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SYMMETRY IN TENSOR ALGEBRAS OVER HILBERT SPACE

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ABSTRACT. This paper deals with three issues: (1) Unitary representations U of a scale of (finite and infinite dimensional) noncompact Lie groups $G(H)$ built on a fixed complex Hilbert space H; and their covariant systems. Our computations for these representations make use of the associated Lie algebras. (2) The covariant representations involve the C^*-algebras going by the names, the Toeplitz algebras, and the Cuntz algebras. (3) An essential result which also is used throughout is our computation of the commutant of the unitary representation U of $G(H)$ mentioned in (1). For a fixed Hilbert space H, we apportion the commutant as a specific projective limit-algebra of operators.

1. Introduction

While our study is motivated by free probability in the sense of Voiculescu (see [27]), our results have wider scope and can be formulated based on just fundamental notions from Hilbert space. Our present purpose is to study a number of symmetries implied by a free tensor operations applied to a fixed complex Hilbert space. The kinds of tensor algebras in turn have roots in the study of quantum fields in mathematical physics; in addition to topics of a more recent vintage: Free probability and the study of noncommutative random variables. The notion of "freeness" or "free independence" in this context is now widely used as an analogue or extension of the better known and classical notion of "independence" as used in statistics.

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Mathematically, it entails free products. This was initiated by Voiculescu and motivated in turn by a number of long standing open problems in operator algebra theory, especially the *free group factor isomorphism problem*, an important unsolved problem dating back to John von Neumann: Given a free group F_n on some number n of generators, and consider the group von Neumann algebras $L(F_n)$, generated by F_n (each is a type II_1-factor). For different values of n, for examples 2 and 3, are the two von Neumann algebras $*$-isomorphic? This problem in turn is analogous to *Tarski's free group problem*: Will two different non-Abelian finitely generated free groups have the same elementary theory?

Until now, the following is known by Radulescu (see [23]): Either (i) or (ii) holds true, where

(i) $L(F_n)$ is $*$-isomorphic to $L(F_\infty)$, for all $n \in \mathbb{N} \setminus \{1\}$,
(ii) $L(F_n)$ and $L(F_m)$ are not $*$-isomorphic, whenever $n \neq m$ in $\mathbb{N}$.

Because of connections to a number of applications ([28], and [25]; for example, random matrix theory, symmetries; [3], [4], and [6]; for example, groupoid theory, and groupoid dynamical systems; [5], and [29]; for example, combinatorics, representation theory, which are our present focus), and large deviations in statistics, these classical questions have found a recent revival.

1.1. Overview. The study of groups of automorphisms of the Toeplitz algebras and their quotients, the Cuntz algebras, was initiated with ideas of Voiculescu (see [26]). Our present study builds on this and explores its significance in the study of representations of certain non-compact Lie groups. Here, we think of these representations by automorphisms, and covariant C^*-algebraic systems.

In Voiculescu's work, the motivation came from free probability theory. But the implications are wider, for example in the study of Krein spaces (see [1], [7], [30]). To see this, consider the non-compact Lie group $U(H,1)$, defined from a fixed $\mathbb{C}$-Hilbert space H. Adjoin one dimension and consider $H \times \mathbb{C}$, and the group of invertible linear transformations action on $H \times \mathbb{C}$, and preserving the quadratic form

$$Q(h,z) = \|h\|^2 - |z|^2 \quad \text{for all } (h,z) \in H \times \mathbb{C}.$$

However, when H is infinite-dimensional, this group and its Lie algebra plays a prominent role in quantum field theory; starting with the paper by D. Shale (see [24]).

It was taken up again in [11] (reprinted in 2008 by Dover), and we refer to this book for additional citations.

Our general framework involves both the theory of C^*-algebras, and a particular covariant system built with a universal representation. The representation is of a certain Lie group $U(H,1)$ and a C^*-algebra $\mathcal{T}(H)$ defined directly from a given Hilbert space H. Our representation system is universal in the sense that it includes more familiar representations such as those arising in the relations known as CCR and CAR from physics. But our framework also allows for deformations. Below is a rough out line, with historical perspective, of some main ideas.

We begin with some general observation about q-relations (e.g., [3], [15], [16]), C^*-algebras (e.g., [17]), and representations (e.g., [11]).

1.2. Applications. Mathematical models in physics (quantum theory, quantum fields, elementary particles, and quantum statistical mechanics) involve systems of generators and relations of operators in Hilbert space, the best known of them are CCRs for Boson, and CARs for Fermions; standing for canonical commutation/anti-commutation relations.

Starting with a choice of a complex Hilbert space H, we build a quantization functor: Fock space, C^*-algebras, and representations. If $\dim H$ is finite, we have the Stone–von Neumann uniqueness theorem (e.g., [11]) from the theory of unitary representations of Lie groups (in this case, Heisenberg groups). Specially, we get uniqueness of the representation up to unitary equivalence. So, when H is fixed, there is a canonical C^*-algebra, and representations given by an assigned value of a quantum number. "Canonical" is the C in CCR: canonical commutation relations. This simple picture fails if $\dim H = \infty$; everything changes.

In more detail, for infinite dimensional Hilbert space H, the conclusion of the Stone–von Neumann uniqueness theorem is then no longer valid. But isomorphism class of the C^*-algebra is the same: Still the CCR-C^*-algebra, often called $\mathrm{CCR}(H)$ is unique as a C^*-algebra up to $*$-isomorphism, even though for infinite dimensional H there is a multitude of inequivalent unitary irreducible representations; starting with all those from quantum field theory. Without this non-uniqueness, there would be no physics of fields and of quantum statistical mechanics!

So, for the q-C^*-algebras, work with the first named author and Werner et al. (e.g., [15], and [16]) yields that for a certain range of q, we have q-commutation relations and an associated family of C^*-algebras $A(q)$. In these papers, they show that when the range of q is suitably restricted, there is just a single C^*-isomorphism class, but as we vary the value of the parameter q, the associated unitary representations are certainly mutually inequivalent in the sense of the familiar notation of unitary equivalence for unitary representations in Hilbert space.

The special feature about C^*-algebras, is that they have their own category of morphisms and isomorphisms (e.g., [17]). One can talk about them independently of their representations; even independently of any mention of Hilbert space if we return to the axioms of Gelfand, Segal, and Kadison.

Isomorphism in the context of C^*-algebras simply means isomorphism in the category of C^*-algebras.

Now, to the unitary representations of the Lie group $U(H,1)$: They act in the unrestricted Fock space $\mathcal{F}(H)$, built over H, and they are not irreducible there. In fact, it is possible to give a canonical isomorphism between a family of certain martingales on the one hand, and the commutant of our $U(H,1)$-unitary representations on $\mathcal{F}(H)$ are reduced by the symmetric tensors inside $\mathcal{F}(H)$, and by the anti-symmetric tensors (see Section 5 below). That yields what we call the CCRs and the CARs. And we can also recover the q-commutation relations by a different projection in the commutant of the global $U(H,1)$-unitary representations (e.g., [12], [13], and [14]).

The role of various Fock spaces in the study of representations of the Cuntz and Toeplitz algebras have been explored in a number of recent papers, but addressing other questions: [2], [8], [9], [10], [18], [19], [21], [22], and [20].

2. Definitions and background

The purpose of this section is to introduce the precise definition of the tensor categories to be used later. Our starting point is a given (and fixed) Hilbert space H. From H, we build the Hilbert space $\mathcal{F}(H)$, as a Hilbert-orthogonal sum of n-fold tensors, with $n=0,1,2,\ldots$. We are concerned with three dual families of operators on $\mathcal{F}(H)$, left tensor multiplication by elements in H, the operators of right-tensoring in $\mathcal{F}(H)$, and finally the annihilatioin operators on $\mathcal{F}(H)$. Each operator in the three families is indexed by the vectors from H; and for each one, there is an adjoint operator, where the adjoint operation is defined from the inner product on $\mathcal{F}(H)$.

In a physics context, the Hilbert space $\mathcal{F}(H)$ is called the Fock space for Boltzmann statistics; unrestricted. This is by contrast to the more familiar two cases of the Fock space of symmetric tensors (Bosons), and antisymmetric tensors (Fermions). After developing our theory for the unrestricted case, we compare to the symmetric and antisymmetric cases.

The first two families consist of bounded operators, and so they generate respective C^*-algebras. However, the annihilation operators and their adjoints, the creation operators, are unbounded. Our main interest is in the unbounded operators, and their commutation relations; but we will be studying them with the use of covariant representations involving the two C^*-algebras.

Throughout this paper, let H be $\mathbb{C}$-Hilbert space, and $B(H)$, an algebra of all (bounded linear) operators on H. Define the Fock space $\mathcal{F}(H)$ by

$$\mathcal{F}(H) \overset{\text{def}}{=} \mathbb{C}\Omega \oplus \left(\bigoplus_{n=1}^{\infty} H^{\otimes n} \right),$$

where $H^{\otimes n}$ means the tensor product Hilbert space of n-copies of H, for $n \in \mathbb{N}$, and where Ω means the vacuum vector, satisfying that $\|\Omega\| = 1$. Then, on this new Hilbert space $\mathcal{F}(H)$, we define the *left tensor operator* $l(h)$ by

$$l(h)t \overset{\text{def}}{=} h \otimes t \quad \text{for all } t \in \mathcal{F}(H)$$

for all $h \in H$. Similarly, we can define the right tensor operator $r(h)$ by

$$r(h)t \overset{\text{def}}{=} t \otimes h \quad \text{for all } t \in \mathcal{F}(H)$$

for all $h \in H$. Then we can construct the C^*-algebra $\mathfrak{A}_l$ by the C^*-algebra

$$C^*\big(\{l(h) : h \in H\}\big),$$

generated by the left tensor operators $l(h)$, for all $h \in H$. Similarly, the C^*-algebra $\mathfrak{A}_r$ is constructed as the C^*-algebra

$$C^*\big(\{r(h) : h \in H\}\big),$$

generated by the right tensor operators $r(h)$, for all $h \in H$. By definition, the C^*-algebras $\mathfrak{A}_l$, and $\mathfrak{A}_r$ are the C^*-subalgebras of $B(\mathcal{F}(H))$.

It is easy to check that:

$$l(h)^*\Omega = r(h)^*\Omega = 0, \quad \text{the zero vector of } H.$$

Under the above settings, define an operator $a(h)$ on $\mathcal{F}(H)$, for a fixed vector $h \in H$, by

$$a(h)\Omega \overset{\text{def}}{=} 0,$$

and

$$a(h)(k_1 \otimes \cdots \otimes k_n) \overset{\text{def}}{=} \sum_{i=1}^{n} \big(\langle h, k_i\rangle k_1 \otimes \cdots \otimes k_{i-1} \otimes k_{i+1} \otimes \cdots \otimes k_n\big)$$

for all $k_1 \otimes \cdots \otimes k_n \in H^{\otimes n} \subset \mathcal{F}(H)$, for all $n \in \mathbb{N}$.

Definition 2.1. The operator $a(h)$, introduced in the above paragraph, is called the annihilation operator on $\mathcal{F}(H)$, induced by $h \in H$.

On the Fock space $\mathcal{F}(H)$, we define the annihilation operator $a(h)$, for a fixed vector $h \in H$. Thus, it has its unique adjoint $a(h)^*$.

Definition 2.2. Let $a(h)$ be the annihilation operator on $\mathcal{F}(H)$, induced by $h \in H$. Denote the adjoint $a(h)^*$ of $a(h)$ by $a^+(h)$. Then this operator $a^+(h)$ is said to be the creation operator induced by h.

By definition, we can check that

$$a^+(h)\Omega = h \quad \text{for all } h \in H,$$

and

$$\begin{aligned} a^+(h)(k_1 \otimes \cdots \otimes k_n) = h \otimes k_1 \otimes \cdots \otimes k_n & \\ + \sum_{i=1}^{n-1} k_1 \otimes \cdots \otimes k_{i-1} \otimes h \otimes k_{i+1} \otimes \cdots \otimes k_n & \\ + k_1 \otimes \cdots \otimes k_n \otimes h & \end{aligned}$$

for all $k_1 \otimes \cdots \otimes k_n \in H^{\otimes n} \subset \mathcal{F}(H)$.

