Illinois Journal of Mathematics
Volume 55, Number 2, Summer 2011, Pages 675–684
S 0019-2082

ON THE FOURIER TRANSFORMS OF INHOMOGENEOUS SELF-SIMILAR MEASURES

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ABSTRACT. The inhomogeneous self-similar measure μ is defined by the relation

$$\mu = \sum_{j=1}^{N} p_j \mu o S_j^{-1} + p\nu,$$

where $(p_1, \dots, p_N, p)$ is a probability vector, $S_j : \mathbb{R}^n \to \mathbb{R}^n, j = 1, \dots, N$ are contracting similarities and ν is a probability measure on $\mathbb{R}^n$ with compact support. The existence of such measures is well known, see (*Math. Proc. Cambridge Philos. Soc.* **144** (2008) 465–493) and the references therein. In (*Math. Proc. Cambridge Philos. Soc.* **144** (2008) 465–493), the authors have studied the Fourier transforms of inhomogeneous self-similar measures and they give relations about the asymptotic behavior of the Fourier transform of ν and μ. Some constructions which are given with precise asymptotic behavior arise from a discrete measure ν. Here we will see that these constructions can be extended with purely continuous measures ν. In order to prove this, we will construct suitable symmetric Bernoulli convolution measures (*Essays in Commutative Harmonic Analysis* (1979) Springer) and will use the results of (*J. Math. Anal. Appl.* **299** (2004) 550–562).

1. Introduction

The study of the Fourier transforms of measures has a long history, see [5], [6], [12], [13], [14], [18], [19]. Recently, many authors have studied the asymptotic behavior of the Fourier transforms of probability measures, see for example [1], [4], [8], [9], [16], [17]. Here, we will discuss the inhomogeneous

Received December 7, 2009; received in final form April 16, 2010.
2010 *Mathematics Subject Classification.* 42A38, 28A80.

self-similar measures μ which are defined by the relation

$$\mu = \sum_{j=1}^{N} p_j \mu o S_j^{-1} + p\nu, \tag{1}$$

where $S_j : \mathbb{R}^n \to \mathbb{R}^n, j = 1, \ldots, N$ are contracting similarities, $(p_1, \ldots, p_N, p)$ is a probability vector and ν is a probability measure on $\mathbb{R}^n$ with compact support. The existence of such measures is well known, see [11]. For a Borel probability measure on $\mathbb{R}^n$ we define its Fourier transform by the relation

$$\widehat{\mu}(x) = \int_{\mathbb{R}^n} e^{-i2\pi xy}\, d\mu(y)$$

and for $q \in (0, \infty)$ we define the qth lower Fourier dimension $\underline{\Delta}_q(\mu)$ by the relation

$$\underline{\Delta}_q(\mu) = \liminf_{R\to\infty} \frac{\log\left(\frac{1}{\mathcal{L}^n(B(0,R))} \int_{B(0,R)} |\widehat{\mu}(x)|^q\, dx\right)^{\frac{1}{q}}}{-\log R},$$

where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure. We also define

$$\underline{\Delta}_\infty(\mu) = \liminf_{R\to\infty} \frac{\log(\sup_{|x|\geq R} |\widehat{\mu}(x)|)}{-\log R}.$$

Similarly are defined the qth upper Fourier dimension and $\overline{\Delta}_\infty(\mu)$. Of course the construction of measures with precise value of $\underline{\Delta}_\infty(\mu)$ or $\overline{\Delta}_\infty(\mu)$ is a hard problem see [5].

In [10], the authors have studied the L^q spectra and Renyi dimensions of inhomogeneous self-similar measures and in the sequel in [11] (see also [15]) the Fourier transforms of these measures and more precisely the quantities

$$\underline{\Delta}_q(\mu), \qquad \overline{\Delta}_q(\mu), \qquad \underline{\Delta}_\infty(\mu), \qquad \overline{\Delta}_\infty(\mu)$$

and their relation with the corresponding quantities of the measure ν. They have also constructed an example of a measure μ such as that the following holds:

Theorem A. *There exists an inhomogeneous measure μ given by* (1) *such that*

$$\underline{\Delta}_q(\nu) = \underline{\Delta}_q(\mu) = 0, \quad q \geq 1.$$

The construction is based to a measure ν which is discrete and as a result it gives discrete part to the measure μ, as one can easily see using Wiener' s characterization of continuous measures, see [7], [21] and [3], so is easy to take the desired results. Here we shall see that the same is possible with a measure ν purely continuous (and so by Wiener's theorem with μ purely continuous). The following proposition is a direct consequence of Wiener's characterization of continuous measures and of the definitions.

