

## THE SZEGÖ KERNEL FOR CERTAIN NON-PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

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ABSTRACT. We consider the Szegö kernel for domains in  $\mathbb{C}^2$  given by

$$\Omega = \{(z, w) : \operatorname{Im}(w) > b(\operatorname{Re}(z))\}$$

for  $b$  a non-convex quartic polynomial with positive leading coefficient. Such domains are not pseudoconvex. We describe a subset of  $\overline{\Omega} \times \overline{\Omega}$  on which the kernel and all of its derivatives are finite. We show that there are points off the diagonal of  $\partial\Omega \times \partial\Omega$  at which the Szegö kernel is infinite as well as points on the diagonal at which it is finite.

### 1. Introduction

Let  $\Omega \subset \mathbb{C}^n$  be a domain with smooth boundary  $\partial\Omega$ . Let  $\mathcal{O}(\Omega)$  denote the space of holomorphic functions on  $\Omega$ . Associated with such a domain are certain operators: the Bergman projection  $\mathcal{B}$  and the Szegö projection  $\mathcal{S}$ . The former is the orthogonal projection of  $L^2(\Omega)$  onto the closed subspace  $L^2(\Omega) \cap \mathcal{O}(\Omega)$ , while the latter is the orthogonal projection of  $L^2(\partial\Omega)$  onto the closed subspace  $\mathcal{H}^2(\Omega)$  of boundary values of elements of  $\mathcal{O}(\Omega)$ . Ultimately, one would like results concerning the mapping properties of these operators (e.g., conditions under which they extend to bounded operators on the appropriate  $L^p$  spaces).

Often, an understanding of these operators begins with an investigation of an associated integral kernel. That is, often one can identify distributions  $B$  and  $S$  so that for  $f \in L^2(\Omega)$  and  $g \in L^2(\partial\Omega)$ ,

$$\mathcal{B}[f](z) = \int_{\Omega} f(w)B(z, w) dw, \quad \mathcal{S}[g](z) = \int_{\partial\Omega} g(w)S(z, w) d\sigma(w).$$

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Much is known about these kernels for pseudoconvex domains of finite type. Kohn’s formula connects the Bergman projection with the  $\bar{\partial}$ -Neumann operator. Kerzmann [Ker72] uses this connection to show that  $B$  is equal to a  $C^\infty$  function on  $(\bar{\Omega} \times \bar{\Omega}) \setminus \Delta$ , where  $\Delta = \{(z, w) \in \partial\Omega \times \partial\Omega : z = w\}$  is the diagonal of the boundary.

Much more is known about the operators in some settings; a number of authors obtain sharp estimates on the kernels and their derivatives near the diagonal as well as results on the mapping properties of the operators. This is done, for example, by Nagel, Rosay, Stein, and Wainger [NRSW89] for finite-type domains in  $\mathbb{C}^2$  and by McNeal and Stein ([McN94], [MS94], [MS97]) for convex domains in  $\mathbb{C}^n$ .

In contrast with the situation for pseudoconvex domains, comparatively little is known about the Szegő kernel for non-pseudoconvex domains. A notable exception is the work of Carracino ([Car05], [Car07]), in which she obtains detailed estimates for the Szegő kernel on the boundary of the (non-smooth) non-pseudoconvex domain of the form

$$(1.1) \quad \Omega = \{(z_1 = x + iy, z_2 = t + i\xi) : \xi > b(x)\}$$

with

$$(1.2) \quad b(x) = \begin{cases} (x + 1)^2, & x < -\frac{1}{2}, \\ -x^2 + \frac{1}{2}, & -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ (x - 1)^2, & \frac{1}{2} < x. \end{cases}$$

She shows that the Szegő kernel has singularities off of  $\Delta$  in this case.

In this paper, we consider certain non-pseudoconvex domains of the form (1.1) for which  $b$  is smooth, real-valued, but not convex. One checks that such a domain fails to be pseudoconvex if and only if  $b$  fails to be convex. More specifically, we take

$$(1.3) \quad b(x) = \frac{1}{4}x^4 + \frac{1}{2}px^2 + qx, \quad p < 0, q \in \mathbb{R}.$$

(Note that the condition on  $p$  is precisely the one required for a quartic of this form to be non-convex.) Our goal is to identify sets in  $\mathbb{C}^2 \times \mathbb{C}^2$  on which the integrals defining the Szegő kernel and its derivatives are absolutely convergent.

## 2. Definitions, notation, and statement of results

We begin with a more precise discussion of the Szegő projection operator and its associated integral kernel for domains in  $\mathbb{C}^2$  having the form (1.1). We take  $b$  smooth so that  $\Omega \subset \mathbb{C}^2$  is smoothly-bounded. As above, let  $\mathcal{O}(\Omega)$  denote the space of functions holomorphic on  $\Omega$ . Define

$$\mathcal{H}^2(\Omega) := \left\{ F \in \mathcal{O}(\Omega) : \sup_{\varepsilon > 0} \int_{\partial\Omega} |F(x + iy, t + ib(x) + i\varepsilon)|^2 dx dy dt < \infty \right\}.$$

$\mathcal{H}^2(\Omega)$  can be identified with the space of  $f \in L^2(\partial\Omega)$  which are solutions (in the sense of distributions) to

$$(2.1) \quad \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} - ib'(x) \frac{\partial}{\partial t} \right) [f] \equiv 0.$$

With this identification, we define the Szegő projection operator  $\mathcal{S}$  to be the orthogonal projection of  $L^2(\partial\Omega)$  onto this (closed) subspace  $\mathcal{H}^2(\Omega)$ .

One can then prove the existence of an integral kernel associated with the operator. This is discussed, for example, in [Ste72], where the approach is as follows: Begin with an orthonormal basis  $\{\phi_j\}$  for  $\mathcal{H}^2(\Omega)$  and form the sum

$$S(z, w) = \sum_{j=1}^{\infty} \phi_j(z) \overline{\phi_j(w)}.$$

One shows that this converges uniformly on compact subsets of  $\Omega \times \Omega$ , that  $\overline{S(z, \cdot)} \in \mathcal{H}^2(\Omega)$  for each  $z \in \Omega$ , and that for  $g \in \mathcal{H}^2(\Omega)$ ,

$$g(z) = \int_{\partial\Omega} S(z, w) g(w) d\sigma(w).$$

$S$  is then the Szegő kernel. From its construction, it is clear that it will be smooth on  $\Omega \times \Omega$ . It may extend to a smooth function on some larger subset of  $\overline{\Omega} \times \overline{\Omega}$ .

For domains of the form (1.1), one can derive an explicit formula for the Szegő kernel. Let  $z = (z_1, z_2)$  and  $w = (w_1, w_2)$  be elements of  $\mathbb{C}^2$ . Set

$$(2.2) \quad N(\eta, \tau) = \int_{-\infty}^{\infty} e^{2\tau[\eta\lambda - b(\lambda)]} d\lambda.$$

Then

$$(2.3) \quad S(z, w) = c \int \int_{\tau > 0} \tau e^{\eta\tau[z_1 + \bar{w}_1] + i\tau[z_2 - \bar{w}_2]} [N(\eta, \tau)]^{-1} d\eta d\tau,$$

where  $c$  is an absolute constant.

REMARK 2.1. See [HNW10] for detailed discussions of  $\mathcal{H}^p$  spaces for unbounded domains, the derivations of such integral formulas, and the identification of  $\mathcal{H}^2(\Omega)$  with  $L^2(\Omega)$  functions satisfying the differential equation (2.1).

REMARK 2.2. Many authors only consider  $S$  as a distribution on  $\partial\Omega \times \partial\Omega$  since  $S$  is smooth on  $\Omega \times \Omega$ . In this situation, one can identify the boundary with  $\mathbb{R}^3$  and consider the integral kernel

$$(2.4) \quad \begin{aligned} &\mathcal{S}[(x, y, t), (r, s, u)] \\ &= c \int_0^{\infty} \int_{-\infty}^{\infty} \tau e^{\tau[i(t-u) + i\eta(y-s) - [b(x) + b(r) - \eta(x+r)]]} [N(\eta, \tau)]^{-1} d\eta d\tau. \end{aligned}$$

This is done, for example, in the work of Nagel [Nag86], Haslinger [Has95], and Carracino [Car05], [Car07].

We may now state our results:

Let  $b$  be as in (1.3) and let

$$(2.5) \quad z = (z_1, z_2) = (x + iy, t + ib(x) + ih),$$

$$(2.6) \quad w = (w_1, w_2) = (r + is, u + ib(r) + ik).$$

Define

$$(2.7) \quad \Sigma = \{(z, w) : x = r \text{ and } |x| > \sqrt{-p}\} \cup \{(z, w) : |x| = |r| = \sqrt{-p}\}.$$

Also, for a continuous function  $b$  on  $\mathbb{R}$ , define the Legendre transform of  $b$  by

$$(2.8) \quad b^*(\eta) := \sup_{\lambda \in \mathbb{R}} [\eta\lambda - b(\lambda)].$$

**THEOREM 2.3.** *The integral defining  $S(z, w)$  is absolutely convergent in the region in which*

$$(2.9) \quad h + k + b(x) + b(r) - 2b^{**}\left(\frac{x+r}{2}\right) > 0.$$

*This is an open neighborhood of  $(\overline{\Omega} \times \overline{\Omega}) \setminus \Sigma$ . More generally, if  $i_1, j_1, i_2,$  and  $j_2$  are non-negative integers, then*

$$(2.10) \quad \partial_{z_1}^{i_1} \partial_{\bar{w}_1}^{j_1} \partial_{z_2}^{i_2} \partial_{\bar{w}_2}^{j_2} S(z, w) = c' \int \int_{\tau > 0} e^{\eta\tau[z_1 + \bar{w}_1] + i\tau[z_2 - \bar{w}_2]} \frac{\eta^{i_1+j_1} \tau^{i_1+j_1+i_2+j_2+1}}{N(\eta, \tau)} d\eta d\tau$$

*is absolutely convergent in the same region.*

**REMARK 2.4.** Compare this with Theorem 3.2 in [HNW10]. In that theorem, the domain is of the form (1.1) for  $b$  convex, and the region in which the integrals converge absolutely is defined by the inequality

$$h + k + b(x) + b(r) - 2b\left(\frac{x+r}{2}\right) > 0.$$

These two theorems are, in fact, analogous since the Legendre transform is an involution on the set of convex functions.

