Illinois Journal of Mathematics
Volume 55, Number 3, Fall 2011, Pages 1083–1103
S 0019-2082

CRITERIA FOR OPTIMAL GLOBAL INTEGRABILITY OF HAJŁASZ–SOBOLEV FUNCTIONS

YUAN ZHOU

Abstract. The author establishes some geometric criteria for a domain of $\mathbb{R}^n$ with $n \geq 2$ to support a $(pn/(n-ps),p)_s$-Hajłasz–Sobolev–Poincaré imbedding with $s \in (0,1]$ and $p \in (n/(n+s), n/s)$ or an s-Hajłasz–Trudinger imbedding with $s \in (0,1]$.

1. Introduction

The study of the Hajłasz spaces $\dot{M}^{1,p}$ was initiated by Hajłasz [15] on arbitrary metric measure spaces, see [15], [16], [17], [21], [22], [23], [35] for further discussions, generalizations and connections with the classical (Hardy–) Sobolev, Besov and Triebel–Lizorkin spaces. In particular, a fractional version $\dot{M}^{s,p}$ with $s \in (0,1)$ was introduced by Yang [35], and a Sobolev-type version $\dot{M}^{1,p}_{\rm ball}$ on domains by Koskela and Saksman [21].

We first recall some definitions and notions. In this paper, we always let $n \geq 2$ and Ω be a domain of $\mathbb{R}^n$. For every $s \in (0,1]$ and measurable function u, denote by $\mathcal{D}^s(u)$ the *collection of all nonnegative measurable functions g* such that

$$|u(x)-u(y)| \leq |x-y|^s \big[g(x)+g(y)\big] \tag{1.1}$$

for all $x, y \in \Omega \setminus E$, where $E \subset \Omega$ with $|E| = 0$. We also denote by $\mathcal{D}^s_{\rm ball}(u)$ the *collection of all nonnegative measurable functions g* such that (1.1) holds for all $x, y \in \Omega \setminus E$ satisfying $|x-y| < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$.

Definition 1.1. Let $s \in (0,1]$ and $p \in (0,\infty)$. Then the *homogeneous Hajłasz space* $\dot{M}^{s,p}(\Omega)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}^{s,p}(\Omega)} \equiv \inf_{g \in \mathcal{D}^s(u)} \|g\|_{L^p(\Omega)} < \infty,$$

Received March 2, 2010; received in final form April 9, 2010.
The author was supported by the Academy of Finland grant 120972.
2010 *Mathematics Subject Classification.* Primary 46E35. Secondary 42B35.

©2013 University of Illinois

and its *Sobolev-type version* $\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \equiv \inf_{g\in\mathcal{D}^s_{\mathrm{ball}}(u)} \|g\|_{L^p(\Omega)} < \infty.$$

Obviously, for all $s\in(0,1]$ and $p\in(0,\infty)$, $\dot{M}^{s,p}(\Omega)\subset\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$. If Ω is a uniform domain, then $\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)=\dot{M}^{s,p}(\Omega)$ for all $s\in(0,1]$ and $p\in(n/(n+s),\infty)$; see [21, Theorem 19]. But, generally, we cannot expect that $\dot{M}^{s,p}(\Omega)=\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$. For example, this fails when $\Omega=B(0,1)\setminus\{(x,0):x\ge 0\}\subset\mathbb{R}^2$.

Hajłasz–Sobolev spaces are closely related to the classical (Hardy–)Sobolev and Triebel–Lizorkin spaces. In this paper, we always denote by $\dot{W}^{1,p}(\Omega)$ with $p\in(1,\infty)$ the *homogeneous Sobolev space*, by $\dot{H}^{1,p}(\Omega)$ with $p\in(0,1]$ the *Hardy–Sobolev space* as in [26], [27], and by $\dot{F}^s_{p,q}(\mathbb{R}^n)$ with $s\in\mathbb{R}$ and $p,q\in(0,\infty]$ the *homogeneous Triebel–Lizorkin spaces* as in [31]. It was proved in [15], [21] that $\dot{W}^{1,p}(\Omega)=\dot{M}^{1,p}_{\mathrm{ball}}(\Omega)$ for $p\in(1,\infty)$ and $\dot{H}^{1,p}(\Omega)=\dot{M}^{1,p}_{\mathrm{ball}}(\Omega)$ for $p\in(n/(n+1),1]$, which together with [31] implies that $\dot{M}^{1,p}(\mathbb{R}^n)=\dot{M}^{1,p}_{\mathrm{ball}}(\mathbb{R}^n)=\dot{F}^1_{p,2}(\mathbb{R}^n)$ for all $p\in(n/(n+1),\infty)$, while for all $s\in(0,1)$ and $p\in(n/(n+s),\infty)$, $\dot{M}^{s,p}(\mathbb{R}^n)=\dot{M}^{s,p}_{\mathrm{ball}}(\mathbb{R}^n)=\dot{F}^s_{p,\infty}(\mathbb{R}^n)$ as proved in [22], [35].

Now we recall some notions on imbeddings. Let Ω be a bounded domain of $\mathbb{R}^n$, $s\in(0,1]$ and $p\in(n/(n+s),n/s)$. Then Ω is said to support a $(pn/(n-ps),p)_s$*-Hajłasz–Sobolev–Poincaré* (for short, $(pn/(n-ps),p)_s$-HSP) *imbedding* if there exists a constant $C>0$ such that for all $u\in\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$,

$$\|u-u_\Omega\|_{L^{pn/(n-ps)}(\Omega)}\le C\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}, \tag{1.2}$$

where $u_\Omega\equiv\frac{1}{|\Omega|}\int_\Omega u(z)\,dz$. Similarly, Ω is said to support an *s-Hajłasz–Trudinger* (for short, s-HT) *imbedding* if there exists a constant $C>0$ such that for all $u\in\dot{M}^{s,n/s}_{\mathrm{ball}}(\Omega)$,

$$\|u-u_\Omega\|_{\phi_s(L)(\Omega)}\le C\|u\|_{\dot{M}^{s,n/s}_{\mathrm{ball}}(\Omega)}, \tag{1.3}$$

where and in what follows, $\phi_s(t)\equiv\exp(t^{n/(n-s)})-1$ and

$$\|u\|_{\phi_s(L)(\Omega)}\equiv\inf\left\{t>0, \int_\Omega \phi_s\left(\frac{|u(x)|}{t}\right)dx\le 1\right\}. \tag{1.4}$$

It should be pointed out that since $\dot{M}^{1,p}_{\mathrm{ball}}(\Omega)=\dot{W}^{1,p}(\Omega)$ for all $p\in(1,\infty)$, then (1.2) with $s=1$ and $p\in[1,n)$ coincides with the classical $(pn/(n-p),p)$-Sobolev–Poincaré imbedding as in [5, (1.1)], and (1.3) with $s=1$ coincides with the classical Trudinger imbedding as in [5, (1.2)].

Recently, some geometric criteria were established in [3], [4], [5] for a domain to support a $(pn/(n-p),p)$-Sobolev–Poincaré imbedding for $p\in[1,n)$ or a Trudinger imbedding. More precisely, Bojarski [3] first proved that a John domain as in Definition 2.1 always supports a $(pn/(n-p),p)$-Sobolev–Poincaré imbedding for all $p\in[1,n)$. Smith and Stegenga [29] proved that a weak carrot domain as in Definition 2.2 always supports the Trudinger imbedding. Conversely, let Ω be a bounded planar domain or a bounded domain in $\mathbb{R}^n$ with $n \geq 3$ satisfying an additional separation property when $p \in (1,n)$ and a slice property when $p = n$; see Definitions 2.3 and 2.4 below. Then Buckley and Koskela [4], [5] proved that if Ω supports a $(pn/(n-p),p)$-Sobolev–Poincaré imbedding for some/all $p \in [1,n)$, then it is a John domain, and if Ω supports the Trudinger imbedding, then it is a weak carrot domain.

The purpose of this paper is to establish some geometric criteria for a domain of $\mathbb{R}^n$ with $n \geq 2$ to support a $(pn/(n-ps),p)_s$-HSP imbedding with $s \in (0,1]$ and $p \in (n/(n+s), n/s)$ or an s-HT imbedding with $s \in (0,1]$.

To this end, we first establish the linear local connectivity (for short, LLC) of a domain that supports the $(pn/(n-ps),p)_s$-HSP imbedding, where the notion of LLC was introduced by Gehring [8]. Recall that a domain Ω is said to have the LLC *property* if there exists a positive constant b such that for all $z \in \mathbb{R}^n$ and $r > 0$,

LLC(1) points in $\Omega \cap B(z,r)$ can be joined in $\Omega \cap B(z,r/b)$;
LLC(2) points in $\Omega \setminus B(z,r)$ can be joined in $\Omega \setminus B(z,br)$.

Then, as proved by Gehring and Martio [10], a $\dot{W}^{1,n}$-extension domain has the LLC property, and by [20, Theorem 6.4], a $\dot{W}^{1,p}$-extension domain with $p \in (n-1,n)$ has the LLC(2) property; see also [12], [13], [14], [34] and their references. Here and in what follows, Ω is called an A-extension domain with $A = \dot{M}^{s,p}_{\mathrm{ball}}$, $\dot{W}^{1,p}$ or $\dot{H}^{1,p}$ if for every $u \in A(\Omega)$, there exists a $v \in A(\mathbb{R}^n)$ such that $v|_\Omega = u$ and $\|v\|_{A(\mathbb{R}^n)} \lesssim \|u\|_{A(\Omega)}$. Here, we extend the results in [10], [20] as follows.

