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EXTREMAL PROBLEMS IN BERGMAN SPACES AND AN EXTENSION OF RYABYKH'S THEOREM

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ABSTRACT. We study linear extremal problems in the Bergman space A^p of the unit disc for p an even integer. Given a functional on the dual space of A^p with representing kernel $k \in A^q$, where $1/p + 1/q = 1$, we show that if the Taylor coefficients of k are sufficiently small, then the extremal function $F \in H^\infty$. We also show that if $q \le q_1 < \infty$, then $F \in H^{(p-1)q_1}$ if and only if $k \in H^{q_1}$.

An analytic function f in the unit disc $\mathbb{D}$ is said to belong to the Bergman space A^p if

$$\|f\|_{A^p} = \left\{ \int_{\mathbb{D}} \left|f(z)\right|^p d\sigma(z) \right\}^{1/p} < \infty.$$

Here σ denotes normalized area measure, so that $\sigma(\mathbb{D}) = 1$. For $1 < p < \infty$, each functional $\phi \in (A^p)^*$ has a unique representation

$$\phi(f) = \int_{\mathbb{D}} f \overline{k} \, d\sigma$$

for some $k \in A^q$, where $q = p/(p-1)$ is the conjugate index. The function k is called the kernel of the functional ϕ.

In this paper, we study the extremal problem of maximizing $\operatorname{Re} \phi(f)$ among all functions $f \in A^p$ of unit norm. If $1 < p < \infty$, then an extremal function always exists and is unique. However, to find it explicitly is in general a difficult problem, and few explicit solutions are known. Here we consider the problem of determining whether the kernel being "well-behaved" implies that the extremal function is also "well-behaved." A known result in this direction is Ryabykh's theorem, which states that if the kernel is actually in the Hardy space H^q, then the extremal function must be in the Hardy space H^p. In [4],

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we gave a proof of Ryabykh's theorem based on general properties of extremal functions in uniformly convex spaces.

In this paper, we obtain a sharper version of Ryabykh's theorem in the case where p is an even integer. Our results are:

- For $q \le q_1 < \infty$, the extremal function $F \in H^{(p-1)q_1}$ if and only if the kernel $k \in H^{q_1}$.
- If the Taylor coefficients of k are "small enough," then $F \in H^\infty$.
- The map sending a kernel $k \in H^q$ to its extremal function $F \in A^p$ is a continuous map from $H^q \setminus 0$ into H^p.

Our proofs rely heavily on Littlewood–Paley theory, and seem to require that p be an even integer. It is an open problem whether the results hold without this assumption.

1. Extremal problems and Ryabykh's theorem

We begin with some notation. If f is an analytic function, $S_n f$ denotes its nth Taylor polynomial at the origin. Lebesgue area measure is denoted by dA, and $d\sigma$ denotes normalized area measure.

If h is a measurable function in the unit disc, the principal value of its integral is

$$\text{p.v.} \int_{\mathbb{D}} h\, dA = \lim_{r \to 1} \int_{r\mathbb{D}} h\, dA,$$

if the limit exists.

We now recall some basic facts about Hardy and Bergman spaces. For proofs and further information, see [2] and [3]. Suppose that f is analytic in the unit disc. For $0 < p < \infty$ and $0 < r < 1$, the integral mean of f is

$$M_p(f, r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta \right\}^{1/p}.$$

If $p = \infty$, we write

$$M_\infty(f, r) = \max_{0 \le \theta < 2\pi} \left| f\left(re^{i\theta}\right)\right|.$$

For fixed f and p, the integral means are increasing functions of r. If $M_p(f, r)$ is bounded we say that f is in the Hardy space H^p. For any function f in H^p, the radial limit $f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ exists for almost every θ. An H^p function is uniquely determined by the values of its boundary function on any set of positive measure. The space H^p is a Banach space with norm

$$\|f\|_{H^p} = \sup_r M_p(f, r) = \left\| f\left(e^{i\theta}\right)\right\|_{L^p}.$$

It is useful to regard H^p as a subspace of $L^p(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle. For $0 < p < \infty$, if $f \in H^p$, then $f(re^{i\theta})$ converges to $f(e^{i\theta})$ in L^p norm as $r \to 1$.

For $1 < p < \infty$, the dual space $(H^p)^*$ is isomorphic to H^q, where $1/p+1/q = 1$, with an element $k \in H^q$ representing the functional ϕ defined by

$$\phi(f) = \frac{1}{2\pi}\int_0^{2\pi} f\big(e^{i\theta}\big)\overline{k\big(e^{i\theta}\big)}\,d\theta.$$

This isomorphism is not an isometry unless $p = 2$, but it is true that $\|\phi\|_{(H^p)^*} \leq \|k\|_{H^q} \leq C\|\phi\|_{(H^p)^*}$ for some constant C depending only on p. If $f \in H^p$ for $1 < p < \infty$, then $S_n f \to f$ in H^p as $n \to \infty$. The Szegő projection S maps each function $f \in L^1(\mathbb{T})$ into a function analytic in $\mathbb{D}$ defined by

$$Sf(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{1 - e^{-it}z}\,dt.$$

It leaves H^1 functions fixed and maps L^p boundedly onto H^p for $1 < p < \infty$. If $f \in L^p$ for $1 < p < \infty$ and $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, then $Sf(z) = \sum_{n=0}^{\infty} a_n z^n$.

For $1 < p < \infty$, the dual of the Bergman space A^p is isomorphic to A^q, where $1/p + 1/q = 1$, and $k \in A^q$ represents the functional defined by $\phi(f) = \int_{\mathbb{D}} f(z)\overline{k(z)}\,d\sigma(z)$. Note that this isomorphism is actually conjugate-linear. It is not an isometry unless $p = 2$, but if the functional $\phi \in (A^p)^*$ is represented by the function $k \in A^q$, then

$$\|\phi\|_{(A^p)^*} \leq \|k\|_{A^q} \leq C_p\|\phi\|_{(A^p)^*}, \tag{1.1}$$

where C_p is a constant depending only on p. We remark that $H^p \subset A^p$, and in fact $\|f\|_{A^p} \leq \|f\|_{H^p}$. If $f \in A^p$ for $1 < p < \infty$, then $S_n f \to f$ in A^p as $n \to \infty$.