Let A, B be operators in $B(H)$. Define the operator $\Gamma(A)$, by the operator on $\mathcal{F}(H)$,

$$\Gamma(A) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \Gamma_n(A),$$

with

$$\Gamma_0(A) \overset{\text{def}}{=} 1 \in \mathbb{C}\Omega \subset \mathcal{F}(H),$$

and

$$\Gamma_n(A)(k_1 \otimes \cdots \otimes k_n) \overset{\text{def}}{=} Ak_1 \otimes \cdots \otimes A_{k_n}$$

for all $k_1 \otimes \cdots \otimes k_n \in H^{\otimes n} \subset \mathcal{F}(H)$, for all $n \in \mathbb{N}$.

Also, define the operator $d\Gamma(B)$ by the operator on $\mathcal{F}(H)$,

$$d\Gamma(B) \overset{\text{def}}{=} \sum_{n=0}^{\infty} d\Gamma_n(B)$$

with

$$d\Gamma_n(B) \overset{\text{def}}{=} 0, \quad \text{the zero operator on } \mathcal{F}(H),$$

and

$$\begin{aligned} &d\Gamma_n(B)(k_1 \otimes \cdots \otimes k_n) \\ &\quad \overset{\text{def}}{=} \sum_{i=1}^{n} \big(k_1 \otimes \cdots \otimes k_{i-1} \otimes (Bk_i) \otimes k_{i+1} \otimes \cdots \otimes k_n\big) \end{aligned}$$

for all $k_1 \otimes \cdots \otimes k_n \in H^{\otimes n} \subset \mathcal{F}(H)$, for all $n \in \mathbb{N}$.

In particular, the operator $d\Gamma(I)$ on $\mathcal{F}(H)$ is called *the number operator*. We denote the number operator $d\Gamma(I)$ by $\mathcal{N}$.

Now, let's consider the Dirac operator formalism. Let $\langle , \rangle$ be the inner product on a given Hilbert space H. For a fixed vector $h \in H$, define the Dirac operators $|h\rangle$, and $\langle h|$ by

$$|h\rangle h' \overset{\text{def}}{=} hh' \quad \text{for all } h' \in H,$$

and hence, this operator $|h\rangle$ is an operator on H.

$$\langle h|h' \overset{\text{def}}{=} \big\langle h, h' \big\rangle \quad \text{for all } h' \in H,$$

and hence, this operator $\langle h|$ is in fact a linear functional contained in the dual space $H^* = B(H, \mathbb{C})$ of H. With the Dirac vectors $|h\rangle$, and $\langle h|$, we can construct the rank-one operator $|h\rangle\langle k|$, for h, $k \in H$, determined by

$$|h\rangle\langle k|x \overset{\text{def}}{=} \langle k, x\rangle h \quad \text{for all } x \in H.$$

We will need the formula,

$$\operatorname{trace}\big(A|h\rangle\langle k|\big) = \langle k, Ah\rangle$$

valid for all $A \in B(H)$.

By definitions, we can obtain the following fundamental lemma.

LEMMA 2.1. *Let h, $k \in H$. Then*

(2.1) $a^+(k)a(h) = \mathcal{N}\, d\Gamma(|k\rangle\langle h|)$,

(2.2) $a(h)a^+(k) = \langle h,\, k\rangle(\mathcal{N} + 1_{\mathfrak{A}_l}) + (\mathcal{N} + 1_{\mathfrak{A}_l})\, d\Gamma(|k\rangle\langle h|)$,

(2.3) $l(h)^* l(k) = \langle h,\, k\rangle 1_{\mathfrak{A}_l}$,

(2.4) $a(h)l(k) = \langle h,\, k\rangle 1_{\mathfrak{A}_l} + d\Gamma(|k\rangle\langle h|)$,

(2.5) $[a^+(h) - a(h), l(k)] = l(h)l(k) - \langle h, k\rangle 1_{\mathfrak{A}_l}$,

where $1_{\mathfrak{A}_l}$ is the identity element in $\mathfrak{A}_l$, where

$$[A, B] \overset{def}{=} AB - BA$$

for A, $B \in \mathfrak{A}_l$.

Proof. All five formuli concern operator identities for operators which act on the Fock space $\mathcal{F}(H)$. We will prove them by induction, starting with the case where $n = 0$ on $\mathcal{F}(H)$, that is, evaluation the operators on the vacuum Ω.

Under our definition, check that the operators on left sides of formuli yield the same when applied to Ω: For (2.1), the left side yield the zero vector 0 in $\mathcal{F}(H)$, i.e.,

$$a^+(k)a(h)\Omega = 0.$$

For (2.2), the two sides yield $\langle h,\, k\rangle\Omega$. Similarly, when the operators in (2.3), and (2.4) are applied to Ω, we get $\langle h,\, k\rangle\Omega$.

Then two operators in (2.5), when applied to Ω, yield

$$h \otimes k - \langle h, k\rangle\Omega.$$

Now, make the induction hypothesis, that the formuli (2.1) through (2.5) hold up to n, that is, that the induction hold on the closed subspaces $\mathcal{F}_m(H)$, for all $m \le n$, where

$$\mathcal{F}_m(H) \overset{\text{def}}{=} H^{\otimes m}.$$

To get to $n+1$, consider vectors $s \otimes t$, where $s \in H$, and $t = t_1 \otimes \cdots \otimes t_n \in \mathcal{F}_n(H)$. Then we compute the left-hand side in (2.1) as follows:

$$\begin{aligned} a^+(k)a(h)(s \otimes t) &= a^+(k)\big(\langle h, s\rangle t + s \otimes a(h)t\big) \\ &= \langle h, s\rangle a^+(k)t + k \otimes s \otimes a(h)(t) + s \otimes a^+(k)a(h)t. \end{aligned}$$

By the induction hypothesis, we may have that:

$$a^+(k)a(h)t = n\, d\Gamma\big(|k\rangle\langle h|\big),$$

and

$$|k\rangle\langle h|s = \langle h, s\rangle k.$$

Thus

$$\begin{aligned} &a^+(k)a(h)(s \otimes t) \\ &\quad = (n+1)\big(\Gamma\big(|k\rangle\langle h|\big)s \otimes t + s \otimes d\Gamma\big(|k\rangle\langle h|t\big)\big) \\ &\quad = (n+1)\, d\Gamma\big(|k\rangle\langle h|\big)(s \otimes t), \end{aligned}$$

which is the desired conclusion.

The same process applies to the induction step in the verification of (2.2), and details are left to the reader.

For the left-hand side in (2.3) applied to $s \otimes t$, we get that

$$\begin{aligned} l(h)^* l(h)(s \otimes t) &= l(h)^*(k \otimes s \otimes t) \\ &= \langle h, k \rangle s \otimes t, \end{aligned}$$

which is the desired conclusion.

We now turn to the left-hand side in (2.5), applied to $s \otimes t$. We get that

$$\begin{aligned} &\big(a^+(h) - a(h)\big) l(k) - l(k)\big(a^+(h) - a(h)\big) - d\Gamma\big(|k\rangle\langle h|\big) \\ &= a^+(h) l(k) - \langle h, k \rangle 1_{\mathfrak{A}_l} - l(k) a^+(h) + l(k) a(h), \end{aligned}$$

by (2.4), and hence

$$\big[a^+(h) - a(h), l(k)\big](s \otimes t) = h \otimes k \otimes s \otimes t - \langle h, k \rangle s \otimes t. \qquad \square$$

Definition 2.3. Let H be a "finite-dimensional" Hilbert space, and let $\mathfrak{A}_l = C^*(\{l(h) : h \in H\})$ be the left-C^*-algebra, generated by the left tensor operators. Let $\{e_j : j = 1, \ldots, n = \dim H\}$ be the orthonormal basis (ONB, or Hilbert basis) in H. Set

$$T \overset{\text{def}}{=} \sum_{i=1}^{n} l(e_i) l(e_i)^*,$$

and let

$$\mathcal{K} \overset{\text{def}}{=} \text{the two-sided ideal in } \mathfrak{A}_l, \text{ generated by } 1_{\mathfrak{A}_l} - T.$$

Let

$$\psi : \mathfrak{A}_l \to \mathfrak{A}_l / \mathcal{K} \tag{2.6}$$

be the canonical (quotient) $*$-homomorphism. We set

$$\mathcal{O}(H) \overset{\text{def}}{=} \mathfrak{A}_l / \mathcal{K},$$

and

$$s(h) \overset{\text{def}}{=} \psi\big(l(h)\big) \quad \text{for all } h \in H.$$

Notice that the C^*-algebra $\mathcal{O}(H)$ is the *Cuntz algebra*, that is, it is a C^*-algebra, determined by two universal axioms:

(2.7) $s(h)^* s(h) = \langle h,\, k \rangle I$, and
(2.8) $\sum_{i=1}^{n} s(e_i) s(e_i)^* = I$,

where I is the identity operator. Remark that, if $\dim H = \infty$, then we set

$$\mathcal{O}(H) \overset{\text{def}}{=} \mathfrak{A}_l.$$

Proposition 2.2. *Let $\mathcal{D} = \mathcal{D}_l$ be the unital $*$-algebra generated by the elements $l(h)$, and $l(k)^*$, for h, $k \in H$, as a $*$-subalgebra of $\mathfrak{A}_l$. For $x \in \mathcal{D}$, define*

$$\delta_h(x) \stackrel{def}{=} \left[a^+(h) - a(h), x\right] \tag{2.9}$$

for a fixed $h \in H$. Then

(i) *δ_h is a $*$-derivation, that is,*

$$\delta_h(1) = 0, \quad and \quad \delta_h\left(x^*\right) = \left(\delta_h(x)\right)^*,$$

and

$$\delta_h(xy) = \delta_h(x)y + x\delta_h(y) \tag{2.10}$$

for $x, y \in \mathcal{D}$.

(ii) *δ_h, for $h \in H$, passes to the quotient $\mathcal{D}/\mathcal{K}$ and defines a densely defined $*$-derivation $\widetilde{\delta_h}$ on the Cuntz algebra $\mathcal{O}(H)$, satisfying that:*

$$\widetilde{\delta_h}\left(\psi(x)\right) = \psi\left(\delta_h(x)\right) \quad for\ all\ x \in \mathcal{D}. \tag{2.11}$$

Proof. Denote the operator $a^+(h) - a(h)$ by H_h, for $h \in H$, as in (2.9). Then this operator H_h is skew-symmetric, in the sense that

$$\langle H_h\xi, \eta\rangle_{\mathcal{F}(H)} = -\langle \xi, H_h\eta\rangle_{\mathcal{F}(H)} \tag{2.12}$$

for all finite tensor elements ξ, $\eta \in \mathcal{F}(H)$. For operator graphs, this reads

$$H_h \subseteq -H_h^*.$$

It follows that

$$\delta_h\left(x^*\right) = \left(\delta_h(x)\right)^* \tag{2.13}$$

for all $x \in \mathcal{D}$. In particular, from (2.5), we obtain that

$$\delta_h\left(l(k)^*\right) = l(k)^* l(h)^* - \overline{\langle h, k\rangle} 1_{\mathfrak{A}_l} \tag{2.14}$$

for all $k \in H$, where $\overline{z}$ means the conjugate of z, for all $z \in \mathbb{C}$. By (2.5),

$$\delta_h(x) = [H_h, x] = adH_h(x) = H_h x - xH_h, \tag{2.15}$$

is well-defined, for $h \in H$. So, we conclude that δ_h passes to the quotient with (2.6), and (2.11). Indeed, we must check that the idea $\mathcal{K}$ is invariant under the commutator, as in (2.15). But, by using (2.9), (2.10), and by the definition,

$$T = \sum_{i=1}^{n} l(e_i) l(e_i)^*,$$

we can obtain that

$$\delta_h(T - 1) = l(h)(T - 1) + (T - 1)l(h)^* \in \mathcal{K},$$

and therefore,

$$\delta_h\left(x(T-1)y\right) = \delta_h(x)(T-1)y + x\delta_h(T-1)y + x(T-1)\psi\delta_h(y),$$

in $\mathcal{K}$, where we used (2.10) in the last derivation.