THEOREM 1.1. *Let $s \in (0,1]$ and $p \in (n/(n+s), n/s)$. If Ω is a bounded $\dot{M}^{s,p}_{\mathrm{ball}}$-extension domain or Ω is a bounded domain that supports a $(pn/(n-ps),p)_s$-HSP imbedding, then Ω has the LLC(2) property.*

The proof of Theorem 1.1 is given in Section 3. We point out that the approach used here is different from that used by Koskela in [20, Theorem 6.4], where he used the p-capacity to prove the LLC(2) property of a $\dot{W}^{1,p}$-extension domain for $p \in (n-1,n)$. In fact, when $1 < p \leq n-1$, as Koskela [20] pointed out, the p-capacity makes no sense since $\mathrm{Cap}_p(K_0, K_1, \mathbb{R}^n) = 0$ for every pair of disjoint continua $K_0, K_1 \subset \mathbb{R}^n$. So some new ideas are required to prove Theorem 1.1 as the result is new even in the case $s = 1$ and $1 < p \leq n-1$. To this end, we will simplify this question, and then combine some of the ideas from [4], [18], [19] and the properties of Hajłasz–Sobolev functions.

Then, as a corollary to Theorem 1.1, we have the following conclusion, which complements the results in [10], [20].

COROLLARY 1.1. *If Ω is a bounded $\dot{W}^{1,p}$-extension domain when $p \in (1,n)$ or bounded $\dot{H}^{1,p}$-extension domain with $p \in (n/(n+1), 1]$, then Ω has the LLC(2) property.*

Applying Theorem 1.1, we further establish some geometric criteria for a domain to support a $(pn/(n-ps),p)_s$-HSP imbedding, which generalizes the criteria in [3], [4].

THEOREM 1.2. (i) *A John domain of* $\mathbb{R}^n$ *as in Definition* 2.1 *always supports a* $(pn/(n-ps),p)_s$*-HSP imbedding as in* (1.2) *for all* $s\in(0,1]$ *and* $p\in(n/(n+s),n/s)$.

(ii) *Assume that* Ω *is a bounded domain of* $\mathbb{R}^n$ *and satisfies the separation property as in Definition* 2.3. *If* Ω *supports a* $(pn/(n-ps),p)_s$*-HSP imbedding for some* $s\in(0,1]$ *and* $p\in(n/(n+s),n/s)$, *then* Ω *is a John domain.*

To prove Theorem 1.2(ii), we will use the LLC(2) property of these domains given in Theorem 1.1. This is slightly different from that of [4]. On the other hand, notice that $(\mathbb{R}^n,d_s,dx)$ is an Ahlfors n/s-regular metric measure spaces, when $d_s(x,y)=|x-y|^s$ for all $x,y\in\mathbb{R}^n$ and dx denotes the Lebesgue measure. Observe that $M^{s,n/s}_{\mathrm{ball}}(\Omega)$ coincides with $\dot{M}^{1,n/s}_{\mathrm{ball}}(\Omega,d_s,dx)$, the Hajłasz–Sobolev space on domains of $(\mathbb{R}^n,d_s,dx)$ defined similarly to Definition 1.1. Then Theorem 1.2(i) can be deduced from results by Chua and Wheeden [7]. For the reader's convenience, we give a short proof, which will use the ideas from Bojarski [3], the chain property of a John domain as proved by Boman [2], and a key imbedding on balls established by Hajłasz [16, Theorem 8.7].

We also establish an analogue of Theorem 1.2 at the end point $p=n/s$ when $s\in(0,1]$, which generalizes the criteria established in [5], [6], [29], and whose proof uses some ideas from [5], [6], [29], [30] and will be given in Section 4. Also see [24] for similar inequalities on balls.

THEOREM 1.3. (i) *A weak carrot domain of* $\mathbb{R}^n$ *as in Definition* 2.2 *always supports an* s*-HT imbedding for all* $s\in(0,1]$.

(ii) *Assume that* Ω *is a bounded domain of* $\mathbb{R}^n$ *and satisfies the slice property as in Definition* 2.4. *If* Ω *supports an* s*-HT imbedding for some* $s\in(0,1]$, *then* Ω *is a weak carrot domain.*

Notice that, as proved in [4], [5], every simply connected domain in $\mathbb{R}^2$ or every domain in $\mathbb{R}^n$ with $n\geq 3$ that is quasiconformally equivalent to a uniform domain satisfies the slice property and the separation property. So, as a corollary to Theorems 1.2 and 1.3, we have the following conclusion.

COROLLARY 1.2. *Let* Ω *be a bounded simply connected domain in* $\mathbb{R}^2$ *or a bounded domain in* $\mathbb{R}^n$ *with* $n\geq 3$ *that is quasiconformally equivalent to a uniform domain. Then*:

(i) Ω *is a John domain if and only if it supports a* $(pn/(n-ps),p)_s$*-HSP imbedding for some/all* $s\in(0,1]$ *and* $p\in(n/(n+s),n/s)$;

(ii) Ω *is a weak carrot domain if and only if it supports an* s*-HT imbedding for some/all* $s\in(0,1]$.

This paper is organized as follows. In Section 2, we recall some basic notions and properties of the domains and Hajłasz–Sobolev spaces. In Section 3, we present the proof of Theorem 1.1. In Section 4, we give the proofs of Theorems 1.2 and 1.3.

2. Preliminaries

In this section, we recall some notions and basic properties of domains and Hajłasz–Sobolev spaces. We begin with the notion of John domain.

Definition 2.1. Let Ω be a bounded domain of $\mathbb{R}^n$ with $n \geq 2$. Then Ω is called a *John domain* with respect to $x_0 \in \Omega$ and $C > 0$ if for every $x \in \Omega$, there exists a rectifiable curve $\gamma : [0,1] \to \Omega$ parametrized by arclength such that $\gamma(0) = x$, $\gamma(1) = x_0$ and $d(\gamma(t), \Omega^{\complement}) \geq Ct$.

Now we recall the notion of a weak carrot domain (or domains satisfying the quasihyperbolic boundary condition). To this end, for every pair of points $x, y \in \Omega$, define their *quasihyperbolic distance* $k_\Omega(x,y)$ by

$$k_\Omega(x,y) \equiv \inf_\gamma \int_\gamma \frac{1}{d(z,\Omega^{\complement})} |dz|,$$

where the infimum is taken over all rectifiable curves $\gamma \subset \Omega$ joining x and y. As proved in [9], k_Ω is a geodesic distance, namely, there exists a curve $\gamma_{x,y} \subset \Omega$ such that

$$k_\Omega(x,y) = \int_{\gamma_{x,y}} \frac{1}{d(z,\Omega^{\complement})} |dz|.$$

Definition 2.2. A domain Ω is said to satisfy a *weak carrot condition* (or *quasihyperbolic boundary condition*) with respect to $x_0 \in \Omega$ and $C \geq 1$ if for all $x \in \Omega$,

$$k_\Omega(x,x_0) \leq C \log\left(\frac{C}{d(x,\Omega^{\complement})}\right). \tag{2.1}$$

It is easy to see that the John and weak carrot conditions are independent of the choice of x_0 in the sense that if Ω is a John or weak carrot domain with respect to x_0 and C, then for any other $x_1 \in \Omega$, there exists a positive constant $\widetilde{C}$ such that Ω is still a John or weak carrot domain with respect to x_1 and $\widetilde{C}$, respectively. See [6] for more details.

The following characterization of a weak carrot domain established by Smith and Stegenga [29] will be used in the proof of Theorem 1.3.

Lemma 2.1. *Let Ω is a proper subdomain of $\mathbb{R}^n$ and let $x_0 \in \Omega$. Then Ω is a weak carrot domain if and only if there exists a positive constant σ such that*

$$\int_\Omega \exp\big(\sigma k_\Omega(x_0,x)\big)\, dx < \infty.$$

We also recall the notions of a separation property and slice property introduced in [4], [5].

Definition 2.3. A domain Ω has a *separation property* with respect to $x_0 \in \Omega$ and $C > 1$ if for every $x \in \Omega$, there exists a curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$, and such that for each $t \in (0,1]$, either $\gamma([0,t]) \subset B \equiv B(\gamma(t), Cd(\gamma(t), \Omega^{\complement}))$ or each $y \in \gamma([0,t]) \setminus B$ belongs to a different component of $\Omega \setminus \partial B$ than x_0.

Definition 2.4. A domain Ω has a *slice property* with respect to $C > 1$ if for every pair of points $x, y \in \Omega$, there exists a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$, and pairwise disjoint collection of open subsets $\{S_i\}_{i=0}^{j}$, $j \geq 0$, of Ω such that

(i) $x \in S_0$, $y \in S_j$ and x and y are in different components of $\Omega \setminus \overline{S_i}$ for $0 < i < j$;

(ii) if $F \subset\subset \Omega$ is a curve containing both x and y, and $0 < i < j$, then $\operatorname{diam}(S_i) \leq C\ell(F \cap S_i)$;

(iii) for $0 \leq t \leq 1$, $B(\gamma(t), C^{-1}d(\gamma(t), \Omega^{\complement})) \subset \bigcup_{i=0}^{j} S_i$;

(iv) if $0 \leq i \leq j$, then $\operatorname{diam} S_i \leq Cd(z, \Omega^{\complement})$ for all $z \in \gamma_i \equiv \gamma \cap S_i$; also, there exists $x_i \in S_i$ such that $x_0 = x$, $x_j = y$ and $B(x_i, C^{-1}d(x_i, \Omega^{\complement})) \subset S_i$.

