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COMPUTATION OF THE KERNELS OF LÉVY FUNCTIONALS AND APPLICATIONS

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Dedicated to Martin Wirsing

Abstract. An effective computation of the kernels of the chaos decomposition of Lévy functionals is used, to prove, among other things, a chain- and product-rule of the Malliavin derivative for a large class of Lévy processes. In case of finite and infinite-dimensional Brownian motion, the well-known rules are obtained, but for Poisson processes, the results are new. The kernels of a Lévy functional can be computed by taking the expected value of the product of this functional and multiple white noise of the Lévy process.

1. Introduction

Chaotic representations of Lévy functionals often serve as basis for the Malliavin calculus for Lévy processes. We refer to the articles [9], [16], [22], [23], [24] and [27] for infinite dimensional Brownian motion, and to [5], [10], [16], [17], [21] and [23] for finite dimensional Lévy processes. Lévy functionals are square integrable random variables and are uniquely determined by a sequence $(f_n)_{n\in\mathbb{N}_0}$ of deterministic functions, where the domain of f_n is the continuous time line $[0,r]^n, r\in\mathbb{R}^+$ or $[0,\infty[^n$.

An effective recipe for the computation of the kernels is used to prove, among other things, a chain- and product-rule of the Malliavin derivative for a large class Lévy processes. In case of finite and infinite-dimensional Brownian motion, the well-known rules are obtained (see Nualart [16]), but for Poisson processes the results and, in any case, the method are new. As far as I know, this recipe has never been used before to prove these rules.

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This recipe can be found in [19] and is an extension of a result, due to Cutland and Ng [4], for finite-dimensional Brownian motion to more general Lévy processes. N. Wiener [26] was the first to develop chaotic representations of Brownian functionals. He used them to obtain better models for telecommunications under Brownian noise (see P. R. Masani [15]). The domain of a Brownian functional φ is the Wiener space $C_{\mathbb{R}}$ of real continuous functions on the continuous time line, endowed with the Wiener measure. Cutland and Ng [4] have pointed out, that it was the intention of Wiener, to think of the kernels f_n of φ as being given by

$$(1) \qquad f_n(t_1,\dots,t_n) = \mathbb{E}(\varphi \dot{b}_{t_1} \cdots \dot{b}_{t_n}),$$

where $\dot{b}_t$ is the derivative of the Brownian motion b at time t. However, $\dot{b}_t$ only exists in the sense of Schwartz distributions (see Walter [25]).

Instead of taking generalized functions, Cutland and Ng [4] use an extension of the notion “finite” in a countably saturated model $\mathfrak{M}$ of mathematics. We refer to the book of S. Albeverio, J. E. Fenstad, R. Høegh-Krohn, T. Lindstrøm [1] for details. Without any loss of generality, Cutland and Ng replace the Wiener space with a *finite (finite in $\mathfrak{M}$) dimensional Euclidean space Ω. The Wiener measure on $C_{\mathbb{R}}$ is replaced with the Loeb measure [13] over the centered Gaußian measure on Ω of infinitely small variance. The continuous time line is replaced with a *finite time line and the Brownian motion b with a *smooth Brownian motion. Cutland and Ng now showed that, using these new entities, Equation (1) becomes a mathematically exact statement.

The literature is full of nice applications of finitization of topics in stochastic analysis. The pioneers are Loeb [13], Anderson [2], Keisler [8], Lindstrøm [11], Hoover and Perkins [7], ...

In our paper finitization is used, to compute the kernels for quite general Lévy functionals, in particular for the product of two Lévy functionals, in order to obtain the product- and chain-rule in standard terms. Infinite dimensional Brownian motion in the context of abstract Wiener spaces (see Gross [6]) and Poisson processes are included, and, of course, finite dimensional Brownian motion.

An abstract Wiener space is a pair $(\mathbb{H},\mathbb{B})$, where $\mathbb{B}$ is a real Banach space and $\mathbb{H}$ is a densely embedded separable Hilbert space. In case of a $\mathbb{B}$-valued Brownian motion, the range of the kernel f_n of a Brownian functional is the n-fold tensor product of $\mathbb{H}$.

If L is a one dimensional Lévy process, then the range of the kernel f_n of an L-functional is a suitable subspace of $\mathbb{R}^{\mathbb{N}_L}$, where $\mathbb{N}_L \subseteq \mathbb{N}$ (see [19]). In case of one dimensional Brownian motion or for Poisson processes $\mathbb{N}_L = \{1\}$, for symmetrized Poisson processes $\mathbb{N}_L = \{1,2\}$ (see [19]). In [19], there are also examples, where $\mathbb{N}_L = \mathbb{N}$.

The computation of the kernels is simple and intuitive. It will be seen that it is also effective. However, its application is rather technical, although, one should better say, because it is constructive. For example, in order to compute the kernels of the product $\varphi \cdot \psi$ of two Lévy functionals φ and ψ, quite complicated finite combinatorics is used. Therefore, for the reader's convenience, we describe the ideas in the much simpler case of, even infinite dimensional, Brownian motion, before going to more general Lévy processes.

Added to proof: In the meantime versions of the results in this article and the theories behind can be found in an introduction to Malliavin calculus ([20]).

2. The main results

First, we present the product- and chain-rules for finite dimensional Lévy processes. Based on Lindstrøm's article [12], it is shown in [19] that each Lévy process L lives on a fixed *finite dimensional sample space Ω; Ω only depends on the dimension of L. The probability measure $\widehat{\mu} = \widehat{\mu}_L$ on Ω characterizes the process L. For simplicity, we study only one-dimensional processes $L : [0, \infty[\times \Omega \to \mathbb{R}$, except for infinite-dimensional Brownian motion, where $\mathbb{R}$ is replaced by any separable Banach space. We assume that L is locally square integrable and equivalent to a process with limited increments. Brownian motion, Poisson processes and many other important Lévy processes (see [19]) meet this demand. It is also fulfilled for truncated Lévy processes.

We take the σ-algebra $\mathcal{D} := \mathcal{D}_L$ on Ω, generated by the Wiener–Lévy integrals, associated to L. Also determined by L, there exists an initial subset $\mathbb{N}_L$ of $\mathbb{N}$ and a sequence $(p_k)_{k \in \mathbb{N}_L}$ of real orthogonal polynomials.

Each square integrable $\mathcal{D}_L$-measurable random variable φ can be expanded to an orthogonal series $\sum_{n=0}^{\infty} I_n(f_n)$ of multiple integrals $I_n(f_n)$, the so called *chaotic representation* of φ. The *kernels* f_n of φ, which are uniquely determined by φ, are deterministic *square summable* real functions, defined on $\mathbb{N}_L^n \times [0, \infty[^n$, that is, $\sum_{k \in \mathbb{N}_L^n} \int_{[0,\infty[^n} f_n^2(k, \cdot)\, d\lambda^n < \infty$, where λ^n is the Lebesgue measure on $[0, \infty[^n$. Moreover, the kernels f_n are *symmetric*, that is, for all permutations σ on $\{1, \ldots, n\}$,

$$f_n(k_1, \ldots, k_n, r_1, \ldots, r_n) = f_n(k_{\sigma_1}, \ldots, k_{\sigma_n}, r_{\sigma_1}, \ldots, r_{\sigma_n}).$$

Using the shorthand $k = (k_1, \ldots, k_n), r = (r_1, \ldots, r_n)$, the integrals $I_n(f_n)$ have the following form (for details, see [19]):

$$I_n(f_n) = \sum_{k \in \mathbb{N}_L^n} \int_{[0,\infty[^n_{\leq}} f_n(k, r)\, dM^{k_1}(r_1, \cdot) \cdots dM^{k_n}(r_n, \cdot),$$

where $(M^k)_{k \in \mathbb{N}_L}$ is a bunch of real square-integrable martingales M^k. Intuitively, the increment $\Delta M^k(r, \cdot)$ of M^k at $r \in [0, \infty[$ is the polynomial p_k

applied to the increment $\Delta L(r,\cdot)$ of L at r. It should be mentioned that for all $k\in\mathbb{N}_L$, the Doleans-measure of M^k is the Lebesgue measure.

Each $\varphi\in L^2_{\mathcal{D}}(\widehat{\mu})$ has the decomposition $\varphi=\sum_{n=0}^{\infty}I_n(f_n)$ ([19]). The Malliavin derivative D is a densely defined operator from the space $L^2_{\mathcal{D}}(\widehat{\mu})$ into the space $L^2(c\otimes\lambda\otimes\widehat{\mu})$ of real square summable stochastic processes, defined on $\mathbb{N}_L\times[0,\infty[\times\Omega$ (see [19]) by

$$(D\varphi)_{l,r}(X)=D\varphi(l,r,X)=\sum_{n\in\mathbb{N}}I_{n-1}\big(f_n(\cdot,l,\cdot,r)\big)(X),$$

if this series converges in $L^2(c\otimes\lambda\otimes\widehat{\mu})$. Then φ is called *Malliavin differentiable.* For each $S\in\mathbb{N}$, define

$$\varphi_S:=\varphi\restriction S:=\sum_{n\in\mathbb{N}_0}I_n(f_n\restriction S),$$

where $(f_n\restriction S)(k,r):=f_n(k,r)$ if $n\le S, |f_n(k,r)|\le S,\ k\in\{1,\dots,S\}^n$ and $r\le S$. Otherwise, $(f_n\restriction S)(k,r):=0$. If $\varphi=\varphi_S$ or $f=f\restriction S$, then we say that φ, f, respectively, are *bounded by S.*

Bounded $\varphi\in L^2_{\mathcal{D}}(\widehat{\mu})$ are Malliavin differentiable. Fix $\varphi\in L^2_{\mathcal{D}}(\widehat{\mu})$. Since I_n is a bounded operator, $\lim_{S\to\infty}\varphi_S=\varphi$ in $L^2_{\mathcal{D}}(\widehat{\mu})$, and, if φ is Malliavin differentiable, then $\lim_{S\to\infty}D\varphi_S=D\varphi$ in $L^2(c\otimes\lambda\otimes\widehat{\mu})$. Here are the main results, where we use constants $\alpha(\kappa,\widetilde{\kappa},l)\in\mathbb{R}$ depending on $\kappa,\widetilde{\kappa},l\in\mathbb{N}_L$. Intuitively,

$$\alpha(\kappa,\widetilde{\kappa},l)=\int_{[0,\infty[\times\Omega}\Delta M_s^{\kappa}\cdot\Delta M_s^{\widetilde{\kappa}}\cdot\Delta M_s^{l}\,d\lambda\otimes\widehat{\mu}(s,X).$$

The precise definition of $\alpha(\kappa,\widetilde{\kappa},l)$ will be given in Equation (2).