It follows that $\widetilde{\delta_h}$, given in (2.11), is a densely defined derivation on $\mathcal{O}(H)$. However, note that the formula of the formula (2.9) is not valid for $\widetilde{\delta_h}$. □

To proceed our works, we introduce the following new concepts.

DEFINITION 2.4. Let X be a normed space, and let L be a linear operator on X. Let $x \in X$ be an element such that $L^n x$ is well defined, for all $n \in \mathbb{N}$, inductively;

$$L^{n+1}x = L\big(L^n x\big) \quad \text{for } n \in \mathbb{N}.$$

We say that x is an analytic element, if there are constants C and D $(< \infty)$ such that

$$\big\|L^n x\big\| \leq C \cdot n! \cdot D^n \quad \text{for all } n \in \mathbb{N}. \tag{2.16}$$

If a fixed element $x \in X$ is analytic, we say that the quantity

$$r \overset{\text{def}}{=} \frac{1}{D}$$

is the radius of convergence for x. Here, notice that the constant value D depends on x.

We can obtain the following theorem.

THEOREM 2.3. *Let $h \in H$ be fixed. Then*

(i) *The two derivations δ_h and $\widetilde{\delta_h}$ have dense algebras consisting of analytic elements.*

(ii) *The operator H_h, in the sense of* (2.9), *has a dense space of analytic vectors in $\mathcal{F}(H)$.*

Proof. (i) It follows from (2.10), and (2.13) that the analytic element for a $*$-derivations form a $*$-algebra. Thus, to prove the statement (i), it is enough to show that every $l(k)$ is an analytic element for the derivation δ_h. The analytic property will pass to the quotient via (2.11) and (2.15). We will establish the following estimate by induction:

$$\big\|\delta_h^m\big(l(k)\big)\big\| \leq m!\big(2\|h\|_H\big)^m \|k\|_H \tag{2.17}$$

for all $k \in H$, and $m \in \mathbb{N}$, where $\|\cdot\|_H$ means the Hilbert norm on H, induced by the inner product $\langle , \rangle$ on H. First, note that the operator $l(k)$ is bounded. For its operator norm, we have that

$$\big\|l(k)\big\|^2 = \big\|l(k)^* l(k)\big\| = \big\|\langle k, k\rangle 1_{\mathfrak{A}_l}\big\| = \|k\|_H^2. \tag{2.18}$$

An application of (2.5) holds

$$\begin{aligned}
\big\|\delta_h\big(l(k)\big)\big\| &= \big\|l(h)l(k) - \langle h, k\rangle 1_{\mathfrak{A}_l}\big\| \\
&\leq \big\|l(h)l(k)\big\| + \big|\langle h, k\rangle\big| \\
&\leq \big\|l(h)\big\|\big\|l(k)\big\| + \|h\|_H\|k\|_H = 2\|h\|_H\|k\|_H,
\end{aligned}$$

which is the desired estimate (2.17), for $m = 1$. But,

$$\begin{aligned}\delta_h^{m+1}\big(l(k)\big) &= \delta_h^m\big(\delta_h\big(l(k)\big)\big) = \delta_h\big(l(h)l(k)\big) \\ &\quad \text{by (2.15)} \\ &= \sum_{j=0}^{m}\binom{m}{j}\delta_h^j\big(l(k)\big)\delta_h^{m-j}\big(l(k)\big)\end{aligned} \tag{2.19}$$

by (2.10), where $\binom{m}{j} \overset{\text{def}}{=} \frac{m!}{j!(m-j)!}$ are the binomial coefficients, for $j \le m \in \mathbb{N}$. By using the hypothesis (2.17), and (2.18), the desired conclusion for $m+1$ follows from a simple norm estimate: Each of the terms makes the summation on the last equality of (2.19) may be estimated by

$$m!\big(2\|h\|_H\big)^{m+1}\|k\|_H,$$

and there are $m+1$ terms in the sum.

(ii) We now turn to the operator $H_h = a^+(h) - a(h)$, for $h \in H$, from (2.9), (2.12), and (2.15). For $x \in \mathcal{D}$, we apply (2.15) to the vacuum vector Ω, then we get that

$$H_h(x\Omega) = xH_h\Omega + \delta_h(x)\Omega = xh + \delta_h(x)\Omega, \tag{2.20}$$

where we use the abbreviated notation xh for $x \otimes h$ in $\mathcal{F}(H)$, or equivalently,

$$x \otimes h = l(x)h = r(h)x.$$

Indeed, the algebra $\mathcal{D} = \mathcal{D}_l$ is naturally acting both as a subalgebra in $\mathfrak{A}_l$ and as a subspace in $\mathcal{F}(H)$. An iteration of (2.20) yields

$$H_h^m(x\Omega) = \sum_{j=0}^{m}\binom{m}{j}\delta_h^{m-j}(x)H_h^j\Omega. \tag{2.21}$$

The right-hand side of (2.21) satisfies that

$$\delta_h^{m-j}(x)H_h^j\Omega = l\big(\delta_h^{m-j}(x)\big),$$

by applying $H_h^j\Omega$ in $\mathcal{F}(H)$. Thus, by Lemma 2.1, we have

$$\big\|\delta_h^{m-j}(x)H_h^j\Omega\big\|_{\mathcal{F}(H)} \le \big\|\delta_h^{m-j}(x)\big\|\big\|H_h^j\Omega\big\|_{\mathcal{F}(H)},$$

where the norms are identified in the subscripts. But, in the statement (i), we proved that there are finite estimate C_1 and D_1 (depending on x and h), such that

$$\big\|\delta_h^{m-j}(x)\big\| \le C_1(m-j)!D_1^{m-j}.$$

A separate induction shows that

$$\big\|H_h^j\Omega\big\|_{\mathcal{F}(H)} \le j!\big(2\|h\|_H\big)^j.$$

Set

$$D_2 \overset{\text{def}}{=} \max\big\{C_1, D_1, 2\|h\|_H\big\}.$$

Then an estimation applied to (2.21) yields

$$\left\|H_h^m(x\Omega)\right\|_{\mathcal{F}(H)} \leq m!D_2^m,$$

and the proof of the statement (ii) is complete. □

3. C^*-algebras and representations

Starting with a fixed Hilbert space H, we build in a natural way two C^*-algebras. Their construction is facilitated with the use of the Fock space $\mathcal{F}(H)$. The two C^*-algebras then are generated by the operations of tensoring respectively on the left, and on the right, by elements from H. This also allows us to realize the Cuntz C^*-algebra as a quotient by an ideal in the left C^*-algebra (also, called the Toeplitz C^*-algebra over H).

All three C^*-algebras, and their covariant representations, will be needed in our analysis of symmetries of the respective infinite tensor algebras, and the details are carried our in subsequent sections.

We already mentioned the three C^*-algebras $\mathfrak{A}_l$, $\mathfrak{A}_r$, and $\mathcal{O}(H)$, built over a fixed Hilbert space H, in Section 2. We will keep using the same definitions throughout this section, too.

Let $\mathcal{A}$ be a unital C^*-algebra, and $\mathcal{H}$, a Hilbert space. Define the set $\mathcal{R}_{ep}(\mathcal{A}, \mathcal{H})$ by the collection of all unital $*$-homomorphisms from $\mathcal{A}$ to $B(\mathcal{H})$, that is,

$$\mathcal{R}_{ep}(\mathcal{A}, \mathcal{H}) \stackrel{\text{def}}{=} \left\{\pi : \mathcal{A} \to B(\mathcal{H}) | \pi \text{ is a unital}, *\text{-homomorphism}\right\}. \tag{3.1}$$

Each element π of $\mathcal{R}_{ep}(\mathcal{A}, \mathcal{H})$ is called a *representation of* $\mathcal{A}$ (*on* $\mathcal{H}$). We shall use the standard Gelfand–Naimak–Segal (GNS) construction between states on $\mathcal{A}$ and cyclic representations (see, for example, [11]).

A representation $\pi \in \mathcal{R}_{ep}(\mathcal{A}, \mathcal{H})$ is said to be *cyclic*, if and only if there is a vector $\Omega \in \mathcal{H}$, such that

$$\left\{\pi(x)\Omega : x \in \mathcal{A}\right\}$$

is dense in $\mathcal{H}$. When a cyclic vector Ω is known, we will assumed it to be normalized, that is, if Ω is a cyclic vector, then it is automatically assumed that

$$\|\Omega\|_{\mathcal{H}} = 1.$$

Let Ω be a cyclic vector. Then we can construct a state ω_Ω on $\mathcal{A}$, defined by

$$\omega_\Omega(x) = \langle \Omega, x\Omega \rangle_{\mathcal{H}} \quad \text{for all } x \in \mathcal{A}, \tag{3.2}$$

where $\langle , \rangle_{\mathcal{H}}$ means the inner product on $\mathcal{H}$. Since we assumed the cyclic vector Ω be normalized,

$$\omega_\Omega(1_{\mathcal{A}}) = 1. \tag{3.3}$$

The GNS-construction states the converse: Every state on $\mathcal{A}$ is defined from a cyclic representation π via the formula (3.2); and the representation is unique up to unitary equivalence.

It is easy to see that the two C^*-algebras $\mathfrak{A}_l$ and $\mathfrak{A}_r$ from Section 2 may be specified via two states, and separate applications of the GNS construction. Note that, if

$$\omega_\Omega(\bullet) = \langle \Omega, \bullet\Omega\rangle_{\mathcal{F}(H)} \tag{3.4}$$

is the *Fock state* for the vacuum vector Ω of $\mathcal{F}(H)$, and if h, $k \in H$, then

$$\omega_\Omega\big(l(h)l(k)^*\big) = 0, \tag{3.5}$$

by definition.

Assume now that $\{e_i\}_{i=1}^n$ is an ONB of the given Hilbert space H, and let

$$T = \sum_{i=1}^n l(e_i)l(e_i)^*$$

be defined as in Section 2. Then we can get that

$$\omega_\Omega(T - 1_{\mathfrak{A}_l}) = -1.$$

The conclusion is that $\omega_\Omega(\bullet)$ does not pass to the quotient (2.6). As a result, we see that the Cuntz algebra $\mathcal{O}(H)$ is not represented on $\mathcal{F}(H)$!

Definition 3.1. Consider a system,

$$\big(\pi, \alpha_t, U(t), \mathcal{H}, \mathcal{A}, t \in \mathbb{R}\big), \tag{3.6}$$

where $\pi \in \mathcal{R}_{ep}(\mathcal{A}, \mathcal{H})$, $\alpha_t \in \mathrm{Aut}(\mathcal{A})$, $U(t) \in B(\mathcal{H})$, such that $(\alpha_t)_{t\in\mathbb{R}}$ is a one-parameter group of $*$-automorphisms of a C^*-algebra $\mathcal{A}$, and $(U(t))_{t\in\mathbb{R}}$ is a one-parameter group of unitary operators on a Hilbert space $\mathcal{H}$, satisfying that: for all $x \in \mathcal{A}$, and for all $\xi \in \mathcal{H}$,

$$\begin{cases} \lim_{t\to 0}\big\|x - \alpha_t(x)\big\| = 0, & \text{and} \\ \lim_{t\to 0}\big\|\xi - U(t)\xi\big\|_{\mathcal{H}} = 0. \end{cases} \tag{3.7}$$

We say that the system (3.6) is a covariant dynamical system (for short, an $\mathbb{R}$-dynamical system), if

$$\pi\big(\alpha_t(x)\big) = U(t)\pi(x)U(t)^* \tag{3.8}$$

for all $x \in \mathcal{A}$, and $t \in \mathbb{R}$.

