K+1)}} (-1)^{\frac{K+1}{j}} j\big(\nu(j)\xi_j^{\frac{K+1}{j}} + \eta_j^{\frac{K+1}{j}}\big) \\
&\quad + (K+1)\big(\nu(K+1)\xi_{K+1} + \eta_{K+1}\big).
\end{aligned} \tag{2.25}
$$

The terms in the sum on the third line of (2.25) have already been chosen, so we can use Lemma 2.5 to choose $\nu(K+1)$, ξ_{K+1}, and η_{K+1} so that $b_K = a_K$.

Proceeding in this way, we match the first J coefficients of the logarithmic derivatives.

It remains to bound $\nu(j)$ so that we can bound the tail of (2.24).

By (2.25) and the fact that $b_K = a_K$ we have

$$
(K+1)\nu(K+1) \le K+1+|a_K| + \sum_{\substack{j|(K+1)\\ 1\le j\le K}} \big(j\nu(j)+j\big). \tag{2.26}
$$

By Lemma 2.8, the above estimate implies that there exists C' so that $n\nu(n)\le C'\kappa^n$ for all $n\ge 1$. This is sufficient to estimate the tail for $|z|<1/\kappa$ because the coefficient of z^n in (2.24) is bounded by

$$
|b_n| \le \sum_{j\le J}\big(j\nu(j)+j\big) \ll J^2\kappa^J, \tag{2.27}
$$

where we use $\ll$ to mean the quantity is bounded by a constant times $J^2\kappa^J$. So

$$
\sum_{n\ge J+1} |b_n|\big|z^n\big| \ll J^2\kappa^J \sum_{n\ge J+1} |z|^n \ll \frac{J^2}{1-|z|}\kappa^J|z|^J, \tag{2.28}
$$

which goes to 0 as $J\to\infty$ because $|z|<1/\kappa<1$.

This shows that $g'(z)/g(z)$ is close to $f'(z)/f(z)$ for $|z|<1/\kappa$. We can antidifferentiate using Cauchy's theorem, so $\log(f)$ is close to $\log(g_J)+c_J$ for $|z|<1/\kappa$, for some constant c_J. Now exponentiate to get that $f(z)$ is close to $e^{c_J}g(z)$. Since $g(0)=1$, choose $c_J=\log(f(0))$.

This completes the proof of Theorem 2.1. $\square$

2.3. Proof of Theorem 2.2. We describe how to modify the the proof of Theorem 2.1 to give a proof of Theorem 2.2.

Proof of Theorem 2.2. Let

$$h(z) = \prod_{j=1}^{J} \left(\frac{1 + \xi_j z^j}{1 + \xi_j R^j z^j} \right)^{\nu(j)} \frac{1 + \eta_j z^j}{1 + \eta_j R^j z^j}. \tag{2.29}$$

Note that

$$\frac{h'}{h}(z) = \frac{g'}{g}(z) - R \frac{g'}{g}(Rz), \tag{2.30}$$

where g is the function (2.23) appearing in the proof of Theorem 2.1. Writing

$$\frac{h'}{h}(z) = \sum_{k=0}^{\infty} c_k z^k \tag{2.31}$$

we have

$$c_k = \big(1 - R^{k+1}\big) b_k, \tag{2.32}$$

where b_k are the Taylor series coefficients of g'/g given in (2.24).

Thus, when matching the coefficients of h'/h and f'/f, everything goes as before if in each equation we replace b_k by c_k and a_k by $a_k/(1-R^{k+1})$. So the choices of ξ_k, η_k, and $\nu(k)$ follow the same steps. The final step of bounding $\nu(K)$ involves replacing inequality (2.26) by

$$(K+1)\nu(K+1) \le K + 1 + \frac{|a_K|}{1 - R^{K+1}} + \sum_{\substack{j|(K+1) \\ 1 \le j \le K}} j\nu(j) + j. \tag{2.33}$$

But that implies the bound we need on $\nu(j)$ because the only fact we used about a_k is $|a_k| \le C\kappa^k$ for some $C > 0$.

This completes the proof of Theorem 2.2. □

3. Corollaries of Theorem 2.1

We deduce some corollaries about approximation on general sub-discs of the unit disc. We will obtain our results by using the relationship between Euclidean discs and the so-called pseudohyperbolic discs in the open unit disc $\mathbb{D}$. Once we have clarified this relationship, we can use Möbius transformations to map one disc to one centered at the origin. In this way, we reduce our results to the previously solved problem.