THEOREM 2.1 (Product rule). *Let $\mathbb{N}_L$ be finite. Fix Malliavin differentiable $\varphi,\psi\in L^2_{\mathcal{D}}(\widehat{\mu})$ such that $(\mathbb{E}_{\widehat{\mu}}(\varphi_S\cdot\psi_S))_{S\in\mathbb{N}}$ converges in $\mathbb{R}$.*

(A) *Suppose that the sequences $(D\varphi_S\cdot\psi_S)_{S\in\mathbb{N}}$, $(\varphi_S\cdot D\psi_S)_{S\in\mathbb{N}}$ converge in $L^2(c\otimes\lambda\otimes\widehat{\mu})$. Then $(D(\varphi_S\cdot\psi_S))_{S\in\mathbb{N}}$ converges in $L^2(c\otimes\lambda\otimes\widehat{\mu})$ iff $((l,r,X)\mapsto\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L}\alpha(\kappa,\widetilde{\kappa},l)\cdot(D\varphi_S)_{\kappa,r}(X)\cdot(D\psi_S)_{\widetilde{\kappa},r}(X))_{S\in\mathbb{N}}$ converges in $L^2(c\otimes\lambda\otimes\widehat{\mu})$, in which case $\varphi\cdot\psi$ is Malliavin differentiable and*

$$\begin{aligned}\big(D(\varphi\cdot\psi)\big)_{(l,r)}&=(D\varphi)_{(l,r)}\cdot\psi+\varphi\cdot(D\psi)_{(l,r)}\\&\quad+\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L}\alpha(\kappa,\widetilde{\kappa},l)\cdot(D\varphi)_{\kappa,r}\cdot(D\psi)_{\widetilde{\kappa},r}\end{aligned}$$

in $L^2(c\otimes\lambda\otimes\widehat{\mu})$. In case, $\mathbb{N}_L=\{1\}$, we have for $\alpha=\alpha(1,1,1)$

$$\big(D(\varphi\cdot\psi)\big)_r=\frac{(\varphi+\alpha\cdot(D\varphi)_r)\cdot(\psi+\alpha\cdot(D\psi)_r)-\varphi\cdot\psi}{\alpha},\quad \textit{if } \alpha\neq 0,$$

in $L^2(\lambda\otimes\widehat{\mu})$. If $\alpha=0$, then $(D(\varphi\cdot\psi))_r=(D\varphi)_r\cdot\psi+\varphi\cdot(D\psi)_r$.

(B) *Suppose that* $(D(\varphi_S \cdot \psi_S))_{S\in\mathbb{N}}$ *converges in* $L^2(c\otimes\lambda\otimes\widehat{\mu})$. *Then* $\varphi\cdot\psi$ *is Malliavin differentiable and for all* $l\in\mathbb{N}_L$ *we have* $\lambda\otimes\widehat{\mu}$*-a.e.*

$$\big(D(\varphi\cdot\psi)\big)_{l,r} = (D\varphi)_{l,r}\cdot\psi + \varphi\cdot(D\psi)_{l,r} + \sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L}\alpha(\kappa,\widetilde{\kappa},l)\cdot(D\varphi)_{\kappa,r}\cdot(D\psi)_{\widetilde{\kappa},r}.$$

If L is the Brownian motion, then $\mathbb{N}_L=\{1\}$ and $\alpha=0$. For the Poisson process with rate β we have $\mathbb{N}_L=\{1\}$ and $\alpha=\frac{1}{\sqrt{\beta}}$. In case of symmetrized Poisson processes, we have $\mathbb{N}_L=\{1,2\}$ (see [19]).

THEOREM 2.2 (Chain rule). *Suppose that* $\mathbb{N}_L=\{1\}$. *Fix* $g:\mathbb{R}^n\to\mathbb{R}$ *and Malliavin differentiable* $\varphi_1,\dots,\varphi_n$. *Assume that the partial derivatives of* g *exist and that there are polynomials* q_j *in* n *variables with* $\lim q_j = g$ *and* $\lim\partial_i q_j=\partial_i g$ *for* $i=1,\dots,n$.

(A) *Fix* $S\in\mathbb{N}$. *Suppose that* $(D(q_j(\varphi_{1,S},\dots,\varphi_{n,S})))_{j\in\mathbb{N}}$ *converges in* $L^2(\lambda\otimes\widehat{\mu})$ *and* $(\mathbb{E}_{\widehat{\mu}}(q_j(\varphi_{1,S},\dots,\varphi_{n,S})))_{j\in\mathbb{N}}$ *converges in* $\mathbb{R}$.

Then $g(\varphi_{1,S},\dots,\varphi_{n,S})$ *is Malliavin differentiable and* $\lambda\otimes\widehat{\mu}$*-a.e.*,

$$\begin{aligned}&\big(D\big(g(\varphi_{1,S},\dots,\varphi_{n,S})\big)\big)_r\\ &\quad=\frac{1}{\alpha}\big(g\big(\varphi_{1,S}+\alpha\cdot(D\varphi_{1,S})_r,\dots,\varphi_{n,S}+\alpha\cdot(D\varphi_{n,S})_r\big)-g(\varphi_{1,S},\dots,\varphi_{n,S})\big),\end{aligned}$$

where this fraction is equal to $\sum_{i=1}^n(\partial_i g)(\varphi_{1,S},\dots,\varphi_{n,S})\cdot(D\varphi_{i,S})_r$, *if* $\alpha=0$.

(B) *Assume that* (A) *is true for all* $S\in\mathbb{N}$, *and* g *and* $\partial_i g$ *are continuous. Moreover, let* $(D(g(\varphi_{1,S},\dots,\varphi_{n,S})))_{S\in\mathbb{N}}$, $\mathbb{E}_{\widehat{\mu}}(g(\varphi_{1,S},\dots,\varphi_{n,S}))_{S\in\mathbb{N}}$ *converge in* $L^2(\lambda\otimes\widehat{\mu})$, *in* $\mathbb{R}$, *respectively. Then* $g(\varphi_1,\dots,\varphi_n)$ *is Malliavin differentiable and we have* $\lambda\otimes\widehat{\mu}$*-a.e.*

$$\begin{aligned}&\big(D\big(g(\varphi_1,\dots,\varphi_n)\big)\big)_r\\ &\quad=\frac{g(\varphi_1+\alpha\cdot(D\varphi_1)_r,\dots,\varphi_n+\alpha\cdot(D\varphi_n)_r)-g(\varphi_1,\dots,\varphi_n)}{\alpha},\end{aligned}$$

where this fraction is equal to $\sum_{i=1}^n(\partial_i g)(\varphi_1,\dots,\varphi_n)\cdot(D\varphi_i)_r$, *if* $\alpha=0$.

In the terminology of Nualart and Schoutens [17], the densely defined operator $D(\cdot)_l$ from $L^2_{\mathcal{D}}(\widehat{\mu})$ into $L^2_{\mathcal{D}}(\lambda\otimes\widehat{\mu})$ is called the *partial derivative for* $l\in\mathbb{N}_L$.

In the work of G. Di Nunno, Th. Meyer-Brandis, B. Øksendal, F. Proske [5] on pure jump processes, the product rule has the form

$$\big(D(\varphi\cdot\psi)\big)_l=(D\varphi)_l\cdot\psi+\varphi\cdot(D\psi)_l+D(\varphi)_l\cdot D(\psi)_l.$$

In case of the chain rule, they prove a corresponding formula via Wick product similar to the formula above for $\alpha=0$. Moreover, in that work and also in the work of J. L. Solé, F. Utzet, J. Vives [21] the set $\mathbb{R}\times[0,\infty[\times\Omega$ is the domain of the Malliavin derivative $D\varphi$ of a Lévy functional φ, where the measure on $\mathbb{R}$ depends on the Lévy process.

In our approach $D\varphi$ is defined on $\mathbb{N}_L \times [0,\infty[\times \Omega$, the measure on $\mathbb{N}_L \times [0,\infty[$ is the product of the counting measure on $\mathbb{N}_L$ and Lebesgue measure on $[0,\infty[$. Later on, we briefly refer to the work of Nualart and Schoutens [17]. They take the power jump processes of a Lévy process to prove a chaos representation result for Lévy functionals. J. A. Léon, J. L. Solé, F. Utzet, J. Vives [10] use their approach to define the directional Malliavin derivative and the directional Skorohod integral.

3. Preliminaries

To keep this paper self-contained to a large extent, we recall some relevant notions in [19]. We are working in a countably saturated model $\mathfrak{M}$ of mathematics, where we have a strict extension ${}^*A$ of each infinite standard set A. However, we can work in $\mathfrak{M}$ as it is common practice in mathematics. For example, the extensions ${}^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*<$ of $\mathbb{R}, +, \cdot, <$ together with $0, 1$ build an ordered field.

It should be mentioned that not each subset of ${}^*A$ is a set in $\mathfrak{M}$. The internal subsets of ${}^*A$ (internal in $\mathfrak{M}$) include the finite subsets and have nice closure properties: each subset of ${}^*A$, which can be defined by using only internal objects, is again internal. For details and for undefined notations in this article, consult the book of S. Albeverio, J. E. Fenstad, R. Høegh Krohn, T. Lindstrøm [1] or the book of P. Loeb and M. Wolff [14]. Recall that $a \in {}^*\mathbb{R}$ is called *limited*, if $|a| < n$ for some $n \in \mathbb{N}$. For limited a there exists a uniquely determined standard number, denoted by ${}^\circ a$, which is *infinitely close* to a, that is, the difference of a and ${}^\circ a$ is smaller than any positive standard real number.