PROPOSITION 1. (i) *The measure ν has discrete part if and only if the same holds for μ.*

(ii) *If μ is a measure with discrete part then $\overline{\Delta}_q(\nu) = \overline{\Delta}_q(\mu) = 0$ and $\overline{\Delta}_\infty(\nu) = \overline{\Delta}_\infty(\mu) = 0$.*

In order to prove our results, we will construct suitable symmetric Bernoulli convolution measures see [5, Section 6.6] and will make use the results of [1]. More precisely, we prove the following.

THEOREM 2. *Let $\beta \in [0,1]$. Then there exists an inhomogeneous measure μ on $[0,1]$ given by (1) with ν purely continuous measure such that $\overline{\Delta}_2(\nu) = \underline{\Delta}_2(\nu) = \frac{\beta}{2}$ and*

(i) $\frac{\beta}{2} \leq \underline{\Delta}_q(\mu) \leq \overline{\Delta}_q(\mu) \leq \frac{\beta}{q}$, *for $0 < q \leq 2$,*

(ii) $\frac{\beta}{q} \leq \underline{\Delta}_q(\mu) \leq \overline{\Delta}_q(\mu) \leq \frac{\beta}{2}$, *for $q \geq 2$.*

REMARK 1. (i) In the above theorem for $\beta = 0$, we take Theorem A with μ continuous measure and $q \in (0, +\infty)$.

(ii) For $q = 2$, we have measures such that $\overline{\Delta}_q(\nu) = \underline{\Delta}_q(\nu) = \underline{\Delta}_2(\mu) = \overline{\Delta}_2(\mu) = \frac{\beta}{2}$.

(iii) The results of theorem agree with the conjectures of [11].

Also for $q = \infty$, we have the following theorem.

THEOREM 3. *There exists an inhomogeneous measure μ given by (1) with ν purely continuous measure such that*

$$\overline{\Delta}_\infty(\nu) = \overline{\Delta}_\infty(\mu) = 0.$$

In view of the above theorems, it is natural to ask the following:

QUESTION. Does there exist an inhomogeneous measure μ given by (1) so as in the above theorems we additionally have that

$$\widehat{\nu}(x) \to 0, \text{ as } |x| \to \infty?$$

In a such case, it is well known that the measure ν is purely continuous and the converse does not hold, see [7].

NOTE 1. We work with Bernoulli convolution measures for which the Fourier–Stieltjes transforms do not tend to zero. If one requires the convergence to zero, then must work with numbers which are not Pisot, [5, Section 6.6] (the number 1/2 is Pisot number), and for such numbers we don't have estimations about the asymptotic behavior of the Fourier transform of the corresponding Bernoulli convolution, in contrary to some Pisot numbers for which we have estimations, see [4].

The proofs are given in the next section.

2. The construction of the measure μ

For the proof of Theorem 2, we need some notations and some well known results which are stated consequently. We observe that it is convenient to construct a continuous measure ν with the property that

$$C_1 R^{1-\beta-\varepsilon} \le \int_{|x|\le R} \left|\widehat{\nu}(x)\right|^2 dx \le C_2 R^{1-\beta+\varepsilon}, \quad R \to +\infty \tag{2}$$

for any $\varepsilon > 0$, where C_1 and C_2 are constants which are dependent only on ε. Then we will have that

$$\begin{aligned}\overline{\Delta}_2(\nu) &= \limsup_{R\to\infty} \frac{\log\left(\frac{1}{2R}\int_{|x|\le R} |\widehat{\nu}(x)|^2\,dx\right)^{\frac{1}{2}}}{-\log R} \\ &\le \limsup_{R\to\infty} \frac{\log\left(\frac{1}{2R} C_1 R^{1-\beta-\varepsilon}\right)^{\frac{1}{2}}}{-\log R} = \frac{\beta+\varepsilon}{2}.\end{aligned}$$

Using the other inequality, we take that

$$\underline{\Delta}_2(\nu) \ge \frac{\beta-\varepsilon}{2}.$$

Since the above holds for any $\varepsilon > 0$, we take that

$$\overline{\Delta}_2(\nu) = \underline{\Delta}_2(\nu) = \frac{\beta}{2}.$$

Also we want to inherit the property (2) to μ. For this, we take $n = 1$, $S_1 x = \frac{1}{2}x$ and

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\nu.$$

It is easy to see that we have

$$\widehat{\mu}(x) = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \widehat{\nu}\left(\frac{x}{2^j}\right).$$

