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TORIC IDEALS FOR HIGH VERONESE SUBRINGS OF TORIC ALGEBRAS

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ABSTRACT. We prove that the defining ideal of a sufficiently high Veronese subring of a toric algebra admits a quadratic Gröbner basis consisting of binomials. More generally, we prove that the defining ideal of a sufficiently high Veronese subring of a standard graded ring admits a quadratic Gröbner basis. We give a lower bound on d such that the defining ideal of dth Veronese subring admits a quadratic Gröbner basis. Eisenbud–Reeves–Totaro stated the same theorem without a proof with some lower bound on d. In many cases, our lower bound is less than Eisenbud–Reeves–Totaro's lower bound.

1. Introduction

We denote by $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ the set of nonnegative integers. For a multi-index $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r$ and variables $\mathbf{x} = x_1, \ldots, x_r$, we write $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_r^{a_r}$ and $|\mathbf{a}| = a_1 + \cdots + a_r$. For a given positive integer d, we set $\mathbb{N}^r_d = \{\mathbf{a} \in \mathbb{N}^r \mid |\mathbf{a}| = d\}$. We denote by $\mathbf{e}_i$ the vector with 1 in the ith position and zeros elsewhere. In this paper, "quadratic" means "of degree at most two."

Let $B = \bigoplus_{i \in \mathbb{N}} B_i$ be a standard $\mathbb{N}$-graded ring (that is, B is generated by B_1 over B_0 as an algebra) with $B_0 = K$ a field. For $d \in \mathbb{N}$, we call $B^{(d)} = \bigoplus_{i \in \mathbb{N}} B_{di}$ the *dth Veronese subring* of B. In this paper, we investigate Gröbner bases of the defining ideal of $B^{(d)}$. We say that a homogeneous ideal admits quadratic Gröbner basis if there exists a Gröbner basis consisting of homogeneous polynomials of degree at most 2 with respect to some term order.

We call a finite collection $\mathcal{A} = \{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(s)}\} \subset \mathbb{Z}^n$, $\mathbf{m}^{(i)} = (m_1^{(i)}, \ldots, m_n^{(i)})$, a *configuration* if there exists a vector $0 \neq \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Q}^n$ such

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that $\boldsymbol{\lambda} \cdot \mathbf{m}^{(i)} = \sum_j \lambda_j \cdot m_j^{(i)} = 1$ for all i. We denote by $K[\mathcal{A}]$ the standard $\mathbb{N}$-graded K-algebra $K[\mathbf{z}^{\mathbf{m}^{(1)}}, \dots, \mathbf{z}^{\mathbf{m}^{(s)}}]$. For a configuration $\mathcal{A}$, let $\phi_{\mathcal{A}}$ be the ring homomorphism

$$\phi_{\mathcal{A}} : K[y_1, \dots, y_s] \to K\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}\right]$$
$$y_i \mapsto \mathbf{z}^{\mathbf{m}^{(i)}} = \prod_{j=1}^{n} z_j^{m_j^{(i)}}.$$

We denote $\operatorname{Ker} \phi_{\mathcal{A}}$ by $P_{\mathcal{A}}$ and call it a *toric ideal* of $\mathcal{A}$. It is known that the toric ideal $P_{\mathcal{A}}$ is a homogeneous ideal generated by the binomials $u - v$ where u and v are monomials of $K[y_1, \dots, y_s]$ with $\phi_{\mathcal{A}}(u) = \phi_{\mathcal{A}}(v)$. We consider the toric ideal $P_{\mathcal{A}^{(d)}}$ which is the defining ideal of the dth Veronese subring $K[\mathcal{A}]^{(d)} = K[\mathcal{A}^{(d)}]$ of $K[\mathcal{A}]$ where

$$\mathcal{A}^{(d)} = \left\{a_1 \mathbf{m}^{(1)} + \dots + a_s \mathbf{m}^{(s)} \mid (a_1, \dots, a_s) \in \mathbb{N}_d^s\right\}.$$

We will prove the following theorem.

THEOREM 1 (Theorem 3.16). *$P_{\mathcal{A}^{(d)}}$ admits a quadratic Gröbner basis for all sufficiently large d.*

We prove this theorem in the more general situation: Let $S = K[y_1, \dots, y_s]$ be a standard graded polynomial rings over a field K, and I a homogeneous ideal of S. Let $R^{[d]} = K[x_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}_d^s]$ be a polynomial ring whose variables correspond to the monomials of degree d in S, and $\phi_d : R^{[d]} \to S$ a ring homomorphism $\phi_d(x_{\mathbf{a}}) = \mathbf{y}^{\mathbf{a}}$. Then $R^{[d]}/\phi_d^{-1}(I) \cong (S/I)^{(d)}$. The main result of this paper is the next theorem.

THEOREM 2 (Theorem 3.14). *Let $I \subset S$ be a homogeneous ideal, and $\prec$ a term order on S. Let $\{\mathbf{y}^{\mathbf{a}^{(1)}}, \dots, \mathbf{y}^{\mathbf{a}^{(r)}}\}$, $\mathbf{a}^{(i)} = (a_{i1}, \dots, a_{is}) \in \mathbb{N}^s$, be the minimal system of generators of $\operatorname{in}_{\prec}(I)$. Then $\phi_d^{-1}(I)$ admits a quadratic Gröbner basis if*

$$d \geq s\big(\max\{a_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq s\} + 1\big)/2.$$