**THEOREM 2.5.** *If  $[(x + i0, 0 + ib(x)), (r + i0, 0 + ib(r))] \in \Sigma$ ,  $\mathcal{S}[(x, 0, 0), (r, 0, 0)]$  is infinite. Also, if  $\delta = h + k > 0$ ,*

$$\lim_{\delta \rightarrow 0^+} S[(x, i(b(x) + h)), (r, i(b(r) + k))] = \infty.$$

Since these theorems show that the nature of the singular set for the Szegő kernel can be different in the non-pseudoconvex case from what has been observed in the pseudoconvex case, we summarize these differences in the following:

**COROLLARY 2.6.** *If the domain  $\Omega$  is not pseudoconvex, there may be points on the diagonal of the boundary at which the Szegő kernel is not singular and points off the diagonal at which the kernel is singular.*

An understanding of the Szegő kernel requires sharp estimates of the integral  $N(\eta, \tau)$ . For fixed  $\eta, \tau$ , this is an integral of the form  $\int_{-\infty}^{\infty} e^{\rho(\lambda)} d\lambda$ , where  $\rho$  satisfies  $\lim_{|\lambda| \rightarrow \infty} \rho(\lambda) = -\infty$ . The heuristic principle that guides the analysis of such integrals is that the main contribution comes from a neighborhood of the point(s) at which the exponent attains its global maximum. If  $\lambda_0$  is such a point,

$$\int_{-\infty}^{\infty} e^{\rho(\lambda)} d\lambda = e^{\rho(\lambda_0)} \int_{-\infty}^{\infty} e^{\rho(\lambda) - \rho(\lambda_0)} d\lambda = e^{\rho(\lambda_0)} \int_{-\infty}^{\infty} e^{-p(x)} dx,$$

where  $p(x) := \rho(\lambda_0) - \rho(x + \lambda_0)$  is non-negative and vanishes to second order at the origin. In Section 3, we focus on understanding the “main contribution” to the integral  $N$ , while in Section 4, we focus on uniform estimates on the integral that remains once we have taken out this contribution. The theorems are established in Sections 5 and 6.

### 3. Behavior of $\eta\lambda - b(\lambda)$

Define

$$B_\eta(\lambda) := \eta\lambda - b(\lambda)$$

and consider  $b^*(\eta) = \sup_\lambda B_\eta(\lambda)$ , the Legendre transform of  $b$ . Since  $B_\eta$  is a polynomial of degree 4 in  $\lambda$  with negative leading coefficient, it tends to  $-\infty$  as  $|\lambda| \rightarrow \infty$ . It follows that the supremum is achieved at some  $\lambda$  at which  $B'_\eta$  vanishes; that is, at some  $\lambda$  satisfying  $\lambda^3 + p\lambda - (\eta - q) = 0$ . Note that this is a depressed cubic equation. Therefore, by considering its discriminant, one finds the following propositions.

PROPOSITION 3.1.

- (1) If  $4(-p)^3 \leq 27(\eta - q)^2$ , there is a single  $\lambda$  at which  $B'_\eta$  changes sign, hence  $B_\eta$  has a single local extremum, which is necessarily the location of the global maximum.
- (2) If  $4(-p)^3 > 27(\eta - q)^2$ ,  $B'_\eta(\lambda) = 0$  has three distinct solutions. Two correspond to local maxima of  $B_\eta$ . We label them  $\lambda_-(\eta)$  and  $\lambda_+(\eta)$ , with  $\lambda_-(\eta) < \lambda_+(\eta)$ .

The next propositions contain more specific information about the location(s) of the global maximum of  $B_\eta$ .

PROPOSITION 3.2. Let  $g(\lambda) = -(\lambda^3 + p\lambda)$ . Then

- (1)  $g$  is positive on  $(-\infty, -\sqrt{-p})$  and  $(0, \sqrt{-p})$ , and  $g$  is negative on  $(-\sqrt{-p}, 0)$  and  $(\sqrt{-p}, \infty)$ .
- (2)  $g$  increases on  $(-\sqrt{-p/3}, \sqrt{-p/3})$  and decreases on  $(-\infty, -\sqrt{-p/3})$  and  $(\sqrt{-p/3}, \infty)$ .

*Proof.* The proof is simple calculus and is omitted. □

PROPOSITION 3.3. Let  $B_\eta(\lambda) = \eta\lambda - b(\lambda)$ , with  $b$  as in (1.3).

(i) If  $\eta - q = 0$ ,  $\lambda_-(\eta) = -\sqrt{-p}$ ,  $\lambda_+(\eta) = \sqrt{-p}$ , and  $B_\eta(\lambda_-(\eta)) = B_\eta(\lambda_+(\eta))$ . In other words, the global maximum of  $B_\eta$  is achieved at two distinct points.

(ii) If  $0 < \eta - q < (\frac{4(-p)^3}{27})^{\frac{1}{2}}$ ,

$$-\sqrt{-p} < \lambda_-(\eta) < 0 < \sqrt{-p} < \lambda_+(\eta)$$

and  $B_\eta(\lambda_+) > B_\eta(\lambda_-)$ .

(iii) If  $(\frac{4(-p)^3}{27})^{\frac{1}{2}} \leq \eta - q$ ,  $B_\eta$  has a single local (hence global) maximum at  $\lambda_+(\eta) > \sqrt{-p}$ , and  $\lambda_+(\eta) \sim \eta^{\frac{1}{3}}$  as  $\eta \rightarrow \infty$ .

(iv) If  $-(\frac{4(-p)^3}{27})^{\frac{1}{2}} < \eta - q < 0$ ,

$$\lambda_-(\eta) < -\sqrt{-p} < 0 < \lambda_+(\eta) < \sqrt{-p}$$

and  $B_\eta(\lambda_-) > B_\eta(\lambda_+)$ .

(v) If  $\eta - q < -(\frac{4(-p)^3}{27})^{\frac{1}{2}} < 0$ ,  $B_\eta$  has a single local (hence global) maximum at  $\lambda_-(\eta) < -\sqrt{-p}$ , and  $\lambda_-(\eta) \sim \eta^{\frac{1}{3}}$  as  $\eta \rightarrow -\infty$ .

*Proof.* (i): If  $\eta = q$ , then the local extrema of  $B_\eta$  occur at solutions to  $g(\lambda) = 0$ . The three solutions are  $\lambda = -\sqrt{-p}, 0, \sqrt{-p}$ , and the local maximum is attained at  $\lambda = \pm\sqrt{-p}$ . Since in this case  $B_\eta(\lambda) = -\frac{1}{4}\lambda^4 - \frac{1}{2}p\lambda^2$  is even, the conclusion follows.

(ii): By Proposition 3.1, the upper bound on  $\eta$  guarantees that  $B_\eta$  in fact has two local maxima. Since  $\eta - q > 0$ , for  $\lambda \in [0, \sqrt{-p}]$ ,

$$B'_\eta(\lambda) = (\eta - q) - (\lambda^3 + p\lambda) = (\eta - q) + g(\lambda) > 0.$$

Since  $g$  is decreasing for  $\lambda > \sqrt{\frac{-p}{3}}$ ,  $(\eta - q) + g(\lambda) = 0$  has precisely one solution in  $(\sqrt{-p}, \infty)$ , and it is the location of a local maximum for  $B_\eta$ . We have named this point  $\lambda_+(\eta)$ . On the other hand, since  $(\eta - q) + g(\lambda)$  is also positive on  $(-\infty, -\sqrt{-p}]$ , the second local maximum  $\lambda_-(\eta)$  is in  $(-\sqrt{-p}, 0)$ .

Now,  $\eta - q > 0$  and  $\lambda_- < 0$  imply  $(\eta - q)\lambda_- < (\eta - q)(-\lambda_-)$ . Since  $\lambda_+$  is the location of the global maximum of the restriction of  $B_\eta$  to the positive real axis,  $B_\eta(\lambda_-) = (\eta - q)\lambda_- - (\frac{1}{4}\lambda_-^4 + \frac{1}{2}p\lambda_-^2) < (\eta - q)(-\lambda_-) - (\frac{1}{4}(-\lambda_-)^4 + \frac{1}{2}p(-\lambda_-)^2) = B_\eta(-\lambda_-) < B_\eta(\lambda_+)$ . This proves (ii).

(iii): By Proposition 3.1, we are in the situation in which  $B'_\eta(\lambda) = 0$  has a single solution. An identical argument to the one used to prove (ii) shows that the solution, which we call  $\lambda_+(\eta)$ , satisfies  $\sqrt{-p} < \lambda_+(\eta)$ .

We now prove the statement about the asymptotic behavior of  $\lambda_+(\eta)$ . Since  $\lambda_+^3 > \lambda_+^3 + p\lambda_+ = \eta - q$ ,  $\lambda_+(\eta) \rightarrow \infty$  as  $\eta \rightarrow \infty$ . Also, since  $\lambda_+^3 = \eta - q - p\lambda_+$ , we have

$$1 = \frac{\eta}{\lambda_+^3} + o(1).$$

Thus  $\lambda_+^3 \sim \eta$ , that is,  $\lambda_+^3 = \eta[1 + o(1)]$  as  $\eta \rightarrow \infty$ . It follows that  $\lambda_+(\eta) \sim \eta^{\frac{1}{3}}$  as  $\eta \rightarrow \infty$ .

The proofs of (iv) and (v) are almost identical to the proofs of (ii) and (iii) and are omitted. □

Define a function

$$(3.1) \quad \lambda(\eta) = \begin{cases} \lambda_-(\eta), & \eta < q, \\ \sqrt{-p}, & \eta = q, \\ \lambda_+(\eta), & \eta > q. \end{cases}$$

Thus, for  $\eta \neq q$ ,  $\lambda(\eta)$  is the location of the global maximum of  $B_\eta$ . For  $\eta = q$ , the global maximum is achieved at two points,  $\pm\sqrt{-p}$ . Which of these we choose for the value of  $\lambda(q)$  is arbitrary.