Fix an unlimited $H \in {}^*\mathbb{N}$ such that, for technical reasons, each $n \in \mathbb{N}$ divides H and for $t \leq H$ let $T_t := \{\frac{i}{H} \mid i \in {}^*\mathbb{N}, \frac{i}{H} \leq t\}$. Set $T := T_H$. On T_t^n, $n \in \mathbb{N}$, we take the counting measure ν^n with

$$\nu_t^n(A) := \frac{|A|}{H^n}$$

for all internal subsets $A \subseteq T_t^n$, where $|A|$ denotes the internal number of elements of T_t^n. Let $(T_t^n, L_{\nu_t^n}, \widehat{\nu_t^n})$ denote the Loeb space over (T_t^n, ν_t^n). It is a nonfinite measure space iff t is unlimited. However, the set of all $(t_1, \ldots, t_n) \in T_t^n$ such that all the ith components t_i for some $i \in \{1, \ldots, n\}$ are unlimited is a $\widehat{\nu_t^n}$-nullset. For unlimited t the set T_t can be seen as an infinitely fine partition of $[0,\infty[$. However, T_t is finite in the sense of $\mathfrak{M}$, $|T_t| = t \cdot H$. Set

$$T_<^n := \{t \in T^n \mid t_1 < \cdots < t_n\}, \qquad T_{\neq}^n := \{t \in T^n \mid t_i \neq t_j \text{ for } i \neq j\}.$$

It is known that the *standard part map* $st : (t_1, \ldots, t_n) \mapsto ({}^\circ t_1, \ldots, {}^\circ t_n)$ is $\widehat{\nu^n}$-a.s. well defined and a measure preserving map from T^n onto $[0,\infty[^n$, where $[0,\infty[^n$ is endowed with the Lebesgue measure λ^n. Let $\mathcal{L}^n \subseteq L_{\nu^n}$ be the σ-algebra, generated by the standard part map, augmented by the $\widehat{\nu^n}$-nullsets, and let for $p \in [0,\infty]$ $L^p_{\mathcal{L}^n}(\widehat{\nu^n})$ be the space of $\mathcal{L}^n$-measurable functions in $L^p(\widehat{\nu^n})$. Then the L^p-spaces $L^p(\lambda^n)$ and $L^p_{\mathcal{L}^n}(\widehat{\nu^n})$ can be identified, because

$$\iota : L^p(\lambda^n) \to L^p_{\mathcal{L}^n}(\widehat{\nu^n}), \qquad \iota(\varphi)(t_1,\ldots,t_n) := \varphi({}^\circ t_1,\ldots,{}^\circ t_n)$$

defines a *canonical*, that is, basis independent, isometric isomorphism between both L^p-spaces. It is more comfortable to work with $L^p_{\mathcal{L}^n}(\widehat{\nu^n})$ than with $L^p(\lambda^n)$, because $\widehat{\nu^n}$ is closely linked to the counting measure ν^n in the following sense: fix $r \in \mathbb{N}$ and $B \in L_{\nu_r^n}$. Then there exists an internal $A \subseteq T_r^n$ such that the symmetric difference of A and B is a $\widehat{\nu_r^n}$-nullset. It follows that $\widehat{\nu_r^n}(B) \approx \nu_r^n(A)$. It should be mentioned that the full Loeb σ-algebra L_{ν^n} is much larger than $\mathcal{L}^n$.

Let $(\Lambda,\mathcal{C},\rho)$ be a measure space, let g, defined on $T^n \times \Lambda$, be $\mathcal{L}^n \otimes \mathcal{C}$-measurable and let f, defined on $[0,\infty[^n \times \Lambda$, be *Leb* $\otimes\, \mathcal{C}$-measurable. Then f and g are called *equivalent*, if $g(t_1,\ldots,t_n,x) = f({}^\circ t_1,\ldots,{}^\circ t_n,x)$ for $\widehat{\nu^n} \otimes \rho$-almost all $(t_1,\ldots,t_n,x)$. Equivalent functions can be identified.

Using the work of Lindstrøm's [12], it is shown in [19] that all (for simplicity one-dimensional) Lévy processes are determined by internal Borel probability measures μ^1 on ${}^*\mathbb{R}$. Let μ be the H^2-fold product of μ^1 on $\Omega := {}^*\mathbb{R}^T$. Let $(\Omega, L_\mu(\mathcal{B}), \widehat{\mu})$ denote the Loeb space over $(\Omega,\mathcal{B},\mu)$, where $\mathcal{B}$ is the internal Borel algebra on the H^2-dimensional Euclidean space Ω. On $\mathcal{B}$ we take the *canonical filtration* $(\mathcal{B}_t)_{t\in T}$, i.e., $\mathcal{B}_t$ is unable to distinguish X from Y in Ω if $X_s = Y_s$ for all $s \le t$. Often we use the filtration $(\mathcal{B}_{t^-})_{t\in T}$ with $t^- := t - \frac{1}{H}$, where $\mathcal{B}_0 := \{\Omega,\emptyset\}$. The conditional expectation with respect to $\mathcal{B}_t$ is denoted by $\mathbb{E}^{\mathcal{B}_t}$.

We assume that the Lévy processes, determined by μ^1, are equivalent to processes with bounded increments. This condition is true for Brownian motion, Poisson processes and many other important Lévy processes. In [19], the reader can find the details and some examples.

While Nualart and Schoutens [17] orthonormalize the full power jump process of the Lévy process, in [19] we have orthonormalized the increments and obtain internal sequences $(p_k)_{k\in\mathbb{N}_L\cup\{0\}}$ of orthogonal polynomials p_k in the internal space $L^2(\mu^1)$ with $p_0 = 1$ and $\mathbb{E}_{\mu^1} p_k^2 = \frac{1}{H}$ for $k \ge 1$. It is assumed that $\mathbb{N}_L$ is an initial segment of $\mathbb{N}$, depending on the Lévy process L. In [19], we have written $\mathbb{N}_{\circ L}$ instead of $\mathbb{N}_L$. The following terms are crucial: set for $l,\kappa,\widetilde{\kappa} \in \mathbb{N}_L$

$$\sigma(\kappa,\widetilde{\kappa},l) := H\mathbb{E}_{\mu^1} p_\kappa \cdot p_{\widetilde{\kappa}} \cdot p_l \quad \text{and} \quad \alpha(\kappa,\widetilde{\kappa},l) := {}^\circ\sigma(\kappa,\widetilde{\kappa},l). \tag{2}$$

A slight modification (see [19]) of the Loeb–Anderson lifting theorem (see [2], [13]) is used: There exists a *finite extension M_L of $\mathbb{N}_L$ such that any measurable $f : \mathbb{N}_L^m \times [0,\infty[^n \times \Omega \to \mathbb{R}$ has a *lifting* $F : M_L^m \times T^n \times \Omega \to {}^*\mathbb{R}$, i.e., $F_k(t,\cdot)$ is $\mathcal{B}$-measurable and $F_k(t,X) \approx f_k({}^\circ t,X)$ for all $k \in \mathbb{N}_L^m$ and $\widehat{\nu^n \otimes \mu}$-almost all (t,X). Then we call f the *standard part* of F, denoted by ${}^\circ F$. The function

F is called *S-square summable* if ${}^\circ\sum_{k\in M_L^m}\|F_k\|^2_{\nu^n\otimes\mu}=\sum_{k\in\mathbb{N}_L^m}\|{}^\circ F_k\|^2_{\lambda^n\otimes\widehat{\mu}}$, in which case we write $F\in SL^2(c^m\otimes\nu^n\otimes\mu)$. If $m=0$, then F is called *S-square integrable.*

4. Infinite-dimensional Brownian motion

In the case of finite dimensional Brownian motion, $\mathbb{N}_L=\{1\}$ and $\alpha=\alpha(1,1,1)=0$. Therefore, the rules for Brownian motion are much simpler than they are for more general Lévy processes, even in the infinite-dimensional case. For the reader's convenience, we first give priority to the computation of the kernels in the chaos decomposition of Gaußian functionals in the infinite dimensional case.

Fix an infinite dimensional separable Hilbert space $\mathbb{H}$. The n-fold tensor product $\mathbb{H}^{\otimes n}$ of $\mathbb{H}$ is the Hilbert-space of continuous multilinear maps $f:\mathbb{H}^n\to\mathbb{R}$ with $\sum_{i_1,\dots,i_n\in\mathbb{N}}f^2(\mathfrak{b}_{i_1},\dots,\mathfrak{b}_{i_1})<\infty$, where $(\mathfrak{b}_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}$. For $n=1$ $\mathbb{H}^{\otimes 1}$ is the topological dual space $\mathbb{H}'=\mathbb{H}$ of $\mathbb{H}$. Set $\mathbb{H}^{\otimes 0}:=\mathbb{R}$. The scalar product on $\mathbb{H}^{\otimes n}$ is denoted by $\langle f,g\rangle$ and the norm by $\|\cdot\|$. In [18], there is a *finite-dimensional representation $\mathbb{F}$ of $\mathbb{H}$. This means the following.

There is a finite-dimensional (in the sense of $\mathfrak{M}$) linear space $\mathbb{F}$ and an embedding $*$ from $\mathbb{H}^{\otimes n}$ into $\mathbb{F}^{\otimes n}$ such that for all $f,g\in\mathbb{H}^{\otimes n}$,

$$\langle f,g\rangle\approx\sum_{i_1,\dots,i_n\le\omega}{}^*f\cdot{}^*g(\mathfrak{e}_{i_1},\dots,\mathfrak{e}_{i_1})=:\langle{}^*f,{}^*g\rangle,$$

where $\mathfrak{E}:=(\mathfrak{e}_i)_{i\in\omega}$ is an internal orthonormal basis of $\mathbb{F}$. The norm on $\mathbb{F}^{\otimes n}$ is also denoted by $\|\cdot\|$. Set $\mathbb{F}^{\otimes 0}:={}^*\mathbb{R}$. If $f\in\mathbb{H}^{\otimes n}$ and $F\in\mathbb{F}^{\otimes n}$, then we define $f\approx_{\mathbb{F}^n}F\Leftrightarrow\|{}^*f-F\|\approx 0$, in which case we call f *the standard part of* F, denoted by ${}^\circ F$. Note that $\|{}^*f-F\|^2=\sum_{i\in\omega^n}({}^*f-F)^2(\mathfrak{e}_{i_1},\dots,\mathfrak{e}_{i_n})$.