Note that Theorem 2 implies Theorem 1. Eisenbud–Reeves–Totaro [7] proved that in the case where the coordinates $y_1, \dots, y_s$ of S are generic, $\phi_d^{-1}(I)$ admits quadratic Gröbner basis for $d \geq \operatorname{reg}(I)/2$. Our lower bound $s(\max\{a_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq s\} + 1)/2$ seems large compared with $\operatorname{reg}(I)/2$, but is easy to compute. Let $\delta(\operatorname{in}_{\prec}(I)) = \max\{a_{i1} + \dots + a_{is} \mid 1 \leq i \leq r\}$. Eisenbud–Reeves–Totaro gave a easily computable rough lower bound $(s\delta(\operatorname{in}_{\prec}(I)) - s + 1)/2$ (with $\prec$ and coordinates $y_1, \dots, y_s$ chosen so that $\delta(\operatorname{in}_{\prec}(I))$ is minimal). In many cases, our lower bound is less than Eisenbud–Reeves–Totaro's rough lower bound. Since coordinate transformation does not preserve the property that an ideal is generated by binomials, we can not use generic coordinates to prove Theorem 1. Our proof does not need any coordinate transformation. Eisenbud–Reeves–Totaro stated that the assertion of Theorem 2 holds true for $d \geq s\lceil \delta(\mathrm{in}_{\prec}(I))/2 \rceil$ without a proof (see [7] comments after Theorem 11). Our lower bound is often less than Eisenbud–Reeves–Totaro's lower bound $s\lceil \delta(\mathrm{in}_{\prec}(I))/2 \rceil$.

One of the reasons why we are interested in whether a given homogeneous ideal admits quadratic Gröbner basis is that this is a sufficient condition for the residue ring to be a *homogeneous Koszul algebra*. We call a graded ring B homogeneous Koszul algebra if its residue field has a linear minimal graded free resolution. Fröberg [8] proved that if I is generated by monomials of degree two then S/I is Koszul. Hence if $I \subset S = K[y_1, \ldots, y_s]$ admits a quadratic initial ideal, then $B = S/I$ is a homogeneous Koszul algebra by a deformation argument. Therefore, Theorem 2 implies the theorem of Backelin.

THEOREM 1.1 (Backelin [1]). *A Veronese subring $B^{(d)}$ of a standard $\mathbb{N}$-graded ring $B = K[B_1]$ over a field K is a homogeneous Koszul algebra for all sufficiently large $d \in \mathbb{N}$.*

In the case of $I = 0$, Barcanescu and Manolache [2] proved that Veronese subrings of polynomial rings are Koszul. We also prove that the defining ideals of Veronese subrings of polynomial rings admit quadratic Gröbner bases with respect to a certain term order (Theorem 3.8). See [3], [7] and [6] for other term orders that give quadratic Gröbner bases.

2. Preliminaries on Gröbner bases

Here, we recall the theory of Gröbner bases. See [4], [5] and [9] for details.

Let $R = K[x_1, \ldots, x_r]$ be a polynomial ring over a field K. The monomial $\mathbf{x}^{\mathbf{a}}$ in R is identified with lattice point $\mathbf{a} \in \mathbb{N}^r$. A total order $\prec$ on $\mathbb{N}^r$ is a *term order* if the zero vector 0 is the unique minimal element, and $\mathbf{a} \prec \mathbf{b}$ implies $\mathbf{a} + \mathbf{c} \prec \mathbf{b} + \mathbf{c}$ for all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{N}^r$. We define $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$ if $\mathbf{a} \prec \mathbf{b}$. We denote by $\mathcal{M}$ the set of all monomials of R. Let $\mathbf{a} = (a_1, \ldots, a_r)$, $\mathbf{b} = (b_1, \ldots, b_r) \in \mathbb{N}^r$.

DEFINITION 2.1 (Lexicographic order). The *lexicographic order* $\prec_{\mathrm{lex}}$ with $x_r \prec \cdots \prec x_1$ is defined as follows: $\mathbf{a} \prec_{\mathrm{lex}} \mathbf{b}$ if $a_j < b_j$ where $j = \min\{i \mid a_i \neq b_i\}$.

DEFINITION 2.2 (Reverse lexicographic order). The *reverse lexicographic order* $\prec_{\mathrm{rlex}}$ with $x_r \prec \cdots \prec x_1$ is defined as follows: $\mathbf{a} \prec_{\mathrm{rlex}} \mathbf{b}$ if $|\mathbf{a}| < |\mathbf{b}|$ or $|\mathbf{a}| = |\mathbf{b}|$ and $a_j > b_j$ where $j = \max\{i \mid a_i \neq b_i\}$.

DEFINITION 2.3. Let $\prec$ be a term order on R, $f \in R$, and I an ideal of R. The *initial term* $\mathrm{in}_{\prec}(f)$ is the highest term of f with respect to $\prec$. We call $\mathrm{in}_{\prec}(I) = \langle \mathrm{in}_{\prec}(f) \mid f \in I \rangle$ the *initial ideal* of I with respect to $\prec$. We say that a finite subset G of I is a *Gröbner basis* of I with respect to $\prec$ if $\mathrm{in}_{\prec}(I) = \langle \mathrm{in}_{\prec}(g) \mid g \in G \rangle$.

We give a criterion for a given finite subset of a toric ideal to be a Gröbner basis.