**PROPOSITION 3.4.** *The function  $\eta \mapsto \lambda(\eta)$  maps  $\mathbb{R}$  onto  $\mathbb{R} \setminus [-\sqrt{-p}, \sqrt{-p}]$ . Furthermore, it is*

- (a) *differentiable on  $\mathbb{R} \setminus \{q\}$ ,*
- (b) *continuous from the right at  $\eta = q$ , and*
- (c) *increasing and injective on  $\mathbb{R}$ .*

*Proof.* The equation  $\eta = q + \lambda^3 + p\lambda$  clearly expresses  $\eta$  as a function of  $\lambda$ . Furthermore, the restriction of this function to  $(-\infty, -\sqrt{-p}) \cup [\sqrt{-p}, \infty)$  is easily seen to be one-to-one with image  $\mathbb{R}$ . Thus, its inverse function is well-defined on  $\mathbb{R}$  and maps this set to  $(-\infty, -\sqrt{-p}) \cup [\sqrt{-p}, \infty)$ . Since  $\lambda$  restricted to  $\mathbb{R} \setminus \{q\}$  is the inverse of a function which is smooth with non-vanishing derivative on its (restricted) domain,  $\lambda$  is itself continuous and differentiable there, with derivative  $\lambda'(\eta) = \frac{1}{3[\lambda(\eta)]^2 + p} > 0$ . The proposition is established. □

**COROLLARY 3.5.** *The function  $\eta \rightarrow b^*(\eta)$  is continuous on  $\mathbb{R}$ .*

*Proof.* This is immediate since  $b^*$  is a real-valued function which is convex on all of  $\mathbb{R}$ . □

#### 4. Estimates on $\int_{-\infty}^{\infty} e^{-p(x)} dx$

**4.1. Definitions and notation.** Let  $p$  be a real polynomial of even degree with positive leading coefficient. We are interested in estimates on

$$(4.1) \quad \int_{-\infty}^{\infty} e^{-p(x)} dx$$

which are uniform in the coefficients of  $p$ . If  $p$  is convex (i.e., if  $p''(x) \geq 0$  for all  $x$ ) with  $p(0) = p'(0) = 0$ , we know that

$$(4.2) \quad \int_{-\infty}^{\infty} e^{-p(x)} dx \approx |\{x : p(x) \leq 1\}|,$$

where we use the notation  $A \approx B$  to mean that there exists a constant  $c$  such that  $cB \leq A \leq \frac{1}{c}B$ . When such inequalities hold, we say that  $A$  and  $B$  are *comparable*. It will be understood whenever this notation is used that the

underlying constant  $c$  is independent of all important parameters. Thus in our case, it is always independent of the coefficients of the polynomial  $p$  and depends only, perhaps, on the degree of  $p$ .

Our goal is to extend the estimate (4.2) to the situation in which  $p$  is a fourth-degree polynomial with positive leading coefficient. By translating, shifting, and reflecting about the  $y$ -axis if necessary, we can arrange it so that the (not necessarily unique) global minimum of the polynomial is zero and occurs at  $x = 0$ , and so that  $p$  is convex for all  $x \leq 0$ . Since  $p''$  has degree 2, if  $p$  fails to be convex on all of  $\mathbb{R}$ , there is a single interval on which  $p''$  is negative.

Thus, suppose  $p''$  has zeros at  $x = A$  and  $x = A + C$  where  $A, C > 0$ . Then there exists  $B > 0$  so that

$$(4.3) \quad p''(x) = B(x - A)(x - (A + C)).$$

REMARK 4.1. One checks easily that if  $A = 0$ ,  $p$  cannot have its global minimum at 0 unless  $C = 0$  as well. In this case, we would have  $p(x) = Bx^4$ , which is convex. Furthermore, if  $A > 0$  but  $C = 0$ ,  $p''$  is never negative, hence  $p$  is convex.

If we anti-differentiate (4.3) twice, using the assumption that  $p(0) = p'(0) = 0$ , we find

$$(4.4) \quad p(x) = \frac{B}{12}x^2[x^2 - 2(2A + C)x + 6A(A + C)].$$

In the analysis that follows, it will be essential to know what relationship, if any, exists between  $A$  and  $C$ . Thus, write  $C = \alpha A$  for  $\alpha > 0$ . Then

$$(4.5) \quad p(x) = \frac{B}{12}x^2[x^2 - 2A(2 + \alpha)x + 6A^2(1 + \alpha)].$$

PROPOSITION 4.2. *Let  $p$  be as in (4.5), with  $A, B, \alpha > 0$ .  $p$  is non-negative if and only if*

$$(4.6) \quad 0 < \alpha \leq 1 + \sqrt{3}.$$

*Proof.*  $p$  is non-negative if and only if the expression  $x^2 - 2A(2 + \alpha)x + 6A^2(1 + \alpha)$  is non-negative for all  $x$ . The conclusion follows by finding those positive  $\alpha$  for which this quadratic has non-positive discriminant.  $\square$

Next, we prove an inequality concerning the value of  $p$  at its inflection points.

PROPOSITION 4.3. *If  $p(x) = \frac{B}{12}[x^4 - 2A(2 + \alpha)x^3 + 6A^2(1 + \alpha)x^2]$  and  $0 < \alpha \leq 1 + \sqrt{3}$ , then there exists  $c > 0$  independent of  $A$  and  $B$  so that  $p((1 + \alpha)A) \geq p(A) \geq cBA^4$ .*

*Proof.* We have  $p((1 + \alpha)A) = \frac{BA^4}{12}(3 + 8\alpha + 6\alpha^2 - \alpha^4)$  and  $p(A) = \frac{BA^4}{12}(3 + 4\alpha)$ . The lower bound on  $p(A)$  follows immediately since for  $0 < \alpha \leq 1 + \sqrt{3}$ ,  $3 + 4\alpha$  is bounded below by a positive constant.



Observe that  $p((1 + \alpha)A) - p(A) = \frac{B}{12}A^4(4\alpha + 6\alpha^2 - \alpha^4)$ . One confirms easily that  $\alpha = -2, 0, 1 \pm \sqrt{3}$  are roots. Finally, since at  $\alpha = 1$ ,  $p((1 + \alpha)A) - p(A) = \frac{B}{12}A^4(4 + 6 - 1) > 0$ , we conclude that this difference is positive for all  $0 < \alpha < 1 + \sqrt{3}$ . This proves the proposition.  $\square$

A convex polynomial clearly has only one local extremum, which is necessarily the location of the global minimum. For non-convex  $p$ , however, it is possible that  $p$  has other extrema. More specifically, if  $p$  is a fourth-degree polynomial,  $p'$  is a polynomial of degree three, hence it has either a single real root or three real roots (counting multiplicities). We have the following propositions.

**PROPOSITION 4.4.** *Let  $p(x) = \frac{B}{12}[x^4 - 2A(2 + \alpha)x^3 + 6A^2(1 + \alpha)x^2]$ , with  $A, B > 0$  and  $0 < \alpha \leq 1 + \sqrt{3}$ . Then  $p'$  has three real roots if and only if  $2 \leq \alpha \leq 1 + \sqrt{3}$ .*

*Proof.* We find that  $p'(x) = \frac{B}{12}x[4x^2 - 6A(2 + \alpha)x + 12A^2(1 + \alpha)]$ . This has three real roots if and only if  $3A^2(3\alpha^2 - 4\alpha - 4) \geq 0$ . This occurs if and only if  $\alpha \leq -\frac{2}{3}$  or  $\alpha \geq 2$ . Since we have assumed  $0 < \alpha \leq 1 + \sqrt{3}$ , the conclusion follows.  $\square$

To analyze the integral (4.1), we begin by writing it as a sum:

$$\begin{aligned}
 (4.7) \quad \int_{-\infty}^{\infty} e^{-p(x)} dx &= \int_{-\infty}^0 e^{-p(x)} dx + \int_0^A e^{-p(x)} dx \\
 &\quad + \int_A^{(1+\alpha)A} e^{-p(x)} dx + \int_{(1+\alpha)A}^{\infty} e^{-p(x)} dx \\
 &= I + II + III + IV.
 \end{aligned}$$

Observe that  $p$  is convex on the intervals of integration for  $I$ ,  $II$ , and  $IV$ . Obtaining sharp estimates on these integrals requires the results of the next subsection.

**4.2. Some estimates on functions on intervals of convexity.** We will use the following results repeatedly. The first gives the size of the integral of  $e^{-p}$  over any interval on which  $p$  is convex. It uses a modification of an argument in Halfpap, Nagel, and Wainger [HNW10] proving an analogous estimate if  $p$  is convex on all of  $\mathbb{R}$ .

**LEMMA 4.5.** *Let  $p$  be a polynomial satisfying  $\lim_{|x| \rightarrow \infty} p(x) = \infty$ .*

- (1) *Suppose  $p'$  is positive and increasing on an interval  $(x_0, x_f)$ , where  $x_f$  may equal  $+\infty$ . Suppose further that in the case in which  $x_f < \infty$ ,  $p(x_f) \geq p(x_0) + 1$ . Then*

$$(4.8) \quad \int_{x_0}^{x_f} e^{-p(x)} dx \approx e^{-p(x_0)} |\{x \in (x_0, x_f) : p(x_0) < p(x) < p(x_0) + 1\}|.$$

(2) Suppose  $p'$  is negative and increasing on an interval  $(x_f, x_0)$ , where  $x_f$  may equal  $-\infty$ . Suppose further that in the case in which  $x_f > -\infty$ ,  $p(x_f) \geq p(x_0) + 1$ . Then

$$(4.9) \quad \int_{x_f}^{x_0} e^{-p(x)} dx \approx e^{-p(x_0)} |\{x \in (x_f, x_0) : p(x_0) < p(x) < p(x_0) + 1\}|.$$

(3) Suppose  $p'$  is (i) positive and increasing on  $I = (x_0, x_f)$  with  $x_f < \infty$  or (ii) negative and increasing on  $I = (x_f, x_0)$  with  $x_f > -\infty$ . Suppose further that  $p(x_0) < p(x_f) < p(x_0) + 1$ . Then

$$(4.10) \quad \int_I e^{-p(x)} dx \approx e^{-p(x_0)} |x_f - x_0|.$$

*Proof.* We sketch the proof of (1). The remaining parts follow in a similar manner.