We point out that, as proved in [4], [5], every simply connected domain in $\mathbb{R}^2$ or every domain in $\mathbb{R}^n$ with $n \geq 3$ that is quasiconformally equivalent to a uniform domain satisfies a slice property and a separation property. Every John domain satisfies both a separation and a slice property; see [6].

The following conclusion is essentially established in [21] and plays an important role in the proofs of Theorems 1.1, 1.2 and 1.3. For every $\rho > 0$, similarly to $\mathcal{D}^s_{\mathrm{ball}}(u)$, we denote by $\mathcal{D}^{s,\rho}_{\mathrm{ball}}(u)$ the collection of all measurable functions g such that (1.1) holds for all $x, y \in \Omega \setminus E$ satisfying $|x-y| < \rho \operatorname{dist}(x, \partial\Omega)$. Notice that $\mathcal{D}^s_{\mathrm{ball}}(u) = \mathcal{D}^{s,1/2}_{\mathrm{ball}}(u)$ and $\mathcal{D}^s(u) = \mathcal{D}^{s,\infty}_{\mathrm{ball}}(u)$.

Lemma 2.2. *Let $s \in (0,1]$ and $p \in (n/(n+s), \infty)$. Then $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ if and only if there exists a $\rho \in (0,1)$ such that $\inf_{g \in \mathcal{D}^{s,\rho}_{\mathrm{ball}}(u)} \|g\|_{L^p(\Omega)} < \infty$. Moreover, for given ρ, there exists a positive constant C such that for all $u \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$,*

$$C^{-1}\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \leq \inf_{g \in \mathcal{D}^{s,\rho}_{\mathrm{ball}}(u)} \|g\|_{L^p(\Omega)} \leq C\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}.$$

We also need the following imbedding, which is essentially established by Hajłasz [16, Theorem 8.7] when $n = 1$ and pointed out by Yang [35] when $s \in (0,1)$.

Lemma 2.3. *Let $s \in (0,1]$ and $p \in (n/(n+s), n/s)$. Then for every $\sigma > 1$, there exists a positive C constant such that for all balls or cubes B and $u \in \dot{M}^{s,p}(\sigma B)$,*

$$\|u - u_B\|_{L^{pn/(n-ps)}(B)} \leq C\|u\|_{\dot{M}^{s,p}(\sigma B)}.$$

By Lemma 2.3, we have the following conclusion.

LEMMA 2.4. *Let $s\in(0,1]$ and $p\in(n/(n+s),n/s)$. Then a bounded $\dot{M}^{s,p}_{\mathrm{ball}}$-extension domain always supports a $(pn/(n-ps),p)_s$-HSP imbedding.*

Proof. Assume that Ω is an $\dot{M}^{s,p}_{\mathrm{ball}}$-extension domain. Let $u\in\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$. Then there exists a $v\in\dot{M}^{s,p}_{\mathrm{ball}}(\mathbb{R}^n)$ such that $v|_\Omega=u$ and $\|v\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\mathbb{R}^n)}\lesssim\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}$. Let B be a ball of $\mathbb{R}^n$ such that $\Omega\subset B$. Then $v\in\dot{M}^{s,p}_{\mathrm{ball}}(2B)$ and thus by Lemma 2.2, we have $v\in L^{pn/(n-ps)}(B)$ and

$$\|v-v_B\|_{L^{pn/(n-ps)}(B)}\lesssim\|v\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\mathbb{R}^n)}\lesssim\|v\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}$$

which further implies that

$$\begin{aligned}\|u-u_\Omega\|_{L^{pn/(n-ps)}(\Omega)}&\le\|v-v_\Omega\|_{L^{pn/(n-ps)}(B)}\\&\lesssim\|v-v_B\|_{L^{pn/(n-ps)}(B)}\lesssim\|v\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}.\end{aligned}$$

This means that Ω supports a $(pn/(n-ps),p)_s$-HSP imbedding and thus finishes the proof of Lemma 2.4. □

Finally, we state some conventions. Throughout the paper, we denote by C a positive constant which is independent of the main parameters, but which may vary from line to line. Constants with subscripts, such as C_0, do not change in different occurrences. The symbol $A\lesssim B$ or $B\gtrsim A$ means that $A\le CB$. If $A\lesssim B$ and $B\lesssim A$, we then write $A\sim B$. For any locally integrable function f, we denote by $⨍_E f$ the average of f on E, namely, $⨍_E f\equiv\frac{1}{|E|}\int_E f\,dx$.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. By Lemma 2.4, it suffices to prove that a domain which supports a $(pn/(n-ps),p)_s$-HSP imbedding has the LLC(2) property. Assume that Ω is a bounded domain that supports a $(pn/(n-ps),p)_s$-HSP imbedding. We want to show that Ω has the LLC(2) property. To this end, let $L\equiv\operatorname{diam}\Omega$ and $x_0\in\Omega$ be such that $r_0\equiv d(x_0,\Omega^{\complement})=\max\{d(x,\Omega^{\complement}):x\in\Omega\}$. Notice that if $u(y)=0$ for all $y\in B(x_0,r_0)$, then the $(pn/(n-ps),p)_s$-HSP imbedding implies that

$$\|u\|_{L^{pn/(n-ps)}(\Omega)}\lesssim\|u\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)},\tag{3.1}$$

where the constant depends on r_0 and $|\Omega|$ but not on u.

We claim that if $x,x_0\in\Omega\setminus B(z,r)$ for $z\in B(x_0,2L)$ and $r\in(0,2L)$, then x,x_0 are contained in the same component of $\Omega\setminus B(z,br)$ for some fixed constant $b\in(0,1)$, which may depend on Ω and x_0 but not on z and x.

Assume that the above claim holds for the moment. Then we deduce Theorem 1.1 from it by the following 2 steps. Let $x,y\in\Omega\setminus B(z,r)$ for $z\in\mathbb{R}^n$ and $r\in(0,\infty)$.

Step 1. There exists a positive constant $\widetilde{b}$ independent of x such that if $x, x_0 \in \Omega \setminus B(z,r)$, then x, x_0 are contained in the same component of $\Omega \setminus B(z, \widetilde{b}r)$. To see this, assume that $x_0 \in \Omega \setminus B(z,r)$. If $z \notin B(x_0, 2L)$, then $\Omega \cap B(z,r) \neq \emptyset$ implies that $r \geq d(z,x_0) - L \geq r/2 \geq L$, and moreover $\Omega \setminus B(z,r) \neq \emptyset$ implies that $\Omega \cap B(z,r/2) = \emptyset$. Thus if $x_0, x \in \Omega \setminus B(z,r)$ with $d(z,x_0) \geq 2L$, then x_0, x are contained in the same component of $\Omega \setminus B(z,r)$ if $\Omega \setminus B(z,r) = \emptyset$ or of $\Omega \setminus B(z,r/2)$ if $\Omega \setminus B(z,r) \neq \emptyset$. If $z \in B(x_0, 2L)$, then by the above claim, it suffices to consider the case $r \geq 2L$. Since $r \geq 2L$ implies $d(z,\Omega) \geq r - L \geq r/2$, which means that $\Omega \cap B(z,r/2) = \emptyset$, then x_0, x are contained in the same component of $\Omega \setminus B(z,r/2)$.

Step 2. There exists a positive constant b independent of x, y such that x, y are contained in the same component of $\Omega \setminus B(z, br)$. To see this, if $x_0 \in \Omega \setminus B(z, \frac{r_0}{10L} r)$, then x_0, x and x_0, y, and thus x, y, are contained in the same component of $\Omega \setminus B(z, \widetilde{b} \frac{r_0}{10L} r)$. If $x_0 \in B(z, \frac{r_0}{10L} r)$, then $r - \frac{r_0}{10L} r \leq L$, which implies that $r \leq 2L$ and thus $|z - x_0| \leq \frac{r_0}{10L} r \leq r_0/5$. Obviously,

$$B\left(z, \frac{r_0}{10L} r\right) \subset B\left(x_0, \frac{r_0}{5L} r\right) \subset B(x_0, r_0) \subset \Omega,$$

which means that $\Omega \setminus B(z, \frac{r_0}{10L} r)$ is connected, and thus x, y are contained in the same component of $\Omega \setminus B(z, \frac{r_0}{10L} r)$.

Therefore, with the aid of the above claim, combining Step 1 and Step 2, we obtain Theorem 1.1. So we have reduced Theorem 1.1 to the above claim. The remainder of the proof of Theorem 1.1 consists of the proof of the above claim.