In this paper, the only Bergman spaces we consider are those with $1 < p < \infty$. For a given linear functional $\phi \in (A^p)^*$ such that $\phi \neq 0$, we investigate the extremal problem of finding a function $F \in A^p$ with norm $\|F\|_{A^p} = 1$ for which

$$\operatorname{Re}\phi(F) = \sup_{\|g\|_{A^p}=1} \operatorname{Re}\phi(g) = \|\phi\|. \tag{1.2}$$

Such a function F is called an extremal function, and we say that F is an extremal function for a function $k \in A^q$ if F solves problem (1.2) for the functional ϕ with kernel k. This problem has been studied by Vukotić [10], Khavinson and Stessin [7], and Ferguson [4], among others. Note that for $p = 2$, the extremal function is $F = k/\|k\|_{A^2}$.

A closely related problem is that of finding $f \in A^p$ such that $\phi(f) = 1$ and

$$\|f\|_{A^p} = \inf_{\phi(g)=1} \|g\|_{A^p}. \tag{1.3}$$

If F solves the problem (1.2), then $\frac{F}{\phi(F)}$ solves the problem (1.3), and if f solves (1.3), then $\frac{f}{\|f\|}$ solves (1.2). When discussing either of these problems, we always assume that ϕ is not the zero functional; in other words, that k is not identically 0.

The problems (1.2) and (1.3) each have a unique solution when $1 < p < \infty$ (see [4], Theorem 1.4). Also, for every function $f \in A^p$ such that f is not identically 0, there is a unique $k \in A^q$ such that f solves problem (1.3) for k (see [4], Theorem 3.3). This implies that for each $F \in A^p$ with $\|F\|_{A^p} = 1$, there is some nonzero k such that F solves problem (1.2) for k. Furthermore, any two such kernels k are positive multiples of each other.

The Cauchy–Green theorem is an important tool in this paper.

Cauchy–Green Theorem. *If Ω is a region in the plane with piecewise smooth boundary and $f \in C^1(\overline{\Omega})$, then*

$$\frac{1}{2i}\int_{\partial\Omega} f(z)\,dz = \int_{\Omega} \frac{\partial}{\partial \overline{z}} f(z)\,dA(z),$$

where $\partial\Omega$ denotes the boundary of Ω.

The next result is an important characterization of extremal functions in A^p for $1 < p < \infty$ (see [9], p. 55).

Theorem A. *Let $1 < p < \infty$ and let $\phi \in (A^p)^*$. A function $F \in A^p$ with $\|F\|_{A^p} = 1$ satisfies*

$$\operatorname{Re}\phi(F) = \sup_{\|g\|_{A^p}=1} \operatorname{Re}\phi(g) = \|\phi\|$$

if and only if

$$\int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F}\,d\sigma = 0$$

for all $h \in A^p$ with $\phi(h) = 0$. If F satisfies the above conditions, then

$$\int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F}\,d\sigma = \frac{\phi(h)}{\|\phi\|}$$

for all $h \in A^p$.

Ryabykh's theorem relates extremal problems in Bergman spaces to Hardy spaces. It says that if the kernel for a linear functional is not only in A^q but also in H^q, then the extremal function is not only in A^p but in H^p as well.

Ryabykh's Theorem. *Let $1 < p < \infty$ and let $1/p + 1/q = 1$. Suppose that $\phi \in (A^p)^*$ and $\phi(f) = \int_{\mathbb{D}} f\overline{k}\,d\sigma$ for some $k \in H^q$. Then the solution F to the extremal problem (1.2) belongs to H^p and satisfies*

$$\|F\|_{H^p} \le \left\{ \left[\max(p-1,1)\right] \frac{C_p\|k\|_{H^q}}{\|k\|_{A^q}} \right\}^{1/(p-1)}, \tag{1.4}$$

where C_p is the constant in (1.1).

Ryabykh [8] proved that $F \in H^p$. The bound (1.4) was proved in [4], by a variant of Ryabykh's proof.

As a corollary Ryabykh's theorem implies that the solution to the problem (1.3) is in H^p as well. Note that the constant $C_p \to \infty$ as $p \to 1$ or $p \to \infty$.

To obtain our results, including a generalization of Ryabykh's theorem, we will need the following technical lemmas. Their proofs, which involve Littlewood–Paley theory, are deferred to the end of the paper.

LEMMA 1.1. *Let p be an even integer. Let $f \in H^p$ and let h be a polynomial. Then*

$$\text{p.v.} \int_{\mathbb{D}} |f|^{p-1} \overline{\operatorname{sgn} f} f' h \, d\sigma = \lim_{n\to\infty} \int_{\mathbb{D}} |f|^{p-1} \overline{\operatorname{sgn} f} (S_n f)' h \, d\sigma.$$

LEMMA 1.2. *Suppose that $1 < p_1 < \infty$ and $1 < p_2, p_3 \le \infty$, and also that*

$$1 = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

Let $f_1 \in H^{p_1}$, $f_2 \in H^{p_2}$, and $f_3 \in H^{p_3}$. Then

$$\left| \text{p.v.} \int_{\mathbb{D}} \overline{f_1} f_2 f_3' \, d\sigma \right| \le C \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}} \|f_3\|_{H^{p_3}},$$

where C depends only on p_1 and p_2. (Implicit is the claim that the principal value exists.) Moreover, if $p_3 < \infty$, then

$$\text{p.v.} \int_{\mathbb{D}} \overline{f_1} f_2 f_3' \, d\sigma = \lim_{n\to\infty} \int_{\mathbb{D}} \overline{f_1} f_2 (S_n f_3)' \, d\sigma.$$

2. The norm-equality

Let p be an even integer and let q be its conjugate exponent. Let $k \in H^q$ and let F be the extremal function for k over A^p. We will denote by ϕ the functional associated with k. Let F_n be the extremal function for k when the extremal problem is posed over P_n, the space of polynomials of degree at most n. Also, let

$$K(z) = \frac{1}{z} \int_0^z k(\zeta) \, d\zeta, \tag{2.1}$$

so that $(zK)' = k$. During proof of Ryabykh's theorem in [4], an important step is to show that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| F_n(e^{i\theta}) \right|^p d\theta = \frac{1}{2\pi \|\phi_{|P_n}\|} \int_0^{2\pi} F_n \left[\left(\frac{p}{2} \right) \overline{k} + \left(1 - \frac{p}{2} \right) \overline{K} \right] d\theta$$

(see [4], p. 2652). We will now derive a similar result for F:

THEOREM 2.1. *Let p be an even integer, let $k \in H^q$, and let $F \in A^p$ be the extremal function for k. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \left| F(e^{i\theta}) \right|^p h(e^{i\theta}) \, d\theta = \frac{1}{2\pi \|\phi\|} \int_0^{2\pi} F \left[\left(\frac{p}{2} \right) h \overline{k} + \left(1 - \frac{p}{2} \right) (zh)' \overline{K} \right] d\theta$$

for every polynomial h.