Suppose  $x_f < \infty$ , and let  $J$  be the largest positive integer such that  $p(x_f) \geq p(x_0) + J$ . Our hypotheses guarantee that such a  $J$  exists. For each positive integer  $j \leq J$ , define  $x_j$  to be the unique element of  $(x_0, x_f)$  for which  $p(x_j) = p(x_0) + j$ . Clearly,

$$e^{-p(x_0)}(x_1 - x_0) \leq \int_{x_0}^{x_f} e^{-p(x)} dx.$$

For the reverse inequality, observe that

$$\begin{aligned} \int_{x_0}^{x_f} e^{-p(x)} dx &= \sum_{j=0}^{J-1} \int_{x_j}^{x_{j+1}} e^{-p(x)} dx + \int_{x_J}^{x_f} e^{-p(x)} dx \\ &\leq \sum_{j=0}^{J-1} e^{-p(x_0)-j}(x_{j+1} - x_j) + e^{-p(x_0)-J}(x_f - x_J) \\ &\leq e^{-p(x_0)} \left[ (x_1 - x_0) + \sum_{j=1}^{J-1} e^{-j}(x_{j+1} - x_1) + e^{-J}(x_f - x_1) \right]. \end{aligned}$$

We now estimate  $x_{j+1} - x_1$  in terms of  $x_1 - x_0$ .

$$j = p(x_{j+1}) - p(x_1) = \int_{x_1}^{x_{j+1}} p'(x) dx \geq p'(x_1)(x_{j+1} - x_1).$$

Since

$$p'(x_1)(x_1 - x_0) \geq \int_{x_0}^{x_1} p'(x) dx = 1,$$

we have  $x_{j+1} - x_1 \leq j(x_1 - x_0)$ . A similar estimate holds for  $x_f - x_1$ . It follows that

$$\int_{x_0}^{x_f} e^{-p(x)} dx \lesssim (x_1 - x_0)e^{-p(x_0)}. \quad \square$$

LEMMA 4.6 (Bruna, Nagel, Wainger [BNW88]). *Let  $p$  be a polynomial of degree  $m$  satisfying  $p(0) = p'(0) = 0$ ; i.e.,  $p(x) = \sum_{k=2}^m a_k x^k$ . If  $p$  is convex on an interval  $[0, A]$ , then there exists a constant  $C_m$ , depending on  $m$  but independent of  $A$ , such that*

$$(4.11) \quad p(x) \geq C_m \sum_{k=2}^m |a_k| x^k \quad \text{for all } x \in [0, A].$$

This lemma is useful to us because it allows us to prove the following.

PROPOSITION 4.7. *Let  $p$  be as in Lemma 4.6. Suppose that  $p(A) > 1$ . Then  $p(x) = 1$  has a unique solution  $\mu$  in  $[0, A]$  and*

$$(4.12) \quad \mu \approx \left[ \sum_{k=2}^m |a_k|^{1/k} \right]^{-1}.$$

*Proof.* This is a standard argument, included here for completeness.

It follows from Lemma 4.6 that there exists  $C_m$  such that for all  $x \in [0, A]$ ,

$$C_m \sum_{k=2}^m |a_k| x^k \leq \sum_{k=2}^m a_k x^k \leq \sum_{k=2}^m |a_k| x^k.$$

Define  $\tilde{p}(x) = \sum_{k=2}^m |a_k| x^k$ . Then if  $y_1$  is the positive solution to  $\tilde{p}(x) = 1$  and  $y_2$  is the positive solution to  $C_m \tilde{p}(x) = 1$ , then  $y_1 \leq \mu \leq y_2$ . It therefore suffices to show that  $y_1$  and  $y_2$  are comparable to the expression on the right of (4.12). We show this for  $y_2$ .

By definition,  $y_2$  satisfies  $\sum_{k=2}^m C_m |a_k| y_2^k = 1$ . Thus for every  $k$ ,  $2 \leq k \leq m$ ,  $C_m |a_k| y_2^k \leq 1$ , and hence  $y_2 \leq [C_m^{1/k} |a_k|^{1/k}]^{-1}$ . Since this is true for any  $k$ , it is true for the  $k_0$  such that  $C_m^{1/k_0} |a_{k_0}|^{1/k_0} = \max_{\{2 \leq k \leq m\}} C_m^{1/k} |a_k|^{1/k}$ . On the other hand,

$$C_m^{1/k_0} |a_{k_0}|^{1/k_0} \geq \frac{1}{m-1} \sum_{k=2}^m C_m^{1/k} |a_k|^{1/k}.$$

It follows that

$$(4.13) \quad y_2 \leq \left[ \frac{1}{m-1} \sum_{k=2}^m C_m^{1/k} |a_k|^{1/k} \right]^{-1} \leq \frac{m-1}{C_m^{1/2}} \left[ \sum_{k=2}^m |a_k|^{1/k} \right]^{-1}.$$

This gives the desired upper bound on  $y_2$ .

To obtain a lower bound, let  $k_1$  be such that

$$C_m |a_{k_1}| y_2^{k_1} = \max_{\{2 \leq k \leq m\}} C_m |a_k| y_2^k.$$

Then

$$(m-1) C_m |a_{k_1}| y_2^{k_1} \geq C_m \tilde{p}(y_2) = 1,$$

and so

$$\begin{aligned} y_2 &\geq [(m-1)^{1/k_1} C_m^{1/k_1} |a_{k_1}|^{1/k_1}]^{-1} \\ &\geq \left(\frac{1}{m-1}\right)^{1/2} \left(\frac{1}{C_m}\right)^{1/m} [|a_{k_1}|^{1/k_1}]^{-1} \\ &\geq \left(\frac{1}{m-1}\right)^{1/2} \left(\frac{1}{C_m}\right)^{1/m} \left[\sum_{k=2}^m |a_k|^{1/k}\right]^{-1}. \end{aligned}$$

We have now proved the desired estimates on  $y_2$ . The estimates on  $y_1$  follow by setting  $C_m = 1$ . □

**4.3. Estimates of the integral (4.1).** In this section, we prove the following lemma.

LEMMA 4.8. *If  $\beta, \delta > 0$  and  $p(x) = \beta x^4 + \gamma x^3 + \delta x^2$  attains its global minimum at the origin, then*

$$(4.14) \quad \int_{-\infty}^{\infty} e^{-[\beta x^4 + \gamma x^3 + \delta x^2]} dx \approx [\beta^{\frac{1}{4}} + |\gamma|^{\frac{1}{3}} + \delta^{\frac{1}{2}}]^{-1}.$$

Since the result is already known for convex  $p$ , it suffice to establish it for non-convex  $p$ , taking  $\beta = \frac{B}{12}$ ,  $\gamma = -\frac{BA(2+\alpha)}{6}$ , and  $\delta = \frac{BA^2(1+\alpha)}{2}$ . As in (4.7), we consider this as a sum of four integrals.

4.3.1. *The integral I.* To estimate  $I$ , note that

$$q(x) = p(-x) = \frac{B}{12} [x^4 + 2A(2 + \alpha)x^3 + 6A^2(1 + \alpha)x^2]$$

is convex on  $(0, \infty)$  with  $q(0) = q'(0) = 0$ . Thus by Lemma 4.5 and Proposition 4.7,  $I$  satisfies the estimate (4.14), that is,

$$(4.15) \quad \begin{aligned} I &\approx \left[ \left(\frac{B}{12}\right)^{\frac{1}{4}} + \left(\frac{BA}{6}(2 + \alpha)\right)^{\frac{1}{3}} + \left(\frac{BA^2}{2}(1 + \alpha)\right)^{\frac{1}{2}} \right]^{-1} \\ &\approx [B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A]^{-1}. \end{aligned}$$

In (4.15), we have also used Proposition 4.2 to conclude that  $2 + \alpha$  and  $1 + \alpha$  are both comparable to 1.

Since clearly  $I \leq I + II + III + IV$ , the lemma will follow if we can show that  $II, III, IV \lesssim I$ .

4.3.2. *The integral II.* We have two cases, depending on whether  $p(A) \geq 1$  or  $p(A) < 1$ .

First, if  $p(A) \geq 1$ , then Lemma 4.5 and Proposition 4.7 imply, as they did in the case of integral  $I$ , that

$$(4.16) \quad II \approx [B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A]^{-1} \approx I,$$

as desired.

Suppose, then, that  $p(A) < 1$ . Then by Lemma 4.5,

$$(4.17) \quad II \approx A.$$

By Proposition 4.3,  $cBA^4 \leq p(A)$ , and so if  $p(A) < 1$ ,  $BA^4 \lesssim 1$ . Thus,

$$\begin{aligned} A[B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A] &= B^{\frac{1}{4}}A + B^{\frac{1}{3}}A^{\frac{4}{3}} + B^{\frac{1}{2}}A^2 \\ &= (BA^4)^{\frac{1}{4}} + (BA^4)^{\frac{1}{3}} + (BA^4)^{\frac{1}{2}} \\ &\lesssim 1. \end{aligned}$$

It follows from (4.17) that  $II \lesssim I$ .

4.3.3. *The integral III.* This is the integral over the interval on which  $p''$  is negative. This forces the minimum of  $p$  on this interval to be either  $p((1 + \alpha)A)$  or  $p(A)$ . By Proposition 4.3, both are bounded below by  $cBA^4$  for some uniform positive constant  $c$ . Therefore,

$$(4.18) \quad III \leq \alpha Ae^{-cBA^4} \lesssim Ae^{-cBA^4}.$$

This contribution is always less than that from the integral  $I$ . Indeed,

$$Ae^{-cBA^4} [B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A] = [(BA^4)^{\frac{1}{4}} + (BA^4)^{\frac{1}{3}} + (BA^4)^{\frac{1}{2}}]e^{-cBA^4}$$

is uniformly bounded since the function  $f(x) = (x^{\frac{1}{4}} + x^{\frac{1}{3}} + x^{\frac{1}{2}})e^{-cx}$  is bounded on the positive real axis.

4.3.4. *The integral IV.* As with integrals  $I$  and  $II$ , we are integrating over an interval on which  $p$  is convex. In order to use Lemma 4.5, we need to know the minimum value of  $p$  on this interval. We distinguish two cases.