By this, we take that

$$\left|\widehat{\mu}(x)\right|^2 = \sum_{j=0}^{\infty} \frac{1}{4^{j+1}} \left|\widehat{\nu}\left(\frac{x}{2^j}\right)\right|^2 + \sum_{\substack{i,j=0 \\ i\ne j}} \frac{1}{2^{i+1}} \frac{1}{2^{j+1}} \widehat{\nu}\left(\frac{x}{2^i}\right) \overline{\widehat{\nu}\left(\frac{x}{2^j}\right)}. \tag{3}$$

We observe that by the relation (3) we have the following estimation:

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{1}{4^{j+1}} \int_{|x|\le R} \left|\widehat{\nu}\left(\frac{x}{2^j}\right)\right|^2 dx &= \sum_{j=0}^{\infty} \frac{1}{2^{j+2}} \int_{|x|\le \frac{R}{2^j}} \left|\widehat{\nu}(y)\right|^2 dx \\ &\ge C \sum_{j=0}^{\infty} \frac{1}{2^{j+2}} \left(\frac{R}{2^j}\right)^{1-\beta-\varepsilon} = C' R^{1-\beta-\varepsilon},\end{aligned} \tag{4}$$

and an analogous inequality holds as an upper bound.

In order for the property (2) to be inherited to the inhomogeneous measure μ, we require the measure ν to be such that we have "cancellation" of the integrals of the terms $\widehat{\nu}(\frac{x}{2^i})\widehat{\nu}(\frac{x}{2^j})$.

In [1], we have studied the asymptotic behavior of the Fourier transforms of coin-tossing measures. These are probability measures on $[0,1]$ of the form

$$\nu' = \mathop{*}_{n=1}^{\infty}\left(\frac{1+a_n}{2}\delta(0) + \frac{1-a_n}{2}\delta\big(1/2^n\big)\right), \tag{5}$$

where $a_n \in [-1,1]$, $\delta(x)$ denotes the probability atom at x and the convergence is in the weak* sense. We proved the following theorem.

Theorem B. *If* $\varepsilon > 0$,

$$\beta = 1 - \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\log_2\big(1+a_n^2\big) \quad \textit{and} \quad \gamma = 1 - \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\log_2\big(1+a_n^2\big),$$

then there exist positive constants c_1 *and* c_2, *which are dependent only on* ε *such that*

$$c_2 R^{1-\gamma-\varepsilon} \le \int_{|x|\le R}\big|\widehat{\nu'}(x)\big|^2\,dx \le c_1 R^{1-\beta+\varepsilon}, \quad R\to+\infty.$$

We construct a measure ν_1' of the form (5) with some conditions. We want (see [5], [2])

$$\sum_{n=1}^{\infty} a_n^2 = +\infty \tag{6}$$

in order to have singular measure and

$$\sum_{n=1}^{\infty}\big(1-|a_n|\big) = +\infty \tag{7}$$

in order to have purely continuous measure.

In order for the relation (2) to hold, we take the value of γ so as to be equal to β. In other words, we need to have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\log_2\big(1+a_n^2\big) = 1-\beta. \tag{8}$$

We take a subsequence n_k of the natural numbers such that for the sequence a_n holds that $a_{n_k} = 0$, the other values of a_n to be 1 and simultaneously satisfy the relations (6), (7) and (8). It is easy for one to see that the measure ν_1' with the above values of the a_n is of Cantor type. The measure ν_1' is the Bernoulli convolution

$$\nu_1' = \mathop{*}_{k=1}^{\infty}\left(\frac{1}{2}\delta(0) + \frac{1}{2}\delta\big(1/2^{n_k}\big)\right)$$

for which we have that

$$\widehat{\nu_1'}(x) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi x}{2^{n_k}}\right) e^{i\frac{\pi x}{2^{n_k}}} \tag{9}$$

and so

$$\left|\widehat{\nu_1'}(x)\right|^2 = \prod_{k=1}^{\infty} \cos^2\left(\frac{\pi x}{2^{n_k}}\right).$$

In order to estimate the integrals of (3), we need to avoid the exponential term of the product in (9). This is possible by doing the convolution of the measure ν_1' with ν_0 where

$$\nu_0 = \mathop{*}_{k=1}^{\infty} \delta\left(-\frac{1}{2^{n_k+1}}\right).$$

The measure

$$\nu = \nu_1' * \nu_0$$

is given by the relation

$$\nu = \mathop{*}_{k=1}^{\infty} \left[\frac{1}{2}\delta\left(-\frac{1}{2^{n_k+1}}\right) + \frac{1}{2}\delta\left(\frac{1}{2^{n_k+1}}\right)\right]$$

a symmetric Bernoulli convolution measure, see [5]. The Fourier transform is given by the formula

$$\widehat{\nu}(x) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi x}{2^{n_k}}\right)$$

and so

$$\left|\widehat{\nu}(x)\right|^2 = \left|\widehat{\nu_1'}(x)\right|^2. \tag{10}$$

We need the following lemmas.