Set $\Omega:=\mathbb{F}^T$, that is, we now replace ${}^*\mathbb{R}$ in Section 3 by the ω-dimensional $\mathbb{F}$ and $\mathbb{R}$ by $\mathbb{H}$. Then Ω is an $H^2\cdot\omega$-dimensional Euclidean space. Let $\gamma^1:=\gamma^1_{\frac{1}{H}}$ be the centered Gaußian measure on ${}^*\mathbb{R}$ of variance $\frac{1}{H}$. This measure was introduced by Cutland [3]; the density is infinitely close to the Dirac δ-function, but a smooth function.

The internal probability measure γ on the internal Borel algebra $\mathcal{B}$ of Ω is the $H^2\cdot\omega$-fold product of γ^1. With the shorthand $x_{s,i}:=\langle X_s,\mathfrak{e}_i\rangle$ we have for all $B\in\mathcal{B}$,

$$\gamma(B):=\gamma(B^{\mathfrak{E}}):=\int_{B^{\mathfrak{E}}}e^{-\frac{H}{2}\sum_{s\in T,i\in\omega}x^2_{s,i}}\,d(x_{s,i})_{s\in T,i\in\omega}\cdot\sqrt{\frac{H}{2\pi}}^{H^2\cdot\omega},$$

where $B^{\mathfrak{E}}:=\{(x_{s,i})_{s\in T,i\le\omega}\mid(\sum_{i=1}^{\omega}x_{s,i}\cdot\mathfrak{e}_i)_{s\in T}\in B\}$. Note that $\gamma(B)$ does not depend on the orthonormal basis of $\mathbb{F}$.

Let $\mathbb{B}$ be an abstract Wiener space over $\mathbb{H}$. Then $\mathbb{B}$ is the Banach space completion of $\mathbb{H}$ with respect to a *Gross measurale norm* $|\cdot|$ on $\mathbb{H}$, which means in nonstandard terms: for all internal subspaces $E \subseteq \mathbb{F}$ with $E \perp {}^*a$ for all $a \in \mathbb{H}$,

$$\widehat{\gamma_1}\big\{x \in E \mid |x| \not\approx 0\big\} = 0,$$

where $\widehat{\gamma_1}$ is the Loeb measure over the internal Gaußian measure of variance 1 on E. The dual space $\mathbb{B}'$ of $\mathbb{B}$ is a dense subspace of $\mathbb{H}' = \mathbb{H}$ in the original norm on $\mathbb{H}$. It is well known that each separable Banach space appears as an abstract Wiener space (see Kuo [9]).

It follows from [18] that there exists a continuous Brownian motion $b_{\mathbb{B}} : \Omega \times [0,\infty[\to \mathbb{B}$ for any abstract Wiener space $\mathbb{B}$ over $\mathbb{H}$, defined by

$$(3) \qquad b_{\mathbb{B}}\big(X, {}^\circ t\big) = {}^\circ B(X,t) \quad \text{with } B(X,t) := \sum_{s \leq t} X_s = \sum_{s \leq t, i \leq \omega} \langle X_s, \mathfrak{e}_i \rangle \mathfrak{e}_i$$

for $\widehat{\gamma}$-almost all $X \in \Omega$ and all limited $t \in T$, where $B(X,t) \in \mathbb{F}$ is fixed and ${}^\circ B(X,t)$ is the standard part of $B(X,t)$ in the topology of $\mathbb{B}$. Note that B is an internal discrete Brownian motion.

The probability space Ω is very rich, although it is finite dimensional: Ω only depends on $\mathbb{H}$ and not on the many quite different abstract Wiener spaces over $\mathbb{H}$. Moreover, it is shown in [18] that the *standard part map* $\kappa_{\mathbb{B}} : X \mapsto b_{\mathbb{B}}(X, \cdot)$ is a surjective measurable mapping from Ω onto the space $C_{\mathbb{B}}$ of continuous functions from $[0,\infty[$ into $\mathbb{B}$. This is true for any abstract Wiener space $\mathbb{B}$ over the fixed $\mathbb{H}$. The image measure $W_{\mathbb{B}}$ of $\widehat{\gamma}$ by $\kappa_{\mathbb{B}}$ is called the *Wiener measure* on the Borel-algbra on $C_{\mathbb{B}}$. Let $\mathcal{W} \subseteq L_\gamma(\mathcal{B})$ be the σ-algebra, generated by $b_{\mathbb{B}}$, augmented by the $\widehat{\gamma}$-nullsets. Again, $\mathcal{W}$ does not depend on $\mathbb{B}$, only on $\mathbb{H}$. In analogy to the case of $L^p_{\mathcal{L}^n}(\widehat{\nu^n})$ and $L^p(\lambda^n)$ we can identify the "nonstandard space" $L^p_{\mathcal{W}}(\widehat{\gamma})$ with the standard space $L^p(W_{\mathbb{B}})$. Each $\varphi \in L^p_{\mathcal{W}}(\widehat{\gamma})$ can be identified with $\psi \in L^p(W_{\mathbb{B}})$ if $\varphi(X) = \psi(b_{\mathbb{B}}(X,\cdot))$. It is easier to work with $L^p_{\mathcal{W}}(\widehat{\gamma})$ than with $L^p(W_{\mathbb{B}})$, because $\widehat{\gamma}$ is closely linked to the Gaußian measure γ on a finite-dimensional space. Moreover, $L^p_{\mathcal{W}}(\widehat{\gamma})$ is independent of $\mathbb{B}$.

Fix a Lebesgue square integrable function $g : [0,\infty[^n_{\lneq} \to \mathbb{H}^{\otimes n}$. By a slight modification of the Loeb–Andersen lifting theorems, there exists an internal function $G : T^n_{\lneq} \to \mathbb{F}^{\otimes n}$ such that $g({}^\circ t)$ is the standard part of $G(t)$ for $\widehat{\nu^n}$-almost all $t \in T^n_{\lneq}$ and G is *S-square integrable*, i.e. ${}^\circ \sum_{t \in T^n_{\lneq}} \|G(t)\|^2 \frac{1}{H^n} = \int_{[0,\infty[^n_{\lneq}}^\infty \|g\|^2 \, d\lambda^n$. The *iterated Itô integral* $I_n(g) : \Omega \to \mathbb{R}$ is $\widehat{\gamma}$-a.s. well defined (see [18]), by setting

$$I_n(g) := {}^\circ I_n(G) \quad \text{with } I_n(G)(X) := \sum_{t_1 < \cdots < t_n \in T} G_{t_1,\ldots,t_n}(X_{t_1}, \ldots, X_{t_n}).$$

Note that

$$G_{t_1,\ldots,t_n}(X_{t_1}, \ldots, X_{t_n}) = \sum_{i_1,\ldots,i_n \in \omega} G_{t_1,\ldots,t_n}(\mathfrak{e}_{i_1}, \ldots, \mathfrak{e}_{i_n}) \cdot x_{t_1,i_1} \cdots x_{t_n,i_n}.$$

Since $I_n(G)$ is a finite sum, we can apply the binomial theorem to prove that

$$\mathbb{E}_\gamma\big(I_n(G)\big)^4 \le (4!)^n\Bigg(\sum_{t\in T^n_<, i\in\omega^n} G_t^2(\mathfrak{e}_{i_1},\dots,\mathfrak{e}_{i_n})\frac{1}{H^n}\Bigg)^2 \tag{4}$$

$$= (4!)^n\Bigg(\sum_{t\in T^n_<}\|G_t\|^2\frac{1}{H^n}\Bigg)^2$$

is limited, thus $I_n(G)$ is S-square integrable. By Loeb theory and the computations (i), (ii), (iii) below, we have

$$\mathbb{E}_{\widehat{\gamma}}\big(\big(I_n(g)\big)^2\big) \approx \mathbb{E}_\gamma\big(\big(I_n(G)\big)^2\big) = \sum_{t\in T^n_<}\big\|G(t)\big\|^2\frac{1}{H^n} = \int_{[0,\infty[^n_\le}^\infty \|g\|^2\,d\lambda^n.$$

The proof, due to Cutland and Ng [4] for the one-dimensional case, can be used to prove the following result (see also Theorem 5.6 in [19]). It is an application of the chaos decomposition theorem in [18].

THEOREM 4.1. *Fix $\varphi \in L^2_{\mathcal{W}}(\widehat{\gamma})$ and an S-square integrable lifting Φ of φ. For all $n\in\mathbb{N}_0$ define $F_n : T^n_< \to \mathbb{F}^{\otimes n}$ by setting*

$$F_n(r_1,\dots,r_n)(\mathfrak{e}_{j_1},\dots,\mathfrak{e}_{j_n}) := H^n\cdot\mathbb{E}_\gamma(\Phi\cdot x_{r_1,j_1}\cdots x_{r_n,j_n}). \tag{5}$$

Then the standard part ${}^\circ F_n : [0,\infty[^n \to \mathbb{H}^{\otimes n}$ of F_n exists and φ has the chaos expansion

$$\varphi = \sum_{n=0}^\infty I_n\big({}^\circ F\big).$$

Let us write Equation (5) in the following form: since the X_r are the increments of the internal Brownian motion B we obtain, using $\Delta t := \frac{1}{H}$,

$$F_n(r_1,\dots,r_n)(\mathfrak{e}_{j_1},\dots,\mathfrak{e}_{j_n}) = \mathbb{E}\Bigg(\Phi\bigg\langle\frac{\Delta B_{r_1}}{\Delta t_1},\mathfrak{e}_{j_1}\bigg\rangle\cdots\bigg\langle\frac{\Delta B_{r_n}}{\Delta t_n},\mathfrak{e}_{j_n}\bigg\rangle\Bigg)$$
$$= \mathbb{E}\big(\Phi\langle\dot B_{r_1},\mathfrak{e}_{j_1}\rangle\cdots\langle\dot B_{r_n},\mathfrak{e}_{j_n}\rangle\big),$$

where $\dot B_t$ may be understood as the "derivative" of B at time t. This is Wiener's Equation (1).