First, suppose the minimum occurs at  $x = (1 + \alpha)A$ . Note that this implies that  $p'((1 + \alpha)A) \geq 0$ . We must find

$$|\{x > (1 + \alpha)A : p[(1 + \alpha)A] \leq p(x) \leq p[(1 + \alpha)A] + 1\}|.$$

If  $y$  is the unique solution to  $p(y) = p[(1 + \alpha)A] + 1$  in this interval, then the desired measure is  $\nu = y - (1 + \alpha)A$ . Recall that  $p$  has an inflection point at  $x = (1 + \alpha)A$  and expand  $p$  about  $(1 + \alpha)A$  to obtain

$$\begin{aligned} p[(1 + \alpha)A] + p'((1 + \alpha)A)(x - (1 + \alpha)A) \\ + \frac{\alpha BA}{6}(x - (1 + \alpha)A)^3 + \frac{B}{12}(x - (1 + \alpha)A)^4. \end{aligned}$$

Thus,  $\nu$  is the solution to

$$(4.19) \quad p'((1 + \alpha)A)\nu + \frac{\alpha BA}{6}\nu^3 + \frac{B}{12}\nu^4 = 1.$$

It follows that the solution to (4.19) is less than the  $\tilde{\nu}$  satisfying

$$(4.20) \quad \frac{\alpha BA}{6}\tilde{\nu}^3 + \frac{B}{12}\tilde{\nu}^4 = 1.$$

Since  $\alpha, A, B, \tilde{\nu} > 0$ ,  $\nu \leq \tilde{\nu} \approx [B^{\frac{1}{4}} + B^{\frac{1}{3}}(\alpha A)^{\frac{1}{3}}]^{-1}$ . Thus,

$$(4.21) \quad IV \lesssim e^{-cBA^4} [B^{\frac{1}{4}} + B^{\frac{1}{3}}(\alpha A)^{\frac{1}{3}}]^{-1}.$$

We claim that  $IV \lesssim I$ . Indeed, since  $\alpha > 0$ ,

$$\begin{aligned} e^{-cBA^4} \frac{B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A}{B^{\frac{1}{4}} + B^{\frac{1}{3}}(\alpha A)^{\frac{1}{3}}} &\leq e^{-cBA^4} \frac{B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A}{B^{\frac{1}{4}}} \\ &= e^{-cBA^4} [1 + (BA^4)^{\frac{1}{12}} + (BA^4)^{\frac{1}{4}}]. \end{aligned}$$

Since  $f(x) = [1 + x^{\frac{1}{12}} + x^{\frac{1}{4}}]e^{-cx}$  is a bounded function on the positive real axis, the conclusion follows.

Suppose, next, that the minimum of  $p$  on  $[(1 + \alpha)A, \infty)$  occurs at some point  $x_0$  interior to the interval at which  $p'$  vanishes. In this case,  $p'$  has three distinct real roots, and so by Proposition 4.4,  $2 < \alpha \leq 1 + \sqrt{3}$ . Precisely the same argument we used above to show that, regardless of the size of  $p(A)$ ,

$$\int_{-\infty}^A e^{-p(x)} dx \approx \int_{-\infty}^0 e^{-p(x)} dx \approx e^{-p(0)} |\{x < 0 : 0 < p(x) < 1\}|$$

shows that, regardless of the size of  $p[(1 + \alpha)A]$ ,

$$(4.22) \quad IV \approx \int_{x_0}^{\infty} e^{-p(x)} dx \approx e^{-p(x_0)} |\{x > x_0 : p(x_0) < p(x) < p(x_0) + 1\}|.$$

Thus we must estimate  $p(x_0)$  and the positive number  $y$  satisfying  $p(x_0) + 1 = p(x_0 + y)$ . Expanding  $p$  in powers of  $y = x - x_0$  yields

$$\begin{aligned} (4.23) \quad p(x) &= p(x_0) + p'(x_0)y + \frac{1}{2}p''(x_0)y^2 + \frac{1}{6}p'''(x_0)y^3 + \frac{1}{24}p^{(4)}(x_0)y^4 \\ &= p(x_0) + \frac{B}{2}[x_0^2 - A(2 + \alpha)x_0 + A^2(1 + \alpha)]y^2 \\ &\quad + \frac{B}{6}[2x_0 - A(2 + \alpha)]y^3 + \frac{B}{12}y^4. \end{aligned}$$

Recall from the proof of Proposition 4.4 that  $p'(x) = \frac{B}{6}x[2x^2 - 3A(2 + \alpha)x + 6A^2(1 + \alpha)]$ . Set

$$(4.24) \quad \varepsilon = 9\alpha^2 - 12\alpha - 12.$$

This is positive since  $\alpha > 2$ . Then

$$(4.25) \quad x_0 = \frac{A}{4}[3(2 + \alpha) + \sqrt{\varepsilon}]$$

and

$$x_0^2 = \frac{A^2}{8}[9\alpha^2 + 12\alpha + 12 + 3(2 + \alpha)\sqrt{\varepsilon}].$$

Substituting (4.25) and (4.24) into (4.23) yields

$$p(x_0 + y) = p(x_0) + \frac{BA^2}{48}[\varepsilon + 3(2 + \alpha)\sqrt{\varepsilon}]y^2 + \frac{BA}{12}[2 + \alpha + \sqrt{\varepsilon}]y^3 + \frac{B}{12}y^4.$$

Thus,

$$1 = \frac{BA^2}{48} [\varepsilon + 3(2 + \alpha)\sqrt{\varepsilon}]y^2 + \frac{BA}{12} [2 + \alpha + \sqrt{\varepsilon}]y^3 + \frac{B}{12}y^4,$$

and so

$$(4.26) \quad y \approx [B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}}(2 + \alpha + \sqrt{\varepsilon})^{\frac{1}{3}} + B^{\frac{1}{2}}A\varepsilon^{\frac{1}{4}}(\sqrt{\varepsilon} + 3(2 + \alpha))^{\frac{1}{2}}]^{-1}.$$

Since  $2 < \alpha \leq 1 + \sqrt{3}$ , for such  $\alpha$ ,  $0 < \varepsilon = 3(3\alpha^2 - 4\alpha - 4) \lesssim 1$ . Hence,

$$(4.27) \quad y \approx [B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A\varepsilon^{\frac{1}{2}}]^{-1}.$$

Recall that we wish to show that  $IV \lesssim I$ , or, equivalently, that

$$(4.28) \quad e^{-p(x_0)} [B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A\varepsilon^{\frac{1}{4}}]^{-1} \lesssim [B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A]^{-1}.$$

Since  $e^{-p(x_0)} \leq 1$  and  $\varepsilon \lesssim 1$ , this follows immediately in the case in which  $\varepsilon$  is also bounded below by an absolute constant  $\beta$ .

To prove (4.28) for all  $\varepsilon$ , therefore, it suffices to find an absolute constant  $\beta$  such that (4.28) holds for all  $0 < \varepsilon \leq \beta$ . Since such an estimate is likely to rely upon the relative smallness of  $e^{-p(x_0)}$  compared to  $BA^4$ , we need more information about the size of  $p(x_0)$ . A calculation using (4.5) shows

$$\begin{aligned} p(x_0) &= \frac{BA^4}{768} (9\alpha^2 + 12\alpha + 12 + 3(2 + \alpha)\sqrt{\varepsilon}) \\ &\quad \times (-3\alpha^2 + 12\alpha + 12 - (2 + \alpha)\sqrt{\varepsilon}) \\ &\approx BA^4(-3\alpha^2 + 12\alpha + 12 - (2 + \alpha)\sqrt{\varepsilon}). \end{aligned}$$

We claim that there exist positive constants  $\beta$  and  $d$  such that for all  $\alpha \in (2, 1 + \sqrt{3}]$ , if  $\varepsilon \leq \beta$ ,  $-3\alpha^2 + 12\alpha + 12 - (2 + \alpha)\sqrt{\varepsilon} \geq d$ , from which it will follow that  $p(x_0) \geq dBA^4$ .

Indeed, it is easy to see that

$$-3\alpha^2 + 12\alpha + 12 - (2 + \alpha)\sqrt{\varepsilon} \geq 6(1 - \sqrt{\varepsilon}).$$

This is bounded below by 3 if  $\varepsilon \leq \frac{1}{4}$ . The claim follows.

To prove (4.28) when  $\varepsilon \leq \frac{1}{4}$ , we must show that

$$(4.29) \quad e^{-dBA^4} \frac{B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A}{B^{\frac{1}{4}} + B^{\frac{1}{3}}A^{\frac{1}{3}} + B^{\frac{1}{2}}A\varepsilon^{\frac{1}{4}}} = e^{-dBA^4} \frac{1 + (BA^4)^{\frac{1}{12}} + (BA^4)^{\frac{1}{4}}}{1 + (BA^4)^{\frac{1}{12}} + (BA^4)^{\frac{1}{4}}(\varepsilon)^{\frac{1}{4}}}$$

is bounded. This is indeed the case since

$$0 \leq f(x) = e^{-dx} \frac{1 + x^{\frac{1}{12}} + x^{\frac{1}{4}}}{1 + x^{\frac{1}{12}} + x^{\frac{1}{4}}\varepsilon^{\frac{1}{4}}} \leq e^{-dx} (1 + x^{\frac{1}{12}} + x^{\frac{1}{4}})$$

and the latter is bounded above on the positive real axis.

4.3.5. *Another interpretation.* Recall that (4.2) holds if  $p$  is convex and  $p(0) = p'(0) = 0$ . We claim that our estimates show that the same is true in the case of any fourth-degree polynomial with positive leading coefficient and global minimum at the origin. Indeed, set

$$\mu = |\{x : p(x) \leq 1\}|, \quad \mu^- = |\{x < 0 : p(x) \leq 1\}|.$$

Clearly

$$e^{-1}\mu \leq \int_{\{x:p(x)\leq 1\}} e^{-p(x)} dx \leq \int_{-\infty}^{\infty} e^{-p(x)} dx.$$

On the other hand, the estimates of the previous section imply the existence of a constant  $C > 0$  such that

$$\int_{-\infty}^{\infty} e^{-p(x)} dx \leq C\mu^-.$$

Since  $\mu^- \leq \mu$ , it follows that  $\int_{-\infty}^{\infty} e^{-p(x)} dx \approx \mu$ , as claimed.

**4.4. Remarks on polynomials of higher degree.** The results of this paper cannot easily be extended to tube domains (1.1) defined by higher-degree non-convex polynomials  $b$  because it is not clear what uniform estimate should replace Lemma 4.8.