Proof. Since Ryabykh's theorem says that $F \in H^p$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^p h\left(e^{i\theta}\right) d\theta = \lim_{r\to 1}\frac{i}{2\pi}\int_{\partial(r\mathbb{D})} \left|F(z)\right|^p h(z) z\, d\overline{z},$$

where h is any polynomial. Apply the Cauchy–Green theorem to transform the right-hand side into

$$\text{p.v.}\, \frac{1}{\pi}\int_{\mathbb{D}} \left((zh)'F + \frac{p}{2} zhF'\right)|F|^{p-1}\overline{\operatorname{sgn} F}\, dA(z).$$

Invoking Lemma 1.1 with zh in place of h shows that this limit equals

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{\mathbb{D}} \left((zh)'F + \frac{p}{2} zh(S_nF)'\right)|F|^{p-1}\overline{\operatorname{sgn} F}\, dA(z).$$

Since $(zh)'F + \frac{p}{2}zh(S_nF)'$ is in A^p, we may apply Theorem A to reduce the last expression to

$$\lim_{n\to\infty}\frac{1}{\pi\|\phi\|}\int_{\mathbb{D}} \left((zh)'F + \frac{p}{2} zh(S_nF)'\right)\overline{k}\, dA(z). \tag{2.2}$$

Recall that we have defined $K(z) = \frac{1}{z}\int_0^z k(\zeta)\, d\zeta$. To prepare for a reverse application of the Cauchy–Green theorem, we rewrite the integral in (2.2) as

$$\frac{1}{\pi\|\phi\|}\int_{\mathbb{D}} \left[\frac{\partial}{\partial\overline{z}}\left\{(zh)'F\overline{zK}\right\} + \frac{p}{2}\frac{\partial}{\partial z}\left\{zhS_n(F)\overline{k}\right\} \right.$$
$$\left. - \frac{p}{2}\frac{\partial}{\partial\overline{z}}\left\{(zh)'S_n(F)\overline{zK}\right\}\right] dA(z).$$

Now this equals

$$\lim_{r\to 1}\frac{1}{\pi\|\phi\|}\int_{r\mathbb{D}} \left[\frac{\partial}{\partial\overline{z}}\left\{(zh)'F\overline{zK}\right\} + \frac{p}{2}\frac{\partial}{\partial z}\left\{zhS_n(F)\overline{k}\right\} \right.$$
$$\left. - \frac{p}{2}\frac{\partial}{\partial\overline{z}}\left\{(zh)'S_n(F)\overline{zK}\right\}\right] dA(z).$$

We apply the Cauchy–Green theorem to show that this equals

$$\lim_{r\to 1}\left[\frac{1}{2\pi i\|\phi\|}\int_{\partial(r\mathbb{D})} (zh)'F\overline{zK}\, dz + \frac{ip}{4\pi\|\phi\|}\int_{\partial(r\mathbb{D})} zhS_n(F)\overline{k}\, d\overline{z} \right.$$
$$\left. - \frac{p}{4\pi i\|\phi\|}\int_{\partial(r\mathbb{D})} (zh)'S_n(F)\overline{zK}\, dz\right].$$

Since F is in H^p and both k and K are in H^q, the above limit equals

$$\frac{1}{2\pi i\|\phi\|}\int_{\partial\mathbb{D}} (zh)'F\overline{zK}\, dz + \frac{ip}{4\pi\|\phi\|}\int_{\partial\mathbb{D}} zhS_n(F)\overline{k}\, d\overline{z}$$
$$- \frac{p}{4\pi i\|\phi\|}\int_{\partial\mathbb{D}} (zh)'S_n(F)\overline{zK}\, dz$$
$$= \frac{1}{2\pi\|\phi\|}\int_0^{2\pi} (zh)'F\overline{K} + S_n(F)\left(\frac{p}{2}h\overline{k} - \frac{p}{2}(zh)'\overline{K}\right) d\theta.$$

We let $n \to \infty$ in the above expression to reach the desired conclusion. $\square$

Taking $h = 1$, we have the following corollary, which we call the "norm-equality."

COROLLARY 2.2 (The norm-equality). *Let p be an even integer, let $k \in H^q$, and let F be the extremal function for k. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^p d\theta = \frac{1}{2\pi\|\phi\|}\int_0^{2\pi} F\left[\left(\frac{p}{2}\right)\overline{k} + \left(1 - \frac{p}{2}\right)\overline{K}\right] d\theta.$$

The norm-equality is useful mainly because it yields the following theorem.

THEOREM 2.3. *Let p be an even integer. Let $\{k_n\}$ be a sequence of H^q functions, and let $k_n \to k$ in H^q. Let F_n be the A^p extremal function for k_n and let F be the A^p extremal function for k. Then $F_n \to F$ in H^p.*

Note that Ryabykh's theorem shows that each $F_n \in H^p$, and that $F \in H^p$. But because the operator taking a kernel to its extremal function is not linear, one cannot apply the closed graph theorem to conclude that $F_n \to F$.

To prove Theorem 2.3 we will use the following lemma involving the notion of uniform convexity. A Banach space X is called *uniformly convex* if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$,

$$\left\|\frac{1}{2}(x+y)\right\| > 1 - \delta \quad \text{implies} \quad \|x - y\| < \varepsilon.$$

An equivalent definition is that if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\|x_n\| = \|y_n\| = 1$ for all n and $\|x_n + y_n\| \to 2$ then $\|x_n - y_n\| \to 0$. This concept was introduced by Clarkson in [1]. See also [4], where it is applied to extremal problems. To apply the lemma, we use the fact that the space H^p is uniformly convex for $1 < p < \infty$. By $x_n \rightharpoonup x$, we mean that x_n approaches x weakly.

LEMMA 2.4. *Suppose that X is a uniformly convex Banach space, that $x \in X$, and that $\{x_n\}$ is a sequence of elements of X. If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ in X.*

This lemma is known. For example, it is contained in Exercise 15.17 in [6].

Proof of Theorem 2.3. We will first show that $F_n \rightharpoonup F$ in H^p (that is, F_n converges to F weakly in H^p). Next, we will use this fact and the norm-equality to show that $\|F_n\|_{H^p} \to \|F\|_{H^p}$. By the lemma, it will then follow that $F_n \to F$ in H^p.