Example. Fix a square integrable $g : [0,\infty[\, \to \mathbb{H}'$ and an S-square integrable lifting $G : T \to \mathbb{F}'$. Set $\varphi := e^{I_1(g)-\frac12\int_{[0,\infty[}\|g\|^2\,d\lambda}$. Then

$$\Phi := e^{I_1(G)-\frac12\sum_{t\in T}\|G(t)\|^2\frac{1}{H}}$$

is an S-square integrable lifting of φ. Therefore the kernel $f_n : [0,\infty[^n_\le \to \mathbb{H}^{\otimes n}$ of $e^{I_1(g)-\frac12\int_{[0,\infty[}\|g\|^2\,d\lambda}$ is the standard part of $F_n : T^n_< \to \mathbb{F}^{\otimes n}$ with

$$F_n(r_1,\dots,r_n)(\mathfrak{e}_{j_1},\dots,\mathfrak{e}_{j_n})$$
$$= H^n\cdot\mathbb{E}_\gamma\big(e^{I_1(G)-\frac12\sum_{t\in T}\|G(t)\|^2\frac{1}{H}}\cdot x_{r_1,j_1}\cdots x_{r_n,j_n}\big).$$

Using the shorthand $c := \sqrt{\frac{H}{2\pi}}^{H^2\cdot\omega}$, we obtain by elementary computation:

$$\begin{aligned}
&F_n(r_1,\dots,r_n)(\mathfrak{e}_{j_1},\dots,\mathfrak{e}_{j_n})\\
&\quad = cH^n \int_{{}^*\mathbb{R}^{H^2\cdot\omega}} e^{\sum_{t\in T, i\le\omega}(G_t(\mathfrak{e}_i)\cdot x_{t,i} - \frac{1}{2H}G_t^2(\mathfrak{e}_i) - \frac{H}{2}x_{t,i}^2)}\\
&\qquad \cdot x_{r_1,j_1}\cdots x_{t_n,j_n}\, d(x_{t,i})_{t\in T, i\le\omega}\\
&\quad = cH^n \int_{{}^*\mathbb{R}^{H^2\cdot\omega}} e^{\sum_{t\in T,i\le\omega} -\frac{H}{2}(x_{t,i}^2 - \frac{G_t(\mathfrak{e}_i)}{H})^2} x_{r_1,j_1}\cdots x_{r_n,j_n}\, d(x_{t,i})_{t\in T,i\le\omega}\\
&\quad = \sqrt{\frac{H}{2\pi}}^n H^n \int_{{}^*\mathbb{R}^n} A\, d(x_{r_i,j_i})_{i\le n}\\
&\quad \left(\text{with } A = e^{\sum_{i=1}^n -\frac{H}{2}x_{r_i,i_i}^2}\left(x_{r_1,j_1} + \frac{G_{r_1}(e_{j_1})}{H}\right)\cdots\left(x_{t_n,j_n} + \frac{G_{r_n}(e_{jn})}{H}\right)\right)\\
&\quad = G_{r_1}(e_{j_1})\cdots G_{r_n}(e_{j_n}).
\end{aligned}$$

It follows that F_n is the n-fold tensor product $G\odot\cdots\odot G$ of G, and therefore,

$$^\circ F_n = g^{\odot n} : [0,\infty[^n \to \mathbb{H}^{\otimes n}$$

with $g^{\odot n}(r_1,\dots,r_n)(a_1,\dots,a_n) = g_{r_1}(a_1)\cdots g_{r_n}(a_n)$. This is an elementary proof of the following well-known result:

PROPOSITION 4.2. $e^{I_1(g) - \frac{1}{2}\int_{[0,\infty[}\|g\|^2\, d\lambda} = 1 + \sum_{n=1}^\infty I_n(g^{\odot n})$.

To obtain the product- and chain rule, we transfer $\varphi,\psi \in L^2_{\mathcal{W}}(\widehat{\gamma}) = L^2(W_{\mathbb{B}})$ into the model $\mathfrak{M}$. Using the chaos decomposition theorem in [18], φ,ψ have S-square integrable liftings Φ,Ψ of the form

$$\Phi = \sum_{n=0}^M I_n(F_n), \qquad \Psi = \sum_{n=0}^M I_n(G_n),$$

where the $F_n, G_n : T^n_{\neq} \to \mathbb{F}^{\otimes n}$ are S-square integrable and symmetric. Now assume that φ,ψ be Malliavin differentiable. The Malliavin derivative $D\varphi$ of φ is a process from $[0,\infty[\times\Omega$ into $\mathbb{H}' = \mathbb{H}$. By results in [18], we may assume that $D\varphi, D\psi$ have S-square integrable liftings of the form

$$D\Phi(r,X) := \sum_{n=1}^M I_{n-1}\big(F_n(\cdot,r)\big), \qquad D\Psi(r,X) := \sum_{n=1}^M I_{n-1}\big(G_n(\cdot,r)\big) \in \mathbb{F}' = \mathbb{F}.$$

Note that for all $a\in\mathbb{F}$,

$$D\Phi_r(X)(a) := D\Phi(r,X)(a) := \sum_{n=1}^M \sum_{t\in T_<^{n-1}} F_n(t,r)(X_{t_1},\dots,X_{t_{n-1}},a).$$

In order to prove the product rule, we want to compute functions $K_n : T^n_< \to \mathbb{F}^{\otimes n}$, such that $\varphi \cdot \psi = \sum_{n=0}^{\infty} I_n({}^\circ K_n)$ and

$$\sum_{n=0}^{M} I_n(K_n) \text{ is an } S\text{-square integrable lifting of } \varphi \cdot \psi.$$

However, $\varphi \cdot \psi$ is not square integrable, in general. Therefore, we assume that φ and ψ belong to *finite chaos levels*, that is, $M \in \mathbb{N}_0$. Then,

$\Phi \cdot \Psi$, $(D\Phi) \cdot \Psi$, $\Phi \cdot D\Psi$ and $D(\Phi \cdot \Psi)$ are S-square integrable liftings of $\varphi \cdot \psi, (D\varphi) \cdot \psi$, $\varphi \cdot D\psi$ and $D(\varphi \cdot \psi)$, respectively.

Now K_n, given by

$$K_n(r_1, \ldots, r_n, \mathfrak{e}_{\rho_1}, \ldots, \mathfrak{e}_{\rho_n}) = H^n \mathbb{E}_\gamma(\Phi \cdot \Psi \cdot x_{r_1,\rho_1} \cdots x_{r_n,\rho_n})$$

has the desired property. We have to compute all possible

$$a := \mathbb{E}_\gamma(x_{t_1,\tau_1} \cdots x_{t_m,\tau_m} \cdot x_{s_1,\sigma_1} \cdots x_{s_k,\sigma_k} \cdot x_{r_1,\rho_1} \cdots x_{r_n,\rho_n})$$

with $t_1 < \cdots < t_m, s_1 < \cdots < s_k, r_1 < \cdots < r_n$. Here are some typical examples:

(i) Let $r_n < t := t_m = s_k$. If $\sigma := \tau_m = \sigma_k$, then

$$\begin{aligned} a &= \mathbb{E}_\gamma\big(x_{t_1,\tau_1} \cdots x_{t_{m-1},\tau_{m-1}} x_{s_1,\sigma_1} \cdots x_{s_{k-1},\sigma_{k-1}} x_{r_1,\rho_1} \cdots x_{r_n,\rho_n} \cdot \mathbb{E}^{\mathcal{B}_{t^-}} x^2_{t,\sigma}\big) \\ &= \mathbb{E}_\gamma\left(x_{t_1,\tau_1} \cdots x_{t_{m-1},\tau_{m-1}} x_{s_1,\sigma_1} \cdots x_{s_{k-1},\sigma_{k-1}} x_{r_1,\rho_1} \cdots x_{r_n,\rho_n} \frac{1}{H}\right), \end{aligned}$$

because $\mathbb{E}^{\mathcal{B}_{t^-}} x^2_{t,\sigma} = \mathbb{E}_{\gamma^1} x^2 = \frac{1}{H}$. If $\tau_m \neq \sigma_k$, then

$$\begin{aligned} a &= \mathbb{E}_\gamma\big(x_{t_1,\tau_1} \cdots x_{t_{m-1},\tau_{m-1}} x_{s_1,\sigma_1} \cdots x_{s_{k-1},\sigma_{k-1}} x_{r_1,\rho_1} \cdots x_{r_n,\rho_n} \\ &\qquad \cdot \mathbb{E}^{\mathcal{B}_{t^-}}(x_{t,\tau_m} x_{t,\sigma_k})\big) \\ &= 0, \end{aligned}$$

because $\mathbb{E}^{\mathcal{B}_{t^-}}(x_{t,\tau_m} \cdot x_{t,\sigma_k}) = \mathbb{E}_{\gamma^2} x \cdot y = 0$. We may continue in the same manner: for example, let $t_{m-1} < s_{k-1} = r_n, \sigma_{k-1} = \rho_n, r_n < t_m = s_k$ and $\tau_m = \sigma_k$. Then

$$a = \mathbb{E}_\gamma\left(x_{t_1,\tau_1} \cdots x_{t_{m-1},\tau_{m-1}} x_{s_1,\sigma_1} \cdots x_{s_{k-2},\sigma_{k-2}} x_{r_1,\rho_1} \cdots x_{r_{n-1},\rho_{n-1}} \frac{1}{H^2}\right).$$

(ii) Let $t_m, s_k < r_n =: r$. Then

$$a = \mathbb{E}_\gamma\big(x_{t_1,\tau_1} \cdots x_{t_m,\tau_m} x_{s_1,\sigma_1} \cdots x_{s_k,\sigma_k} x_{r_1,\rho_1} \cdots x_{r_{n-1},\rho_{n-1}} \mathbb{E}^{\mathcal{B}_{r^-}} x_{r,\rho_n}\big) = 0$$

because $\mathbb{E}^{\mathcal{B}_{r^-}} x_{r,\rho_n} = \mathbb{E}_{\gamma^1} x = 0$.