Consider, for a moment, the analogue of Lemma 4.8 for convex polynomials:

LEMMA 4.9. *Let  $n$  be a positive integer and define  $p(x) = \sum_{j=2}^{2n} \beta_j x^j$ . Suppose  $p$  is convex on  $\mathbb{R}$ . Then*

$$(4.30) \quad I := \int_{-\infty}^{\infty} e^{-p(x)} dx \approx \left[ \sum_{j=2}^{2n} |\beta_j|^{\frac{1}{j}} \right]^{-1}.$$

This lemma is not new; it follows easily from the results of Bruna, Nagel, and Wainger discussed above. We saw in Lemma 4.8 that this same result holds if  $n = 2$  even if we replace the hypothesis that  $p$  is convex with the weaker hypotheses that  $p$  attains its global minimum at 0 and  $\beta_{2n} > 0$ . We claim that such a result does *not* hold if  $n = 3$ .

Indeed, consider  $p(x) = x^2(x-a)^4 = x^6 - 4ax^5 + 6a^2x^4 - 4a^3x^3 + a^4x^2$ , with  $a > 1$ . Clearly  $p$  is non-negative, attains its global minimum at the origin, and is convex for  $x \leq 0$ . If (4.30) were true, we would have both

$$\frac{1}{a^2} \approx [1 + a^{\frac{1}{5}} + a^{\frac{1}{2}} + a + a^2]^{-1} \approx I$$

and

$$I \geq \int_a^{\infty} e^{-x^2(x-a)^4} dx = \int_0^{\infty} e^{-(y+a)^2y^4} dy \approx [1 + a^{\frac{1}{5}} + a^{\frac{1}{2}}]^{-1} \approx \frac{1}{a^{\frac{1}{2}}}.$$

(We have used in the above the observation that  $q(y) = (y+a)^2y^4$  is convex on the positive real axis with global minimum at the origin.) Since there is no positive  $C$  independent of  $a > 1$  such that  $\frac{1}{a^2} \geq \frac{C}{a^{\frac{1}{2}}}$ , our claim is established.



It is not hard to see what is going on; in the case of a non-convex fourth-degree polynomial, if there are two competing global minima, they are both points at which the polynomial vanishes to order two. A higher-degree polynomial can have different orders of vanishing at different competing global minima. Thus order of vanishing must be taken into account in the higher-degree case.

**5. Proof of Theorem 2.3**

We now return to the analysis of the integral  $N$  in (2.2).

$$\begin{aligned}
 &= e^{2\tau b^*(\eta)} \int_{-\infty}^{\infty} e^{2\tau[\eta\lambda - b(\lambda) - B_\eta(\lambda(\eta))]} d\lambda \\
 &= e^{2\tau b^*(\eta)} \int_{-\infty}^{\infty} e^{2\tau[\eta\lambda - b(\lambda) - \eta\lambda(\eta) + b(\lambda(\eta))]} d\lambda \\
 &= e^{2\tau b^*(\eta)} \int_{-\infty}^{\infty} e^{2\tau[-b''(\lambda(\eta))\frac{(\lambda-\lambda(\eta))^2}{2} - b'''(\lambda(\eta))\frac{(\lambda-\lambda(\eta))^3}{6} - \frac{(\lambda-\lambda(\eta))^4}{4}]} d\lambda \\
 &= e^{2\tau b^*(\eta)} \int_{-\infty}^{\infty} e^{-[2\tau b''(\lambda(\eta))\frac{y^2}{2} + 2\tau b'''(\lambda(\eta))\frac{y^3}{6} + 2\tau\frac{y^4}{4}]} dy \\
 &\approx e^{2\tau b^*(\eta)} \left[ \left(\frac{\tau}{2}\right)^{\frac{1}{4}} + \left|\frac{\tau b'''(\lambda(\eta))}{3}\right|^{\frac{1}{3}} + (\tau b''(\lambda(\eta)))^{\frac{1}{2}} \right]^{-1} \\
 &\approx e^{2\tau b^*(\eta)} \left[ \tau^{\frac{1}{4}} + \tau^{\frac{1}{3}} |\lambda(\eta)|^{\frac{1}{3}} + \tau^{\frac{1}{2}} (3\lambda(\eta)^2 + p) \right]^{-1},
 \end{aligned}$$

where we have used the result of the previous section in the second-last line.

We now prove Theorem 2.3. If we show that each integral

$$(5.1) \quad \int \int_{\tau > 0} e^{\eta\tau[z_1 + \bar{w}_1] + i\tau[z_2 - \bar{w}_2]} \frac{\eta^{i_1 + j_1} \tau^{i_1 + j_1 + i_2 + j_2 + 1}}{N(\eta, \tau)} d\eta d\tau$$

is absolutely convergent when

$$h + k + b(x) + b(r) - 2b^{**}\left(\frac{x+r}{2}\right) > 0,$$

it will follow that the integral in fact is equal to  $\partial_{z_1}^{i_1} \partial_{\bar{w}_1}^{j_1} \partial_{z_2}^{i_2} \partial_{\bar{w}_2}^{j_2} S(z, w)$ .

Set  $\delta = h + k$ ,  $(z_1, z_2, w_1, w_2) = (x + iy, t + ib(x) + ih, r + is, u + ib(r) + ik)$ ,  $i_1 + j_1 = n$ ,  $i_1 + j_1 + i_2 + j_2 = m$  (so that  $m \geq n$ ). The integral becomes

$$(5.2) \quad S^{n,m,\delta} := \int \int_{\tau > 0} e^{\eta\tau[x+r+i(y-s)] + i\tau[t-u+i(b(x)+b(r)+\delta)]} \frac{\eta^n \tau^{m+1}}{N(\eta, \tau)} d\eta d\tau,$$

which converges absolutely if and only if

$$(5.3) \quad \tilde{S}^{n,m,\delta} := \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\tau[\delta + b(x) + b(r) - \eta(x+r)]} \frac{|\eta|^n \tau^{m+1}}{N(\eta, \tau)} d\tau d\eta < \infty.$$

We see that

$$\begin{aligned} \tilde{\mathcal{S}}^{n,m,\delta} &\approx \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)+2b^*(\eta)]} \\ &\quad \times \left[ \tau^{\frac{1}{4}} + \tau^{\frac{1}{3}} |\lambda(\eta)|^{\frac{1}{3}} + \tau^{\frac{1}{2}} (3\lambda(\eta)^2 + p)^{\frac{1}{2}} \right] |\eta|^n \tau^{m+1} d\tau d\eta. \\ &:= \mathcal{I}_1^{n,m,\delta} + \mathcal{I}_2^{n,m,\delta} + \mathcal{I}_3^{n,m,\delta}. \end{aligned}$$

(Since the superscripts are cumbersome, we will often omit them.) Furthermore, let  $\mathcal{I}_i(\eta)$  denote the integrand of the  $\eta$  integral defining  $\mathcal{I}_i$ , so that

$$\mathcal{I}_i = \int_{-\infty}^{\infty} \mathcal{I}_i(\eta) d\eta.$$

Set

$$(5.4) \quad A(x, r, \eta) = b(x) + b(r) - \eta(x + r) + 2b^*(\eta).$$

Since

$$(5.5) \quad A(x, r, \eta) = \sup_{\lambda} [\eta\lambda - b(\lambda)] - [\eta x - b(x)] + \sup_{\lambda} [\eta\lambda - b(\lambda)] - [\eta r - b(r)],$$

$A$  is non-negative.

Each  $\mathcal{I}_i(\eta)$  involves an integral in  $\tau$  of the form

$$(5.6) \quad \int_0^{\infty} e^{-\tau[\delta+A(x,r,\eta)]} \tau^a d\tau$$

which equals

$$(5.7) \quad c_a \frac{1}{[\delta + A(x, r, \eta)]^{a+1}} \quad \text{if } \delta + A(x, r, \eta) > 0.$$

It is now clear that there are two potential barriers to the convergence of the full integrals  $\mathcal{I}_i$ :

- (1) insufficient growth of  $A$  in  $\eta$  at infinity, and
- (2) vanishing of  $\delta + A(x, r, \eta)$  for some finite  $\eta$  for certain choices of  $x, r, \delta$ .

The next subsections explore these issues and in so doing establish the theorem.

**5.1. Behavior of  $A(x, r, \eta)$  for large  $\eta$ .**

LEMMA 5.1. *Fix  $x, r \in \mathbb{R}$ . Then  $A(x, r, \eta) \sim \frac{3}{2}\eta^{\frac{4}{3}}$  as  $|\eta| \rightarrow \infty$ .*

*Proof.* Recall from Proposition 3.3 that  $\lambda(\eta) \sim \eta^{\frac{1}{3}}$  as  $|\eta| \rightarrow \infty$ , i.e.,  $\lambda(\eta) = \eta^{\frac{1}{3}}(1 + o(1))$  as  $|\eta| \rightarrow \infty$ . Thus as  $|\eta| \rightarrow \infty$ ,

$$\begin{aligned} A(x, r, \eta) &= b(x) + b(r) - \eta(x + r) + 2(\eta\lambda(\eta) - b[\lambda(\eta)]) \\ &= b(x) + b(r) - \eta(x + r) \\ &\quad + 2 \left[ \eta^{\frac{4}{3}}(1 + o(1)) - \frac{1}{4}\eta^{\frac{4}{3}}(1 + o(1))^4 \right] \end{aligned}$$

$$\begin{aligned}
 & \left. - \frac{1}{2} p \eta^{\frac{2}{3}} (1 + o(1))^2 - q \eta^{\frac{1}{3}} (1 + o(1)) \right] \\
 &= \frac{3}{2} \eta^{\frac{4}{3}} + \eta^{\frac{4}{3}} o(1) + O(|\eta|) \\
 &= \frac{3}{2} \eta^{\frac{4}{3}} (1 + o(1)). \quad \square
 \end{aligned}$$

REMARK 5.2. Our arguments can be extended to obtain a generalized asymptotic expansions for  $\lambda(\eta)$  and  $A(x, r, \eta)$ . See Olver [Olv97], Section 1.5, for a detailed discussion of such techniques.