In the following argument, we let $x \in \Omega$, $z \in B(x_0, 2L)$ and $r \in (0, 2L)$ be fixed such that $x, x_0 \in \Omega \setminus B(z,r)$ as in the claim. Let $b_z \in (0,1]$ be the supremum of $b \in (0,1)$ such that x, x_0 are contained in the same component of $\Omega \setminus \overline{B(z,br)}$. Without loss of generality, we assume that $b_z \leq 1/10$. Denote by Ω_x the component of $\Omega \setminus \overline{B(z,b_0 r)}$ with $b_0 = 2b_z$ containing x. Take $b_1 \in (b_0, 1]$ such that

$$\big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_1 r)\big)\big| = \frac{1}{2}\big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_0 r)\big)\big| = \frac{1}{2}\big|\Omega_x \cap B(z,r)\big|.$$

Define a function u on Ω by setting

$$u(y) \equiv \begin{cases} 0, & y \in \Omega \setminus \Omega_x; \\ \frac{d(y, B(z,b_0 r))}{b_1 r - b_0 r}, & y \in \Omega_x \cap B(z, b_1 r); \\ 1, & y \in \Omega_x \setminus B(z, b_1 r). \end{cases} \tag{3.2}$$

Then we have the following conclusion, whose proof will be given below.

LEMMA 3.1. *Let u be as in (3.2) and $s \in (0,1]$. Then*

$$g \equiv C(b_1 r - b_0 r)^{-s} \chi_{\Omega_x \cap B(z,r)}$$

is an element of $\mathcal{D}^{s,1/8}_{\rm ball}(u)$, *where* C *is a positive constant independent of* u, x, b_0, b_1, r.

By Lemma 2.2, Lemma 3.1 and (3.1), we further have $u \in \dot{M}^{s,p}_{\rm ball}(\Omega)$ and

$$\|u\|_{L^{pn/(n-ps)}(\Omega)} \lesssim \|u\|_{\dot{M}^{s,p}_{\rm ball}(\Omega)} \lesssim \|g\|_{L^p(\Omega)} \\ \lesssim (b_1 r - b_0 r)^{-s} \big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_0 r)\big)\big|^{1/p},$$

which together with

$$\|u\|_{L^{pn/(n-ps)}(\Omega)} \gtrsim \big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_1 r)\big)\big|^{(n-ps)/pn} \\ \gtrsim \big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_0 r)\big)\big|^{(n-ps)/pn}$$

implies that

$$b_1 r - b_0 r \lesssim \big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_0 r)\big)\big|^{1/n}. \tag{3.3}$$

Hence, if $b_1 \geq 1/2$, then (3.3) implies that

$$(1/2 - b_0) r \lesssim \big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_0 r)\big)\big|^{1/n}. \tag{3.4}$$

If $b_1 < 1/2$, then following the above procedure, we can find a sequence $\{b_j\}_{j=1}^{j_0}$ such that $b_{j_0} \geq 1/2$ and for all $0 \leq j \leq j_0 - 1$, $b_j < 1/2$,

$$\big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_{j+1} r)\big)\big| = \frac{1}{2}\big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_j r)\big)\big|,$$

and

$$b_{j+1} r - b_j r \lesssim \big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_j r)\big)\big|^{1/n}.$$

This implies that

$$\sum_{j=0}^{j_0-1} (b_{j+1} r - b_j r) \lesssim \big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_0 r)\big)\big|^{1/n},$$

and hence (3.4). To control $|\Omega_x \cap (B(z,r) \setminus B(z,b_0 r))|^{1/n}$ via $b_0 r$, define function

$$v(y) \equiv \inf_{\gamma(x_0,y)} \ell\big(\gamma \cap B(z,b_0 r)\big) \tag{3.5}$$

for all $y \in \Omega$, where the infimum is taken over all the rectifiable curves γ joining x_0 and y in Ω. Observe that for all y in the component of $\Omega \setminus \overline{B(z,b_0 r)}$ containing x_0, $v(y) = 0$; for all y in the component $\Omega_x \setminus \overline{B(z,b_0 r)}$ which contains x and does not contain x_0, $v(y)$ is a constant larger than or equal to $b_0 r$. Moreover, we have the following conclusion, whose proof will be given below.

LEMMA 3.2. *Let* v *be as in* (3.5) *and* $s \in (0,1]$. *Then*

$$h \equiv C(b_0 r)^{1-s} \chi_{\Omega \cap B(z,b_0 r)}$$

is an element of $\mathcal{D}^{s,1/8}_{\rm ball}(v)$, *where* C *is a positive constant independent of* v, z, b_0, r.

By Lemma 2.2, Lemma 3.2 and (3.1), we have that $v \in \dot{M}^{s,p}_{\mathrm{ball}}(\Omega)$ and

$$(b_0 r)\big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_0 r)\big)\big|^{(n-ps)/pn} \lesssim \big|\Omega \cap B(z,b_0 r)\big|^{1/p}(b_0 r)^{1-s}$$

which implies that

$$\big|\Omega_x \cap \big(B(z,r) \setminus B(z,b_0 r)\big)\big| \lesssim (b_0 r)^n.$$

By this and (3.4), we have $(1/2 - b_0)r \lesssim b_0 r$, which implies that $b_0 \geq C$ for some fixed constant $C \in (0,1)$ independent of x. This gives the above claim by taking $b = C/4$ and thus finishes the proof of Theorem 1.1. □

Proof of Lemma 3.1. It suffices to check that for every pair of $y, w \in \Omega$ such that $|y - w| < \mathrm{dist}(y, \Omega^{\complement})/8$,

$$\big|u(y) - u(w)\big| \lesssim \frac{|y-w|^s}{(b_1 r - b_0 r)^s}\big[\chi_{\Omega_x \cap B(z,r)}(y) + \chi_{\Omega_x \cap B(z,r)}(w)\big]. \tag{3.6}$$

To prove (3.6), without loss of generality, we may assume that $u(w) < u(y)$. Then $u(y) > 0$ implies that $y \in \Omega_x$ and $u(w) < 1$ implies that $w \notin \Omega_x \setminus B(z, b_1 r)$. We will consider the following three cases for w: (i) $w \in \Omega_x \cap B(z, b_1 r)$; (ii) $w \in \Omega \cap \overline{B(z, b_0 r)}$; (iii) $w \in \Omega \setminus (\Omega_x \cup \overline{B(z, b_0 r)})$.

Case (i). If $y \in \Omega_x \setminus B(z, b_1 r)$, then by $w \in \Omega_x \cap B(z, b_1 r)$, we have

$$d\big(w, B(z, b_0 r)\big) = |w - z| - b_0 r \geq |z - y| - |w - y| - b_0 r \geq (b_1 r - b_0 r) - |w, y|,$$

and thus

$$\begin{aligned}\big|u(y) - u(w)\big| &= \left|1 - \frac{d(w, B(z, b_0 r))}{b_1 r - b_0 r}\right| \\ &\leq \left|1 - \frac{d(w, B(z, b_0 r))}{b_1 r - b_0 r}\right|^s \leq \frac{|w-y|^s}{(b_1 r - b_0 r)^s},\end{aligned}$$

which gives (3.6). If $y \in \Omega_x \cap B(z, b_1 r)$, then by $|w - y| \leq b_1 r - b - 0r$,

$$\begin{aligned}\big|u(y) - u(w)\big| &= \left|\frac{d(y, B(z, b_0 r)) - d(w, B(z, b_0 r))}{b_1 r - b_0 r}\right| \\ &\leq \frac{|w-y|}{b_1 r - b_0 r} \leq \frac{|w-y|^s}{(b_1 r - b_0 r)^s},\end{aligned}$$

which gives (3.6).

Case (ii). If $y \in \Omega_x \cap B(z, r)$, then

$$d\big(y, B(z, b_0 r)\big) \leq |y - w| \leq b_1 r - b_0 r$$

and thus

$$\begin{aligned}\big|u(y) - u(w)\big| &= \left|\min\left\{1, \frac{d(y, B(z, b_0 r))}{b_1 r - b_0 r}\right\}\right| \\ &\leq \left|\min\left\{1, \frac{|w-y|}{b_1 r - b_0 r}\right\}\right|^s \leq \frac{|w-y|^s}{(b_1 r - b_0 r)^s},\end{aligned}$$

which gives (3.6). If $y \in \Omega_x \setminus B(z,r)$, then $|w-y| \geq (1-b_0)r \geq b_0 r$. Since

$$d\big(w,\Omega^{\complement}\big) \leq |w-z| + d\big(z,\Omega^{\complement}\big) \leq 2b_0 r,$$

we have that $|w-y| \geq 2d(w,\Omega^{\complement})$. Moreover, since

$$d\big(y,\Omega^{\complement}\big) \leq |y-w| + d\big(w,\Omega^{\complement}\big) \leq 2|y-w|,$$

by the definition of $\mathcal{D}^{s,1/8}_{\mathrm{ball}}(u)$, we do not need to check (3.6) for $y \in \Omega_x \setminus B(z,r)$.