(iii) Let $t := t_m = s_k = r_n$. Then

$$\begin{aligned} a &= \mathbb{E}_\gamma\big(x_{t_1,\tau_1} \cdots x_{t_{m-1},\tau_{m-1}} x_{s_1,\sigma_1} \cdots x_{s_{k-1},\sigma_{k-1}} x_{r_1,\rho_1} \cdots x_{r_{n-1},\rho_{n-1}} \\ &\qquad \cdot \mathbb{E}^{\mathcal{B}_{t^-}}(x_{t,\tau_m} x_{t,\sigma_k} x_{t,\rho_n})\big) \\ &= 0, \end{aligned}$$

because $\mathbb{E}_{\gamma^1} x^3 = \mathbb{E}_{\gamma^2} x^2 \cdot y = \mathbb{E}_{\gamma^3} x \cdot y \cdot z = 0$. Here is a difference to more general Lévy processes. In general, $\mathbb{E}_{\mu^1} x^3 \neq 0$.

In order to gain control of all the summands in K_n, we proceed as follows: let $n \in \mathbb{N}_0$ and $k \leq n$ We identify $k \in \mathbb{N}_0$ with the set $\{1, \ldots, k\}$, thus $0 = \emptyset$. If π is a strictly monotone increasing function from k into n, then we will write $\pi : k \uparrow n$. For $\pi : k \uparrow n$, let $\overline{\pi} : n - k \uparrow n \setminus \operatorname{range}(\pi)$, that is, $\overline{\pi}$ enumerates the numbers in n, that are different from $\pi_1, \ldots, \pi_k$. For example, if $k = 0$, then $\pi : k \uparrow n = \emptyset$ and $\overline{\pi} : k \uparrow k = id_k$. Calculating $H^n \mathbb{E}_\gamma(\Phi\Psi \cdot \prod_{i=1}^n x_{r_i,\rho_i})$, we obtain the following formula:

$$K_n(r_1, \ldots, r_n, a_1, \ldots, a_n) = \sum_{k=0}^{n} \sum_{\pi : k \uparrow n} \sum_{m=0}^{M^2} \sum_{t \in T^m_<} \sum_{i \in \omega^m} A \cdot B \cdot \frac{1}{H^m},$$

where, with $(a_1, \ldots, a_n) := (\mathfrak{e}_{\rho_1}, \ldots, \mathfrak{e}_{\rho_n})$,

$$A = F_{m+k}(t, r_{\pi_1}, \ldots, r_{\pi_k}, \mathfrak{e}_{i_1}, \ldots, \mathfrak{e}_{i_m}, a_{\pi_1}, \ldots, a_{\pi_k}),$$
$$B = G_{m+n-k}(t, r_{\overline{\pi}_1}, \ldots, r_{\overline{\pi}_{n-k}}, \mathfrak{e}_{i_1}, \ldots, \mathfrak{e}_{i_m}, a_{\overline{\pi}_1}, \ldots, a_{\overline{\pi}_{n-k}}).$$

In order to prove the product rule

$$D(\varphi \cdot \psi)_{\circ r} = D\varphi_{\circ r} \cdot \psi + \varphi \cdot D\psi_{\circ r},$$

we write $K_n(r_1, \ldots, r_{n-1}, r, a_1, \ldots, a_{n-1}, a)$ in a slightly different way:

$$\begin{aligned} &K_n(r_1, \ldots, r_{n-1}, r, a_1, \ldots, a_{n-1}, a) \\ &\quad = K_n^{D\Phi_r \cdot \Psi}(r_1, \ldots, r_{n-1}, r, a_1, \ldots, a_{n-1}, a) \\ &\qquad + K_n^{\Phi \cdot D\Psi_r}(r_1, \ldots, r_{n-1}, r, a_1, \ldots, a_{n-1}, a), \end{aligned}$$

where the first summand is equal to

$$\sum_{k=0}^{n-1} \sum_{\pi : k \uparrow n-1} \sum_{m=0}^{M^2} \sum_{t \in T^m_<} \sum_{i \in \omega^m} A_u B_u \frac{1}{H^m}$$

with

$$A_u = F_{m+k+1}(t, r_{\pi_1}, \ldots, r_{\pi_k}, r, \mathfrak{e}_{i_1}, \ldots, \mathfrak{e}_{i_m}, a_{\pi_1}, \ldots, a_{\pi_k}, a),$$
$$B_u = G_{m+n-1-k}(t, r_{\overline{\pi}_1}, \ldots, r_{\overline{\pi}_{n-1-k}}, \mathfrak{e}_{i_1}, \ldots, \mathfrak{e}_{i_m}, a_{\overline{\pi}_1}, \ldots, a_{\overline{\pi}_{n-1-k}}).$$

The second summand is equal to

$$\sum_{k=0}^{n-1} \sum_{\pi : k \uparrow n-1} \sum_{m=0}^{M^2} \sum_{t \in T^m_<} \sum_{i \in \omega^m} A_v B_v \frac{1}{H^m}$$

with

$$A_v = F_{m+k}(t, r_{\pi_1}, \ldots, r_{\pi_k}, \mathfrak{e}_{i_1}, \ldots, \mathfrak{e}_{i_m}, a_{\pi_1}, \ldots, a_{\pi_k}),$$
$$B_v = G_{m+n-k}(t, r_{\overline{\pi}_1}, \ldots, r_{\overline{\pi}_{n-1-k}}, r, \mathfrak{e}_{i_1}, \ldots, \mathfrak{e}_{i_m}, a_{\overline{\pi}_1}, \ldots, a_{\overline{\pi}_{n-1-k}}, a).$$

Note that for $r \in T$ and $a \in \mathbb{F}$,

$$K_n^{D\Phi_r \cdot \Psi}(r_1, \ldots, r_{n-1}, r, a_1, \ldots, a_{n-1}, a) = H^{n-1}\mathbb{E}_\gamma\left(D\Phi_r(a)\Psi \cdot \prod_{i=1}^{n-1} x_{r_i,\rho_i}\right),$$

$$K_n^{\Phi \cdot D\Psi_r}(r_1, \ldots, r_{n-1}, r, a_1, \ldots, a_{n-1}, a) = H^{n-1}\mathbb{E}_\gamma\left(\Phi D\Psi_r(a) \cdot \prod_{i=1}^{n-1} X_{r_i,\rho_i}\right).$$

Therefore, $K_n^{D\Phi_r \cdot \Psi}, K_n^{\Phi \cdot D\Psi_r}$ build the kernels under $D\Phi_r \cdot \Psi, \Phi \cdot D\Psi_r$ for $D\varphi_{\circ r} \cdot \psi$, $\varphi \cdot D\psi_{\circ r}$, respectively. To sum up, we obtain for $\widehat{\nu \otimes \gamma}$ almost all (r, X)

$$\begin{aligned} D(\varphi \cdot \psi)_{\circ r}(X) &\approx_{\mathbb{F}} \sum_{n=1}^{M^2} I_{n-1}\big(K_n(\cdot, r)\big)(X) \\ &= \sum_{n=1}^{M^2} I_{n-1}\big(K_n^{D\Phi_r \cdot \Psi}(\cdot, r)\big)(X) \\ &\quad + \sum_{n=1}^{M^2} I_{n-1}\big(K_n^{\Phi \cdot D\Psi_r}(\cdot, r)\big)(X) \\ &\approx_{\mathbb{F}} D\varphi_{\circ r}(X) \cdot \psi(X) + \varphi(X) \cdot D\psi_{\circ r}(X). \end{aligned}$$

The proof of the product-rule is finished for functions in finite chaos levels. The proof of more general results, according to Theorems 2.1, 2.2, is similar to the proof in the following section.

5. The proof of the rules

Now we turn back to Sections 2 and 3. Some additional difficulties appear in the case of more general Lévy processes. The set $\mathbb{N}_L$ is often different from $\{1\}$ and the multiple integrals are only square integrable. It follows that, in contrast to Brownian motion, the product of two Lévy functionals even in finite chaos levels is, in general, not square integrable. However, the most unpleasant difference is the fact that $\mathbb{E}_{\mu^1} p_1^3 \neq 0$, for example, in the case of Poisson processes. In the Gaußian case $p_1(x) = x$.

In order to get over some of the difficulties, we replace functionals in finite chaos levels by bounded functions (see Section 2). An internal function $\Phi : \Omega \to {}^*\mathbb{R}$ is called a *polynomial*, *bounded by* $S \in {}^*\mathbb{N}$, if

$$\Phi(X) = \sum_{n \in {}^*\mathbb{N}_0} \sum_{k \in M_L^n} \sum_{t \in T_<^n} F_n(k, t) \prod_{i=1}^n p_{k_i}(X_{t_i}),$$

where $F_n : M_L^n \times T_{\neq}^n \to {}^*\mathbb{R}$ is internal and symmetric and Φ and therefore also the F_n are bounded by S. The functions F_n are called the *kernels of* Φ. By Theorem 6.1 in [19], we can assume that each $\varphi : \Omega \to \mathbb{R}$ in $L^2_{\mathcal{D}}(\widehat{\mu})$ has a polynomial lifting $\Phi : \Omega \to {}^*\mathbb{R} \in SL^2(\mu)$, bounded by some $S \in {}^*\mathbb{N}$. If φ is Malliavin differentiable, then we can, in addition, assume that

$$D\Phi : (l,r,X) \mapsto \sum_{n\in {}^*\mathbb{N}} \sum_{k\in M_L^n} \sum_{t\in T_<^{n-1}} F_n(k,l,t,r) \prod_{i=1}^{n-1} p_{k_i}(X_{t_i})$$

belongs to $SL^2(c\otimes\nu\otimes\mu)$ and is a lifting of the Malliavin derivative of φ.