Lemma 1. *Let $a_j \in \mathbb{R}, j = 1, \dots, n, n \in \mathbb{N}$. Then we have that*

$$\sum_{\substack{\varepsilon_j \in \{\pm 1\} \\ j=1,\dots,n}} \cos\left(\sum_{j=1}^{n} \varepsilon_j a_j\right) = 2^n \prod_{j=1}^{n} \cos a_j.$$

Proof. Elementary properties of the trigonometric functions. □

Lemma 2. *The following equality holds*

$$\int_{|x|\le R} \widehat{\nu}\left(\frac{x}{2^i}\right)\widehat{\nu}\left(\frac{x}{2^j}\right) dx = \lim_{M\to\infty} \int_{|x|\le R} \prod_{k=1}^{M} \cos\left(\frac{\pi x}{2^{n_k+i}}\right) \prod_{k=1}^{M} \cos\left(\frac{\pi x}{2^{n_k+j}}\right) dx.$$

Proof. The proof is left to the reader. □

Proof of Theorem 2. We suppose that $i<j$. Now using Lemma 1 and trigonometric identities, we take that

$$\begin{aligned}
&\int_{|x|\le R}\prod_{k=1}^{M}\cos\Big(\frac{\pi x}{2^{n_k+i}}\Big)\prod_{k=1}^{M}\cos\Big(\frac{\pi x}{2^{n_k+j}}\Big)\,dx\\
&=\frac{1}{4^M}\int_{|x|\le R}\Bigg(\sum_{\substack{\varepsilon_k\in\{\pm1\}\\ k=1,\dots,M}}\cos\Big(\sum_{k=1}^{M}\varepsilon_k\frac{\pi x}{2^{n_k+i}}\Big)\Bigg)\\
&\quad\times\Bigg(\sum_{\substack{\varepsilon_k\in\{\pm1\}\\ k=1,\dots,M}}\cos\Big(\sum_{k=1}^{M}\varepsilon_k\frac{\pi x}{2^{n_k+j}}\Big)\Bigg)\,dx\\
&=\frac{1}{4^M}\sum_{\substack{\varepsilon_k,\varepsilon'_k\in\{\pm1\}\\ k=1,\dots,M}}\int_{|x|\le R}\cos\Big(\sum_{k=1}^{M}\varepsilon_k\frac{\pi x}{2^{n_k+i}}\Big)\cos\Big(\sum_{k=1}^{M}\varepsilon'_k\frac{\pi x}{2^{n_k+j}}\Big)\,dx\\
&=\frac{1}{4^M}\sum_{\substack{\varepsilon_k,\varepsilon'_k\in\{\pm1\}\\ k=1,\dots,M}}\int_{|x|\le R}\frac{1}{2}\cos\Big(\sum_{k=1}^{M}\varepsilon_k\frac{\pi x}{2^{n_k+i}}+\sum_{k=1}^{M}\varepsilon'_k\frac{\pi x}{2^{n_k+j}}\Big)\,dx\\
&\quad+\frac{1}{4^M}\sum_{\substack{\varepsilon_k,\varepsilon'_k\in\{\pm1\}\\ k=1,\dots,M}}\int_{|x|\le R}\frac{1}{2}\cos\Big(\sum_{k=1}^{M}\varepsilon_k\frac{\pi x}{2^{n_k+i}}-\sum_{k=1}^{M}\varepsilon'_k\frac{\pi x}{2^{n_k+j}}\Big)\,dx.
\end{aligned}$$

The absolute value of each of the above integrals is bounded from above by the quantity $c'2^{n_1+i}$, where c' is a constant, as one can see supposing that n_2 is sufficient large that n_1. Using Lemma 2 and the above, we take that

$$\begin{aligned}
&\Bigg|\sum_{\substack{i,j=0\\ i\ne j}}^{\infty}\int_{|x|\le R}\frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\widehat{\nu}\Big(\frac{x}{2^i}\Big)\widehat{\nu}\Big(\frac{x}{2^j}\Big)\,dx\Bigg|\\
&\quad=2\Bigg|\sum_{\substack{i,j=0\\ i<j}}^{\infty}\int_{|x|\le R}\frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\widehat{\nu}\Big(\frac{x}{2^i}\Big)\widehat{\nu}\Big(\frac{x}{2^j}\Big)\,dx\Bigg|\\
&\quad\le 2c'\sum_{\substack{i,j=0\\ i<j}}^{\infty}\frac{1}{2^{i+1}}\frac{1}{2^{j+1}}2^{n_1+i}\\
&\quad=c'2^{n_1}\sum_{j=1}^{\infty}\frac{j}{2^{j+1}}=C(n_1).
\end{aligned}\tag{11}$$

By the construction of the measure ν and combining the relations (3), (4), (10), (11), we have our result for $q=2$.