To prove that $F_n \rightharpoonup F$ in H^p, note that Ryabykh's theorem says that $\|F_n\|_{H^p} \le C(\|k_n\|_{H^q}/\|k_n\|_{A^q})^{1/(p-1)}$. Let $\alpha = \inf_n \|k_n\|_{A^q}$ and $\beta = \sup_n \|k_n\|_{H^q}$. Here $\alpha > 0$ because by assumption none of the k_n are identically zero, and they approach k, which is not identically 0. Therefore $\|F_n\|_{H^p} \le C(\beta/\alpha)^{1/(p-1)}$, and the sequence $\{F_n\}$ is bounded in H^p norm.

Now, suppose that $F_n \not\rightharpoonup F$. Then there is some $\psi \in (H^p)^*$ such that $\psi(F_n) \not\rightharpoonup \psi(F)$. This implies $|\psi(F_{n_j}) - \psi(F)| \geq \varepsilon$ for some $\varepsilon > 0$ and some subsequence $\{F_{n_j}\}$. But since the sequence $\{F_n\}$ is bounded in H^p norm, the Banach–Alaoglu theorem implies that some subsequence of $\{F_{n_j}\}$, which we will also denote by $\{F_{n_j}\}$, converges weakly in H^p to some function $\widetilde{F}$. Then $|\psi(\widetilde{F}) - \psi(F)| \geq \varepsilon$. Now $k_n \to k$ in A^q, and it is proved in [4] that this implies $F_n \to F$ in A^p, which implies $F_n(z) \to F(z)$ for all $z \in \mathbb{D}$. Since point evaluation is a bounded linear functional on H^p, we have that $F_{n_j}(z) \to \widetilde{F}(z)$ for all $z \in \mathbb{D}$, which means that $\widetilde{F}(z) = F(z)$ for all $z \in \mathbb{D}$. But this contradicts the assumption that $\psi(\widetilde{F}) \neq \psi(F)$. Hence, $F_n \rightharpoonup F$.

Let ϕ_n be the functional with kernel k_n, and let ϕ be the functional with kernel k. To show that $\|F_n\|_{H^p} \to \|F\|_{H^p}$, recall that the norm-equality says

$$\frac{1}{2\pi} \int_0^{2\pi} \left|F_n\left(e^{i\theta}\right)\right|^p d\theta = \frac{1}{2\pi\|\phi_n\|} \int_0^{2\pi} F_n \left[\left(\frac{p}{2}\right)\overline{k_n} + \left(1 - \frac{p}{2}\right)\overline{K_n} \right] d\theta.$$

But, if h is any function analytic in $\mathbb{D}$ and $H(z) = (1/z)\int_0^z h(\zeta)\,d\zeta$, it can be shown that $\|H\|_{H^q} \leq \|h\|_{H^q}$ (see [4], proof of Theorem 4.2). Since $k_n \to k$ in H^q, it follows that $K_n \to K$ in H^q. Also, $k_n \to k$ in A^p implies that $\|\phi_n\| \to \|\phi\|$. In addition, $\|F_n\|_{H^p} \leq C$ for some constant C, and $F_n \rightharpoonup F$, so the right-hand side of the above equation approaches

$$\frac{1}{2\pi\|\phi\|} \int_0^{2\pi} F \left[\left(\frac{p}{2}\right)\overline{k} + \left(1 - \frac{p}{2}\right)\overline{K} \right] d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^p d\theta.$$

In other words, $\|F_n\|_{H^p} \to \|F\|_{H^p}$, and so by Lemma 2.4 we conclude that $F_n \to F$ in H^p. □

3. Fourier coefficients of $|F|^p$

Theorem 2.1 can also be used to gain information about the Fourier coefficients of $|F|^p$, where F is the extremal function. In particular, it leads to a criterion for F to be in L^∞ in terms of the Taylor coefficients of the kernel k.

THEOREM 3.1. *Let p be an even integer. Let $k \in H^q$, let F be the A^p extremal function for k, and define K by equation (2.1). Then for any integer $m \geq 0$,*

$$\frac{1}{2\pi} \int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^p e^{im\theta}\, d\theta$$
$$= \frac{1}{2\pi\|\phi\|} \int_0^{2\pi} F e^{im\theta} \left[\left(\frac{p}{2}\right)\overline{k} + \left(1 - \frac{p}{2}\right)(m+1)\overline{K} \right] d\theta.$$

Proof. Take $h(e^{i\theta}) = e^{im\theta}$ in Theorem 2.1. □

This last formula can be applied to obtain estimates on the size of the Fourier coefficients of $|F|^p$.

THEOREM 3.2. *Let p be an even integer. Let $k \in A^q$, and let F be the A^p extremal function for k. Let*

$$b_m = \frac{1}{2\pi} \int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^p e^{-im\theta}\, d\theta,$$

and let

$$k(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then, for each $m \geq 0$,

$$|b_m| = |b_{-m}| \leq \frac{p}{2\|\phi\|} \|F\|_{H^2} \left[\sum_{n=m}^{\infty} |c_n|^2\right]^{1/2}.$$

Proof. The theorem is trivially true if $k \notin H^2$, so we may assume that $k \in A^2 \subset A^q$. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $F \in H^p$, and $p \geq 2$, we have $F \in H^2$. Now, using Theorem 3.1, we find that

$$\begin{aligned}
b_{-m} &= \frac{1}{2\pi} \int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^p e^{im\theta}\, d\theta \\
&= \frac{1}{2\pi\|\phi\|} \int_0^{2\pi} \left(F e^{im\theta}\right) \left[\left(\frac{p}{2}\right)\overline{k} + \left(1 - \frac{p}{2}\right)(m+1)\overline{K}\right] d\theta \\
&= \frac{1}{2\pi\|\phi\|} \int_0^{2\pi} \left[\sum_{n=0}^{\infty} a_n e^{i(n+m)\theta}\right] \\
&\qquad \times \left[\sum_{j=0}^{\infty} \left(\left(\frac{p}{2}\right)\overline{c_j} + \frac{m+1}{j+1}\left(1 - \frac{p}{2}\right)\overline{c_j}\right) e^{-ij\theta}\right] d\theta \\
&= \frac{1}{\|\phi\|} \left|\sum_{n=0}^{\infty} a_n \left(\left(\frac{p}{2}\right)\overline{c_{n+m}} + \frac{m+1}{n+m+1}\left(1 - \frac{p}{2}\right)\overline{c_{n+m}}\right)\right|.
\end{aligned}$$

The Cauchy–Schwarz inequality now gives

$$\begin{aligned}
|b_{-m}| &\leq \frac{1}{\|\phi\|} \left[\sum_{n=0}^{\infty} |a_n|^2\right]^{1/2} \left[\sum_{n=m}^{\infty} \left|\left(\frac{p}{2}\right)\overline{c_n} + \frac{m+1}{n+1}\left(1 - \frac{p}{2}\right)\overline{c_n}\right|^2\right]^{1/2} \\
&\leq \frac{p}{2\|\phi\|} \left[\sum_{n=0}^{\infty} |a_n|^2\right]^{1/2} \left[\sum_{n=m}^{\infty} |c_n|^2\right]^{1/2}.
\end{aligned}$$

Since

$$\left[\sum_{n=0}^{\infty} |a_n|^2\right]^{1/2} = \|F\|_{H^2}$$

the theorem follows. $\square$

The estimate in Theorem 3.2 can be used to obtain information about the size of $|F|^p$ and F, as in the following corollary.