We use the following notations; compare this notation with the notation in the Brownian motion case. Fix $m\in\mathbb{N}$, $\rho\in m\cup\{0\}$, a strictly increasing ρ-tuple $\beta_1<\cdots<\beta_\rho$ in m and $i\in\{\rho,\dots,m\}$. Let $\tau : i-\rho\uparrow m\setminus\{\beta_1,\dots,\beta_\rho\}$ be a strictly monotone increasing function from $i-\rho$ into $m\setminus\{\beta_1,\dots,\beta_\rho\}$. Then $\overline{\tau}$ denotes the *complement* of τ, i.e., $\overline{\tau}$ is the uniquely determined strictly monotone increasing function from $m-i$ onto $m\setminus(\text{range}(\tau)\cup\{\beta_1,\dots,\beta_\rho\})$. Account that τ and $\overline{\tau}$ depend on m. Here is the key to both rules:

THEOREM 5.1. *Suppose that $\varphi,\psi\in L^2_{\mathcal{D}}(\widehat{\mu})$ and the kernels in the chaos decomposition* (*see Theorem* 6.1 *in* [19]) *of φ and ψ are bounded by some standard $S\in\mathbb{N}$. Then we have in $L^2(c\otimes\lambda\otimes\widehat{\mu})$.*

$$\begin{aligned}\big(D(\varphi\cdot\psi)\big)_{(l,r)} &= (D\varphi)_{(l,r)}\cdot\psi+\varphi\cdot(D\psi)_{(l,r)}\\ &\quad+\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L}\alpha(\kappa,\widetilde{\kappa},l)\cdot(D\varphi)_{\kappa,r}\cdot(D\psi)_{\widetilde{\kappa},r}.\end{aligned}$$

Proof. By the chaos expansion result (see Theorem 6.1 in [19]), φ and ψ have polynomial liftings Φ and Ψ. Since φ,ψ are bounded by S, we can assume that Φ and Ψ are also bounded by S. Therefore, $\Phi\cdot\Psi$ belongs to $SL^2(\mu)$ and is a lifting of $\varphi\cdot\psi\in L^2_{\mathcal{D}}(\widehat{\mu})$. Moreover, $D(\Phi\cdot\Psi), D\Phi\cdot\Psi, \Phi\cdot D\Psi\in SL^2(c\otimes\nu\otimes\mu)$ are liftings of $D(\varphi\cdot\psi), D\varphi\cdot\psi, \varphi\cdot D\psi$, respectively. Moreover, $\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L}\sigma(\kappa,\widetilde{\kappa},\cdot)\cdot D\Phi_\kappa\cdot D\Psi_{\widetilde{\kappa}}\in SL^2(c\otimes\nu\otimes\mu)$ and is a lifting of $\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L}\alpha(\kappa,\widetilde{\kappa},l)\cdot D\varphi_\kappa\cdot D\psi_{\widetilde{\kappa}}$. By the recipe for the computation of the kernels of the chaos decomposition (see Theorem 5.6 in [19]), the kernels of $\varphi\cdot\psi$ are the standard parts of the kernels K_m under $\Phi\cdot\Psi$, given by $K_0=\mathbb{E}_\mu(\Phi\cdot\Psi)$ and for $m\ge 1$,

$$K_m(l,r)=H^m\mathbb{E}_\mu\big(\Phi\cdot\Psi\cdot p_{l_1}(X_{r_1})\cdots p_{l_m}(X_{r_m})\big)$$

with $l\in\mathbb{N}_L^m$, $r\in T_<^m$. Let F_n, G_n, be the kernels of Φ,Ψ, respectively. Elementary finite combinatorics tells us that $K_m(l,r)$ is the finite sum:

$$\begin{aligned}&K_m(l,r)\\ &\quad=\sum_{\rho=0}^m\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L^\rho}\sum_{\beta\in m_<^\rho}\sum_{i=\rho}^m\sum_{\tau:i-\rho\uparrow m\setminus\{\beta_1,\dots,\beta_\rho\}}\sum_{n\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_L^n}\sum_{t\in T_<^n}\frac{1}{H^n}\cdot\Pi\end{aligned}$$

with

$$\Pi = F_{n+i}(k,\varkappa,l_{\tau_1},\dots,l_{\tau_{i-\rho}},t,r_{\beta_1},\dots,r_{\beta_\rho},r_{\tau_1},\dots,r_{\tau_{i-\rho}}) \\ \cdot G_{n+m-i+\rho}(k,\widetilde{\varkappa},l_{\overline{\tau}_1},\dots,l_{\overline{\tau}_{m-i}},t,r_{\beta_1},\dots,r_{\beta_\rho},r_{\overline{\tau}_1},\dots,r_{\overline{\tau}_{m-i}}) \\ \cdot \sigma(\kappa_1,\widetilde{\kappa}_1,l_{\beta_1})\cdots\sigma(\kappa_\rho,\widetilde{\kappa}_\rho,l_{\beta_\rho}).$$

It is easy to see that

$$K_m(l,r) = A + B + C(l_m,r_m)$$

with

$$A = \sum_{\rho=0}^{m}\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L^\rho}\sum_{\beta\in(m-1)_<^\rho}\sum_{i=\rho}^{m}\sum_{\substack{\tau:i-\rho\uparrow m\setminus\{\beta_1,\dots,\beta_\rho\},\\ \tau(i-\rho)=m}}\sum_{n\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_L^n}\sum_{t\in T_<^n}\frac{1}{H^n}\cdot\Pi,$$

$$B = \sum_{\rho=0}^{m}\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L^\rho}\sum_{\beta\in(m-1)_<^\rho}\sum_{i=\rho}^{m}\sum_{\substack{\tau:i-\rho\uparrow m\setminus\{\beta_1,\dots,\beta_\rho\},\\ \widetilde{\tau}(m-i)=m}}\sum_{n\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_L^n}\sum_{t\in T_<^n}\frac{1}{H^n}\cdot\Pi,$$

$$C(l_m,r_m) = \sum_{\rho=0}^{m}\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L^\rho}\sum_{\beta\in m_<^\rho,\beta_\rho=m}\sum_{i=\rho}^{m}\sum_{\tau:i-\rho\uparrow m\setminus\{\beta_1,\dots,\beta_\rho\}}\sum_{n\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_L^n}\sum_{t\in T_<^n}\frac{1}{H^n}\cdot\Pi.$$

In the same way, computing the kernel $K_{m-1}^{D\Phi_{l_m,r_m}\cdot\Psi}$ under $D\Phi_{l_m,r_m}\cdot\Psi$, we obtain for $l=(l_1,\dots,l_{m-1}), r=(r_1,\dots,r_{m-1})$,

$$K_{m-1}^{D\Phi_{l_m,r_m}\cdot\Psi}(l,r) \\ = H^{m-1}\mathbb{E}_\mu\big(D\Phi_{l_m,r_m}\cdot\Psi\cdot p_{l_1}(X_{r_1})\cdots p_{l_{m-1}}(X_{r_{m-1}})\big) \\ = \sum_{\rho=0}^{m-1}\sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L^\rho}\sum_{\beta\in m-1_<^\rho}\sum_{i=\rho}^{m-1}\sum_{\tau:i-\rho\uparrow m-1\setminus\{\beta_1,\dots,\beta_\rho\}}\sum_{n\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_L^n}\sum_{t\in T_<^n}\frac{1}{H^n} \\ \cdot F_{n+i+1}(k,\varkappa,l_{\tau_1},\dots,l_{\tau_{i-\rho}},l_m,t,r_{\beta_1},\dots,r_{\beta_\rho},r_{\tau_1},\dots,r_{\tau_{i-\rho}},r_m) \\ \cdot G_{n+m-1-i+\rho}(k,\widetilde{\varkappa},l_{\overline{\tau}_1},\dots,l_{\overline{\tau}_{m-1-i}},t,r_{\beta_1},\dots,r_{\beta_\rho},r_{\overline{\tau}_1},\dots,r_{\overline{\tau}_{m-1-i}}) \\ \cdot\sigma(\kappa_1,\widetilde{\kappa}_1,l_{\beta_1})\cdots\sigma(\kappa_\rho,\widetilde{\kappa}_\rho,l_{\beta_\rho}).$$

It is easy to see that $K_{m-1}^{D\Phi_{l_m,r_m}\cdot\Psi}(l,r)=A$ and

$$B = K_{m-1}^{\Phi\cdot D\Psi_{l_m,r_m}}, \qquad C(l_m,r_m) = \sum_{\eta,\widetilde{\eta}\in\mathbb{N}_L} K_{m-1}^{D\Phi_{\eta,r_m}\cdot D\Psi_{\widetilde{\eta},r_m}}\sigma(\eta,\widetilde{\eta},l_m),$$

where $K_{m-1}^{\Phi\cdot D\Psi_{l_m,r_m}}$ and $K_{m-1}^{D\Phi_{\eta,r_m}\cdot D\Psi_{\widetilde{\eta},r_m}}$ are the kernels under $\Phi\cdot D\Psi_{l_m,r_m}$ and under $D\Phi_{\eta,r_m}\cdot D\Psi_{\widetilde{\eta},r_m}$, respectively. This proves that for $l\in\mathbb{N}_L$ and $r\in T$:

$$\big(D(\Phi\cdot\Psi)\big)_{(l,r)} = (D\Phi)_{(l,r)}\cdot\Psi + \Phi\cdot(D\Psi)_{(l,r)} + \sum_{\kappa,\widetilde{\kappa}\in\mathbb{N}_L}\sigma(\kappa,\widetilde{\kappa},l)\cdot D\Phi_{\kappa,r}\cdot D\Psi_{\widetilde{\kappa},r}.$$

Taking standard parts, we obtain the desired result. □

The product-rule (Theorem 2.1) now follows from Theorem 5.1 and Proposition 6.1 below. The chain-rule (Theorem 2.2) follows from Theorem 5.1, the following lemma and Proposition 6.1 below.