Now, let's replace the additive group $\mathbb{R} = (\mathbb{R}, +)$ of a covariant dynamical system (3.6) to an arbitrary Lie group.

Definition 3.2. Let

$$\big(\pi, \alpha_g, U(g), \mathcal{H}, \mathcal{A}, g \in G\big)$$

be a certain covariant dynamical system, where G is a Lie group, satisfying: $\pi \in \mathcal{R}_{ep}(\mathcal{A}, \mathcal{H})$, $\alpha_g \in \mathrm{Aut}(\mathcal{A})$, $U(g) \in B(\mathcal{H})$, and

$$U(g)^* = U\big(g^{-1}\big) \quad \text{for all } g \in G, \tag{3.9}$$

where g^{-1} means the unique (group-)inverse of g in G, and

$$\begin{cases} \lim_{g \to e} \big\| x - \alpha_g(x) \big\| = 0, & \text{and} \\ \lim_{g \to e} \big\| \xi - U(g)\xi \big\|_{\mathcal{H}} = 0 \end{cases} \tag{3.10}$$

for all $x \in \mathcal{A}$, $\xi \in \mathcal{H}$, where $e \in G$ is the group identity. In this case, we assume the covariance relation;

$$\pi\big(\alpha_g(x)\big) = U(g)\pi(x)U(g)^* \tag{3.11}$$

for all $x \in \mathcal{A}$, and $g \in G$. Then the system is called a G-dynamical system.

In our setting, we can obtain the following lemma.

Lemma 3.1 (See [11]). *Let H be a finite-dimensional Hilbert space, and $\mathfrak{A}_l$ be the left C^*-algebra over H, with its Fock space representation π, and vacuum vector Ω of $\mathcal{F}(H)$, where, in particular,*

$$\pi(h) = l(h) \quad \textit{for all } h \in H.$$

For a fixed element $h \in H$, set

$$\delta_h(\bullet) = \big[a^+(h) - a(h), \bullet\big]. \tag{3.12}$$

Then there exists an $\mathbb{R}$-dynamical system

$$\big(\pi, \alpha_t, U(t), \mathcal{F}(H), \mathfrak{A}_l, t \in \mathbb{R}\big) \tag{3.13}$$

such that

$$\lim_{t \to 0} t^{-1}\big(\alpha_t(x) - x\big) = \delta_h(x), \quad \textit{and} \tag{3.14}$$

$$\lim_{t \to 0} t^{-1}\big(U(t)\xi - \xi\big) = \big(a^+(h) - a(h)\big)\xi \tag{3.15}$$

hold, for all $x \in \mathcal{D} \subset \mathfrak{A}_l$, and $\xi \in \pi(\mathcal{D})\Omega \subseteq \mathcal{F}(H)$.

Remark 3.1. With the use of analytic elements, it is possible to make precise sense of the following two formal power series expressions:

$$\alpha_t(x) = x + t\delta_h(x) + \frac{t^2}{2}\delta_h^2(x) + \cdots = \sum_{m=0}^{\infty} \frac{t^m}{m!}\delta_h^m(x), \quad \text{and} \tag{3.16}$$

$$U(t)\xi = \xi + tH_h\xi + \frac{t^2}{2}H_h^2\xi + \cdots = \sum_{m=0}^{\infty} \frac{t^m}{m!}H_h^m\xi, \tag{3.17}$$

where $H_h = a^+(h) - a(h)$.

The point is that if $x \in \mathcal{D}$, $\xi \in \pi(\mathcal{D})\Omega$, and $t \in \mathbb{R}$, sufficiently close to 0, then the two formal series (3.16), and (3.17) converge in the respective norms.

But, when evaluated an analytic elements, and for sufficiently small $t \in \mathbb{R}$, the expression

$$U(t)\pi(x)U(-t)$$

can be computed as a power series, and a direct verification yields

$$U(t)\pi(x)U(-t) = \pi\big(\alpha_t(x)\big),$$

where $\alpha_t(x)$ is as in (3.16). We leave the details of the proof of the previous lemma to the readers (see [11]). The ingredient of the proof is the existence of dense families of analytic elements, and analytic vectors.

Under our settings and observations, we obtain the following lemma, too.

LEMMA 3.2 (See [11]). *Let H be a Hilbert space, and let U_H denote the group of all unitaries on H.*

For $A \in U_H$, there is a unique automorphism α_A on $\mathfrak{A}_l$, such that

$$\begin{cases} \alpha_A\big(l(h)\big) = l(Ah), \quad \textit{and} \\ \alpha_A\big(l(h)^*\big) = \big(l(Ah)\big)^* \end{cases} \tag{3.18}$$

for all $h \in H$.

Assume that a given Hilbert space H in the above lemma is 1-dimensional. Then we have

$$U_H = U_{\mathbb{C}} \overset{\text{denote}}{=} U_1 = \big\{z \in \mathbb{C} : |z| = 1\big\}.$$

For $z \in U_1$, we can get that:

$$\alpha_z\big(l(h)\big) = zl(h) \quad \text{for } h \in H. \tag{3.19}$$

We get an action of $U_H \times U_1$ by automorphisms. For the group

$$G = U_H \times U_1,$$

there is a G-dynamical system $(\pi, \alpha_A, \mathcal{F}(H), \mathfrak{A}_l, A \in G)$, such that

$$\pi\big(\alpha_A(x)\big) = \Gamma(A)\pi(x)\Gamma(A)^* \tag{3.20}$$

holds for all $A \in U_H$ (regarded as $(A, 1) \in G$) and $x \in \mathfrak{A}_l$, and

$$\pi\big(\alpha_{e^{i\theta}}(x)\big) = e^{i\theta\mathcal{N}}\pi(x)e^{-i\theta\mathcal{N}} \tag{3.21}$$

holds for all $\theta \in \mathbb{R}$, and for all $x \in \mathfrak{A}_l$.

Note that U_1 is also understood as the unit circle $\mathbb{T}$ of $\mathbb{C}$, and we will use it as a group with Haar measure, and dual group $\widehat{\mathbb{T}}$ of $\mathbb{T}$ is group-isomorphic to $\mathbb{Z}$. Hence, the factor $e^{i\theta\mathcal{N}}$ in (3.21) is a unitary representation of $\mathbb{T}$, and

$$P_n \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{i\theta\mathcal{N}}\,d\theta \tag{3.22}$$

is a projection, for $n \in \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N} \cup \{0\}$, the operator P_n is the projection onto the eigenspaces of $\mathcal{N}$, and hence onto the closed subspaces in the decomposition of $\mathcal{F}(H)$.

REMARK 3.2. If $A(t) = e^{tB}$, for $t \in \mathbb{R}$, where B satisfies $B^* = -B$, then

$$\pi\big(\alpha_A(x)\big) = \Gamma(A)\pi(x)\Gamma(A)^*,$$

and

$$\pi\big(\delta_A(x)\big) = \big[d\Gamma(B), \pi(x)\big].$$

4. The Lie group $U(H, 1)$

On starting point is a fixed Hilbert space H. The other objects will be generated from H. We are concerned with a canonical group action in three distinct contexts: First, as a unitary representation U, acting on the Fock space $\mathcal{F}(H)$ of Boltzmann statistics-particles from physics; and at the same time, actions on two C^*-algebras; the Toeplitz C^*-algebra $\mathcal{T}(H)$, and its quotient $\mathcal{O}(H)$ by the ideal $\mathcal{K}$, we introduced in Section 2. The actions by automorphisms comes about via a covariant system built from the unitary representation U.

The group $G = U(H, 1)$ represented and acting in all three instances depends functorially on H. But it will be necessary to make use of its Lie algebra in order to account for the representations and their induced groups of automorphisms. More generally, starting with representations of Lie group G, we get derived representations of the corresponding Lie algebra: If the representation of G is unitary, then it follows that the individual operators (typically unbounded) are generators of strongly continuous unitary one-parameter groups of operators. And since we build real Lie algebras from skew-adjoint operators, we must necessarily work with "real" Lie algebras, even though the initial Hilbert space H is complex. Rationale: We are studying the global unitary representation U of G (from Section 3 above), acting on the Fock space.

But we caution that the two operators $a(h)$ and $a^+(h)$ are "not" in the Lie algebra of G. Rather, in the differentiated representation dU applied to the Lie algebra of G we get the difference $a^+(h) - a(h)$. The differences are skew-adjoint, but not $a(h)$ be itself, hence, the need for Lie algebras over reals.

When working with dU applied to the Lie algebra of G, the framework dictates that each $dU(x)$ is a skew-adjoint operator on $\mathcal{F}(H)$, for x in the "real" Lie algebra. This is so even if H at the outset is complex.

Fix some vector h in H. Then we show that the two elements of the form $dU(x)$:

$$a^+(h) - a(h), \quad \text{and} \quad i\big(a^+(h) + a(h)\big),$$

and their commutators span a copy of the 3-dimensional Heisenberg Lie algebra, with $i = \sqrt{-1}$. Without the i as a factor in front of the sum-operator, we would not be in the Lie algebra of skew-adjoint operators dU (the Lie algebra). The convention is that the operators in this Lie algebra must generate unitary one-parameter groups, which each of these two do.

By contrast, in Section 6 below, we have a discussion of multiplicity of the global unitary representation U of G. We compute the commutant of U as it acts on $\mathcal{F}(H)$. Our answer is that each element in the commutant is given by a sequence of functions (t_n). Specifically, each function t_n is the system is defined on the symmetric group S_n. We prove that such a system (t_n) of functions correspond to an operator in the commutant if and only if they satisfy a certain consistency condition as the index n varies; the consistency condition makes (t_n) a martingale. Moreover, we prove that the correspondence between elements in the commutant and martingales is bijective.

Before introducing the group $U(H,1)$, we recall some preliminaries. Let G be a fixed Lie group generating the Lie algebra $\mathfrak{G}$. And let

$$\exp_G : \mathfrak{G} \to G$$

be the exponential mapping from Lie theory. If G is represented as a matrix group, then $\exp_G$ is the exponential mapping from linear algebra.

REMARK 4.1. The properties of $\exp_G$ we need are the following: It maps the Lie algebra $\mathfrak{G}$ onto an open connected neighborhood of the group-identity e of G;

$$\exp_G(0) = e,$$

and

$$\exp_G\big((s+t)X\big) = \exp_G(sX)\exp_G(sX)$$

for all s, $t \in \mathbb{R}$, for a fixed element $X \in \mathfrak{G}$. But, in general,

$$\exp_G(X+Y) \neq (\exp_G X)(\exp_G Y)$$

for X, $Y \in \mathfrak{G}$.

If ρ is a strongly continuous representation of G, acting on a normed space $\mathcal{X}$, then there is a dense subspace $\mathcal{X}_\infty$ of $\mathcal{X}$, such that the operators

$$d\rho(X)_u \stackrel{\text{def}}{=} \lim_{t\to 0} t^{-1}\big(\rho\big(\exp_G(tX)\big)u - u\big)$$

is well-defined, for all $u \in \mathcal{X}_\infty$, and

$$\big[d\rho(X), d\rho(Y)\big] = d\rho\big([X,Y]\big)$$

holds on $\mathcal{X}_\infty$, for all X, $Y \in \mathfrak{G}$.