Case (iii). We will prove that in this case,

$$|y-w| \geq \frac{1}{8}\max\big\{d\big(y,\Omega^{\complement}\big), d\big(w,\Omega^{\complement}\big)\big\}. \tag{3.7}$$

Thus, we do not need to check (3.6) by the definition of $\mathcal{D}^{s,1/8}_{\mathrm{ball}}(u)$. To prove (3.7), notice that $y \in \Omega_x$ and $w \notin \Omega_x \cup \overline{B(z,b_0 r)}$ implies that y and w are in different components of $\Omega \setminus \overline{B(z,b_0 r)}$. If $|y-w| < d(y,\Omega^{\complement})/4$, then $B(y,2|w-y|) \subset B(y,d(y,\Omega^{\complement})) \subset \Omega$. Observe that either $B(y,2|w-y|) \setminus \overline{B(z,b_0 r)} = \emptyset$ or $B(y,2|w-y|) \setminus \overline{B(z,b_0 r)}$ is connected. So $y,w \in B(y,2|w-y|) \setminus \overline{B(z,b_0 r)} \subset \Omega$ means that $B(y,2|w-y|) \setminus \overline{B(z,b_0 r)}$ are connected and thus y,w are in the same component of $\Omega \setminus \overline{B(z,b_0 r)}$, which is a contradiction. If $|y-w| < d(y,\Omega^{\complement})/8$, then for all $z \neq \Omega$,

$$d(z,y) \geq d(z,w) - d(w,y) \geq d\big(w,\Omega^{\complement}\big)/2,$$

which implies that $d(y,\Omega^{\complement}) \geq 7d(w,\Omega^{\complement})/8$ and thus $|y-w| < d(y,\Omega^{\complement})/4$. Thus, (3.7) holds. This finishes the proof of Lemma 3.1. □

Proof of Lemma 3.2. This is quite similar to the proof of Lemma 3.1. We sketch the proof. It suffices to check that for every pair of $y,w \in \Omega$ such that $|y-w| < \mathrm{dist}(y,\Omega^{\complement})/8$,

$$\big|v(y)-v(w)\big| \lesssim (b_0 r)^{1-s}\big[\chi_{\Omega_x \cap B(z,b_0 r)}(y) + \chi_{\Omega_x \cap B(z,b_0 r)}(w)\big]. \tag{3.8}$$

If both y and w are in the same component of $\Omega \setminus \overline{B(z,b_0 r)}$, then (3.8) holds. If y,w are in different components of $\Omega \setminus \overline{B(z,b_0,r)}$, by an argument similar to that of (3.7), we can prove that (3.7) still holds, and thus we do not need to check (3.8) for such y,w. So we can assume that one of w,y is in $\Omega \cap \overline{B(z,b_0 r)}$. Notice that in this case $I(w,y) \subset \Omega$, where $I(w,y)$ denotes the line segment joining w and y. Then

$$\begin{aligned}\big|u(y)-u(w)\big| &\leq \ell\big(I(w,y) \cap B(z,b_0 r)\big) \leq \min\big\{2b_0 r, |y-w|\big\}\\ &\lesssim (b_0 r)^{1-s}|y-w|^s,\end{aligned}$$

which implies (3.8). This finishes the proof of Lemma 3.2. □

4. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. (i) Assume that Ω is a bounded John domain. Then, as proved by Boman [2], Ω enjoys the following *chain property*: there exist a positive constant $\widetilde{C}$ and a sequence of subcubes of Ω, which is denoted by $\mathcal{F}$, such that:

(a) $\chi_\Omega(x) \le \sum_j \chi_{Q_i}(x) \le \sum_j \chi_{2Q_i}(x) \le \widetilde{C}\chi_\Omega(x)$ for all $x \in \mathbb{R}^n$;

(b) for a fixed subcube $Q_0 \in \mathcal{F}$ and any other $Q \in \mathcal{F}$, there exists a subsequence $\{Q_j\}_{j=1}^N \subset \mathcal{F}$ satisfying that $Q = Q_N \subset \widetilde{C}Q_j$, $\widetilde{C}^{-1}|Q_{j+1}| \le |Q_j| \le \widetilde{C}|Q_{j+1}|$ and $|Q_j \cap Q_{j+1}| \ge \widetilde{C}^{-1}\min\{|Q_j|, |Q_{j+1}|\}$ for all $j = 0, \dots, N-1$.

Let $u \in \dot{M}^{s,p}_{\rm ball}(\Omega)$ and $g \in \mathcal{D}^s_{\rm ball}(u)$ with $\|g\|_{L^p(\Omega)} \lesssim \|u\|_{\dot{M}^{s,p}(\Omega)}$. Then

$$\begin{aligned}
&\int_\Omega \big|u(z) - u_{Q_0}\big|^{pn/(n-ps)}\,dz \\
&\quad\lesssim \sum_{Q\in\mathcal{F}} \int_Q \big|u(z) - u_Q\big|^{pn/(n-ps)}\,dz + \sum_{Q\in\mathcal{F}} |Q||u_Q - u_{Q_0}|^{pn/(n-ps)} \\
&\quad\equiv I_1 + I_2.
\end{aligned}$$

Then by Lemma 2.3, $n/(n-ps) > 1$ and the above chain property, we have

$$\begin{aligned}
I_1 &\lesssim \sum_{Q\in\mathcal{F}} \left(\int_{2Q} \big[g(z)\big]^p\,dz\right)^{n/(n-ps)} \\
&\lesssim \left(\sum_{Q\in\mathcal{F}} \int_{2Q} \big[g(z)\big]^p\,dz\right)^{n/(n-ps)} \\
&\lesssim \left(\int_\Omega \big[g(z)\big]^p\,dz\right)^{n/(n-ps)} \lesssim \|u\|^{pn/(n-ps)}_{\dot{M}^{s,p}_{\rm ball}(\Omega)}.
\end{aligned}$$

To estimate I_2, for every $Q \in \mathcal{F}$, let $\{Q_j\}_{j=1}^N$ be as in (b). Then we have

$$\begin{aligned}
|u_Q - u_{Q_0}| &\lesssim \sum_{j=0}^{N-1} \left(|u_{Q_j} - u_{Q_j\cap Q_{j+1}}| + |u_{Q_j\cap Q_{j+1}} - u_{Q_{j+1}}|\right) \\
&\lesssim \sum_{j=0}^{N} ⨍_{Q_j} \big|u(z) - u_{Q_j}\big|\,dz \\
&\lesssim \sum_{j=0}^{N} |Q_j|^{s/n} \left(⨍_{2Q_j} \big[g(z)\big]^p\,dz\right)^{1/p} \\
&\lesssim \sum_{\widetilde{Q}\in\mathcal{F}: Q\subset 2\widetilde{Q}} |\widetilde{Q}|^{s/n} \left(⨍_{2\widetilde{Q}} \big[g(z)\big]^p\,dz\right)^{1/p}.
\end{aligned}$$

Thus, by $Q \subset \widetilde{C}\widetilde{Q}$ for all $\widetilde{Q} \in \mathcal{F}$, we obtain

$$\begin{aligned} I_2 &\lesssim \sum_{Q\in\mathcal{F}} |Q| \left\{ \sum_{\widetilde{Q}\in\mathcal{F}: Q\subset \widetilde{C}\widetilde{Q}} |\widetilde{Q}|^{s/n} \left(\fint_{2\widetilde{Q}} [g(z)]^p\, dz \right)^{1/p} \right\}^{pn/(n-ps)} \\ &\lesssim \sum_{Q\in\mathcal{F}} \int_Q \left\{ \sum_{\widetilde{Q}\in\mathcal{F}} |\widetilde{Q}|^{s/n} \left(\fint_{2\widetilde{Q}} [g(z)]^p\, dz \right)^{1/p} \chi_{\widetilde{C}\widetilde{Q}}(x) \right\}^{pn/(n-ps)} dx \\ &\lesssim \int_\Omega \left\{ \sum_{\widetilde{Q}\in\mathcal{F}} \left[\mathcal{M}\left(\left[|\widetilde{Q}|^{s/n} \left(\fint_{2\widetilde{Q}} [g(z)]^p\, dz \right)^{1/p} \chi_{\widetilde{Q}} \right]^{1/2} \right)(x) \right]^2 \right\}^{pn/(n-ps)} dx, \end{aligned}$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator. Then by the vector-valued inequality of $\mathcal{M}$ (see, for example, [33]), we have

$$\begin{aligned} I_2 &\lesssim \int_\Omega \left\{ \sum_{\widetilde{Q}\in\mathcal{F}} |\widetilde{Q}|^{s/n} \left(\fint_{2\widetilde{Q}} [g(z)]^p\, dz \right)^{1/p} \chi_{\widetilde{Q}}(x) \right\}^{pn/(n-ps)} dx \\ &\lesssim \left(\sum_{\widetilde{Q}\in\mathcal{F}} \int_{2\widetilde{Q}} [g(z)]^p\, dz \right)^{n/(n-ps)} \\ &\lesssim \left(\int_\Omega [g(z)]^p\, dz \right)^{n/(n-ps)} \lesssim \|u\|^{pn/(n-ps)}_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)}. \end{aligned}$$

This estimate finishes the proof of Theorem 1.2(i).

(ii) Assume that Ω is bounded domain and has a separation property with respect to $x_0 \in \Omega$ and $C_0 \geq 1$. For any fixed point $x \in \Omega$, let γ be a curve as in Definition 2.3. We claim that $d(\gamma(t), \Omega^{\complement}) \gtrsim \operatorname{diam} \gamma([0,t])$ for all $t \in [0,1]$. Assume this claim holds for the moment. Then, as pointed out in [4], even though the claim is not enough to ensure that γ is a John curve for x, it is known that the claim is enough to guarantee that γ can be modified to yield a John curve for x by the arguments in [25, pp. 385–386] and [28, pp. 7–8].