LEMMA 5.2. *Suppose that* $\mathbb{N}_L = \{1\}$. *Fix* $g : \mathbb{R}^n \to \mathbb{R}$ *with* $g(x_1, \ldots, x_n) = x_1^{k_1} \cdots x_n^{k_n}$ *and* $\varphi_1, \ldots, \varphi_n \in L^2_{\mathcal{D}}(\widehat{\mu})$ *bounded by some* $S \in \mathbb{N}$. *Then in* $L^2(\lambda \otimes \widehat{\mu})$

$$D\big(g(\varphi_1, \ldots, \varphi_n)\big) = \begin{cases} \frac{1}{\alpha}(g(\varphi_1 + \alpha \cdot D\varphi_1, \ldots, \varphi_n + \alpha \cdot D\varphi_n) - g(\varphi_1, \ldots, \varphi_n)), & \text{if } \alpha \neq 0, \\ \sum_{i=1}^n (\partial_i g)(\varphi_1, \ldots, \varphi_n) \cdot D\varphi_i, & \text{if } \alpha = 0. \end{cases}$$

Proof. By induction on n, using the product-rule. In the case $n = 1$, apply induction on k_1. □

6. A commutation rule for derivative and limit

The following commutation rule, which we have used, reminds of a result in elementary analysis: we can interchange derivative and limit if the sequence of derivatives converges uniformly, and the original sequence converges in at least one point. Here we have:

PROPOSITION 6.1. *Suppose that* (φ^i) *is a sequence of Malliavin differentiable functions such that* $(D\varphi^i)$ *converges in* $L^2_{\mathcal{L}\otimes\mathcal{D}}(c \otimes \widehat{\nu \otimes \mu})$ *and suppose that* $(\mathbb{E}_{\widehat{\mu}}\varphi^i)$ *converges in the real numbers. Then* (φ^i) *converges to a Malliavin differentiable function and*

$$D\Big(\lim_{i\to\infty} \varphi^i\Big) = \lim_{i\to\infty} D\varphi^i \ \text{ in } L^2_{\mathcal{L}\otimes\mathcal{D}}(c \otimes \widehat{\nu \otimes \mu}).$$

Proof. Let $\varphi^i = \sum_{n=0}^\infty I_n(f_n^i)$. By the assumption,

$$\begin{aligned} 0 &= \lim_{i,j\to\infty} \big\|D\varphi^i - D\varphi^j\big\|^2_{c\otimes\widehat{\nu\otimes\mu}} \\ &= \lim_{i,j\to\infty} \sum_{n=1}^\infty \sum_{k\in\mathbb{N}_L^n} \int_{T_<^{n-1}\times T} \big(f_n^i(k,\cdot) - f_n^j(k,\cdot)\big)^2 \, d\widehat{\nu^n}. \end{aligned}$$

Since the f_n^i are symmetric, $(f_n^i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^2_{\mathcal{L}^n}(c^n \otimes \widehat{\nu^n})$ for all $n \in \mathbb{N}$. Let $\lim_{i\to\infty} f_n^i = f_n$ in $L^2_{\mathcal{L}^n}(c^n \otimes \widehat{\nu^n})$. Then

$$\lim_{i\to\infty} \sum_{k\in\mathbb{N}_L^n} \int_{T_<^{n-1}\times T} \big(f_n^i(k,\cdot) - f_n(k,\cdot)\big)^2 \, d\widehat{\nu^n} = 0.$$

It follows that in $L^2_{\mathcal{D}}(\widehat{\mu})$, $L^2_{\mathcal{L}\otimes\mathcal{D}}(c \otimes \widehat{\nu \otimes \mu})$ respectively,

$$\lim_{i\to\infty} I_n\big(f_n^i\big) = I_n(f_n) \quad \text{and} \quad \lim_{i\to\infty} DI_n\big(f_n^i\big) = DI_n(f_n).$$

Since this is true and (φ^i) and $(D\varphi^i)$ are Cauchy sequences and $I_n(f_n^i) \perp I_m(f_m^j)$, $DI_n(f_n^i) \perp DI_m(f_m^j)$ for $n \neq m$, the following limits exist

$$\lim_{i\to\infty} \sum_{n=1}^{\infty} I_n\big(f_n^i\big) = \sum_{n=1}^{\infty} I_n(f_n) \quad \text{in } L^2_{\mathcal{D}}(\widehat{\mu}),$$

$$\lim_{i\to\infty} D \sum_{n=0}^{\infty} I_n\big(f_n^i\big) = \lim_{i\to\infty} \sum_{n=0}^{\infty} DI_n\big(f_n^i\big) = \sum_{n=1}^{\infty} DI_n(f_n) = D \sum_{n=0}^{\infty} I_n(f_n)$$

in $L^2_{\mathcal{L}\otimes\mathcal{D}}(c \otimes \widehat{\nu \otimes \mu})$. Define

$$\varphi := \sum_{n=1}^{\infty} I_n(f_n) + \lim_{i\to\infty} \mathbb{E}_{\widehat{\mu}} \varphi^i.$$

Then we have $\lim_{i\to\infty} \varphi^i = \varphi$ and $\lim_{i\to\infty} D\varphi^i = D\varphi$. □

References

[1] S. Albeverio, J. E. Fenstad, R. Høegh Krohn and T. Lindstrøm, *Nonstandard methods in stochastic analysis and mathematical physics*, Academic Press, Orlando, FL, 1986. MR 0859372

[2] R. M. Anderson, *A nonstandard representation of Brownian motion and Itô integration*, Israel J. Math. **25** (1976), 15–46. MR 0464380

[3] N. Cutland, *Infinitesimals in action*, J. Lond. Math. Soc. (2) **35** (1987), 202–216. MR 0881511

[4] N. Cutland and S.-A. Ng, *A nonstandard approach to the Malliavin calculus*, Advances in analysis, probability and mathematical physics (Blaubeuren 1992) (S. Albeverio et al., eds.), Math. Appl., vol. 314, Kluwer Acad. Publ., Dordrecht, 1995, pp. 149–170. MR 1344705

[5] G. Di Nunno, Th. Meyer-Brandis, B. Øksendal and F. Proske, *Malliavin calculus and anticipative Itô formulae for Lévy processes*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **8** (2005), 235–258. MR 2146315

[6] L. Gross, *Abstract Wiener spaces*, Proc. 5th Berkeley symp. math. stat. prob. Part I, University of California Press, Berkeley, 1965, pp. 31–41. MR 0212152

[7] D. L. Hoover and E. A. Perkins, *Nonstandard construction of the stochastic integral and applications to stochastic differential equations I and II*, Trans. Amer. Math. Soc. **275** (1983), 1–36, 37–58. MR 0678335

[8] H. J. Keisler, *An infinitesimal approach to stochastic analysis*, Mem. Amer. Math. Soc. **48** (1984). MR 0732752

[9] H. H. Kuo, *Gaussian measures on Banach spaces*, Lecture Notes in Mathematics, vol. 463, Springer, Berlin, 1975. MR 0461643

[10] J. A. Léon, J. L. Solé, F. Utzet and J. Vives, *On Lévy processes, Malliavin calculus and market models with jumps*, Finance Stoch. **6** (2002), 197–225. MR 1897959

[11] T. Lindstrøm, *Hyper-finite stochastic integration I, II, III, and Addendum*, Math. Scand. **46** (1980), 265–333.

[12] T. Lindstrøm, *Hyperfinite Lévy processes*, Stoch. Stoch. Rep. **76** (2004), 517–548. MR 2100020

[13] P. A. Loeb, *Conversion from nonstandard to standard measure spaces and applications in probability theory*, Trans. Amer. Math. Soc. **211** (1975), 113–122. MR 0390154

[14] P. A. Loeb and M. Wolff (eds.), *Nonstandard-analysis for the working mathematician*, Kluwer Scientific, Dordrecht, 2000. MR 1790871

[15] P. R. Masani, *Norbert Wiener*, Vita Mathematica, vol. 5, Birkhäuser Verlag, Basel, 1990.
[16] D. Nualart, *The Malliavin calculus and related topics*, Springer-Verlag, New York, 1995. MR 1344217
[17] D. Nualart and W. Schoutens, *Chaotic and predictable representations for Lévy processes*, Stochastic Process. Appl. **90** (2000), 109–122. MR 1787127
[18] H. Osswald, *Malliavin calculus in abstract Wiener spaces using infinitesimals*, Adv. Math. **176** (2003), 1–37. MR 1978339
[19] H. Osswald, *A smooth approach to Malliavin calculus for Lévy processes*, J. Theoret. Probab. **22** (2009), 441–473. MR 2501329
[20] H. Osswald, *Malliavin calculus for Lévy processes and infinite dimensional Brownian motion*, Tracts in Mathematics, vol. 191, Cambridge University Press, Cambridge, 2012. MR 2918805
[21] J. L. Solé, F. Utzet and J. Vives, *Canonical Lévy process and Malliavin calculus*, Stochastic Process. Appl. **117** (2007), 165–187. MR 2290191
[22] A. S. Üstünel and M. Zakai, *Transformations of Wiener measure under anticipative flows*, Probab. Theory Related Fields **93** (1992), 91–136. MR 1172941
[23] A. S. Üstünel and M. Zakai, *Embedding the abstract Wiener space in a probability space*, J. Funct. Anal. **171** (2000), 124–138. MR 1742861
[24] A. S. Üstünel and M. Zakai, *Transforms of measure on a Wiener space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000. MR 1736980
[25] W. Walter, *Einführung in die Theorie der Distributionen*, Bibliographisches Institut, Mannheim/Wien/Zürich, 1974. MR 0467294
[26] N. Wiener, *The homogeneous chaos*, Amer. J. Math. **60** (1938), 897–936. MR 1507356
[27] M. Zakai, *The Malliavin calculus*, Acta Appl. Math. **3** (1985), 175–207. MR 0781585

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