LEMMA 2.4. *Let $\prec$ be a term order of R, $P_{\mathcal{A}} = \operatorname{Ker}\phi_{\mathcal{A}} \subset R$ a toric ideal, and G a finite subset of $P_{\mathcal{A}}$. Then G is a Gröbner basis of $P_{\mathcal{A}}$ with respect to $\prec$ if and only if for any monomial $u \notin \langle \mathrm{in}_\prec(g) \mid g \in G\rangle$,*

$$u = \min_\prec\bigl\{v \in \mathcal{M} \mid \phi_{\mathcal{A}}(u) = \phi_{\mathcal{A}}(v)\bigr\}.$$

Proof. Let u be a monomial. For a monomial v satisfying $\phi_{\mathcal{A}}(u) = \phi_{\mathcal{A}}(v)$, we have $u - v \in P_{\mathcal{A}}$. Therefore $u = \min_\prec\{v \in \mathcal{M} \mid \phi_{\mathcal{A}}(u) = \phi_{\mathcal{A}}(v)\}$ if and only if $u \notin \mathrm{in}_\prec(P_{\mathcal{A}})$. Since G is a Gröbner basis of $P_{\mathcal{A}}$ if and only if $u \notin \mathrm{in}_\prec(P_{\mathcal{A}})$ for any monomial $u \notin \langle \mathrm{in}_\prec(g) \mid g \in G\rangle$, we conclude the assertion. □

For a weight vector $\omega = (\omega_1, \ldots, \omega_r) \in \mathbb{N}^r$, we can grade the ring R by associating weights ω_i to x_i. To distinguish this grading from the standard one, we say that polynomials or ideals of R are *ω-homogeneous* if they are homogeneous with respect to the graded structure given by ω.

DEFINITION 2.5. Given a polynomial $f \in R$ and a weight vector ω, the *initial form* $\mathrm{in}_\omega(f)$ is the sum of all monomials of f of the highest weight with respect to ω. We call $\mathrm{in}_\omega(I) = \langle \mathrm{in}_\omega(f) \mid f \in I\rangle$ the *initial ideal* of I with respect to ω. If $\mathrm{in}_\omega(I)$ is a monomial ideal, we call G a Gröbner basis of I with respect to ω.

We define a new term order constructed from ω and a term order.

DEFINITION 2.6. For a weight vector ω and a term order $\prec$, we define a new term order $\prec_\omega$ constructed from ω with $\prec$ a tie-breaker as following; $\mathbf{x}^{\mathbf{a}} \prec_\omega \mathbf{x}^{\mathbf{b}}$ if $\omega \cdot \mathbf{a} < \omega \cdot \mathbf{b}$, or $\omega \cdot \mathbf{a} = \omega \cdot \mathbf{b}$ and $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$.

We end this section with the following useful propositions about weighted order. See [9] for the proofs.

PROPOSITION 2.7. $\mathrm{in}_\prec(\mathrm{in}_\omega(I)) = \mathrm{in}_{\prec_\omega}(I)$.

PROPOSITION 2.8. *For any term order $\prec$ and any ideal $I \subset R$, there exists a weight vector $\omega \in \mathbb{N}^r$ such that* $\mathrm{in}_\prec(I) = \mathrm{in}_\omega(I)$.

3. Proof of the main theorem

3.1. Quadratic Gröbner bases of $\operatorname{Ker}\phi_d$**.** Let $S = K[y_1, \ldots, y_s]$ be a standard graded polynomial rings over a field K, $R^{[d]} = K[x_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}^s_d]$ a polynomial ring whose variables correspond to the monomials of degree d in S, and $\phi_d : R^{[d]} \to S$ the ring homomorphism $\phi_d(x_{\mathbf{a}}) = \mathbf{y}^{\mathbf{a}}$. We denote by $\mathcal{M}$ the set of all monomials of $R^{[d]}$. In this section, we prove that $\operatorname{Ker}\phi_d$ has a quadratic Gröbner basis with respect to a certain reverse lexicographic order.

DEFINITION 3.1. Let $\prec$ be a term order on $R^{[d]}$. For a monomial $u \in R^{[d]}$, we define

$$\mathrm{mv}_\prec(u) = \min_\prec\bigl\{x_{\mathbf{a}} \in R^{[d]} \mid \mathbf{y}^{\mathbf{a}} \text{ divides } \phi_d(u)\bigr\}.$$

LEMMA 3.2. *Let $\prec$ a reverse lexicographic order on $R^{[d]}$, and $u \in R^{[d]}$ a monomial. Then*

$$\begin{aligned}&\min_{\prec}\{v \in \mathcal{M} \mid \phi_d(u) = \phi_d(v)\}\\ &\quad = \mathrm{mv}_{\prec}(u) \cdot \min_{\prec}\left\{v' \in \mathcal{M} \;\middle|\; \frac{\phi_d(u)}{\phi_d(\mathrm{mv}_{\prec}(u))} = \phi_d(v')\right\}.\end{aligned}$$

Proof. Since $\phi_d(u)/\phi_d(\mathrm{mv}_{\prec}(u))$ is a monomial whose degree is divisible by d, there exists a monomial $v' \in \mathcal{M}$ such that $\phi_d(v') = \phi_d(u)/\phi_d(\mathrm{mv}_{\prec}(u))$. Let $u_0 = \min_{\prec}\{v \in \mathcal{M} \mid \phi_d(u) = \phi_d(v)\}$. Since $\prec$ is a reverse lexicographic order and $u_0 \prec \mathrm{mv}_{\prec}(u) \cdot v'$, u_0 is divided by $\mathrm{mv}_{\prec}(u)$. Hence, the assertion follows. □

We will give a criterion for a finite subset of $\operatorname{Ker}\phi_d$ to be a Gröbner basis of $\operatorname{Ker}\phi_d$ with respect to a reverse lexicographic order.

PROPOSITION 3.3. *Let $\prec$ be a reverse lexicographic order on $R^{[d]}$, and G a finite subset of $\operatorname{Ker}\phi_d$. Then G is a Gröbner basis of $\operatorname{Ker}\phi_d$ with respect to $\prec$ if and only if $\mathrm{mv}_{\prec}(u)$ divides u for any monomial $u \notin \langle \mathrm{in}_{\prec}(g) \mid g \in G\rangle$.*

Proof. Suppose that G is a Gröbner basis, and let $u \notin \langle \mathrm{in}_{\prec}(g) \mid g \in G\rangle = \mathrm{in}_{\prec}(I)$ be a monomial. Then $\mathrm{mv}_{\prec}(u)$ divides u by Lemmas 2.4 and 3.2.