To prove the above claim, let $N = 2 + C_0/b$, where b is the constant for which LLC(2) holds. For $t \in (0,1]$, if $d(\gamma(t), \Omega^{\complement}) \geq d(x_0, \Omega^{\complement})/N$, then

$$\gamma\big([0,t]\big) \subset \Omega \subset B\left(\gamma(t), \frac{N \operatorname{diam} \Omega}{d(x_0, \Omega^{\complement})} d\big(\gamma(t), \Omega^{\complement}\big) \right),$$

which implies the above claim. Assume that $d(\gamma(t), \Omega^{\complement}) < d(x_0, \Omega^{\complement})/N$. Now it suffices to prove that $\gamma([0,t]) \subset B(\gamma(t), (N-1)d(\gamma(t), \Omega^{\complement}))$. To this end, if $y \in \gamma([0,t]) \setminus B(\gamma(t), (N-1)d(\gamma(t), \Omega^{\complement}))$, since $d(x_0, \gamma(t)) \geq (N-1)d(\gamma(t), \Omega^{\complement})$, then by Theorem 1.1(iii), we know that x_0 and y are contained in the same component of $\Omega \setminus B(\gamma(t), b(N-1)d(\gamma(t), \Omega^{\complement}))$, by $b(N-1) > C_0$, we further know that x_0 and y are in the same component of $\Omega \setminus \partial B(\gamma(t), C_0 d(\gamma(t), \Omega^{\complement}))$, which is a contradiction with the separation property. This verifies the above claim and thus finishes the proof of Theorem 1.2(ii). □

To prove Theorem 1.3, we first establish the following result, which is an improvement on [16, Theorem 8.7(ii)].

LEMMA 4.1. *Let* $s \in (0,1]$. *Then there exist positive constants* $0 < C_1 < 1 < C_2$ *such that for all balls* $B \subset \mathbb{R}^n$ *and* $u \in \dot{M}^{s,n/s}(4B)$,

$$\fint_B \exp\left(C_1 \frac{|u(x) - u_B|}{\|u\|_{\dot{M}^{s,n/s}(4B)}}\right)^{n/(n-s)} dx \le C_2. \tag{4.1}$$

Proof. Assume that $B \equiv B(x_0, 2^{-k_0})$ for some $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$. Let $u \in \dot{M}^{s,p}(4B)$ and $g \in \mathcal{D}^s(u)$ such that $\|g\|_{L^{n/s}(4B)} \le 2\|u\|_{\dot{M}^{s,p}(4B)}$. We extend g to the whole $\mathbb{R}^n$ by setting $g(z) = 0$ for all $z \in \mathbb{R}^n \setminus 4B$. For every Lebesgue point x of u, we have

$$\begin{aligned}
&\left|u(x) - u_{B(x,2^{-k_0-1})}\right| \\
&\quad \le \sum_{j \ge k_0+1} |u_{B(x,2^{-j-1})} - u_{B(x,2^{-j})}| + |u_{B(x,2^{-k_0+1})} - u_B| \\
&\quad \lesssim \sum_{j \ge k_0+1} \fint_{B(x,2^{-j})} \left|u(z) - u_{B(x,2^{-j})}\right| dz \\
&\quad \lesssim \sum_{j \ge k_0+1} \fint_{B(x,2^{-j})} \left|u(z) - u_{B(x,2^{-j+2}) \setminus B(x,2^{-j+1})}\right| dz \\
&\quad \lesssim \sum_{j \ge k_0+1} \fint_{B(x,2^{-j})} \fint_{B(x,2^{-j+2}) \setminus B(x,2^{-j+1})} \left|u(z) - u(y)\right| dy\, dz \\
&\quad \lesssim \sum_{j \ge k_0+1} \fint_{B(x,2^{-j})} \int_{B(x,2^{-j+2}) \setminus B(x,2^{-j+1})} \frac{|g(z) + g(y)|}{|y - x|^{n-s}}\, dy\, dz \\
&\quad \lesssim \sum_{j \ge k_0+1} \int_{B(x,2^{-j+2}) \setminus B(x,2^{-j+1})} \frac{|\mathcal{M}(g)(y)|}{|y - x|^{n-s}}\, dy \\
&\quad \lesssim \int_{B(x_0,2^{-k_0+2})} \frac{|\mathcal{M}(g)(y)|}{|y - x|^{n-s}}\, dy.
\end{aligned}$$

Similarly,

$$|u_B - u_{B(x,2^{-k_0-1})}| \lesssim \fint_B \left|u(z) - u_B\right| dz \lesssim \int_{B(x_0,2^{-k_0+2})} \frac{|\mathcal{M}(g)(y)|}{|y - x|^{n-s}}\, dy.$$

Thus,

$$\left|u(x) - u_B\right| \lesssim \int_{B(x_0,2^{-k_0+2})} \frac{|\mathcal{M}(g)(y)|}{|y - x|^{n-s}}\, dy.$$

Then by [11, Lemma 7.2], for all $q \geq n/s$,

$$\|u-u_B\|_{L^q(B)} \leq q^{1-s/n+1/q}\big|B(0,1)\big|^{1-s/n}|B|^{1/q}\big\|\mathcal{M}(g)\big\|_{L^{n/s}(4B)},$$

which together with the $L^{n/s}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$ implies that

$$\fint_B \big|u(z)-u_B\big|^q\,dz \lesssim q^{1+(n-s)/(nq)}\big|B(0,1)\big|^{nq/(n-s)}\|g\|^q_{L^{n/s}(4B)},$$

and hence for all $q \geq n/s-1$,

$$\fint_B \big|u(z)-u_B\big|^{qn/(n-s)}\,dz \lesssim \frac{nq}{n-s}\bigg(\big|B(0,1)\big|\frac{nq}{n-s}\|g\|^{n/(n-s)}_{L^{n/s}(4B)}\bigg)^q.$$

Then taking $\sigma > [e|B(0,1)|n/(n-s)]^{(n-s)/n}$, we have

$$\begin{aligned}&\fint_B \sum_{j\geq \lfloor n/s\rfloor}\frac{1}{j!}\bigg(\frac{|u(x)-u_B|}{\sigma\|g\|_{L^{n/s}(4B)}}\bigg)^{jn/(n-s)}\,dx\\ &\quad\lesssim \sum_{j\geq 1}\bigg(\frac{n|B(0,1)|}{(n-s)\sigma^{n/(n-s)}}\bigg)^j\frac{j^j}{(j-1)!}\lesssim 1.\end{aligned}$$

Notice that by the Hölder inequality, we have

$$\begin{aligned}&\fint_B \sum_{j=0}^{\lfloor n/s\rfloor}\frac{1}{j!}\bigg(\frac{|u(x)-u_B|}{\sigma\|g\|_{L^{n/s}(4B)}}\bigg)^{jn/(n-s)}\,dx\\ &\quad\lesssim \sum_{j=0}^{\lfloor n/s\rfloor}\bigg(\fint_B \frac{|u(x)-u_B|^{n/s}}{\|g\|^{n/s}_{L^{n/s}(4B)}}\,dx\bigg)^{(n-s)/(j-s)}\lesssim 1.\end{aligned}$$

This gives (4.1) and thus finishes the proof of Lemma 4.1. □

Proof of Theorem 1.3. (i) Assume that Ω is a weak carrot domain. Since Ω is bounded (see [29, Corollary 1]), we may assume that $|\Omega|=1$. Let

$$\widetilde{\phi}_s(t)=\exp\big(t^{n/(n-s)}\big)-\sum_{j=0}^{j_0}\frac{1}{j!}t^{jn/(n-s)}$$

for $t\in(0,\infty)$, where j_0 denotes the maximal integer no more than $n/s-1$. Since $\widetilde{\phi}_s(t)\sim\phi_s(t)$ for $t\geq 1$, so we only need to prove (1.3) for $\widetilde{\phi}_s$; see [1]. It further suffices to prove that there exists a $\sigma\in(0,1)$ such that for all $u\in \dot{M}^{s,n/s}_{\mathrm{ball}}(\Omega)$ with $\|u\|_{\dot{M}^{s,n/s}_{\mathrm{ball}}(\Omega)}=1$,

$$\int_\Omega \widetilde{\phi}_s\big(\sigma\big|u(x)-u_\Omega\big|\big)\,dx \leq 1. \tag{4.2}$$

Let $u\in \dot M^{s,n/s}_{\rm ball}(\Omega)$ with $\|u\|_{\dot M^{s,n/s}_{\rm ball}(\Omega)}=1$ and write

$$\int_\Omega \widetilde{\phi}_s\big(\sigma\big|u(x)-u_\Omega\big|\big)\,dx=\sum_{j=j_0+1}\frac{1}{j!}\sigma^{jn/(n-s)}\int_\Omega\big|u(x)-u_\Omega\big|^{jn/(n-s)}\,dx.$$

Since $\Omega\subset\bigcup_{z\in\Omega}B(z,d(z,\Omega^{\complement})/10)$, then by the standard 1/5-covering theorem, there exist points $\{z_i\}_i\subset\Omega$ such that $\{B(z_i,d(z,\Omega^{\complement})/50)\}_i$ are pairwise disjoint,

$$1\le\sum_i\chi_{B(z_i,d(z_i,\Omega^{\complement})/10)}\le\widetilde{C}\chi_\Omega$$

for some fixed positive constant $\widetilde{C}$. Denote by W the collections of balls $\{B(z_i,d(z_i,\Omega^{\complement})/10)\}$. Let B_0 be a fixed ball in W with the largest radius. Denote by x_B the center of ball B and specially x_0 of B_0. Then

$$\begin{aligned}\int_\Omega\big|u(x)-u_\Omega\big|^{jn/(n-s)}\,dx&\le 2^{jn/(n-s)}\int_\Omega\big|u(x)-u_{B_0}\big|^{jn/(n-s)}\,dx\\&\le 2^{jn/(n-s)}\sum_{B\in W}\int_B\big|u(x)-u_{B_0}\big|^{jn/(n-s)}\,dx\\&\le 4^{jn/(n-s)}\sum_{B\in W}\int_B\big|u(x)-u_B\big|^{jn/(n-s)}\,dx\\&\quad+4^{jn/(n-s)}\sum_{B\in W}|B||u_B-u_{B_0}|^{jn/(n-s)}\\&\equiv I_1(j)+I_2(j).\end{aligned}$$