Next, we will see that the above is enough for $q>0$. Using Hölder's inequality for $q\geq 2$ we take that

$$\int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^2 dx \leq \left(\int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^q dx\right)^{2/q} \left(\int_{|x|\leq R} dx\right)^{1-\frac{2}{q}}$$

and so that

$$\int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^q dx \geq C(\varepsilon) R^{1-\frac{q}{2}(\beta+\varepsilon)}.$$

Also we have the obvious inequality that

$$\int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^q dx \leq \int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^2 dx \leq C(\varepsilon) R^{1-(\beta+\varepsilon)}.$$

That is for $q\geq 2$, we have that

$$C(\varepsilon) R^{1-\frac{q}{2}(\beta+\varepsilon)} \leq \int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^q dx \leq C(\varepsilon) R^{1-(\beta+\varepsilon)}. \tag{12}$$

For $0<q\leq 2$, we have

$$\int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^q dx \leq \left(\int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^2 dx\right)^{q/2} \left(\int_{|x|\leq R} dx\right)^{1-\frac{q}{2}}$$

and so that

$$\int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^q dx \leq C(\varepsilon) R^{1-\frac{q}{2}(\beta+\varepsilon)}.$$

Also we have that (since $|\widehat{\mu}(x)|\leq 1$)

$$\int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^q dx \geq \int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^2 dx \geq C(\varepsilon) R^{1-\beta-\varepsilon}.$$

That is for $0<q\leq 2$ we have that

$$C(\varepsilon) R^{1-(\beta+\varepsilon)} \leq \int_{|x|\leq R} \left|\widehat{\mu}(x)\right|^q dx \leq C(\varepsilon) R^{1-\frac{q}{2}(\beta+\varepsilon)}. \tag{13}$$

The relations (12) and (13) give our result. □

NOTE 2. The measure ν_1' is concentrated on a set with Hausdorff dimension equal to β, see [2]. Using the fact that the Hausdorff dimension is translation invariant, we also have that the measure ν is concentrated on a set with Hausdorff dimension equal to β, that is $\dim(\nu)=\beta$. Using the inequality (12) and [20, Corollary 8.7], we have that the inhomogeneous measure μ is concentrated on a set with Hausdorff dimension at least equal to β, that is $\dim(\mu)\geq\beta$.

Proof of Theorem 3. It is well known and easy to prove that the tetraedric Cantor measure ν_4 has the form

$$\nu_4 = \mathop{*}_{n=1}^{\infty}\left(\frac{1}{2}\delta(0) + \frac{1}{2}\delta\big(2/4^n\big)\right),$$

is purely continuous measure and it's Fourier–Stieltjes transform does not tend to 0 at infinity. It's Fourier transform is given by the formula

$$\widehat{\nu_4}(x) = e^{i\frac{x}{3}}\prod_{j=1}^{\infty}\cos\left(\frac{\pi x}{4^j}\right).$$

Since $|\widehat{\nu_4}(4^n)| = c \neq 0$ for any $n \in \mathbb{N}$, we have that $\underline{\Delta}_{\infty}(\nu) = \overline{\Delta}_{\infty}(\nu) = 0$. We take the inhomogeneous measure μ with $n = 1$ and $S_1 x = \frac{1}{4}x$ by the relation

$$\mu = \frac{1}{2}\mu o S_1^{-1} + \frac{1}{2}\nu.$$

Then we have that

$$\widehat{\mu}(x) = \sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\widehat{\nu_4}\left(\frac{x}{4^j}\right).$$

By the above and since the series converges uniformly, we take that

$$\lim_{n\to\infty}\widehat{\mu}\big(4^n\big) \to \widehat{\nu_4}(1) = e^{i\frac{\pi}{3}}\prod_{j=1}^{\infty}\cos\left(\frac{\pi}{4^j}\right) = c' \neq 0.$$

Using this, we take that $\underline{\Delta}_{\infty}(\mu) = \overline{\Delta}_{\infty}(\mu) = 0$. □

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