Conversely, suppose that $\mathrm{mv}_{\prec}(u)$ divides u for any monomial $u \notin \langle \mathrm{in}_{\prec}(g) \mid g \in G\rangle$. We will prove $u = \min_{\prec}\{v \in \mathcal{M} \mid \phi_d(u) = \phi_d(v)\}$ by induction on the degree of u. Since $u/\mathrm{mv}_{\prec}(u)$ is also not in $\langle \mathrm{in}_{\prec}(g) \mid g \in G\rangle$, it holds that

$$u/\mathrm{mv}_{\prec}(u) = \min_{\prec}\{v \in \mathcal{M} \mid \phi_d(u/\mathrm{mv}_{\prec}(u)) = \phi_d(v)\}$$

by the assumption of induction. By Lemma 3.2, we conclude

$$\begin{aligned}u &= \mathrm{mv}_{\prec}(u) \cdot (u/\mathrm{mv}_{\prec}(u)) = \mathrm{mv}_{\prec}(u) \cdot \min_{\prec}\{v \in \mathcal{M} \mid \phi_d(u/\mathrm{mv}_{\prec}(u)) = \phi_d(v)\}\\ &= \min_{\prec}\{v \in \mathcal{M} \mid \phi_d(u) = \phi_d(v)\}.\end{aligned}$$

Hence, G is a Gröbner basis of I by Lemma 2.4. □

DEFINITION 3.4. Let $\mathbf{a} = (a_1, \dots, a_s) \in \mathbb{N}^s$, and $\sigma \in \mathfrak{S}_s$ be a permutation of indices such that $a_{\sigma(1)} \le a_{\sigma(2)} \le \cdots \le a_{\sigma(s)}$. We define $\Gamma(\mathbf{a}) \in \mathbb{N}^s$ to be

$$\Gamma(\mathbf{a}) = (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(s)}).$$

DEFINITION 3.5. Let $\prec_\Gamma$ be a reverse lexicographic order on $R^{[d]}$ such that the order on variables is defined as follows: $x_{\mathbf{a}} \prec_\Gamma x_{\mathbf{b}}$ if $\Gamma(\mathbf{b}) \prec_{\mathrm{lex}} \Gamma(\mathbf{a})$ or $\Gamma(\mathbf{b}) = \Gamma(\mathbf{a})$ and $\mathbf{b} \prec_{\mathrm{lex}} \mathbf{a}$.

EXAMPLE 3.6. In the case of $s = 2$ and $d = 4$,

$$x_{(2,2)} \prec_\Gamma x_{(3,1)} \prec_\Gamma x_{(1,3)} \prec_\Gamma x_{(4,0)} \prec_\Gamma x_{(0,4)}.$$

In the case of $s=3$ and $d=3$,

$$x_{(1,1,1)} \prec_\Gamma x_{(2,1,0)} \prec_\Gamma x_{(2,0,1)} \prec_\Gamma x_{(1,2,0)} \prec_\Gamma x_{(1,0,2)} \prec_\Gamma x_{(0,2,1)} \prec_\Gamma x_{(0,1,2)} \prec_\Gamma x_{(3,0,0)} \prec_\Gamma x_{(0,3,0)} \prec_\Gamma x_{(0,0,3)}.$$

The following are some typical and important properties of the term order $\prec_\Gamma$.

LEMMA 3.7. *Let* $\mathbf{a}=(a_1,\dots,a_s), \mathbf{b}=(b_1,\dots,b_s) \in \mathbb{N}_d^s$.

(1) $\Gamma(\mathbf{a})$ *is the minimal element of* $\{(a_{\sigma(1)},a_{\sigma(2)},\dots,a_{\sigma(s)}) \mid \sigma \in \mathfrak{S}_s\}$ *with respect to the lexicographic order.*
(2) *If* $\#\{i \mid a_i \neq 0\} > \#\{i \mid b_i \neq 0\}$ *where* $\#F$ *is the cardinality of the set* F, *then* $\mathbf{a} \prec_\Gamma \mathbf{b}$.
(3) *Suppose that* $a_j - a_i \geq 2$ *for some* $1 \leq i,j \leq s$, *and let* $\mathbf{a}' = \mathbf{a} + \mathbf{e}_i - \mathbf{e}_j$. *Then* $x_{\mathbf{a}'} \prec_\Gamma x_{\mathbf{a}}$.
(4) *Suppose that* $a_j - a_i = 1$ *for some* $1 \leq i,j \leq s$, *and let* $\mathbf{a}' = \mathbf{a} + \mathbf{e}_i - \mathbf{e}_j$. *Then* $x_{\mathbf{a}'} \prec_\Gamma x_{\mathbf{a}}$ *if and only if* $i<j$.
(5) *Let* $u \in R^{[d]}$ *be a monomial, and suppose that* $x_{\mathbf{a}} = \mathrm{mv}_{\prec_\Gamma}(u)$. *If* $a_j - a_i \geq 2$ *for some* $1 \leq i,j \leq s$, *then the degree of* $\phi_d(u)$ *in the variable* y_i *is* a_i.

Proof. The assertions of (1), (2), and (3) follow immediately from the definition of the term order $\prec_\Gamma$.