3.1. Approximation on pseudohyperbolic discs. As a corollary to Theorem 2.1, we prove a result about approximating on other discs contained in the unit disc. These other discs, the so-called pseudohyperbolic discs, will allow us to present a result about approximation by Blaschke products. The statement of approximation by Blaschke products appears in Theorem 3.2.

Our result will make use of the fact that Euclidean discs are pseudohyperbolic discs. To see this, recall first that the pseudohyperbolic distance between two points z and w in $\mathbb{D}$ is defined to be the distance

$$\rho(z,w)=\left|\frac{z-w}{1-\overline{w}z}\right|. \tag{3.1}$$

For $a\in\mathbb{D}$ and r with $0<r<1$ we let $D_\rho(a,r)=\{z:\rho(a,z)<r\}$. Given a Euclidean disc $D(a_0,r_0)$, we may rotate it so that the center lies on the positive real axis, and let x and y, with $|x|<y$, denote the points in which the bounding circle $\mathcal{C}$ intersects the real line. Let φ_a be the Möbius function

$$\varphi_a(z)=\frac{z+a}{1+\overline{a}z} \tag{3.2}$$

and let $R=\frac{1+xy}{x+y}$. Then $R>1$ and if $a=R-\sqrt{R^2-1}$ then $r=-\varphi_a^{-1}(x)=\varphi_a^{-1}(y)$. Since a is real, φ_a^{-1} maps $\mathcal{C}$ onto a circle $\mathcal{C}_1$ passing through r and $-r$ and since the real line is orthogonal to $\mathcal{C}$, the real line must be orthogonal to $\mathcal{C}_1$. Therefore, φ_a^{-1} maps $\mathcal{C}$ onto $\{z:|z|=r\}$. Thus, the disc $D(a_0,r_0)$ is rotation of a pseudohyperbolic disc $D_\rho(a,r)$ for some a, r. This means that

$$D_\rho(a,r)=\varphi_a\big(D(0,r)\big). \tag{3.3}$$

For basic information about automorphisms of the disc, see Garnett [4].

Theorem 3.1. *Let f be a function that is analytic and nonvanishing in a neighborhood of the disc $\overline{D(a_0,r_0)}\subset\mathbb{D}$. Then f can be uniformly approximated on $D(a_0,r_0)$ by a polynomial that has all of its zeros lying on the unit circle.*

Proof. Suppose f has no zeros in a neighborhood of the closure of a pseudohyperbolic disc $D_\rho(a,r)$. Then $f\circ\varphi_a$ has no zeros in a neighborhood of the disc $D(0,r)$. By Theorem 2.1, there is a polynomial p with all of its zeros on the unit circle such that

$$\|f\circ\varphi_a-p\|_{D(0,r)}<\varepsilon. \tag{3.4}$$

Therefore by (3.3) and a change of variables,

$$\big\|f-p\circ{\varphi_a}^{-1}\big\|_{D_\rho(a,r)}<\varepsilon. \tag{3.5}$$

Now, letting $z_1,\dots,z_N$ denote the zeros of p, all of which satisfy $|z_j|=1$, we see that

$$p\circ\varphi_a^{-1}(z)=\prod_{j=1}^{N}\big({\varphi_a}^{-1}(z)-z_j\big). \tag{3.6}$$

This is a rational function with poles outside the (closed) unit disc and zeros at $\varphi_a(z_j)$ for $j=1,\ldots,N$. Thus, the zeros of this rational function also lie on the unit circle.

Now choose $s<1$ so that $D_\rho(a,r)\subset D(0,s)$. Since $p\circ\varphi_a^{-1}(z)$ is analytic and nonvanishing in a neighborhood of $D(0,s)$, we can apply Theorem 2.1 again to get a polynomial q so that

$$\left\|q-p\circ{\varphi_a}^{-1}\right\|_{D(0,s)}<\varepsilon, \tag{3.7}$$

which implies

$$\left\|q-p\circ{\varphi_a}^{-1}\right\|_{D_\rho(a,r)}<\varepsilon. \tag{3.8}$$

Combining (3.5) and (3.8) gives $\|f-q\|_{D_\rho(a,r)}<2\varepsilon$, as required. □

There is a Blaschke product version of this result that can be obtained in a similar, but simpler manner; that is, if we use Corollary 2.3 in place of Theorem 2.1 and note that $B\circ\varphi_a^{-1}$ is a finite Blaschke product (see [4, p. 6]) with zeros on the boundary of $D_\rho(a,r)$ whenever the zeros of B lie on $\{z:|z|=r\}$, we obtain the following result.