COROLLARY 3.3. *If $c_n = O(n^{-\alpha})$ for some $\alpha > 3/2$, then $F \in H^\infty$.*

Proof. First observe that

$$\sum_{n=m}^{\infty} \left(n^{-\alpha}\right)^2 \le \int_{m-1}^{\infty} x^{-2\alpha}\, dx = \frac{(m-1)^{1-2\alpha}}{2\alpha - 1}.$$

By hypothesis it follows that

$$\left[\sum_{n=m}^{\infty} |c_n|^2\right]^{1/2} = O\big(m^{(1-2\alpha)/2}\big).$$

Thus, Theorem 3.2 shows that $b_m = O(m^{(1-2\alpha)/2})$. Therefore $\{b_m\} \in \ell^1$ if $\alpha > 3/2$. But $\{b_m\} \in \ell^1$ implies $|F|^p \in L^\infty$, which implies $F \in H^\infty$. □

In fact, $\{b_m\} \in \ell^1$ implies that $|F|^p$ is continuous in $\overline{\mathbb{D}}$, but this does not necessarily mean F will be continuous in $\overline{\mathbb{D}}$. There is a result similar to Corollary 3.3 in [7], where the authors show that if the kernel k is a polynomial, or even a rational function with no poles in $\overline{\mathbb{D}}$, then F is Hölder continuous in $\overline{\mathbb{D}}$. Their technique relies on deep regularity results for partial differential equations. Our result only shows that $F \in H^\infty$, but it applies to a broader class of kernels.

4. Relations between the size of the kernel and extremal function

In this section, we show that if p is an even integer and $q \le q_1 < \infty$, then the extremal function $F \in H^{(p-1)q_1}$ if and only if the kernel $k \in H^{q_1}$. For $q_1 = q$ the statement reduces to Ryabykh's theorem and its previously unknown converse. The following theorem is crucial to the proof.

THEOREM 4.1. *Let p be an even integer and let $q = p/(p-1)$ be its conjugate exponent. Let $F \in A^p$ be the extremal function corresponding to the kernel $k \in A^q$. Suppose that $k \in H^{q_1}$ for some q_1 with $q \le q_1 < \infty$, and that $F \in H^{p_1}$, for some p_1 with $p \le p_1 < \infty$. Define p_2 by*

$$\frac{1}{q_1} + \frac{1}{p_1} + \frac{1}{p_2} = 1.$$

If $p_2 < \infty$, then for every trigonometric polynomial h we have

$$\left| \int_0^{2\pi} |F|^p h\big(e^{i\theta}\big)\, d\theta \right| \le C \frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \|F\|_{H^{p_1}} \|h\|_{L^{p_2}},$$

where C is some constant depending only on p, p_1, and q_1.

The excluded case $p_2 = \infty$ occurs if and only if $q = q_1$ and $p = p_1$. The theorem is then a trivial consequence of Ryabykh's theorem.

Proof of Theorem 4.1. First, let h be an analytic polynomial. In the proof of Theorem 2.1, we showed that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^p h\left(e^{i\theta}\right) d\theta \\ = \lim_{n\to\infty} \frac{1}{\pi\|\phi\|} \int_{\mathbb{D}} \left((hz)'F + \frac{p}{2}hz(S_nF)'\right)\overline{k}\, dA(z). \end{aligned} \tag{4.1}$$

An application of Lemma 1.2 gives

$$\lim_{n\to\infty} \int_{\mathbb{D}} hz(S_nF)'\overline{k}\, dA = \text{p.v.} \int_{\mathbb{D}} hzF'\overline{k}\, dA,$$

so that the right-hand side of equation (4.1) becomes

$$\frac{1}{\pi\|\phi\|}\ \text{p.v.} \int_{\mathbb{D}} \left((hz)'F + \frac{p}{2}hzF'\right)\overline{k}\, dA(z).$$

Apply Lemma 1.2 separately to the two parts of the integral to conclude that its absolute value is bounded by

$$C\frac{1}{\|\phi\|}\|k\|_{H^{q_1}}\|f\|_{H^{p_1}}\|h\|_{H^{p_2}},$$

where C is a constant depending only on p_1 and q_1. Since

$$\frac{1}{\|\phi\|} \le \frac{C_p}{\|k\|_{A^q}}$$

by equation (1.1), this gives the desired result for the special case where h is an analytic polynomial.

Now let h be an arbitrary trigonometric polynomial. Then $h = h_1 + \overline{h_2}$, where h_1 and h_2 are analytic polynomials, and $h_2(0) = 0$. Note that the Szegő projection S is bounded from L^{p_2} into H^{p_2} because $1 < p_2 < \infty$. Thus,

$$\|h_1\|_{H^{p_2}} = \left\|S(h)\right\|_{H^{p_2}} \le C\|h\|_{L^{p_2}}.$$

Also,

$$\|h_2\|_{H^{p_2}} = \left\|zS\left(e^{-i\theta}\overline{h}\right)\right\|_{H^{p_2}} = \left\|S\left(e^{-i\theta}\overline{h}\right)\right\|_{H^{p_2}} \le C\left\|e^{-i\theta}\overline{h}\right\|_{L^{p_2}} = C\|h\|_{L^{p_2}},$$

and so

$$\|h_1\|_{H^{p_2}} + \|h_2\|_{H^{p_2}} \le C\|h\|_{L^{p_2}}.$$

Therefore, by what we have already shown,

$$\begin{aligned} \left|\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^p h\left(e^{i\theta}\right) d\theta\right| &= \left|\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^p \left(h_1\left(e^{i\theta}\right) + \overline{h_2\left(e^{i\theta}\right)}\right) d\theta\right| \\ &\le \left|\int_0^{2\pi} |f|^p h_1\, d\theta\right| + \left|\overline{\int_0^{2\pi} |f|^p h_2\, d\theta}\right| \end{aligned}$$

$$
\begin{aligned}
&\le C\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\|f\|_{H^{p_1}}\big(\|h_1\|_{H^{p_2}}+\|h_2\|_{H^{p_2}}\big)\\
&\le C\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\|f\|_{H^{p_1}}\|h\|_{L^{p_2}}. \qquad \square
\end{aligned}
$$

For a given q_1, we will apply the theorem just proved with p_1 chosen as $p_1 = pp_2'$, where p_2' is the conjugate exponent to p_2. This will allow us to bound the H^{p_1} norm of f solely in terms of $\|\phi\|$ and $\|k\|_{H^{q_1}}$.