Observe that $\sum_{B\in W}\|u\|^{n/s}_{\dot M^{s,n/s}(4B)}\le\widetilde{C}\|u\|_{\dot M^{s,p}(\Omega)}\le\widetilde{C}$. Choose $0<\sigma<C_1/8$ such that $C_2\widetilde{C}[(4\sigma)(C_1)^{-1}]^{(j_0+1)n/(n-s)}\le 1/2$, where C_1 and C_2 are the constants from Lemma 4.1. Then by $|\Omega|=1$ and Lemma 4.1, we have

$$\begin{aligned}\sum_{j=j_0+1}^\infty\frac{1}{j!}\sigma^{jn/(n-s)}I_1(j)&\le\sum_{B\in W}\big[4\sigma(C_1)^{-1}\big]^{(j_0+1)n/(n-s)}\\&\qquad\times\int_B\sum_{j=j_0+1}\frac{1}{j!}(C_1)^{jn/(n-s)}\big|u(x)-u_B\big|^{jn/(n-s)}\,dx\\&\le\sum_{B\in W}\big[\|u\|_{\dot M^{s,n/s}(4B)}(4\sigma)(C_1)^{-1}\big]^{(j_0+1)n/(n-s)}\\&\qquad\times\int_B\exp\bigg(C_1\frac{|u(x)-u_B|}{\|u\|_{\dot M^{s,n/s}(4B)}}\bigg)^{n/(n-s)}\,dx\\&\le C_2\big[(4\sigma)(C_1)^{-1}\big]^{(j_0+1)n/(n-s)}\sum_{B\in W}\|u\|^{n/s}_{\dot M^{s,n/s}(4B)}\\&\le 1/2.\end{aligned}$$

To estimate $I_2(j)$, for each $B \in W \setminus \{B_0\}$, let γ be the geodesic joining x_0 and x_B. By using the Bescovitch covering lemma (see [33]) and some arguments similar to these in the proofs of [5, Theorem 4.1] and [32, Lemma 3.2], we can find a family of balls $\mathcal{B} \equiv \{B_i\}_{i=0}^N$ such that

(a) $B_i \equiv B(w_i, d(w_i, \Omega^{\complement})/10)$ with $w_i \in \gamma$ for all $i = 0, \ldots, N$, $w_0 = x_0$ and $w_N = x_B$;

(b) $B_i \cap B_{i+1} \neq \emptyset$ for all $i = 0, \ldots, N-1$;

(c) $\sum_{i=1}^N \chi_{2B_i}(z) \le \overline{C}$ for all $z \in \Omega$, where the constant $\overline{C}$ only depends on the dimension n.

Let $\widetilde{w}_i \in B_i \cap B_{i+1}$ and $\widetilde{r}_i \equiv \min\{d(w_i, \Omega^{\complement}), d(w_{i+1}, \Omega^{\complement})\}$ for all $i = 0, \ldots, N-1$. Notice that (b) implies that

$$\frac{9}{11} d\big(w_{i+1}, \Omega^{\complement}\big) \le d\big(w_i, \Omega^{\complement}\big) \le \frac{11}{9} d\big(w_i + 1, \Omega^{\complement}\big)$$

for all $i = 0, \ldots, N-1$. Since $d(z, \Omega^{\complement}) \sim d(w_i, \Omega^{\complement})$ for all $z \in B_i$, so by (c), we have

$$N \lesssim \sum_{i=0}^N \int_{B_i \cap \gamma} \frac{1}{d(z, \Omega^{\complement})} |dz| \lesssim k_\Omega(x_B, x_0).$$

Thus, by these, (c) and the Hölder inequality, we have

$$\begin{aligned}
|u_B - u_{B_0}| &\lesssim \sum_{j=0}^{N-1} |u_{B_i} - u_{B(\widetilde{w}_i, \widetilde{r}_i)}| + |u_{B(\widetilde{w}_i, \widetilde{r}_i)} - u_{B_{i+1}}| \\
&\lesssim \sum_{j=0}^N \fint_{2B_i} \big|u(z) - u_{2B_i}\big| \, dz \\
&\lesssim \sum_{j=0}^N \big[d\big(w_i, \Omega^{\complement}\big)\big]^{-n+s} \int_{2B_i} g(x)\, dx \\
&\lesssim \sum_{j=0}^N \bigg\{ \int_{2B_i} \big[g(x)\big]^{n/s} dx \bigg\}^{s/n} \\
&\lesssim N^{(n-s)/n} \lesssim \big[k_\Omega(x_B, x_0)\big]^{(n-s)/n}.
\end{aligned}$$

Let C_3 be the constant from the preceding inequality and notice that there exists a positive constant C_4 such that for all $x \in B$ and $B \in W \setminus \{B_0\}$, $k_\Omega(x_B, x_0) \le C_4 k_\Omega(x, x_0)$. By $|\Omega| = 1$, we have

$$\begin{aligned}
&\sum_{j=j_0+1}^\infty \frac{1}{j!} \sigma^{jn/(n-s)} I_2(j) \\
&\quad \le \sum_{j=j_0+1}^\infty \frac{1}{j!} (4\sigma C_3)^{jn/(n-s)} \sum_{j=j_0+1}^\infty \sum_{B \in W \setminus \{B_0\}} |B| \big[k_\Omega(x_B, x_0)\big]^{j(n-s)/n}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=j_0+1}^{\infty} \frac{1}{j!}(4\sigma C_3C_4)^{jn/(n-s)} \int_{\Omega\setminus B_0} \big[k_\Omega(x,x_0)\big]^{j(n-s)/n}\,dx \\
&\leq \sum_{j=j_0+1} \frac{1}{j!}(4\sigma C_3C_4)^{jn/(n-s)} \sigma^{jn/(n-s)} \int_{\Omega} \big[k_\Omega(x_0,x)\big]^{j}\,dx \\
&\leq \int_\Omega \big[\exp\big((4\sigma C_3C_4)^{n/(n-s)} k_\Omega(x_0,x)\big) - 1\big]\,dx.
\end{aligned}$$

Then by Lemma 2.1, we can choose σ small enough such that

$$\sum_{j=j_0+1}^{\infty} \frac{1}{j!}\sigma^{jn/(n-s)} I_2(j) \leq 1/2,$$

which together with $\sum_{j=j_0+1}^{\infty} \frac{1}{j!}\sigma^{jn/(n-s)} I_1(j) \leq 1/2$ implies (4.2). This finishes the proof of Theorem 1.3(i).

(ii) Assume that Ω has the slice property with respect to y and $C_5 \geq 1$ as in Definition 2.4. Then for every $x\in\Omega$, by Definition 2.4, there exist a rectifiable curve γ and a sequence of $\{S_i\}_{i=0}^{j}$ for some $j\geq 0$, satisfying (i) through (iv) of Definition 2.4. Without loss of generality, we may assume that $j\geq 2$. In fact, as pointed out by Buckley and Koskela [5, p. 890], Definition 2.4(iii) and (iv) implies that $j+1 \geq k_\Omega(x,y)/C_5$. Observe that if $k(x,y)\leq 2C_5$, then (2.1) is clearly satisfied. So we only need to consider the case $j\geq 2$.

For each $i=1,\dots,j-1$, define the function u_i by setting

$$u_i(z) \equiv \inf_{\widetilde{\gamma}} \ell(\widetilde{\gamma}\cap S_i)$$

for all $z\in\Omega$, where the infimum is taken over all rectifiable curves $\widetilde{\gamma}$ joining x and z. Obviously, $u_i(z)=0$ for $z\in\bigcup_{k=0}^{i-1} S_k$ and $u_i(z)$ is a constant for $z\in \bigcup_{k=i+1}^{j} S_k$. Then by an argument similar to the one in the proof of Lemma 3.2, we can prove that $u\in \dot{M}^{s,n/s}_{\mathrm{ball}}(\Omega)$ and $g_i \equiv r_i^{1-s}\chi_{B(x_i,2C_5 d(x_i,\Omega^{\complement}))}$ is a constant multiple of an element of $\mathcal{D}^{s,1/(16C_5)}_{\mathrm{ball}}(u)$, where $r_i = \operatorname{diam} S_i \sim d(x_i,\Omega^{\complement})$ by Definition 2.4(iii). We omit the details. Then $\|u_i\|_{\dot{M}^{s,p}_{\mathrm{ball}}(\Omega)} \lesssim r_i^{1-s+n/p}$. Notice that Definition 2.4(ii) implies that

$$\big|u_i(x)-u_i(y)\big| \geq C_5^{-1} d\big(x_i,\Omega^{\complement}\big) \gtrsim r_i.$$