(4) Note that $\Gamma(\mathbf{a}) = \Gamma(\mathbf{a}')$, and $\mathbf{a}'$ is the vector obtained from $\mathbf{a}$ by swapping the ith and jth components. Hence $x_{\mathbf{a}'} \prec_\Gamma x_{\mathbf{a}}$ if and only if $\mathbf{a} \prec_{\mathrm{lex}} \mathbf{a}'$, which is equivalent to $i<j$.

(5) Assume, to the contrary, that the degree of $\phi_d(u)$ in the variable y_i is strictly greater than a_i. Then $\mathbf{y}^{\mathbf{a}'}$ divides $\phi_d(u)$ where $\mathbf{a}' = \mathbf{a} + \mathbf{e}_i - \mathbf{e}_j$. By (3), we have $x_{\mathbf{a}'} \prec_\Gamma x_{\mathbf{a}}$ which contradicts the definition of $\mathrm{mv}_{\prec_\Gamma}(u)$. □

THEOREM 3.8. *Let*

$$G_\Gamma = \left\{ x_{\mathbf{a}+\mathbf{e}_i} x_{\mathbf{b}+\mathbf{e}_j} - x_{\mathbf{a}+\mathbf{e}_j} x_{\mathbf{b}+\mathbf{e}_i} \mid \mathbf{a},\mathbf{b} \in \mathbb{N}_{d-1}^s, 1 \leq i < j \leq s \right\}.$$

Then G_Γ *is a Gröbner basis of* $\operatorname{Ker}\phi_d$ *with respect to* $\prec_\Gamma$.

Proof. As $(\mathbf{a}+\mathbf{e}_i)+(\mathbf{b}+\mathbf{e}_j) = (\mathbf{a}+\mathbf{e}_j)+(\mathbf{b}+\mathbf{e}_i)$ for $\mathbf{a},\mathbf{b} \in \mathbb{N}_{d-1}^s$, G_Γ is a finite subset of $\operatorname{Ker}\phi_d$. Let $u \in R^{[d]}$ be a monomial such that $x_{\mathbf{a}} = \mathrm{mv}_{\prec_\Gamma}(u)$ does not divide u. To conclude the assertion, it is enough to show that $u \in \langle \mathrm{in}_{\prec_\Gamma}(g) \mid g \in G_\Gamma \rangle$ by Proposition 3.3. Take $\mathbf{a}=(a_1,\dots,a_s), \mathbf{b}=(b_1,\dots,b_s) \in \mathbb{N}_d^s$ such that

$$x_{\mathbf{a}} = \mathrm{mv}_{\prec_\Gamma}(u),$$
$$x_{\mathbf{b}} = \min_{\prec_\Gamma}\left\{ x_{\mathbf{c}} \in R^{[d]} \mid x_{\mathbf{c}} \text{ divides } u \right\}.$$

Note that $\mathbf{a} \neq \mathbf{b}$ and $x_{\mathbf{a}} \prec_\Gamma x_{\mathbf{b}}$. Let $\tau \in \mathfrak{S}_s$ be a permutation of indices such that $b_{\tau(1)} \leq \dots \leq b_{\tau(s)}$. Since

$$(b_{\tau(1)},b_{\tau(2)},\dots,b_{\tau(s)}) = \Gamma(\mathbf{b}) \preceq_{\mathrm{lex}} \Gamma(\mathbf{a}) \preceq_{\mathrm{lex}} (a_{\tau(1)},a_{\tau(2)},\dots,a_{\tau(s)})$$

by Lemma 3.7(1), and $(b_{\tau(1)},\dots,b_{\tau(s)}) \neq (a_{\tau(1)},\dots,a_{\tau(s)})$, there exists $1 \le j \le s$ such that $a_{\tau(i)} = b_{\tau(i)}$ for all $i < j$ and

$$b_{\tau(j)} < a_{\tau(j)}.$$

As $|\mathbf{a}| = |\mathbf{b}| = d$, there exists $k > j$ such that

$$b_{\tau(k)} > a_{\tau(k)}.$$

As $\mathbf{y}^{\mathbf{b}}$ divides $\phi_d(u)$ and $b_{\tau(k)} > a_{\tau(k)}$, we have

$$a_{\tau(j)} - a_{\tau(k)} \le 1$$

by Lemma 3.7(5). As

$$a_{\tau(j)} - a_{\tau(k)} = (a_{\tau(j)} - b_{\tau(j)}) + (b_{\tau(j)} - b_{\tau(k)}) + (b_{\tau(k)} - a_{\tau(k)}),$$

and $a_{\tau(j)} - b_{\tau(j)}, b_{\tau(k)} - a_{\tau(k)} > 0$, we have

$$b_{\tau(k)} - b_{\tau(j)} > 0.$$

Since $\mathbf{y}^{\mathbf{a}}$ divides $\phi_d(u)$, the degree of $\phi_d(u/x_{\mathbf{b}})$ in the variable $y_{\tau(j)}$ is not less than $a_{\tau(j)} - b_{\tau(j)} > 0$, and thus there exists $\mathbf{c} = (c_1,\dots,c_s) \in \mathbb{N}_d^s$ such that $c_{\tau(j)} > 0$ and $x_{\mathbf{c}}$ divides $u/x_{\mathbf{b}}$. We set

$$\mathbf{b}' = \mathbf{b} + \mathbf{e}_{\tau(j)} - \mathbf{e}_{\tau(k)},$$
$$\mathbf{c}' = \mathbf{c} - \mathbf{e}_{\tau(j)} + \mathbf{e}_{\tau(k)}.$$