THEOREM 4.2. *Let p be an even integer, and let q be its conjugate exponent. Let $F \in A^p$ be the extremal function for a kernel $k \in A^q$. If, for q_1 such that $q \le q_1 < \infty$, the kernel $k \in H^{q_1}$, then $F \in H^{p_1}$ for $p_1 = (p-1)q_1$. In fact,*

$$\|F\|_{H^{p_1}} \le C\left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\right)^{1/(p-1)},$$

where C depends only on p and q_1.

Proof. The case $q_1 = q$ is Ryabykh's theorem, so we assume $q_1 > q$. Set $p_1 = (p-1)q_1$. Then $p_1 > p = (p-1)q$. Choose p_2 so that

$$\frac{1}{q_1}+\frac{1}{p_1}+\frac{1}{p_2}=1.$$

This implies that $p_2 = p_1/(p_1 - p)$, and so its conjugate exponent $p_2' = p_1/p$. Note that $1 < p_2 < \infty$. Let F_n denote the extremal function corresponding to the kernel S_nk, which does not vanish identically if n is chosen sufficiently large. Since S_nk is a polynomial, F_n is in H^∞ (and thus $F_n \in H^{p_1}$) by Corollary 3.3. Hence for any trigonometric polynomial h, Theorem 4.1 yields

$$\left|\frac{1}{2\pi}\int_0^{2\pi}|F_n|^p h(e^{i\theta})\,d\theta\right| \le C\frac{\|S_nk\|_{H^{q_1}}}{\|S_nk\|_{A^q}}\|F_n\|_{H^{p_1}}\|h\|_{L^{p_2}}.$$

Since the trigonometric polynomials are dense in $L^{p_2}(\partial\mathbb{D})$, taking the supremum over all trigonometric polynomials h with $\|h\|_{L^{p_2}} \le 1$ gives

$$\big\||F_n|^p\big\|_{L^{p_2'}} \le C\frac{\|S_nk\|_{H^{q_1}}}{\|S_nk\|_{A^q}}\|F_n\|_{H^{p_1}},$$

which implies

$$
\begin{aligned}
\|F_n\|_{H^{p_1}}^p &= \left\{\frac{1}{2\pi}\int_0^{2\pi}\big(|F_n(e^{i\theta})|^p\big)^{p_2'}\,d\theta\right\}^{1/p_2'} = \big\||F_n|^p\big\|_{L^{p_2'}}\\
&\le C\frac{\|S_nk\|_{H^{q_1}}}{\|S_nk\|_{A^q}}\|F_n\|_{H^{p_1}},
\end{aligned}
$$

since $pp_2' = p_1$. Because $\|F_n\|_{H^{p_1}} < \infty$, we may divide both sides of the inequality by $\|F_n\|_{H^{p_1}}$ to obtain

$$\|F_n\|_{H^{p_1}}^{p-1} \le C\frac{\|S_nk\|_{H^{q_1}}}{\|S_nk\|_{A^q}},$$

where C depends only on p and q_1. In other words,

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|F_n\left(re^{i\theta}\right)\right|^{p_1}d\theta\right)^{(p-1)/p_1}\leq C\frac{\|S_nk\|_{H^{q_1}}}{\|S_nk\|_{A^q}}$$

for all $r<1$ and for all n sufficiently large. Note that $S_nk\to k$ in H^{q_1} and in A^q. Since $S_nk\to k$ in A^q, Theorem 3.1 in [4] says that $F_n\to F$ in A^p, and thus $F_n\to F$ uniformly on compact subsets of $\mathbb{D}$. Thus, letting $n\to\infty$ in the last inequality gives

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|F\left(re^{i\theta}\right)\right|^{p_1}d\theta\right)^{(p-1)/p_1}\leq C\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}$$

for all $r<1$. In other words,

$$\|F\|_{H^{p_1}}\leq\left(C\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\right)^{1/(p-1)}. \qquad \square$$

Recall from Section 1 that a function $F\in A^p$ with unit norm has a corresponding kernel $k\in A^q$ such that F is the extremal function for k, and this kernel is uniquely determined up to a positive multiple. Theorem 4.2 says that if p is an even integer and a kernel k belongs not only to the Bergman space A^q but also to the Hardy space H^{q_1} for some q_1 where $q\leq q_1<\infty$, then the A^p extremal function F associated with it is actually in H^{p_1} for $p_1=(p-1)q_1\geq p$. It is natural to ask whether the converse is true. In other words, if $F\in H^{p_1}$ for some p_1 with $p\leq p_1<\infty$, must it follow that the corresponding kernel belongs to H^{q_1}? The following theorem says that this is indeed the case.

THEOREM 4.3. *Suppose p is an even integer and let q be its conjugate exponent. Let $F\in A^p$ with $\|F\|_{A^p}=1$, and let k be a kernel such that F is the extremal function for k. If $F\in H^{p_1}$ for some p_1 with $p\leq p_1<\infty$, then $k\in H^{q_1}$ for $q_1=p_1/(p-1)$, and*

$$\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\leq C\|F\|_{H^{p_1}}^{p-1},$$

where C is a constant depending only on p and p_1.

Proof. Let h be a polynomial and let ϕ be the functional in $(A^p)^*$ corresponding to k. Then by Theorem A,

$$\begin{aligned}\frac{1}{\|\phi\|}\int_{\mathbb{D}}\overline{k(z)}\big(zh(z)\big)'\,d\sigma &= \int_{\mathbb{D}}\left|F(z)\right|^{p-1}\operatorname{sgn}\left(\overline{F(z)}\right)\big(zh(z)\big)'\,d\sigma\\ &= \int_{\mathbb{D}}\overline{F^{p/2}}F^{(p/2)-1}\big(zh(z)\big)'\,d\sigma.\end{aligned}$$