Let V be a vector space over $\mathbb{R}$, or over $\mathbb{C}$, and let J be an invertible transformation on V. Pick a non-degenerated bi-linear, or sequilinear form φ on V, such that $V_\varphi^* \stackrel{\text{iso}}{=} V$, where $\stackrel{\text{iso}}{=}$ means "being vector-space isomorphic." Set

$$Q(v) \stackrel{\text{def}}{=} \varphi(v, Jv) \quad \text{for all } v \in V. \tag{4.1}$$

Then

$$G_Q \stackrel{\text{def}}{=} \big\{g : V \to V \,|\, Q(gv) = Q(v), \forall v \in V\big\} \tag{4.2}$$

is a Lie group. We shall take a Hilbert space $V = H \times \mathbb{C}$, where H is a Hilbert space, with

$$Q\big((h,z)\big) = \|h\|_H^2 - |z|^2 \quad \text{for all } (h,z) \in V. \tag{4.3}$$

Or, equivalently, with

$$J = \begin{pmatrix} 1_H & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.4}$$

$$Q(v) = \langle v, Jv \rangle_V \quad \text{for all } v \in V, \tag{4.5}$$

where $\langle , \rangle_V$ means the inner product on V.

The following lemma is needed for our works.

LEMMA 4.1 (See [11]). (i) *The Lie group* $G = U(H,1)$ *of the form* Q, *from* (4.3) *through* (4.5), *consists of the following block matrices*,

$$\begin{pmatrix} A & h_1 \\ \langle h_2, \bullet \rangle_H & z \end{pmatrix}, \tag{4.6}$$

where $A \in B(H)$, *and* $h_1,\ h_2 \in H$, *and* $z \in \mathbb{C}$, *satisfying*

$$\begin{cases} A^* h_1 = z h_2, \\ A^* A - |h_2\rangle\langle h_2| = 1_H, \\ \|h_1\|_H^2 - |z|^2 = -1. \end{cases} \tag{4.7}$$

(ii) *The group* $U_H \times U_1$ *embeds into* $U(H,1)$, *via*

$$\begin{pmatrix} A & 0 \\ 0 & z \end{pmatrix}.$$

(iii) *The Lie algebra* $\mathfrak{G}$, *generated by the group* $U(H,1)$ *consists of the following block matrices*

$$\begin{pmatrix} B & h \\ \langle h, \bullet \rangle_H & w \end{pmatrix}, \tag{4.8}$$

where $B : H \to H$ *is linear*, $h \in H$, *and* $w \in \mathbb{C}$, *where* B, *and* w *satisfy*:

$$B^* = -B, \quad \textit{and} \quad \overline{w} = -w,$$

that is, B *is skew-Hermitian on* H, *and* w *is pure imaginary in* $\mathbb{C}$.

(iv) *Inside the Lie algebra in the sense of* (iii) *has two subspaces* $\mathfrak{K}$ *and* $\mathfrak{P}$, *where*

$$\mathfrak{K} = \left\{ \begin{pmatrix} B & 0 \\ 0 & w \end{pmatrix} \,\middle|\, B^* = -B, \text{ and } \overline{w} = -w \right\}, \tag{4.9}$$

$$\mathfrak{P} = \left\{ \begin{pmatrix} 0 & h \\ \langle h, \bullet \rangle_H & 0 \end{pmatrix} \,\middle|\, h \in H \right\} \overset{\text{Hilbert}}{=} H, \tag{4.10}$$

where $\overset{Hilbert}{=}$ *means "being Hilbert-space isomorphic." With the Lie commutator* $[X,Y]=XY-YX$, *we can get the followings*:

$$[\mathfrak{K},\mathfrak{K}]\subseteq\mathfrak{K},\qquad [\mathfrak{K},\mathfrak{P}]\subseteq\mathfrak{P},\quad \textit{and}\quad [\mathfrak{P},\mathfrak{P}]\subseteq\mathfrak{K}. \tag{4.11}$$

(v) *The relation* (4.11) *spells out as follows*:

$$\begin{aligned}\left[\begin{pmatrix}0 & h_1\\ \langle h_1,\bullet\rangle_H & 0\end{pmatrix},\begin{pmatrix}0 & h_2\\ \langle h_2,\bullet\rangle_H & 0\end{pmatrix}\right]\\ =\begin{pmatrix}|h_1\rangle\langle h_2|-|h_1\rangle\langle h_2|^* & 0\\ 0 & \langle h_1,h_2\rangle_H-\overline{\langle h_1,h_2\rangle_H}\end{pmatrix}\end{aligned} \tag{4.12}$$

for all $h_1,h_2\in H\overset{Hilbert}{=}\mathfrak{P}$.

By the previous lemma, we can obtain the following corollary.

Corollary 4.2. *Let* H *be a finite-dimensional Hilbert space and* $\mathfrak{A}_l$, $\mathcal{F}(H)$, *given as above*, *and let* G *be a Lie group* $U(H,1)$, *that is*,

$$G=U(H,1).$$

Then there exist the groups of automorphisms

$$(\alpha_g)_{g\in G}\subset \operatorname{Aut}(\mathfrak{A}_l),\quad \textit{and} \tag{4.13}$$

$$\big(U(g)\big)_{g\in G}\subset B\big(\mathcal{F}(H)\big), \tag{4.14}$$

and a unitary representation π *of* G, *such that*

$$\pi\big(\alpha_g(x)\big)=U(g)\pi(x)U(g)^* \tag{4.15}$$

for all $g\in G$, *and* $x\in\mathfrak{A}_l$, *that is*, *the homomorphisms in* (4.13) *and* (4.14) *define a* G*-dynamical system*

$$\big(\pi,\alpha_g,\mathcal{F}(H),\mathfrak{A}_l,g\in U(H,1)\big).$$

Moreover, *for each one-parameter subgroup of* G, *the corresponding* $\mathbb{R}$*-dynamical system coincides with that of* (3.13). *Specifically*,

$$\alpha\left(\exp_G t\begin{pmatrix}0 & h\\ \langle h,\bullet\rangle_H & 0\end{pmatrix}\right)=e^{t\delta_h}, \tag{4.16}$$

and

$$U\left(\exp_G t\begin{pmatrix}0 & h\\ \langle h,\bullet\rangle_H & 0\end{pmatrix}\right)=e^{t(a^+(h)-a(h))} \tag{4.17}$$

for all $t\in\mathbb{R}$, *and* $h\in H$.

Proof. This follows from the existence of analytic vectors, and the existence of the $\mathbb{R}$-dynamical system in the sense of (3.13), and the above lemma. We further rely on standard tools from representation theory of Lie algebras (e.g., [11]). □

COROLLARY 4.3. *The G-dynamical system in the sense of the above corollary passes to the quotient $\mathcal{O}(H)$ in the short exact sequence,*

$$0 \to \mathcal{K} \to \mathfrak{A}_l \xrightarrow{\psi} \mathcal{O}(H) \to 0, \tag{4.18}$$

where ψ is the natural quotient map, where $\mathcal{O}(H)$ is the Cuntz algebra, and $\mathcal{K}$ is the two-sided ideal of $\mathfrak{A}_l$, generated by the single element $T - 1_{\mathfrak{A}_l}$, where

$$T = \sum_{e \in ONB} l(e)l(e)^*.$$

We then have an induced action

$$\widetilde{\alpha} : U(H,1) \to \operatorname{Aut}\big(\mathcal{O}(H)\big),$$

by automorphisms as follows:

$$\widetilde{\alpha}\left(\exp_{U(H,1)} t \begin{pmatrix} 0 & h \\ \langle h, \bullet \rangle & 0 \end{pmatrix}\right) = e^{t\widetilde{\delta_h}}, \tag{4.19}$$

where δ_h is given in (4.17), and

$$\widetilde{\delta_h}\big(\psi(x)\big) = \psi\big(\delta_h(x)\big) \quad \text{for all } x \in \mathfrak{A}_l. \tag{4.20}$$

Proof. This follows directly from an application on the covarience relation (4.15). $\square$

Observe the following lemma.

LEMMA 4.4. (i) *Let H be a fixed finite dimensional Hilbert space, and $U(H,1)$, the Lie group from above, and let $\mathcal{U}$ be the Lie algebra generated by $U(H,1)$. Let $\lambda : H \to \mathcal{U}$ be the mapping, defined by*

$$\lambda(h) \overset{def}{=} \begin{pmatrix} 0 & h \\ \langle h, \bullet \rangle_H & 0 \end{pmatrix},$$

then we have

$$A\lambda(h)A^* = \lambda(Ah) \tag{4.21}$$

for all $A \in U_H$, and $h \in H$.

(ii) *If ρ is a representation of $U(H,1)$, then*

$$\rho(A)d\rho\big(\lambda(h)\big)\rho(A)^{-1} = d\rho\big(\lambda(Ah)\big) \tag{4.22}$$

holds on the space of differentiable vectors for the representations.

Proof. Let A and h be as in the statement of (i). Then $A^* = -A$, since $A \in U_H$, and

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & h \\ \langle h, \bullet \rangle & 0 \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & Ah \\ \langle Ah, \bullet \rangle & 0 \end{pmatrix}, \tag{4.23}$$

where we used the formula

$$\big(\langle h|\big)A^* = \langle Ah|$$

for composition. The consequence (4.22) in (ii) follows from representation theory (see, for example, [11]). □

By the previous lemma, we can obtain the following corollary.

COROLLARY 4.5. *Let H and $\lambda : H \to \mathcal{U}$ be as above, and let*

$$\begin{pmatrix} B & 0 \\ 0 & \beta \end{pmatrix}$$

be in the Lie algebra of $U_H \times U_1$. Then

$$\left[\begin{pmatrix} B & 0 \\ 0 & \beta \end{pmatrix}, \lambda(h)\right] = \lambda\big((B - \beta I)h\big)$$

holds for all $h \in H$. And the corresponding operator formula holds in any Lie algebra representation of $\mathcal{U}$. □

The above corollary can be illustrated by the following example.

EXAMPLE 4.1. Here, we work our the subspace $\mathcal{P} \subset \mathcal{U}$ in the special case where $\dim H = 2$. The case where $\dim H = 1$ is not of interest here, since the Cuntz algebra $\mathcal{O}(H)$ degenerates when $\dim H = 1$. Let

$$\lambda : H \to \mathcal{P}$$

be the mapping from (4.21), given by

$$\lambda(h) = \begin{pmatrix} 0 & h \\ \langle h, \bullet \rangle_H & 0 \end{pmatrix} \quad \text{for all } h \in H.$$

Notice that λ is only linear over $\mathbb{R}$. With $\dim H = 2$, we pick the standard ONB. A computation (4.21) and (4.22) show that

$$\exp_G\big(t\lambda(e_1)\big) = \begin{pmatrix} \mathrm{ch}(t) & 0 & \mathrm{sh}(t) \\ 0 & 1 & 0 \\ \mathrm{sh}(t) & 0 & \mathrm{ch}(h) \end{pmatrix} \in U(H, 1),$$

and

$$\exp_G\big(t\lambda(e_2)\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{ch}(t) & \mathrm{sh}(t) \\ 0 & \mathrm{sh}(t) & \mathrm{ch}(t) \end{pmatrix} \in U(H, 1)$$

for all $t \in \mathbb{R}$, where $\mathrm{ch}(t)$, and $\mathrm{sh}(t)$ are the cosine-hypobolic, and the sine-hypobolic functions, respectively, that is,

$$\mathrm{ch}(t) = \frac{1}{2}\big(e^t + e^{-t}\big), \quad \text{and} \quad \mathrm{sh}(t) = \frac{1}{2}\big(e^t - e^{-t}\big).$$

Using (4.12), for the Lie commutator of $\lambda(e_1)$, and $\lambda(e_2)$, we have that

$$\big[\lambda(e_1), \lambda(e_2)\big] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{\text{denote}}{=} E,$$

and

$$\exp_G(tE) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U_H \times U_1,$$

in $U(H,1)$.

Thus, generally, we can obtain the following proposition.