Then $x_{\mathbf{b}}x_{\mathbf{c}} - x_{\mathbf{b}'}x_{\mathbf{c}'} \in G_\Gamma$ and $x_{\mathbf{b}}x_{\mathbf{c}}$ divides u. To complete the proof, we will show that $x_{\mathbf{b}}x_{\mathbf{c}}$ is the initial term of $x_{\mathbf{b}}x_{\mathbf{c}} - x_{\mathbf{b}'}x_{\mathbf{c}'}$. Since $x_{\mathbf{b}} \prec_\Gamma x_{\mathbf{c}}$ and $\prec_\Gamma$ is a reverse lexicographic order, it is enough to show that $x_{\mathbf{b}'} \prec_\Gamma x_{\mathbf{b}}$. In the case of $b_{\tau(k)} - b_{\tau(j)} \ge 2$, we have $x_{\mathbf{b}'} \prec_\Gamma x_{\mathbf{b}}$ by Lemma 3.7(3). In the case of $b_{\tau(k)} - b_{\tau(j)} = 1$, we have

$$a_{\tau(j)} - a_{\tau(k)} = (a_{\tau(j)} - b_{\tau(j)}) - 1 + (b_{\tau(k)} - a_{\tau(k)}) \ge 1,$$

and hence $a_{\tau(j)} - a_{\tau(k)} = 1$. Let $\mathbf{a}' = \mathbf{a} - \mathbf{e}_{\tau(j)} + \mathbf{e}_{\tau(k)}$. Then $\mathbf{y}^{\mathbf{a}'}$ divides $\phi_d(u)$ as $b_{\tau(k)} > a_{\tau(k)}$. Thus $x_{\mathbf{a}} \prec_\Gamma x_{\mathbf{a}'}$ by the definition of $\mathrm{mv}_{\prec_\Gamma}(u)$. This implies $\tau(j) < \tau(k)$ by Lemma 3.7(4). Therefore, $x_{\mathbf{b}'} \prec_\Gamma x_{\mathbf{b}}$ again by Lemma 3.7(4). □

See [3], [7] and [6] for other term orders that give quadratic Gröbner bases of $\operatorname{Ker}\phi_d$.

We already proved Theorem 2 in the case of $I = 0$. In the rest of this paper, we prove that there exists a term order on $R^{[d]}$ such that the initial ideal of $\phi_d^{-1}(I)$ is generated by quadratic monomials for all sufficiently large d for any homogeneous ideal $I \subset S$. First, we will prove this in the case where I is a monomial ideal, and then reduce the general case to the monomial ideal case.

3.2. In the case of monomial ideals.

DEFINITION 3.9. Let $I \subset S$ be a monomial ideal, and $\prec$ any term order on $R^{[d]}$. We define

$$L_\prec(I) = \left\langle \mathcal{M} \cap \left(\phi_d^{-1}(I) \setminus \mathrm{in}_\prec(\mathrm{Ker}\,\phi_d)\right)\right\rangle$$

to be the monomial ideal of $R^{[d]}$ generated by all monomials in $\phi_d^{-1}(I) \setminus \mathrm{in}_\prec(\mathrm{Ker}\,\phi_d)$, and $M_\prec(I)$ to be the minimal system of generators of $L_\prec(I)$ consisting of monomials.

LEMMA 3.10. *Let $I \subset S$ be a monomial ideal, and $\prec$ any term order on $R^{[d]}$. Let G be a Gröbner basis of* $\mathrm{Ker}\,\phi_d$ *with respect to $\prec$. Then $G \cup M_\prec(I)$ is a Gröbner basis of $\phi_d^{-1}(I)$ with respect to $\prec$.*

Proof. First, we note that $G \cup M_\prec(I) \subset \phi_d^{-1}(I)$. Take $f \in \phi_d^{-1}(I)$, and let g be the remainder on division of f by G. Then any term of g is not in $\mathrm{in}_\prec(\mathrm{Ker}\,\phi_d)$. Hence different monomials appearing in g map to different monomials under ϕ_d. Since I is a monomial ideal, it follows that all terms of g are in $L_\prec(I)$. Thus, the remainder on division of g by $M_\prec(I)$ is zero. Therefore, a remainder on division of f by $G \cup M_\prec(I)$ is zero. This implies that $G \cup M_\prec(I)$ is a Gröbner basis of $\phi_d^{-1}(I)$. □

PROPOSITION 3.11. *Let $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{N}^s$, and $a = \max\{a_i \mid i = 1, \ldots, s\}$. Assume that $d \geq s(a+1)/2$. Let $u \in (\phi_d^{-1}(I) \setminus \mathrm{in}_\prec(\mathrm{Ker}\,\phi_d)$ be a monomial of degree ≥ 2, and set $x_\mathbf{b} = \mathrm{mv}_{\prec_\Gamma}(u)$, $\mathbf{b} = (b_1, \ldots, b_s) \in \mathbb{N}^s_d x_\mathbf{c} = \mathrm{mv}_{\prec_\Gamma}(u/x_\mathbf{b})$, $\mathbf{c} = (c_1, \ldots, c_s) \in \mathbb{N}^s_d$. Then $x_\mathbf{b} x_\mathbf{c} \in \phi_d^{-1}(\mathbf{y^a})$. In particular, $L_{\prec_\Gamma}(\mathbf{y^a})$ is generated by quadratic monomials.*