By hypothesis, $F^{p/2} \in H^{(2p_1)/p}$ and $F^{(p/2)-1} \in H^{2p_1/(p-2)}$. A simple calculation shows that

$$\frac{1}{q_1'} = \frac{q_1 - 1}{q_1} = \frac{p_1 - p + 1}{p_1}$$

and thus

$$\frac{p}{2p_1} + \frac{p-2}{2p_1} + \frac{1}{q_1'} = 1.$$

Now we will apply the first part of Lemma 1.2 with $f_1 = F^{p/2}$ and $f_2 = F^{(p/2)-1}$ and $f_3 = zh$, and with $2p_1/p$ in place of p_1, and $2p_1/(p-2)$ in place of p_2, and q_1' in place of p_3. Note that this is permitted since $1 < 2p_1/p < \infty$, and $1 < q_1' < \infty$, and $1 < 2p_1/(p-2) \leq \infty$. (In fact, we even know that $2p_1/(p-2) < \infty$ unless $p = 2$, which is a trivial case since then $F = k/\|k\|_{A^2}$.) With these choices, Lemma 1.2 gives

$$\begin{aligned}
\left| \int_{\mathbb{D}} \overline{F^{p/2}} F^{(p/2)-1} \big(zh(z)\big)' \, d\sigma \right| &\leq C \big\|F^{p/2}\big\|_{H^{2p_1/p}} \big\|F^{p/2-1}\big\|_{H^{2p_1/(p-2)}} \|zh\|_{H^{q_1'}} \\
&= C\|F\|_{H^{p_1}}^{p/2} \|F\|_{H^{p_1}}^{(p-2)/2} \|h\|_{H^{q_1'}} \\
&= C\|F\|_{H^{p_1}}^{p-1} \|h\|_{H_{q_1'}}.
\end{aligned}$$

Since

$$\left| \int_{\mathbb{D}} \overline{k(z)} \big(zh(z)\big)' \, d\sigma \right| \leq C\|\phi\| \|F\|_{H^{p_1}}^{p-1} \|h\|_{H_{q_1'}}$$

for all polynomials h, we may define a continuous linear functional ψ on $H^{q_1'}$ such that

$$\psi(h) = \int_{\mathbb{D}} \overline{k(z)} \big(zh(z)\big)' \, d\sigma$$

for all analytic polynomials h. Then ψ has an associated kernel in H^{q_1}, which we will call $\widetilde{k}$. Thus, for all $h \in H^{q_1'}$, we have

$$\psi(h) = \frac{1}{2\pi} \int_0^{2\pi} \overline{\widetilde{k}\big(e^{i\theta}\big)} h\big(e^{i\theta}\big) \, d\theta.$$

But then the Cauchy–Green theorem gives

$$\begin{aligned}
(4.2) \qquad & \int_{\mathbb{D}} \overline{k(z)} \big(zh(z)\big)' \, d\sigma \\
&= \psi(h) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \overline{\widetilde{k}\big(e^{i\theta}\big)} h\big(e^{i\theta}\big) \, d\theta = \frac{i}{2\pi} \int_{\partial\mathbb{D}} \overline{\widetilde{k}(z)} h(z) z \, d\overline{z} \\
&= \lim_{r \to 1} \frac{i}{2\pi} \int_{\partial(r\mathbb{D})} \overline{\widetilde{k}(z)} h(z) z \, d\overline{z} = \lim_{r \to 1} \int_{r\mathbb{D}} \overline{\widetilde{k}(z)} \big(zh(z)\big)' \, d\sigma \\
&= \int_{\mathbb{D}} \overline{\widetilde{k}(z)} \big(zh(z)\big)' \, d\sigma,
\end{aligned}$$

where h is any analytic polynomial.

Now, for any polynomial $h(z)$, define the polynomial $H(z)$ so that

$$H(z) = \frac{1}{z}\int_0^z h(\zeta)\,d\zeta.$$

Then substituting $H(z)$ for $h(z)$ in equation (4.2), and using the fact that $(zH)' = h$, we have

$$\int_{\mathbb{D}} \overline{\widetilde{k}(z)}h(z)\,d\sigma = \int_{\mathbb{D}} \overline{k(z)}h(z)\,d\sigma$$

for every polynomial h. But since the polynomials are dense in A^p, and k and $\widetilde{k}$ are both in A^q, which is isomorphic to the dual space of A^p, we must have that $k = \widetilde{k}$, and thus $k \in H^{q_1}$.

Now for any polynomial h,

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{k\big(e^{i\theta}\big)}h\big(e^{i\theta}\big)\,d\theta \le C\|\phi\|\|F\|_{H^{p_1}}^{p-1}\|h\|_{H^{q_1'}},$$

and so

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{k\big(e^{i\theta}\big)}h\big(e^{i\theta}\big)\,d\theta \le C\|k\|_{A^q}\|F\|_{H^{p_1}}^{p-1}\|h\|_{H^{q_1'}}$$

by inequality (1.1). But if h is any trigonometric polynomial,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \overline{k\big(e^{i\theta}\big)}h(\theta)\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \overline{k\big(e^{i\theta}\big)}\big[S(h)\big(e^{i\theta}\big)\big]\,d\theta \\ &\le C\|k\|_{A^q}\|F\|_{H^{p_1}}^{p-1}\big\|S(h)\big\|_{H^{q_1'}} \\ &\le C\|k\|_{A^q}\|F\|_{H^{p_1}}^{p-1}\|h\|_{L^{q_1'}},\end{aligned}$$

where S denotes the Szegő projection. Taking the supremum over all trigonometric polynomials h with $\|h\|_{L^{q_1'}} \le 1$ and dividing both sides of the inequality by $\|k\|_{A^q}$, we arrive at the required bound. □

The main results of this section can be summarized in the following theorem.

THEOREM 4.4. *Suppose that p is an even integer with conjugate exponent q. Let $k \in A^q$ and let F be the A^p extremal function associated with k. Let p_1, q_1 be a pair of numbers such that $q \le q_1 < \infty$ and*

$$p_1 = (p-1)q_1.$$

Then $F \in H^{p_1}$ if and only if $k \in H^{q_1}$. More precisely,

$$C_1\left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\right)^{1/(p-1)} \le \|F\|_{H^{p_1}} \le C_2\left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\right)^{1/(p-1)}$$

where C_1 and C_2 are constants that depend only on p and p_1.

Note that if $p_1 = (p-1)q_1$, then $q \le q_1 < \infty$ is equivalent to $p \le p_1 < \infty$.

5. Proof of the lemmas

We now give the proofs of Lemmas 1.1 and 1.2. These proofs are rather technical and require applications of maximal functions and Littlewood–Paley theory.

Definition 5.1. For a function f analytic in the unit disc, the Hardy–Littlewood maximal function is defined on the unit circle by

$$f^*\big(e^{i\theta}\big) = \sup_{0\le r<1} \big|f\big(re^{i\theta}\big)\big|.$$

The following is the simplest form of the Hardy–Littlewood maximal theorem (see, for instance, [2], p. 12).