Theorem 3.2. *Let f be a function that is analytic and nonvanishing in a neighborhood of the disc $\overline{D(a_0,r_0)}\subset\mathbb{D}$. Then, for δ with $0<\delta<r_0$, the function f can be uniformly approximated on $D(a_0,r_0-\delta)$ by a constant times a Blaschke product with all of its zeros on the circle $\{z:|z-a_0|=r_0\}$.*

From these results, we obtain a corollary about functions with zeros. For $0<p<\infty$ and f an analytic function on $\mathbb{D}$, we say that $f\in H^p$ if

$$\sup_r \frac{1}{2\pi}\int\left|f\left(re^{i\theta}\right)\right|^p d\theta=\|f\|_{H^p}^p<\infty. \tag{3.9}$$

It is well known that given a nonzero function $f\in H^p$ the zero sequence of f, denoted (z_n), is a Blaschke sequence. Letting C_1 denote the (possibly infinite) Blaschke product with zeros (z_n) there exists a function g that is analytic on $\mathbb{D}$ and has no zeros in $\mathbb{D}$ such that $f=C_1g$. Applying the previous theorem to g, we obtain the following.

Corollary 3.3. *Let $0<p<\infty$ and let $f\in H^p$. If $\overline{D(a_0,r_0)}\subset\mathbb{D}$ and $0<\delta<r_0$, then f can be uniformly approximated on $D(a_0,r_0-\delta)$ by functions of the form $c_0C_1C_2$ where c_0 is a constant, C_1 is the Blaschke factor of f, and C_2 is a Blaschke product with zeros on the circle $\{z:|z-a_0|=r_0\}$.*

4. The Riemann zeta function and random matrix theory

One of the motivations for the work in this paper was to understand possible consequences of Voronin's universality result [16], [14] for the Riemann zeta function. Combining the universality result with the principle that the zeta-function can be modeled by the characteristic polynomials of a random unitary matrices [5], suggests the theorem that nonvanishing analytic functions can be approximated by polynomials having all zeros on the unit circle.

We recall Voronin's theorem and then discuss consequences for random unitary matrices.

THEOREM 4.1 (Voronin [16], [14]). *Let $0 < r < \frac{1}{4}$ and suppose g is a nonvanishing continuous function on the disc $|s| \le r$ which is analytic in the interior. Then for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0,T] : \max_{|s|<r} \left| \zeta\left(\frac{3}{4} + i\tau + s\right) - g(s) \right| < \varepsilon \right\} > 0. \tag{4.1}$$

Here $\zeta(s)$ is the Riemann zeta function. The standard reference is Titchmarsh [15].

Note that the theorem says a *positive proportion* of shifts of $\zeta(\frac{3}{4}+s)$ approximate the given function $g(s)$. Below we formulate a random matrix analogue of this observation.

If $U \in U(N)$ is an $N \times N$ unitary matrix, we let

$$\Lambda_U(x) = \det\bigl(I - U^* z\bigr) \tag{4.2}$$

denote its characteristic polynomial, where U^* is the conjugate transpose of U. Note that this is a slightly different normalization than commonly used for the characteristic polynomial of a matrix; it is defined in this way so that $\Lambda_U(0) = 1$. Recasting Proposition 2.4 in terms of characteristic polynomials of unitary matrices, we have:

COROLLARY 4.2. *Suppose $(c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$. If N is sufficiently large then there exists a matrix $U \in U(N)$ such that*

$$\bigl(\Lambda_U'(0), \Lambda_U''(0), \ldots, \Lambda_U^{(n)}(0)\bigr) = (c_1, c_2, \ldots, c_n). \tag{4.3}$$

We suggest that there should be a new proof of Theorem 2.1, based on the idea of providing an explicit lower bound for the probability that a given function is closely approximated by the characteristic polynomial of a random unitary matrix. We formulate the problem as follows:

CONJECTURE 4.3. *Suppose f is a nonvanishing analytic function on the unit disc with $f(0) = 1$, and suppose $0 < r < 1$ and $\varepsilon > 0$ are given. If e^N matrices $U \in U(N)$ are chosen randomly with respect to Haar measure, then the probability that at least one of those e^N matrices satisfies*

$$\bigl|\det\bigl(I - U^* z\bigr) - f(z)\bigr| < \varepsilon \quad \textit{for all } |z| < r, \tag{4.4}$$

is positive and bounded below independent of N.

See [3] for an explanation of why one uses e^N matrices from $U(N)$ to model the zeta function on $[0, T]$.

If this random matrix approach is successful, it will produce polynomials of a very different form than those in Theorem 2.1. The characteristic polynomial of a random unitary matrix has, with probability 1, only simple zeros, and those zeros tend to be very evenly spaced on the unit circle. This is in sharp contrast to the polynomials produced in our construction.

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