This lemma, equation (5.7), and parts (iii) and (v) of Proposition 3.3 allow us to conclude the following:

- (1)  $\mathcal{I}_1^{n,m,\delta}(\eta) \sim c(\eta^{\frac{4}{3}})^{-(\frac{5}{4}+m+1)} |\eta|^n = c|\eta|^{-3-\frac{4}{3}m+n}$ . Since  $m \geq n \geq 0$ ,  $-3 - \frac{4}{3}m + n \leq -3$ , and so for any fixed  $n, m$ , and  $\delta$ ,  $\mathcal{I}_1^{n,m,\delta}$  is convergent at infinity.
- (2)  $\mathcal{I}_2^{n,m,\delta}(\eta) \sim c(\eta^{\frac{4}{3}})^{-(\frac{4}{3}+m+1)} \cdot |\eta|^{\frac{1}{3}|\frac{1}{3}} = c|\eta|^{-3-\frac{4}{3}m+n}$ , and so each  $\mathcal{I}_2^{n,m,\delta}$  is convergent at infinity.
- (3)  $\mathcal{I}_3^{n,m,\delta}(\eta) \sim c(\eta^{\frac{4}{3}})^{-(\frac{3}{2}+m+1)} \cdot |\eta|^{\frac{1}{3}} = c|\eta|^{-3-\frac{4}{3}m+n}$ , and so each  $\mathcal{I}_3^{n,m,\delta}$  is convergent at infinity.

**5.2. Vanishing of  $\delta + A(x, r, \eta)$ .** The estimates of the previous sections show that whether or not the integrals  $\mathcal{I}_i$  converge depends upon whether or not for some fixed  $x, r$ , and  $\delta$  the function  $\eta \mapsto \delta + A(x, r, \eta)$  vanishes for some finite  $\eta_0$  and, if so, the behavior of this function near such a point. In particular, we have proved the following proposition.

PROPOSITION 5.3. *If for some  $x, r$ , and  $\delta$  fixed*

$$\inf_{\eta} [\delta + A(x, r, \eta)] > 0,$$

*then each  $\mathcal{I}_i^{n,m,\delta}$  is finite.*

Note, moreover, that

$$\begin{aligned}
 \inf_{\eta} [\delta + A(x, r, \eta)] &= \delta + b(x) + b(r) - 2 \sup_{\eta} \left[ \eta \left( \frac{x+r}{2} \right) - b^*(\eta) \right] \\
 &= \delta + b(x) + b(r) - 2b^{**} \left( \frac{x+r}{2} \right).
 \end{aligned}$$

(The convexity of  $b^*$  guarantees the finiteness of the supremum in the first line.) We have thus proved that the integrals defining the Szegő kernel and all its derivatives converge absolutely in the region

$$(5.8) \quad \delta + b(x) + b(r) - 2b^{**} \left( \frac{x+r}{2} \right) > 0.$$

We do not yet know which  $x, r,$  and  $\delta$  are in this set. We claim first that if  $z = (z_1, z_2) = (x + iy, t + ib(x) + ih) \in \Omega$  and  $w = (w_1, w_2) = (r + is, t + ib(r) + ik) \in \Omega,$   $(z, w)$  is in the region in  $\mathbb{C}^2$  defined by (5.8). Indeed,  $z, w \in \Omega$  implies  $h, k > 0,$  and hence  $\delta = h + k > 0.$  It follows that  $\delta + A(x, r, \eta) \geq \delta,$  and hence its infimum over  $\eta$  is bounded below by  $\delta$  as well. Thus the inequality in (5.8) is satisfied.

To prove the remainder of Theorem 2.3, we must determine which  $(z, w) \in \partial\Omega \times \partial\Omega$  are in the region (5.8). For such  $(z, w), \delta = 0.$  We thus need to determine all (fixed)  $x$  and  $r$  for which  $A(x, r, \eta)$  is bounded away from zero independent of  $\eta.$

By (5.5),  $A$  is a sum of two non-negative functions

$$(5.9) \quad A_x(\eta) := \sup_{\lambda}(\eta\lambda - b(\lambda)) - (\eta x - b(x)) = b^*(\eta) - (\eta x - b(x)),$$

$$(5.10) \quad A_r(\eta) := \sup_{\lambda}(\eta\lambda - b(\lambda)) - (\eta r - b(r)) = b^*(\eta) - (\eta r - b(r)).$$

Thus for fixed  $x$  and  $r, A$  vanishes at some  $\eta_0$  if and only if both  $A_x(\eta_0)$  and  $A_r(\eta_0)$  vanish. Furthermore, by Corollary 3.5,  $\eta \rightarrow A(x, r, \eta)$  is continuous, and by Lemma 5.1,  $A(x, r, \eta) \sim c\eta^{\frac{4}{3}}$  as  $|\eta| \rightarrow \infty.$  Thus if for some fixed  $x$  and  $r, A(x, r, \cdot)$  never vanishes, it is bounded below by a positive constant for all  $\eta.$  We thus identify  $(z, w)$  in the region (5.8) by identifying pairs  $x$  and  $r$  for which  $A(x, r, \cdot)$  never vanishes.

*Case 1:*  $|x| < \sqrt{-p}$  or  $|r| < \sqrt{-p}.$  For definiteness, suppose  $|x| < \sqrt{-p}.$   $A_x$  could only vanish if  $x$  were such that, for some value of  $\eta,$  the supremum of  $B_{\eta}(\lambda) = \eta\lambda - b(\lambda)$  were achieved at  $x.$  But Proposition 3.3 shows that the supremum of  $B_{\eta}$  is always achieved at one or more points *outside of*  $(-\sqrt{-p}, \sqrt{-p}).$  This completes the proof in this case.

*Case 2:*  $|x|, |r| > \sqrt{-p},$  and  $x \neq r.$  Since the map  $\eta \mapsto \lambda(\eta)$  maps  $\mathbb{R} \setminus \{q\}$  onto  $\mathbb{R} \setminus [-\sqrt{-p}, \sqrt{-p}]$  and is injective, there exists a unique  $\eta_1 \neq q$  and a unique  $\eta_2 \neq q$  such that  $\lambda(\eta_1) = x$  and  $\lambda(\eta_2) = r.$  Since  $x \neq r, \eta_1 \neq \eta_2.$  It follows that in this case  $A(x, r, \cdot)$  never vanishes.

*Case 3:*  $|x| = \sqrt{-p}$  but  $|r| > \sqrt{-p}.$  (A symmetric argument covers the case  $|r| = \sqrt{-p}$  but  $|x| > \sqrt{-p}.)$  Then  $A_x(\eta) = 0$  only at  $\eta = q$  whereas one easily computes that  $A_r(q) = \frac{1}{4}(r^2 + p)^2 > 0.$  Thus,  $A(x, r, \cdot)$  does not vanish.

This completes the proof of Theorem 2.3. □

### 6. Proof of Theorem 2.5

We begin by observing that

$$\begin{aligned} & S[(x, i(b(x) + h)), (r, i(b(r) + k))] \\ &= c \int \int_{\tau > 0} \tau e^{\eta\tau(x+r) - \tau[b(x)+b(r)+h+k]} N(\eta, \tau)^{-1} d\eta d\tau \\ &= \tilde{S}^{0,0,\delta}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}[(x, 0, 0), (r, 0, 0)] &= c \int_{\tau>0} \tau e^{\eta\tau(x+r) - \tau[b(x)+b(r)]} N(\eta, \tau)^{-1} d\eta d\tau \\ &= \tilde{\mathcal{S}}^{0,0,0}, \end{aligned}$$

where  $\tilde{\mathcal{S}}^{n,m,\delta}$  is as defined in (5.3). We will shorten the notation for these integrals to  $\tilde{\mathcal{S}}^\delta$ . Thus to prove Theorem 2.5, we must show that

- (i)  $\tilde{\mathcal{S}}^0$  is divergent, and
- (ii)  $\lim_{\delta \rightarrow 0^+} \tilde{\mathcal{S}}^\delta = \infty$ .

It is immediately clear that (ii) will follow from (i) since the integrand of  $\tilde{\mathcal{S}}^\delta$  is non-negative and converges pointwise and monotonically to the integrand of  $\tilde{\mathcal{S}}^0$  as  $\delta \rightarrow 0^+$ . Furthermore, (i) will follow if the corresponding statement holds for any of the three integrals  $\mathcal{I}_i^{0,0,0}$  (again, abbreviated  $\mathcal{I}_i^0$ ). We will show that

- (iii)  $\mathcal{I}_1^0$  is divergent.

As we saw in the previous section,  $\mathcal{I}_1^0$  converges if  $x$  and  $r$  are chosen in such a way that  $A(x, r, \cdot)$  never vanishes. Thus in order to establish (iii), we need detailed information about the behavior of  $A$  near values  $\eta_0$  for which  $A(x, r, \eta_0) = 0$ . Recall that if  $A(x, r, \eta) \neq 0$ , (5.7) shows that the integrand of  $\mathcal{I}_1^0$  is comparable to

$$(6.1) \quad [A(x, r, \eta)]^{-\frac{9}{4}}.$$

We prove (iii) by considering the behavior of  $A$  in three subcases.

**6.1. Case 1:  $x = r$  and  $|x| > \sqrt{-p}$ .** In this case, there exists a unique  $\eta_0 \neq q$  such that  $x = r = \lambda(\eta_0)$ .

Suppose  $\eta \neq \eta_0$  and recall that  $\eta = [\lambda(\eta)]^3 + p\lambda(\eta) + q$  and  $\eta_0 = x^3 + px + q$  so that

$$(6.2) \quad \eta_0 - \eta = (x - \lambda(\eta))(x^2 + x\lambda(\eta) + [\lambda(\eta)]^2 + p).$$

Then (suppressing the dependence of  $\lambda$  on  $\eta$ )

$$\begin{aligned} A(x, x, \eta) &= 2A_x(\eta) \\ &= 2 \left[ \eta\lambda - \frac{1}{4}\lambda^4 - \frac{p}{2}\lambda^2 - q\lambda - \eta x + \frac{1}{4}x^4 + \frac{p}{2}x^2 + qx \right] \\ &= 2(x - \lambda) \left[ \frac{1}{4}(x + \lambda)(x^2 + \lambda^2) + \frac{p}{2}(x + \lambda) - (\eta - q) \right] \\ &= 2(x - \lambda) \left[ \frac{1}{4}(x + \lambda)(x^2 + \lambda^2) + \frac{p}{2}(x + \lambda) - \lambda^3 - p\lambda \right] \\ &= \frac{1}{2}(x - \lambda)^2(x^2 + 2\lambda x + 3\lambda^2 + 2p). \end{aligned}$$

We are concerned with how this function varies with  $\eta$ . We have the following proposition and corollary.