PROPOSITION 4.6. *Let* $\dim H = 2$, *and let*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \textit{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli spin matrices. Then, for the Lie algebra $\mathcal{U}$ *of* $U(H,1)$, *we have the following commutator rules*:

$$\big[\lambda(e_1), \lambda(e_2)\big] = i\sigma_y,$$
$$\big[\lambda(e_1), \lambda(ie_2)\big] = i\sigma_x,$$

and

$$\left[\lambda\left(\frac{e_1 + ie_2}{\sqrt{2}}\right), \lambda\left(\frac{ie_1 + e_2}{\sqrt{2}}\right)\right] = i\sigma_z.$$

In particular,

$$\big[\lambda(H), \lambda(H)\big] = u(H),$$

where $u(H)$ *is the whole Lie algebra of the unitary group* U_2.

Also, we can get the following proposition.

PROPOSITION 4.7. *Let* H *be a Hilbert space, and consider the group* $U(H,1)$, *and its Lie algebra* $\mathcal{U}$. *The vector part* $\mathcal{P} = \lambda(H)$ *in* $\mathcal{U}$ *is viewed as a real Lie algebra. If* V *is a real form of* H, *then, for* $v \in V$, *we have*

$$dU\big(\lambda(iv)\big) = i\big(a^+(v) + a(v)\big), \tag{4.24}$$

where dU *is the derived Lie algebra representation.*

Proof. Recall the general formula

$$dU\big(\lambda(h)\big) = a^+(h) - a(h)$$

for $h \in H$. Now, apply this to $h = iv$, then we get

$$dU\big(\lambda(iv)\big) = a^+(iv) - a(iv) = ia^+(v) + ia(v),$$

which is the desired conclusion. □

5. Heisenberg Lie algebras

As before, let H be a fixed complex Hilbert space of dimension at least 2, but possibly infinite-dimensional, and let $\mathcal{F}(H)$ be the Fock space over H. We will study the representations of the generally infinite-dimensional Lie group $U(H,1)$.

The purpose of this section is to introduce two finite-dimensional Lie algebras. They turn out to play an important role in our analysis of the representation of the infinite-dimensional Lie algebra of $U(H,1)$. The reason for this is two-fold; first, the $U(H,1)$-Lie algebra contains isomorphic copies of these "small" Lie algebras. Secondly, we can use known theorems for the "small" Lie algebras in order to infer spectral theoretic properties of the representations of $U(H,1)$.

In this section, we consider certain Lie algebras over real, $\mathbb{R}$.

Definition 5.1. (i) A 3-dimensional Lie algebra $\mathfrak{H}$ over $\mathbb{R}$ is called a Heisenberg Lie algebra, if it has a basis $\{x,y,z\} \subset \mathfrak{H}$, satisfying

$$[x,y] = z, \quad \text{and} \quad [z,x] = [z,y] = 0, \tag{5.1}$$

i.e., the center in $\hbar$ is one-dimensional.

(ii) A real Lie algebra $\mathcal{G}$, spanned by 3-elements x_1, x_2, x_3 is said to be an isomorphic copy of the Lie algebra $\mathrm{sl}_2(\mathbb{R})$, if

$$[x_1,x_2] = x_3, \qquad [x_3,x_1] = -x_2, \quad \text{and} \quad [x_3,x_2] = x_1, \tag{5.2}$$

where $[\cdot,\cdot]$ denotes the Lie bracket in $\mathcal{G}$.

By Section 4, we can obtain the following proposition, with help of above definition.

Proposition 5.1. *Let H be a complex Hilbert space and $U(H,1)$, the Lie group introduced as above, and let $\mathcal{G}$ be the Lie algebra generated by $U(H,1)$. Let*

$$\lambda : H \to \mathcal{G}$$

be

$$\lambda(h) \overset{def}{=} \begin{pmatrix} 0 & h \\ \langle h, \bullet \rangle_H & 0 \end{pmatrix} \quad \text{for } h \in H, \tag{5.3}$$

where the 0 *in the left top coner denotes the zero operator on H. Then every $\lambda(h)$, for $h \in H \setminus \{0\}$, is contained in a copy of the Lie algebra* $\mathrm{sl}_2(\mathbb{R})$.

Proof. We may assume, without loss of generality, that $\|h\|_H = 1$. Now, let $h = e_1$ be the first vector in some ONB in H. Set

$$x \overset{\text{def}}{=} \lambda(e_1), \quad \text{and} \quad y \overset{\text{def}}{=} \lambda(ie_1), \tag{5.4}$$

and

$$E_1 \overset{\text{def}}{=} |e_1\rangle\langle e_1|, \quad \text{on } H.$$

In H, we work with the splitting:

$$H = \mathbb{C}e_1 \oplus (H \ominus \mathbb{C}e_1). \tag{5.5}$$

Finally, relative to the representation, set

$$\gamma_0 \overset{def}{=} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}. \tag{5.6}$$

Now, using Proposition 4.6, we get that:

$$[x, y] = 2 \begin{pmatrix} -iE_1 & 0 \\ 0 & i \end{pmatrix} = 2(-iE_1 + \gamma_0),$$

with a slight abuse of notation, where x and y are introduced in (5.4). Again, with Proposition 4.6, we can get that:

$$[-iE_1 + \gamma_0, x] = -2y, \quad \text{and} \quad [iE_1 + \gamma_0, y] = 2x. \tag{5.7}$$

As a result, we conclude that the real space of the three elements

$$\{-iE_1 + \gamma_0, x, y\}$$

is a Lie-isomorphic copy of $\mathrm{sl}_2(\mathbb{R})$. The spectral projections of the two operators

$$dU(x) = a^+(h) - a(h),$$

and

$$dU(y) = i\big(a^+(h) + a(h)\big)$$

follows from [11]. Note here that iE_1 is contained in the Lie algebra of U_H, while γ_0 is contained in the Lie algebra of U_1. Further, we view $U_H \times U_1$ as a compact subgroup in $U(H, 1)$. So, the desired conclusion follows. □

By the previous proposition, we can obtain the following two corollaries.

Corollary 5.2. *With the raising and lowering operators in $\mathcal{F}(H)$, we have that*

$$\begin{aligned} \big[a^+(e_1) - a(e_1), i\big(a^+(e_1) + a(e_1)\big)\big] &= 2i\big[a^+(e_1), a(e_1)\big] \\ &= 2\big(d\Gamma(iE_1) + i(\mathcal{N} + 1)\big), \end{aligned} \tag{5.8}$$

where $\mathcal{N}$ is the number operator on $\mathcal{F}(H)$, in the sense of Section 1.

Proof. The formula (5.8) is gotten immediately from the results of Section 3, and 4, above. We apply the differentiated representation dU and its restriction $d\Gamma$ to the two sides in (5.6). □

Remark 5.1. For some purposes, it is convenient to rewrite (5.8) in the form,

$$\big[a^+(e_1) - a(e_1), i\big(a^+(e_1) + a(e_1)\big)\big] = -2\big(d\Gamma(iE_1) + (\mathcal{N} + 1)\big), \tag{5.9}$$

but note that, in (5.9), the first operator $a^+(e_1) - a(e_1)$ is skew-Hermitian while the second operator $a^+(e_1) + a(e_1)$ is Hermitian.

Corollary 5.3. *The operator $a^+(e_1) - a(e_1)$ in (5.9) is essentially skew-adjoint while $a^+(e_1) + a(e_1)$ is essentially self-adjoint. Both operators have Lebesgue spectrum, that is, the spectral resolution measure for both operators is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$; and the spectrum of the corresponding two self-adjoint operators is equal to $\mathbb{R}$.*

Proof. We check that the two operators in (5.9) have deficiency indices $(0,0)$, then the stated conclusions regarding their spectra follows immediately from the Stone–von Neumann uniqueness theorem.

But with a use of Nelson's theorem on analytic vectors (for representations of Lie algebras, see [11]), we further note that it is enough to prove essential self-adjointness of

$$\Delta_s \stackrel{\text{def}}{=} dU\big(\lambda(e_1)\big)^2 + dU\big(\lambda(ie_1)\big)^2, \tag{5.10}$$

where we introduce the notation Δ_s for self-Laplacian. We now compute Δ_s in detail; a direct computation using the above corollary yields

$$\begin{aligned} \Delta_s &= (-2)\big(a^+(e_1)a(e_1) + a(e_1)a^+(e_1)\big) \\ &= (-2)\big\{a^+(e_1), a(e_1)\big\}, \end{aligned} \tag{5.11}$$

where $\{\cdot,\cdot\}$ is the anti-commutator. Since the operator $\{a^+(e_1), a(e_1)\}$ commutes with $\mathcal{N}$, its essential self-adjointness may be checked on the individual eigenspaces for $\mathcal{N}$, as it acts on $\mathcal{F}(H)$, that is, recall that each n-subspace, we have

$$P_n\mathcal{F}(H) = \mathcal{F}_n(H) = H^{\otimes n}. \tag{5.12}$$

And the desired conclusion is immediate. Indeed, we may read off an exact formula for $\{a^+(e_1), a(e_1)\}$ directly from Lemma 2.1, specifically,

$$\big\{a^+(e_1), a(e_1)\big\} = (\mathcal{N} + 1_{\mathfrak{A}_l}) + (2\mathcal{N} + 1_{\mathfrak{A}_l})\, d\Gamma(E_1). \tag{5.13}$$

□

6. Martingales on the permutation groups

Here, we returned to the representations in $\mathcal{F}(H)$ of the Lie group $U(H,1)$. As noted, in this context, the unitary representations of $U(H,1)$ can best be understood via representations of the corresponding Lie algebras. Each of the operators in the Lie algebra are skew-adjoint, and so the question of spectral type is natural. To resolve this question, it is also helpful to identify finite-dimensional Lie subalgebras. Two reasons: $U(H,1)$ is in general infinite-dimensional, so it must be understood as an inductive limit in such a way that the limit considerations carry over to the operators. Secondly, the standard results on spectral type apply to specific finite-dimensional Lie algebras. The two Lie algebras we will need are the Heisenberg Lie algebra, and the $\mathrm{sl}_2(\mathbb{R})$-Lie algebra.

For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set $S_0 = \{\varnothing\}$, $S_1 = \{1\}$, and

$$S_n = \text{the group of permutations of } \{1, \dots, n\},$$

where a permutation in S_n is a bijective map from $\{1,\dots,n\}$ onto $\{1,\dots,n\}$, for $n \in \mathbb{N}$. Thus, if $n > 1$, we identify elements $\pi \in S_n$ with their graphs

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}.$$

We will consider the set-disjoint union $\bigsqcup_{n\in\mathbb{N}_0} S_n$, and a $\mathbb{C}$-valued function t on this set. Notice that a function t on $\bigsqcup_{n\in\mathbb{N}_0} S_n$ means a system $(t_n)_{n\in\mathbb{N}_0}$ of functions

$$t_n : S_n \to \mathbb{C} \quad \text{for all } n \in \mathbb{N}_0. \tag{6.1}$$

Given a specific function t, for every $n \in \mathbb{N}_0$, and subsets $A \subset S_n$, we define

$$\mathbb{E}_A^{(n)}(t) \stackrel{\text{def}}{=} \sum_{a\in A} t_n(a). \tag{6.2}$$

We view (6.2) as an *expectation.*

For every $\rho \in S_n$, and $j \in \{1,\dots,n\}$, remove the line

$$\begin{pmatrix} \rho^{-1}(j) \\ j \end{pmatrix}$$

in the graph of ρ. Then, for $\pi \in S_{n-1}$, we can construct the conditional expectation,

$$\mathbb{E}_j^{(n)}(t:\pi) \stackrel{\text{def}}{=} \sum_{\begin{pmatrix} \rho^{-1}(j) \\ j \end{pmatrix} \cup \pi = \rho} \mathbb{E}_{\{\rho\}}^{(n-1)}(t),$$

that is, $\mathbb{E}_j^{(n)}(t:\pi)$ is the summation $\mathbb{E}_A^{(n)}(t)$ in (6.2), where A is the set of permutations ρ, which agree with π, where some line

$$\begin{pmatrix} \rho^{-1}(j) \\ j \end{pmatrix}$$

for $j \in \{1,\dots,n\}$, is removed from its graph of ρ.