Proof. First, note that $x_\mathbf{c}$ is well defined since $x_\mathbf{b}$ divides u by Lemma 3.2. Assume, to the contrary, that $x_\mathbf{b} x_\mathbf{c} \notin \phi_d^{-1}(\mathbf{y^a})$. Since $\mathbf{y^a}$ does not divide $\mathbf{y^{b+c}} = \phi_d(x_\mathbf{b} x_\mathbf{c})$, we have $b_i + c_i < a_i$ for some $1 \leq i \leq s$. On the other hand, since $|\mathbf{b} + \mathbf{c}| = 2d \geq s(a+1)$, there exists $1 \leq j \leq s$ $(j \neq i)$ such that $b_j + c_j > a + 1$. Hence, $(b_j + c_j) - (b_i + c_i) = (b_j + c_j - a) + (a - b_i + c_i) \geq (b_j + c_j - a) + (a_i - b_i + c_i) \geq 3$. Thus, we have $b_j - b_i \geq 2$ or $c_j - c_i \geq 2$. Since $\mathbf{y^a}$ divides $\phi_d(u)$ and $b_i + c_i < a_i$, the degree of $\phi_d(u)$ in the variable y_i is strictly greater than b_i, and the degree of $\phi_d(u/x_\mathbf{b})$ in the variable y_i is strictly greater than c_i. This contradicts to Lemma 3.7(5). Hence, $x_\mathbf{b} x_\mathbf{c} \in \phi_d^{-1}(\mathbf{y^a})$. Since $x_\mathbf{b} x_\mathbf{c}$ divides u if $u \neq x_\mathbf{b}$ by Lemma 3.2, $L_{\prec_\Gamma}(\mathbf{y^a})$ is generated by quadratic monomials. □

THEOREM 3.12. *Let $I \subset S$ be a monomial ideal with a system of generators $\{\mathbf{y}^{\mathbf{a}^{(1)}}, \ldots, \mathbf{y}^{\mathbf{a}^{(r)}}\}$, $\mathbf{a}^{(i)} = (a_{i1}, \ldots, a_{is}) \in \mathbb{N}^s$, and set*

$$a = \max\{a_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq s\}.$$

If $d \geq s(a+1)/2$, then $\mathrm{in}_{\prec_\Gamma}(\phi_d^{-1}(I))$ is generated by quadratic monomials.

Proof. Let $u \in \phi_d^{-1}(I)$ be a monomial. Then $u \in \phi_d^{-1}(\mathbf{y}^{\mathbf{a}^{(i)}})$ for some i. Thus it follows that $L_{\prec_\Gamma}(I) = \sum_{i=1}^r L_{\prec_\Gamma}(\mathbf{y}^{\mathbf{a}^{(i)}})$. Hence, $M_{\prec_\Gamma}(I)$ consists of quadratic monomials by Proposition 3.11. By Lemma 3.10, the Gröbner basis of $\phi_d^{-1}(I)$ with respect to $\prec_\Gamma$ is the union of $M_{\prec_\Gamma}(I)$ and G_Γ in Theorem 3.8. This proves the assertion. □

3.3. In the case of homogeneous ideals. Let $I \subset S$ be a homogeneous ideal, and fix a weight vector ω of S such that $\mathrm{in}_\omega(I)$ is a monomial ideal. We denote by $\phi_d^*\omega$ the weight vector on $R^{[d]}$ that assign $\omega \cdot \mathbf{a}$ to the weight of $x_\mathbf{a}$ for $\mathbf{a} \in \mathbb{N}_d^s$. Since the weight of $x_\mathbf{a}$ coincides with the weight of $\mathbf{y}^\mathbf{a} = \phi_d(x_\mathbf{a})$, ϕ_d is a homogeneous homomorphism of degree zero with respect to the graded ring structures on $R^{[d]}$ and S defined by $\phi_d^*\omega$ and ω. For the simplicity of the notation, we regard zero-polynomial as a homogeneous (ω-homogeneous) polynomial of any degree (weight).

LEMMA 3.13. *With the notation as above*, $\mathrm{in}_{\phi_d^*\omega}(\phi_d^{-1}(I)) = \phi_d^{-1}(\mathrm{in}_\omega(I))$.

Proof. First, note that $\mathrm{in}_{\phi_d^*\omega}(\phi_d^{-1}(I))$ and $\phi_d^{-1}(\mathrm{in}_\omega(I))$ are both homogeneous and $\phi_d^*\omega$-homogeneous. Since ϕ_d sends $\phi_d^*\omega$-homogeneous polynomials to ω-homogeneous polynomials, we have $\phi_d(\mathrm{in}_{\phi_d^*\omega}(g)) = \mathrm{in}_\omega(\phi_d(g))$ for all $g \in R^{[d]}$. Hence it follows that $\mathrm{in}_{\phi_d^*\omega}(\phi_d^{-1}(I)) \subset \phi_d^{-1}(\mathrm{in}_\omega(I))$.

We will prove the converse inclusion. Let $\{f_1, \dots, f_r\}$ be a Gröbner basis of I with respect to ω consisting of homogeneous polynomials. Let $g \in \phi_d^{-1}(\mathrm{in}_\omega(I))$ be a homogeneous and $\phi_d^*\omega$-homogeneous polynomial. We set ℓ and m to be the degree and the weight of g. Then $\phi_d(g)$ is a homogeneous and ω-homogeneous polynomial of degree $d\ell$ and of weight m. Since $\phi_d(g) \in \mathrm{in}_\omega(I)$, there exist homogeneous and ω-homogeneous polynomials $h_1, \dots, h_r$ such that $\phi_d(g) = \sum_{i=1}^r h_i \cdot \mathrm{in}_\omega(f_i)$, and $h_i \cdot \mathrm{in}_\omega(f_i)$ is of degree $d\ell$ and of weight m. We set $q = \sum_{i=1}^r h_i f_i$. Then q is a homogeneous polynomial of degree $d\ell$ satisfying $q \in I$, and $\mathrm{in}_\omega(q) = \sum_{i=1}^r h_i \cdot \mathrm{in}_\omega(f_i) = \phi_d(g)$. We write $q = \mathrm{in}_\omega(q) + \sum_{i<m} q_i$ where q_i is a homogeneous and $\phi_d^*\omega$-homogeneous polynomial of degree $d\ell$ and of weight i. For $i < m$, there exists $g_i \in R^{[d]}$ a homogeneous and ω-homogeneous polynomial of degree ℓ and of weight i such that $\phi_d(g_i) = q_i$. Then $\phi_d(g + \sum_{i<m} g_i) = q$ and $\mathrm{in}_{\phi_d^*\omega}(g + \sum_{i<m} g_i) = g$. Therefore, we have $g \in \mathrm{in}_{\phi_d^*\omega}(\phi_d^{-1}(I))$. □

Now, we are ready to prove the main theorem of this paper.