Moreover, define $u = j^{-s/n}\sum_{i=1}^{j-1} r_i^{-1}u_i$. Then the function

$$g \equiv j^{-s/n}\sum_{i=1}^{j-1} r_i^{-1} g_i = j^{-s/n}\sum_{i=1}^{j-1} r_i^{-s}\chi_{B(x_i,2C_5 d(x_i,\Omega^{\complement}))}$$

is a constant multiple of an element of $\mathcal{D}^{s,1/(16C_5)}_{\mathrm{ball}}(u)$, which together with the Fefferman–Stein vector-valued inequality of the Hardy–Littlewood maximal function $\mathcal{M}$ (see, for example, [33]) and Definition 2.4 imply that

$$\|u\|^{n/s}_{\dot{M}^{s,n/s}_{\rm ball}(\Omega)} \tag{4.3}$$
$$\lesssim j^{-1}\int_\Omega\left(\sum_{i=1}^{j-1} r_i^{-s}\chi_{B(x_i,2C_5 d(x_i,\Omega^{\complement}))}(z)\right)^{n/s}dz$$
$$\lesssim j^{-1}\int_\Omega\left(\sum_{i=1}^{j-1}\left[\mathcal{M}\left(\left[r_i^{-s}\chi_{B(x_i,C_5^{-1}d(x_i,\Omega^{\complement}))}\right]^{1/2}\right)(z)\right]^2\right)^{n/s}dz$$
$$\lesssim j^{-1}\int_\Omega\left(\sum_{i=1}^{j-1} r_i^{-s}\chi_{B(x_i,C_5^{-1}d(x_i,\Omega^{\complement}))}(z)\right)^{n/s}dz$$
$$\lesssim j^{-1}\int_\Omega\sum_{i=1}^{j-1} r_i^{-n}\chi_{B(x_i,C_5^{-1}d(x_i,\Omega^{\complement}))}(z)\,dz\lesssim 1.$$

On the other hand, since $u(z)\geq(j-1)^{1-s/n}$ for $z\in S_j$, then for

$$t^{n/(n-s)}\leq(j-1)/\log\left(1+\left[C_5^{-1}d\left(x,\Omega^{\complement}\right)\right]^{-n}\right),$$

we have

$$\int_\Omega\phi_s\left(\frac{u(z)}{t}\right)dz\geq\int_{S_j}\phi_s\left(\frac{u(z)}{t}\right)dz\geq\left[C_5^{-1}d\left(x,\Omega^{\complement}\right)\right]^{-n}|S_j|>1,$$

which implies that

$$\|u\|_{\phi_s(L)(\Omega)}\gtrsim\left\{\frac{j-1}{\log(1+[C_5^{-1}d(x,\Omega^{\complement})]^{-n})}\right\}^{(n-s)/n}$$
$$\gtrsim\left\{\frac{j}{\log(1+[d(x,\Omega^{\complement})]^{-1})}\right\}^{(n-s)/n}.$$

From this, (4.3) and s-Trudinger inequality, it follows that

$$j\lesssim\log\left(1+\left[d\left(x,\Omega^{\complement}\right)\right]^{-1}\right),$$

which further implies that

$$\int_\gamma\frac{1}{d(z,\Omega^{\complement})}|dz|\lesssim\log\left(1+\left[d\left(x,\Omega^{\complement}\right)\right]^{-1}\right).$$

This means that Ω is a weak carrot domain and thus finishes the proof of Theorem 1.3(ii). □

Acknowledgments. The author would like to thank Professor Pekka Koskela for his kind suggestions and significant discussions on this topic. The author would like to thank Professor Dachun Yang for his kind suggestions.

References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, Elsevier, Amsterdam, 2003. MR 2424078

[2] J. Boman, *L^p-estimates for every strongly elliptic systems*, umpublished manuscript.

[3] B. Bojarski, *Remarks on Sobolev imbedding inequalities*, Complex analysis, Joensuu 1987, Lecture Notes in Math., vol. 1351, Springer, Berlin, 1988, pp. 52–68. MR 0982072

[4] S. M. Buckley and P. Koskela, *Sobolev–Poincaré implies John*, Math. Res. Lett. **2** (1995), 577–594. MR 1359964

[5] S. M. Buckley and P. Koskela, *Criteria for imbeddings of Sobolev–Poincaré type*, Internat. Math. Res. Notices **18** (1996), 881–901. MR 1420554

[6] S. M. Buckley and J. O'Shea, *Weighted Trudinger-type inequalities*, Indiana Univ. Math. J. **48** (1999), 85–114. MR 1722194

[7] S.-K. Chua and R. L. Wheeden, *Self-improving properties of inequalities of Poincaré type on measure spaces and applications*, J. Funct. Anal. **255** (2008), 2977–3007. MR 2464568

[8] F. W. Gehring, *Univalent functions and the Schwarzian derivative*, Comment. Math. Helv. **52** (1977), 561–572. MR 0457701

[9] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Anal. Math. **36** (1979), 50–74. MR 0581801

[10] F. W. Gehring and O. Martio, *Quasiextremal distance domains and extension of quasiconformal mappings*, J. Anal. Math. **45** (1985), 181–206. MR 0833411

[11] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 2001. MR 1814364

[12] V. M. Gol'dshteĭn, *Quasiconformal, quasi-isometric mappings and the Sobolev spaces*, Complex analysis and applications '81 (Varna, 1981), Publ. House Bulgar, Acad. Sci., Sofia, 1984, pp. 202–212. MR 0883236

[13] V. M. Gol'dshteĭn and Y. G. Reshetnyak, *Introduction to the theory of functions with generalized derivatives, and quasiconformal mappings* (in Russian), "Nauka", Moscow, 1983. MR 0738784

[14] V. M. Gol'dshteĭn and S. K. Vodop'anov, *Prolongement de fonctions différentiables hors de domaines plans*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), 581–584. MR 0647686

[15] P. Hajłasz, *Sobolev spaces on an arbitrary metric spaces*, Potential Anal. **5** (1996), 403–415. MR 1401074

[16] P. Hajłasz, *Sobolev spaces on metric-measure spaces*, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), Contemp. Math., vol. 338, Amer. Math. Soc., Providence, RI, 2004, pp. 173–218. MR 2039955

[17] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), 1–101. MR 1683160

[18] P. Hajłasz, P. Koskela and H. Tuominen, *Sobolev imbeddings, extensions and measure density condition*, J. Funct. Anal. **254** (2008), 1217–1234. MR 2386936

[19] P. Hajłasz, P. Koskela and H. Tuominen, *Measure density and extendability of Sobolev functions*, Rev. Mat. Iberoam. **24** (2008), 645–669. MR 2459208

[20] P. Koskela, *Capacity extension domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. Diss. **73** (1990), 1–42. MR 1039115

[21] P. Koskela and E. Saksman, *Pointwise characterizations of Hardy–Sobolev functions*, Math. Res. Lett. **15** (2008), 727–744. MR 2424909

[22] P. Koskela, D. Yang and Y. Zhou, *A characterization of Hajłasz–Sobolev and Triebel–Lizorkin spaces*, J. Funct. Anal. **258** (2010), 2637–2661. MR 2593336

[23] P. Koskela, D. Yang and Y. Zhou, *Pointwise characterizations of Besov and Triebel–Lizorkin spaces and quasiconformal mappings*, Adv. Math. **226** (2011), 3579–3621. MR 2764899

[24] P. MacManus and C. Pérez, *Trudinger inequalities without derivatives*, Trans. Amer. Math. Soc. **354** (2002), 1997–2012. MR 1881027

[25] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1979), 383–401. MR 0565886

[26] A. Miyachi, *Hardy–Sobolev spaces and maximal functions*, J. Math. Soc. Japan **42** (1990), 73–90. MR 1027541

[27] A. Miyachi, *Extension theorems for real variable Hardy and Hardy–Sobolev spaces*, Harmonic analysis (Sendai, 1990), ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, pp. 170–182. MR 1261438

[28] R. Näkki and J. Väisälä, *John disks*, Expo. Math. **9** (1991), 3–44. MR 1101948

[29] W. Smith and D. A. Stegenga, *Hölder domains and Poincaré domains*, Trans. Amer. Math. Soc. **319** (1990), 67–100. MR 0978378

[30] W. Smith and D. A. Stegenga, *Sobolev imbeddings and integrability of harmonic functions on Hölder domains*, Potential theory (Nagoya, 1990), de Gruyter, Berlin, 1992, pp. 303–313. MR 1167248

[31] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983. MR 0781540

[32] P. Shvartsman, *On Sobolev extension domains in* R^n, J. Funct. Anal. **258** (2010), 2205–2245. MR 2584745

[33] E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, 1993. MR 1232192

[34] S. K. Vodop'janov, V. M. Gol'dšteĭn and T. G. Latfullin, *A criterion for the extension of functions of the class* L_2^1 *from unbounded plane domains*, Sibirsk. Mat. Zh. **20** (1979), 416–419. MR 0530508

[35] D. Yang, *New characterizations of Hajłasz–Sobolev spaces on metric spaces*, Sci. China Ser. A **46** (2003), 675–689. MR 2025934

Yuan Zhou, Department of Mathematics, Beihang University, Beijing 100191, P. R. China, and Department of Mathematics and Statistics, P. O. Box 35 (MaD), FI-40014, University of Jyväskylä, Finland

E-mail address: yuanzhou@buaa.edu.cn