PROPOSITION 6.1. For  $|x| > \sqrt{-p}$  fixed,

(a)

$$x^2 + x\lambda(\eta) + [\lambda(\eta)]^2 + p \geq \begin{cases} -2p, & |x| \geq 2\sqrt{-p}, \\ |x|(|x| - \sqrt{-p}), & \sqrt{-p} < |x| < 2\sqrt{-p}. \end{cases}$$

(b)

$$x^2 + 2x\lambda(\eta) + 3[\lambda(\eta)]^2 + 2p \geq \begin{cases} -4p, & |x| \geq 3\sqrt{-p}, \\ (|x| - \sqrt{-p})^2, & \sqrt{-p} < |x| < 3\sqrt{-p}. \end{cases}$$

Thus both expressions are bounded below by a positive constant independent of  $\eta$ .

*Proof.* Recall that  $|\lambda(\eta)| \geq \sqrt{-p}$ . Our task in part (a) is thus to find the global minimum of  $f(\lambda) = x^2 + x\lambda + \lambda^2 + p$  on  $\{\lambda : |\lambda| \geq \sqrt{-p}\}$ . There are two cases to consider depending on whether  $f$  attains its minimum at a critical point or at  $\lambda = \pm\sqrt{-p}$ .

Observe that  $f'(\lambda) = x + 2\lambda = 0$  when  $\lambda = -\frac{1}{2}x$ . If  $\frac{1}{2}|x| \geq \sqrt{-p}$ , this indeed is the location of the global minimum, which is then seen to be  $\frac{3}{4}x^2 + p \geq -2p$ . If  $\frac{1}{2}|x| < \sqrt{-p}$ , the global minimum is one of the two quantities  $x^2 \pm x\sqrt{-p} - p + p$ , which is in turn  $\geq x^2 - |x|\sqrt{-p} = |x|(|x| - \sqrt{-p})$ . This proves (a). The proof of (b) is similar and is omitted. □

COROLLARY 6.2. If  $|x| > \sqrt{-p}$ ,

$$A(x, x, \eta) \approx (\eta - \eta_0)^2 (1 + |\eta|)^{-\frac{2}{3}}.$$

*Proof.* By Proposition 6.1, we may write

$$A(x, x, \eta) = (\eta - \eta_0)^2 \frac{x^2 + 2x\lambda(\eta) + 3[\lambda(\eta)]^2 + 2p}{2(x^2 + x\lambda(\eta) + [\lambda(\eta)]^2 + p)^2} := (\eta - \eta_0)^2 g(\eta).$$

The proposition shows that  $g$  is finite for all  $\eta$  and bounded away from zero. Thus for  $\eta$  on any fixed interval  $[-K, K]$ ,  $g(\eta) \approx 1$ . On the other hand, Proposition 3.3 shows that  $\lambda(\eta) \sim \eta^{\frac{1}{3}}$  as  $|\eta| \rightarrow \infty$ , and so  $g(\eta) \approx |\eta^{\frac{1}{3}}|^{-2}$  for  $|\eta| > K$ . Thus for all  $\eta$ ,  $g(\eta) \approx (1 + |\eta|)^{-\frac{2}{3}}$ . □

By (6.1),

$$(6.3) \quad \mathcal{I}_1^0(\eta) \approx [(\eta - \eta_0)^2 (1 + |\eta|)^{-\frac{2}{3}}]^{-\frac{9}{4}},$$

and thus  $\mathcal{I}_1^0$  is divergent, establishing (iii) in this case.

**6.2. Case 2:**  $x = r$  and  $|x| = \sqrt{-p}$ . Here, the analysis is slightly more delicate because the discontinuity of  $\lambda$  occurs at  $\eta = q$ , and one of  $\pm x - \lambda(\eta)$  vanishes at  $\eta = q$ . In this situation, the analogue of (6.2) above is the relationship

$$(6.4) \quad \eta - q = \lambda(\eta) [\lambda(\eta) - x] [\lambda(\eta) + x].$$

Furthermore, since  $x^2 = -p$ , in this case  $A(x, x, \eta) = \frac{1}{2}(x - \lambda)^2(3\lambda - x)(\lambda + x)$ .

PROPOSITION 6.3. *Let A be as above.*

1. *If  $x = \sqrt{-p}$ , then for  $\eta > q$ ,*

$$A(x, x, \eta) \approx (\eta - q)^2(1 + |\eta|)^{-\frac{2}{3}}.$$

2. *If  $x = -\sqrt{-p}$ , then for  $\eta < q$ ,*

$$A(x, x, \eta) \approx (\eta - q)^2(1 + |\eta|)^{-\frac{2}{3}}.$$

*Proof.* In both cases, it is enough to observe that for the values of  $\eta$  indicated, both  $|x + \lambda(\eta)| \geq 2|x| > 0$  and  $|3\lambda(\eta) - x| \geq 2|x| > 0$ . We may thus solve (6.4) for  $x - \lambda$  and substitute into the expression for  $A$ . The estimate then follows, again using the fact (Proposition 3.3) that  $\lambda(\eta) \sim \eta^{\frac{1}{3}}$  as  $|\eta| \rightarrow \infty$ .  $\square$

Thus in this case, as above,  $\mathcal{I}_1^0$  is divergent.

**6.3. Case 3:**  $|x| = |r| = \sqrt{-p}$  but  $x = -r$ . The point here is that although  $x \neq r$ , there is an  $\eta_0$  for which  $B_{\eta_0}$  achieves its global maximum at both  $x$  and  $r$ : when  $\eta = q$  and  $x = \pm\sqrt{-p}$  and  $r = \mp\sqrt{-p}$  (see Proposition 3.3). In this case,  $A_x(\eta)$  vanishes if and only if  $\eta = q$ . For  $\eta \neq q$

$$(6.5) \quad A_{\pm\sqrt{-p}}(\eta) = (\eta - q)(\lambda(\eta) \mp \sqrt{-p}) - \frac{1}{4}([\lambda(\eta)]^2 + p)^2.$$

PROPOSITION 6.4. *If  $|x| = \sqrt{-p}$ ,*

$$A(x, -x, \eta) = (\eta - q)h(\eta),$$

where

$$|h(\eta)| \approx (1 + |\eta|)^{\frac{1}{3}}.$$

*Proof.* It follows from (6.4) that

$$[\lambda(\eta)]^2 + p = \frac{\eta - q}{\lambda(\eta)}$$

and so

$$\begin{aligned} A(\pm\sqrt{-p}, \mp\sqrt{-p}, \eta) &= 2(\eta - q) \left[ \lambda(\eta) - \frac{[\lambda(\eta)]^2 + p}{4\lambda(\eta)} \right] \\ &= (\eta - q) \frac{3[\lambda(\eta)]^2 - p}{2\lambda(\eta)} \\ &:= (\eta - q)h(\eta). \end{aligned}$$

Since the numerator of  $h$  is bounded below by  $-4p > 0$  and the denominator is bounded in absolute value away from zero, it follows that if we fix an interval  $[-K, K]$ ,  $|h(\eta)| \approx 1$  on the interval.

On the other hand, since  $\lambda(\eta) \sim \eta^{\frac{1}{3}}$  as  $|\eta| \rightarrow \infty$ , for sufficiently large  $K$ ,

$$|h(\eta)| \approx |\eta|^{\frac{1}{3}} \quad \text{for } |\eta| > K.$$

The proposition follows.  $\square$

Since the integrand of  $\mathcal{I}_1^0$  is  $\approx [|\eta - q|(1 + |\eta|)^{\frac{1}{3}}]^{-\frac{9}{4}}$ ,  $\mathcal{I}_1^0$  diverges in this case as well. The proof of Theorem 2.5 is now complete.

#### REFERENCES

- [BNW88] J. Bruna, A. Nagel and S. Wainger, *Convex hypersurfaces and Fourier transforms*, Ann. of Math. **127** (1988), 333–365. MR 0932301
- [Car05] C. Carracino, *Estimates for the Szegő kernel on a model non-pseudoconvex domain*, Ph.D. thesis, University of Wisconsin, 2005. MR 2707696
- [Car07] C. Carracino, *Estimates for the Szegő kernel on a model non-pseudoconvex domain*, Illinois J. Math. **51** (2007), 1363–1396. MR 2417433
- [Has95] F. Haslinger, *Singularities of the Szegő kernel for certain weakly pseudoconvex domains in  $\mathbb{C}^2$* , J. Funct. Anal. **129** (1995), 406–427. MR 1327185
- [HNW10] J. Halfpap, A. Nagel and S. Wainger, *The Bergman and Szegő kernels near points of infinite type*, Pac. J. Math. **246** (2010), 75–128. MR 2645880
- [Ker72] N. Kerzman, *The Bergman kernel function. Differentiability at the boundary*, Math. Ann. **195** (1972), 149–158. MR 0294694
- [McN94] J. D. McNeal, *Estimates on the Bergman kernels of convex domains*, Adv. Math. **109** (1994), 108–139. MR 1302759
- [MS94] J. D. McNeal and E. M. Stein, *Mapping properties of the Bergman projection on convex domains of finite type*, Duke Math. J. **73** (1994), 177–199. MR 1257282
- [MS97] J. D. McNeal and E. M. Stein, *The Szegő projection on convex domains*, Math. Z. **224** (1997), 519–553. MR 1452048
- [Nag86] A. Nagel, *Vector fields and non-isotropic metrics*, Beijing lectures in harmonic analysis (E. M. Stein, ed.), Princeton University Press, Princeton, NJ, 1986. MR 0864374
- [NRSW89] A. Nagel, J. P. Rosay, E. M. Stein and S. Wainger, *Estimates for the Bergman and Szegő kernels in  $\mathbb{C}^2$* , Ann. of Math. **129** (1989), 113–149. MR 0979602
- [Olv97] F. W. J. Olver, *Asymptotics and special functions*, A. K. Peters, Wellesley, MA, 1997. MR 1429619
- [Ste72] E. M. Stein, *Boundary behavior of holomorphic functions of several complex variables*, Princeton University Press, Princeton, NJ, 1972. MR 0473215

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