THEOREM 3.14. *Let $I \subset R^{[d]}$ be a homogeneous ideal, and fix a weight vector ω of S such that* $\mathrm{in}_\omega(I)$ *is a monomial ideal. Let* $\{\mathbf{y}^{\mathbf{a}^{(1)}}, \dots, \mathbf{y}^{\mathbf{a}^{(r)}}\}$, $\mathbf{a}^{(i)} = (a_{i1}, \dots, a_{is}) \in \mathbb{N}^s$, *be the minimal system of generators of* $\mathrm{in}_\omega(I)$ *and set*

$$a = \max\{a_{ij} \mid 1 \le i \le r, 1 \le j \le s\}.$$

Let $\prec_{\Gamma_\omega}$ *be the term order on* $R^{[d]}$ *constructed from* $\phi_d^*\omega$ *with* $\prec_\Gamma$ *a tie-breaker as in Definition* 2.6. *Then* $\mathrm{in}_{\prec_{\Gamma_\omega}}(\phi_d^{-1}(I))$ *is generated by quadratic monomials if* $d \geq s(a+1)/2$.

Proof. By Proposition 2.7 and Lemma 3.13, we have

$$\mathrm{in}_{\prec_{\Gamma_\omega}}\big(\phi_d^{-1}(I)\big) = \mathrm{in}_{\prec_\Gamma}\big(\mathrm{in}_{\phi_d^*\omega}\big(\phi_d^{-1}(I)\big)\big) = \mathrm{in}_{\prec_\Gamma}\big(\phi_d^{-1}\big(\mathrm{in}_\omega(I)\big)\big).$$

Since $\mathrm{in}_\omega(I)$ is a monomial ideal, the assertion follows from Theorem 3.12. □

Observation 3.15. Let the notation be as in Theorem 3.14. We will compare our lower bound on d with Eisenbud–Reeves–Totaro's lower bound. We set $\delta(\mathrm{in}_\omega(I)) = \max\{a_{i1} + \cdots + a_{is} \mid 1 \leq i \leq r\}$.

Eisenbud–Reeves–Totaro [7] proved that $\phi_d^{-1}(I)$ has quadratic initial ideal for $d \geq \mathrm{reg}(I)/2$ in the case where the coordinates $y_1, \ldots, y_s$ of S are generic. Our lower bound $s(a+1)/2$ seems large compared with $\mathrm{reg}(I)/2$, but is easy to compute. Eisenbud–Reeves–Totaro also gave a easily computable rough lower bound $(s\delta(\mathrm{in}_\omega(I)) - s + 1)/2$. Our lower bound is less than Eisenbud–Reeves–Totaro's rough lower bound if and only if $a + 2 \leq \delta(\mathrm{in}_\omega(I))$. Thus, there exist a lot of examples in which our lower bound is less than Eisenbud–Reeves–Totaro's rough lower bound.

If the coefficient field K is finite, or we are interested in Gröbner bases consisting of binomials, we can not deal with generic coordinates. Eisenbud–Reeves–Totaro stated without a proof that $\phi_d^{-1}(I)$ has quadratic initial ideal if $d \geq s\lceil\delta(\mathrm{in}_\omega(I))/2\rceil$ (with $\prec$ and coordinates $y_1, \ldots, y_s$ chosen so that $\delta(\mathrm{in}_\prec(I))$ is minimal, see [7] comments after Theorem 11). If $\delta(\mathrm{in}_\prec(I))$ is odd, our lower bound is always not greater than Eisenbud–Reeves–Totaro's lower bound $s\lceil\delta(\mathrm{in}_\prec(I))/2\rceil$. If $\delta(\mathrm{in}_\prec(I))$ is even, our lower bound is greater than Eisenbud–Reeves–Totaro's lower bound only in the case where the inequality $a \geq \delta(\mathrm{in}_\omega(I))$ holds. This inequality holds if and only if there exist $1 \leq i \leq r$, $1 \leq j \leq s$ and $N \in \mathbb{N}$ such that $\mathbf{y}^{\mathbf{a}^{(i)}} = y_j^N$ and $\deg \mathbf{y}^{\mathbf{a}^{(k)}} \leq N$ for all $1 \leq k \leq r$ which does not occur very often.

Applying Theorem 3.14 to toric ideals, we obtain the next theorem.

Theorem 3.16. *With the notation as in the introduction,* $P_{\mathcal{A}^{(d)}}$ *admits a quadratic Gröbner basis for sufficiently large* d.

Remark 3.17. It is easy to show that if I admits a squarefree initial ideal, then $\phi_d^{-1}(I)$ also admits a squarefree initial ideal using Lemma 3.10 and Lemma 3.13 (the lexicographic order in [6] gives squarefree initial ideal of $\mathrm{Ker}\,\phi_d$, and if I is a squarefree monomial ideal then so is $L_\prec(I)$ in Definition 3.9 for any term order $\prec$). However, $\mathrm{in}_{\prec_\Gamma}(\mathrm{Ker}\,\phi_d)$ is not squarefree, and it seems to be an open question whether $\phi_d^{-1}(I)$ admits a quadratic squarefree initial ideal if I has a squarefree initial ideal.

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