Definition 6.1. We say that a function t on $\bigsqcup_{n\in\mathbb{N}_0} S_n$ is a martingale, if

$$t_{n-1}(\pi) = \mathbb{E}_j^{(n)}(t:\pi) \tag{6.3}$$

for all $j \in \{1,\dots,n\}$, and for all $n \in \mathbb{N}$.

The convention for the initial terms in the definition (6.3) of a martingale t are the following: Denote the two elements in S_2 by e and s. Then the first steps in (6.3) are

$$t_0 = t_1 = t_2(e) + t_2(s). \tag{6.4}$$

Remark 6.1. (i) The set of all martingales is an infinite-dimensional variety.

(ii) The set of all martingales is not closed under pointwise product.

Indeed, the statement (i) holds, by (6.3) and by induction. Also, the statement (ii) follows from even (6.4) that the product of two martingales t and t' does not satisfy (6.4); Note the cross terms in

$$\big(t_2(e)+t_2(s)\big)\big(t_2'(e)+t_2'(s)\big).$$

PROPOSITION 6.1. (i) *The set* $\bigsqcup_{n\in\mathbb{N}_0} S_n$ *of all permutations is a group and it has a natural unitary representation on*

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H), \quad \textit{where} \tag{6.5}$$

$$\mathcal{F}_0(H) \overset{def}{=} \mathbb{C}, \qquad \mathcal{F}_1(H) \overset{def}{=} H, \quad \textit{and} \quad \mathcal{F}_n(H) \overset{def}{=} H^{\otimes n}. \tag{6.6}$$

In the first two cases $n=0$, *and* $n=1$, *the action is trivial. If* $n>1$, *and* $\pi\in S_n$, *we set*

$$\pi\cdot(k_1\otimes k_2\otimes\cdots\otimes k_n) \overset{def}{=} k_{\pi(1)}\otimes k_{\pi(2)}\otimes\cdots\otimes k_{\pi(n)}. \tag{6.7}$$

(ii) *The algebra of the two representations*

$$\Gamma_n(A) : k_1\otimes\cdots\otimes k_n \mapsto Ak_1\otimes\cdots\otimes A_{k_n} \tag{6.8}$$

for all $A\in U_H$, *and that of* (6.7) *are each other's commutant.*

Proof. The statements (i) and (ii) are followed by the Weyl's theorem in standard representation theory. □

By the previous discussions, we can obtain the following theorem.

THEOREM 6.2. *There is a bijective correspondence between the set of martingales and the commutant of the unitary representation* U *of the group* $U(H,1)$. *If* T *is in the commutant of* $U(H,1)$, *and if* t *is a martingale, then the correspondence* $t\leftrightarrow T$ *is fixed by*

$$P_nTP_n = \sum_{\pi\in S_n} t_n(\pi)\pi, \tag{6.9}$$

where P_n *is the projection in* (5.12), *and where the action on* $\mathcal{F}_n(H)$ *on the right-hand side of* (6.9) *is given by* (6.7).

Proof. Assume that some bounded linear operator T on $\mathcal{F}(H)$ commutes with the unitary $U(H,1)$ representation in Corollary 4.2. Using Lemma 3.5, we conclude that T must commute with the projections $P_0,P_1,P_2,\ldots$. Set

$$T_n \overset{\text{def}}{=} P_nTP_n = TP_n \quad \text{for } n\in\mathbb{N}_0.$$

Then we can get the representation,

$$T = \bigoplus_{n=0}^{\infty} T_n. \tag{6.10}$$

An application of Proposition 5.1 shows that, for each $n \in \mathbb{N}_0$, T_n has the representation (6.9), for some function t_n on S_n.

It remains to show that T commutes with the rest of $U(H,1)$, if and only if t is martingale. (It is clear that the martingale is bounded, if and only if the operator T in (6.10) and (6.9) is bounded on $\mathcal{F}(H)$.)

Using Lemma 4.4, it is clear that

$$\big[T, a^+(h) - a(h)\big] = 0 \quad \text{for all } h \in H. \tag{6.11}$$

Combining (6.11), and (6.10), we see that (6.11) amounts to

$$\big[T, a(h)\big] = 0 \quad \text{for all } h \in H. \tag{6.12}$$

To see this, use:

$$P_m a^+(h) P_n = \begin{cases} a^+(h) P_n & \text{if } m = n+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P_m a(h) P_n = \begin{cases} a(h) P_n & \text{if } m = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

It remains to verify that (6.12) is equivalent to the martingale property for $t = (t_n)_{n \in \mathbb{N}_0}$. And this is a direct computation which we leave to the readers. $\square$

Now, let's define the convolution on the group C^*-algebra $C^*(S_n)$, generated by S_n, which is the operator multiplication on $C^*(S_n)$, precisely.

DEFINITION 6.2. Fix $n \in \mathbb{N}$, and S_n, the permutation group, and let $C^*(S_n)$ be the group C^*-algebra generated by S_n. Define the product $A * B$ of operators A, and B in $C^*(S_n)$ by

$$(A * B)(\rho) \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} A(\sigma) B\big(\sigma^{-1}\rho\big) \tag{6.13}$$

for all $\rho \in S_n$.

By the previous theorem, we can obtain the following corollary, with help of the above definition.

COROLLARY 6.3. (i) *The commutant in Corollary* 5.2 *is a projective limit as follows*: *An operator* T *on* $\mathcal{F}(H)$,

$$T \leftrightarrow (t_n), \quad t_n \in C^*(S_n),$$

is in the commutant, if and only if, we have

$$\mathbb{E}_j^{(n)}(t : \pi) = t_{n-1}(\pi) \quad \textit{for } \pi \in S_{n-1}. \tag{6.14}$$

(ii) *For elements*, $T = (t_n)$, *and* $W = (w_n)$ *in the commutant, we have the formulas*,

$$\mathbb{E}_j^{(n)}(t * w : \pi) = (t_{n-1} * w_{n-1})(\pi) \tag{6.15}$$

are valid, for all $\pi \in S_{n-1}$*, for all* $j = 1, \ldots, n$*, and* $n \in \mathbb{N}$.

Proof. The statement (i) is a direct consequence of the above theorem. So, the main point is proving the multiplication formula (6.15) in (ii). Let $T = (t_n)$, and $W = (w_n)$ be as specified. We now compute the left-hand side in (6.15) as follows: Let $\pi \in S_{n-1}$. Then

$$\begin{aligned}
\mathbb{E}_i^{(n)}(t * w : \pi) &= \sum_{\rho \in A_i(\pi)} (t_n * w_n)(\rho) \\
&= \sum_{\rho \in A_i(\pi)} \sum_{\sigma \in S_n} t_n(\sigma) w_n\left(\sigma^{-1}\rho\right) \\
&\quad \text{by (6.13)} \\
&= \sum_{\rho \in A_i(\pi)} \sum_{\sigma^* \in S_{n-1}} \sum_{\sigma \in A_j(\sigma^*)} t_n(\sigma) w_n\left(\sigma^{-1}\rho\right) \\
&\quad \text{by the partition property} \\
&= \sum_{\sigma^* \in S_{n-1}} t_{n-1}\left(\sigma^*\right) w_{n-1}\left(\left(\sigma^*\right)^{-1}\pi\right) \\
&\quad \text{by (6.14)} \\
&= (t_{n-1} * w_{n-1})(\pi) \\
&\quad \text{by (6.13).}
\end{aligned}$$

□

The purpose of the next two results is to show that, when the Hilbert space H is fixed, the commutant of the associated unitary representation of $U(H, 1)$ contains a special element in its center. Secondly, we point out that the commutant is non-Abelian. Our results are more specific: The element in the center is special; and the nature of the noncommutativity is further illustrated.

COROLLARY 6.4. *Consider the correspondence* (6.9) *between the martingales* (t_n) *and the operators in the commutant* $\mathcal{C}_H$ *of the unitary representation* U *of* $U(H, 1)$*. Then* P_{sym}*, given by* $t_n^{(\mathrm{sym})}(\bullet) = \frac{1}{n!}$*, is in the center of* $\mathcal{C}_H$.

Proof. Using Corollary 6.3, we get

$$\begin{aligned}
\left(t_n^{(\mathrm{sym})} * w_n\right)(\rho) &= \left(w_n * t_n^{(\mathrm{sym})}\right)(\rho) \\
&= \sum_{\sigma \in S_n} w_n(\sigma) t_n^{(\mathrm{sym})}\left(\sigma^{-1}\rho\right) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} w_n(\sigma)
\end{aligned}$$

for all $W=(w_n)\in\mathcal{C}_H$, $n\in\mathbb{N}$, and $\rho\in S_n$. The last expression is the normalized Haar measure applied to w_n, that is,

$$h_n(w_n)=\frac{1}{n!}\sum_{\sigma\in S_n} w_n(\sigma),$$

and we obtain

$$t_n^{(\mathrm{sym})} * w_n = h_n(w_n)t_n^{(\mathrm{sym})}. \qquad \square$$

By the previous proposition, we can have the following corollary.

COROLLARY 6.5. *If* $\dim H\geq 2$, *then the commutant* $\mathcal{C}_H$ *is non-Abelian.*

Proof. Note that, for all $n\in\mathbb{N}$,

$$P_n\mathcal{C}_H P_n\subseteq C^*(S_n)$$

(see the Appendix below). Setting $n=3$, we see that

$$t_3=\rho_2+\rho_6, \quad \text{and} \quad w_3=\rho_3+\rho_5$$

are both in $P_3\mathcal{C}_H P_3$, where the four elements in S_3 are

$$\rho_2=(1,2), \qquad \rho_3=(2,3), \qquad \rho_5=(1,2)(2,3),$$

and

$$\rho_6=(2,3)(1,2),$$

where (i,j) means the permutations, sending i to j (also, see the Appendix below). The noncommutativity now follows from the commutator formula

$$t_3 * w_3 - w_3 * t_3 = \rho_3-\rho_2+\rho_5-\rho_6\neq 0. \qquad \square$$

Also, we can obtain the following corollary.

COROLLARY 6.6. *Let* $P^{(\mathrm{sym})}$ *denote the projection from* $\mathcal{F}(H)$ *onto the closed subspace* $\mathcal{F}_{\mathrm{sym}}(H)$. *Then* $P^{(\mathrm{sym})}$ *is in the commutant* $\mathcal{C}_H$ *of the* $U(H,1)$*-representation, and the restriction*

$$U(H,1)\ni g\longmapsto U(g)|_{\mathcal{F}_{\mathrm{sym}}(H)} \tag{6.16}$$

is an irreducible unitary representation.

Proof. We show that, for all $n\in\mathbb{N}$, we have that

$$P^{(\mathrm{sym})}P_n=P_nP^{(\mathrm{sym})}=\frac{1}{n!}\sum_{\sigma\in S_n}\sigma, \tag{6.17}$$

where

$$\sigma\cdot k_1\otimes\cdots\otimes k_n=k_{\sigma(1)}\otimes\cdots\otimes k_{\sigma(n)}. \tag{6.18}$$

If $W:\mathcal{F}_{\mathrm{wym}}(H)\to\mathcal{F}_{\mathrm{sym}}(H)$ is in the commutant of the restricted representation (6.16), then

$$WP^{(\mathrm{sym})}=P^{(\mathrm{sym})}W,$$

and the functions $w_n : S_n \to \mathbb{C}$ correspond to W must satisfy

$$t_n * w_n = w_n, \quad \text{on } S_n, \text{ for all } n \in \mathbb{N},$$

where

$$t_n(\sigma) \stackrel{\text{def}}{=} \frac{1}{n!} \quad \text{for all } \sigma \in S_n. \tag{6.19}$$

If we define $\varphi = (\varphi_n)_{n\in\mathbb{N}}$, by

$$\varphi_n(W) \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} w_n(\sigma) \quad \text{for all } n \in \mathbb{N},$$

then we can obtain that

$$w_n(\rho) = \frac{1}{n!}\varphi(w_n). \tag{6.20}$$

But, if $\pi \in S_{n-1}$, then

$$w_{n-1}(\pi) = \sum_{\rho \in S(j,\pi)} w_n(\rho).$$

Therefore,

$$\varphi(w_n) = \varphi(w_{n-1}) = \cdots = w_1 = w_0, \quad \text{in } \mathbb{C},$$

by Theorem 6.2. Hence,

$$W = w_0 P^{(\text{sym})}, \tag{6.21}$$

where we used (6.17), (6.19), and (6.20) in the derivation of (6.21). Thus, the commutant of the restricted representation (6.16) is 1-dimensional; and so this representation is irreducible. □

7. The action of $U(H,1)$ by automorphisms

Our purpose here is to examine the decomposition theory for the canonical unitary representation of $U(H,1)$. This representation is reducible. In fact, we prove that operators in its commutant are in bijective correspondence with a specific family of martingales on the system of permutation groups S_n, as n varies. Our second result is that the action of $U(H,1)$ by automorphisms on the Cuntz algebra is ergodic.

The formulas (4.16), and (4.17) show the existence of the action in covariant systems by $*$-automorphisms of

$$\mathfrak{A}_l = C^*\big(\{l(h) : h \in H\}\big),$$

and its induced action on

$$\mathcal{O}(H) = \mathfrak{A}_l / \mathcal{K}. \tag{7.1}$$

Here, we are referring to the formulas (4.13), (4.15), and (4.17), which specify the action. For $\mathcal{O}(H)$, the relevant formulas are (4.19), and (4.20). Also, we can obtain the following lemma.

LEMMA 7.1. *Let H be a Hilbert space with* $\dim H \geq 2$, *and let* $U(H,1)$ *and* $\mathfrak{A}_l$ *be given as above. Let* $t = (t_n)_{n\in\mathbb{N}_0}$ *be a bounded martingale and let* T *be the corresponding operator in the sense of* (6.9) *in the commutant,*

$$T \in U\big(U(H,1)\big)'.$$

Then T *commutes with the family* $\{r(h) : h \in H\}$, *if and only if* T *is a constant multiple of the identity operator on* $\mathcal{F}(H)$.

Proof. The idea is to prove that t is constant. Specifically, for every $n \in \mathbb{N}_0$,

$$t_n(\pi) = 0 \quad \text{for all } \pi \in S_n \setminus \{id\}, \tag{7.2}$$

and $t_0 = t_1 = t_2 = \cdots$, where id means the identity element of S_n.

We study the condition

$$\big[T, r(h)\big] = 0 \quad \text{for all } h \in H. \tag{7.3}$$

Applying this to $\mathcal{F}_1(H) = H$, we get

$$\begin{aligned} Tr(h)k &= T(k \otimes h) \\ &= t_2(e)k \otimes h + t_2(s)h \otimes k \\ &= t_1 k \otimes h. \end{aligned}$$

Since $\dim H > 1$, we may pick $k \in H \ominus \mathbb{C}h$, which is nonzero. Then the two vectors $k \otimes h$, and $h \otimes k$ are orthogonal in $\mathcal{F}(H)$. As a result,

$$t_2(e) = t_1, \quad \text{and} \quad t_2(s) = 0.$$

Compare this with (7.4) in Section 5. This is the first step in the proof of (7.2); and the rest follows from a simple induction. □

By the previous lemma, we can get the following proposition.

PROPOSITION 7.2. *The action* α *of* $G = U(H,1)$ *on* $\mathfrak{A}_l$, *and on* $\mathcal{O}(H)$ *is ergodic.*

Proof. We must show that, if $T \in \mathfrak{A}_l$, and

$$\alpha_g(T) = T \quad \text{for all } g \in G, \tag{7.4}$$

then T is a multiple of $1_{\mathfrak{A}_l} \in \mathfrak{A}_l$; and that the analogues assertion holds in $\mathcal{O}(H)$.

However, the formula (7.4) follows from the above lemma, and the formula (7.5) below;

$$\alpha_g(T) = U(g)TU(g)^{-1}, \tag{7.5}$$

observed in (4.15). For the two C^*-algebras $\mathfrak{A}_l$ and $\mathfrak{A}_r$, we have the following two commutator rules;

$$\begin{aligned} \big[l(k), r(h)\big] &= 0, \quad \text{and} \\ \big[l(k)^*, r(h)\big] &= \langle k, h\rangle_H P_0. \end{aligned} \tag{7.6}$$

It follows from that $[T, r(h)]$ is a compact operator for all $h \in H$. If $[T, r(h)] = 0$, then desired conclusion is immediate from the above lemma.

But, even if $[T, r(h)]$ is nonzero, $t_0 = 0$, by the compactness, and an induction argument yields the desired conclusion. $\square$

Appendix: Computations for martingales

In this Appendix, we collect a number of computations regarding the correspondence between martingales and operators in the commutant of the unitary representation of $U(H, 1)$ discussed above.

Let $t = (t_n)_{n\in\mathbb{N}_0}$ be a system of $\mathbb{C}$-valued functions with $t_0 = t_1 \in \mathbb{C}$,

$$t_n : S_n \to \mathbb{C} \quad \text{for all } n > 1$$

on $\bigsqcup_{n\in\mathbb{N}_0} S_n$, such that

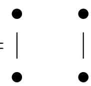
$$t_{n-1}(\pi) = \sum_{\rho \in S(j,\pi)} t_n(\rho) \quad \text{for } j = 1, 2, \ldots, n, \tag{A.1}$$

where

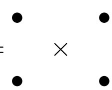
$$S(j, \pi) = \left\{ \rho \in S_n \middle| \pi = \rho \setminus \begin{pmatrix} \rho^{-1}(j) \\ j \end{pmatrix} \right\}, \tag{A.2}$$

where $\rho \setminus \binom{\rho^{-1}(j)}{j}$ is the element in S_{n-1} resulting from ρ by extraction of the edge $\binom{\rho^{-1}(j)}{j}$.

For instance, if S_2 is the permutation group over $\{1, 2\}$, then we can have two elements in S_2, depicted by

$$e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} =$$  ,

and

$$s = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} =$$  .

Let $S_3 = \{\rho_1, \rho_2, \ldots, \rho_6\}$ be the permutation group over $\{1, 2, 3\}$, where

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \qquad \rho_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$
$$\rho_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad \rho_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \rho_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

If t is martingale, then we obtain

$$t_0 = t_1 = t_2(e) + t_2(s),$$

and, for $e \in S_2$

$$\begin{aligned} t_2(e) &= t_3(\rho_1) + t_3(\rho_2) + t_3(\rho_5) \quad (j=1) \\ &= t_3(\rho_1) + t_3(\rho_2) + t_3(\rho_3) \quad (j=2) \\ &= t_3(\rho_1) + t_3(\rho_3) + t_3(\rho_6) \quad (j=3). \end{aligned}$$

And, for the element $s \in S_2$, we have

$$\begin{aligned} t_2(s) &= t_3(\rho_3) + t_3(\rho_4) + t_3(\rho_6) \quad (j=1) \\ &= t_3(\rho_4) + t_3(\rho_5) + t_3(\rho_6) \quad (j=2) \\ &= t_3(\rho_2) + t_3(\rho_4) + t_3(\rho_5) \quad (j=3). \end{aligned}$$

For the general case, note the following partitioning formula:

$$S_n = \bigsqcup_{\pi \in S_{n-1}} S(j, \pi) \quad \text{for all } n > 1. \tag{A.3}$$

The readers may check the following recursive system for $\mathcal{C}_H$:

LEMMA. *For every $n \in \mathbb{N}$, $n > 1$, fixed, consider the system of equations,*

$$t_{n-1}(\pi) = \sum_{\rho \in S(j,\pi)} t_n(\rho),$$

with $t_{n-1}(\bullet)$ given. The rank *of this system is $n-1$, when we consider $(t_n(\rho))_{\rho \in S_n}$, as undetermined variables.*

EXAMPLE (The projection from $\mathcal{F}(H)$ onto the subspace $\mathcal{F}_{\text{sym}}(H)$ of all symmetric tensors). Set

$$t_0 = t_1 = 1, \quad \text{and} \quad t_n(\pi) = \frac{1}{n!} \tag{A.4}$$

for all $\pi \in S_n$, for $n \in \mathbb{N}$. Recall the action of $\pi \in S_n$ on $H^{\otimes n}$,

$$\pi \cdot k_1 \otimes \cdots \otimes k_n = k_{\pi(1)} \otimes \cdots \otimes k_{\pi(n)}. \tag{A.5}$$

Hence, t in (A.4) satisfies (A.1), and

$$P_{\text{sym}} \overset{\text{def}}{=} \sum_{n=0}^{\infty} P_n^{(\text{sym})} \tag{A.6}$$

is well-defined, where

$$P_n^{(\text{sym})} = \frac{1}{n!} \sum_{\pi \in S_n} \pi(\bullet), \tag{A.7}$$

that is, (A.7) is the special case of

$$P_n = \sum_{\pi \in S_n} t_n(\pi)\pi(\bullet),$$

then t is the family of functions specified in (A.4).

Now, let H be a complex Hilbert space with $\dim H = 2$.

PROPOSITION. *Let U be the unitary representation of Lie group $U(H,1)$ in $\mathcal{F}(H)$, and let*

$$\mathcal{C}_H \overset{def}{=} \big\{T \in B\big(\mathcal{F}(H)\big) | TU(g) = U(g)T, \forall g \in U(H,1)\big\} \tag{A.8}$$

be the commutatant. Then a subalgebra $\mathcal{A}$ in $\mathcal{C}_H$ is Abelian, if and only if $P_n\mathcal{A}P_n$ is Abelian, for all $n \in \mathbb{N}$.

Proof. Let $\mathcal{A}$ be as stated in the proposition, and let

$$t_n : S_n \to \mathbb{C} \quad \text{for } n \in \mathbb{N} \cup \{0\}, \tag{A.9}$$

be the family of functions defined in (A.1). If $A \in \mathcal{A}$, then

$$P_nAP_n = \sum_{\pi \in S_n} t_n^{(A)}(\pi)\pi \tag{A.10}$$

specifies the correspondence $A \longleftrightarrow (t_n^{(A)})_n$.

We make the following observations.

$$(P_nAP_n)(P_nBP_n) = P_nABP_n \tag{A.11}$$

holds for all A, $B \in \mathcal{C}_H$, and for all $n \in \mathbb{N} \cup \{0\}$. Also,

$$t_n^{(AB)}(\pi) = \sum_{\sigma \in S_n} t_n^{(A)}(\sigma) t_n^{(B)}\big(\sigma^{-1}\pi\big) \tag{A.12}$$

for all $n \in \mathbb{N} \cup \{0\}$, and for all $\pi \in S_n$. □

COROLLARY. *The correspondence in the above Proposition gives a bijection between Abelian subalgebras $\mathcal{A}$ in the commutant $\mathcal{C}_H$, and families of commutative subalgebras in $C^*(S_n)$, where the operator multiplication on $C^*(S_n)$ is defined to be the convolution, defined in (A.11). The system (t_n) of functions in (A.9) satisfies the martingale condition:*

$$t_{n-1}(\pi) = \mathbb{E}_j^{(n)}(t_n : \pi) \quad \text{for } \pi \in S_{n-1}, \tag{A.13}$$

of Definition 5.1, where $j \in \{1, \dots, n\}$.

COROLLARY. *With the embedding,*

$$\mathcal{O}_H^{(n-1)} \hookrightarrow \mathcal{O}_H^{(n)}, \tag{A.14}$$

specified by (A.13), we see that the rank of $\mathcal{O}_H^{(n)}$ over $\mathcal{O}_H^{(n-1)}$ is $n-1$, for all $n \in \mathbb{N}$.

Proof. This follows by induction, making use of the previous results. In particular, the martingale condition (A.13). □

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