Theorem B (Hardy–Littlewood). *If $f \in H^p$ for $0 < p \le \infty$, then $f^* \in L^p$ and*

$$\big\|f^*\big\|_{L^p} \le C\|f\|_{H^p},$$

where C is a constant depending only on p.

Further results of a similar type may be found in [5].

Definition 5.2. For a function f analytic in the unit disc, the Littlewood–Paley function is

$$g(\theta, f) = \left\{\int_0^1 (1-r)\big|f'\big(re^{i\theta}\big)\big|^2\,dr\right\}^{1/2}.$$

A key result of Littlewood–Paley theory is that the Littlewood–Paley function, like the Hardy–Littlewood maximal function, belongs to L^p if and only if $f \in H^p$. Formally, the result may be stated as follows (see [11], Volume 2, Chapter 14, Theorems 3.5 and 3.19).

Theorem C (Littlewood–Paley). *For $1 < p < \infty$, there are constants C_p and B_p depending only on p so that*

$$\big\|g(\cdot, f)\big\|_{L^p} \le C_p\|f\|_{H^p}$$

for all functions f analytic in $\mathbb{D}$, and

$$\|f\|_{H^p} \le B_p\big\|g(\cdot, f)\big\|_{L^p}$$

for all functions f analytic in $\mathbb{D}$ such that $f(0) = 0$.

We now apply the Littlewood–Paley theorem to obtain the following result, from which Lemmas 1.1 and 1.2 will follow.

Theorem 5.3. *Suppose $1 < p_1, p_2 \le \infty$, and let p be defined by $1/p = 1/p_1 + 1/p_2$. Suppose furthermore that $1 < p < \infty$. If $f_1 \in H^{p_1}$ and $f_2 \in H^{p_2}$, and h is defined by*

$$h(z) = \int_0^z f_1(\zeta) f_2'(\zeta)\,d\zeta,$$

then $h \in H^p$ *and* $\|h\|_{H^p} \le C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$, *where* C *depends only on* p_1 *and* p_2.

Proof. By the definitions of the Littlewood–Paley function and the Hardy–Littlewood maximal function,

$$\begin{aligned} g(\theta,h) &= \left\{\int_0^1 (1-r)\left|f_1\left(re^{i\theta}\right)f_2'\left(re^{i\theta}\right)\right|^2 dr\right\}^{1/2} \\ &\le f_1^*(\theta)\left\{\int_0^1 (1-r)\left|f_2'\left(re^{i\theta}\right)\right|^2 dr\right\}^{1/2} \\ &= f_1^*(\theta)g(\theta,f_2). \end{aligned}$$

Therefore, since $h(0)=0$, Theorem C gives

$$\|h\|_{H^p} \le C\big\|g(\cdot,h)\big\|_{L^p} \le C\big\|f_1^* g(\cdot,f_2)\big\|_{L^p}.$$

Applying first Hölder's inequality and then Theorem B, we infer that

$$\|h\|_{H^p} \le C\big\|f_1^*\big\|_{L^{p_1}}\big\|g(\cdot,f_2)\big\|_{L^{p_2}} \le C\|f_1\|_{H^{p_1}}\big\|g(\cdot,f_2)\big\|_{L^{p_2}}.$$

If $p_2<\infty$, Theorem C allows us to conclude that

$$\|h\|_{H^p} \le C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}.$$

This proves the claim under the assumption that $p_2<\infty$.

If $p_2=\infty$, then $p_1<\infty$ by assumption. Integration by parts gives

$$h(z) = f_1(z)f_2(z) - f_1(0)f_2(0) - \int_0^z f_2(\zeta)f_1'(\zeta)\,d\zeta.$$

The H^p norm of the first term is bounded by $\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$, by Hölder's inequality. The second term is bounded by $C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$ for some C, since point evaluation is a bounded functional on Hardy spaces. The H^p norm of the last term is bounded by $C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$, by what we have already shown, and thus $\|h\|_{H^p} \le C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$. □

Theorem 5.3 will now be used together with the Cauchy–Green theorem to prove Lemmas 1.2 and 1.1.

Proof of Lemma 1.2. Define

$$I_r = \int_{r\mathbb{D}} \overline{f_1} f_2 f_3'\,dA \quad\text{and}\quad H(z) = \int_0^z f_2(\zeta)f_3'(\zeta)\,d\zeta.$$

Then Theorem 5.3 says that $H \in H^q$ and that $\|H\|_{H^q} \le C\|f_2\|_{H^{p_2}}\|f_3\|_{H^{p_3}}$, where $\frac{1}{q} = \frac{1}{p_2} + \frac{1}{p_3}$. By the Cauchy–Green formula,

$$I_r = \frac{i}{2}\int_{\partial(r\mathbb{D})} \overline{f_1(z)}H(z)\,d\overline{z}.$$

Since $1/p_1+1/q=1$, Hölder's inequality gives

$$|I_r| = \frac{1}{2}\left|\int_{\partial(r\mathbb{D})} \overline{f_1(z)}H(z)\,d\overline{z}\right| \le \pi M_{p_1}(f_1,r)M_q(H,r).$$

But since $\|H\|_{H^q} \le C\|f_2\|_{H^{p_2}}\|f_3\|_{H^{p_3}}$, this shows that

$$|I_r| \le C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}\|f_3\|_{H^{p_3}},$$

which bounds the principal value in question, assuming it exists.

To show that it exists, note that for $0<s<r$, the Cauchy–Green formula gives

$$\begin{aligned} 2|I_r - I_s| &= \left|\int_{\partial(r\mathbb{D}-s\mathbb{D})} \overline{f_1(z)}H(z)\,d\overline{z}\right| \\ &= \left|\int_0^{2\pi} \left[r\overline{f_1(re^{i\theta})}H(re^{i\theta}) - s\overline{f_1(se^{i\theta})}H(se^{i\theta})\right]e^{-i\theta}\,d\theta\right| \\ &\le \left|\int_0^{2\pi} \overline{f_1(re^{i\theta})}\left(rH(re^{i\theta}) - sH(se^{i\theta})\right)e^{-i\theta}\,d\theta\right| \\ &\quad + \left|\int_0^{2\pi} s\left(\overline{f_1(re^{i\theta})} - \overline{f_1(se^{i\theta})}\right)H(se^{i\theta})e^{-i\theta}\,d\theta\right|. \end{aligned}$$

We let $f_r(z)=f(rz)$. Then Hölder's inequality shows that the expression on the right of the above inequality is at most

$$M_{p_1}(f_1,r)\|rH_r - sH_s\|_{H^q} + s\big\|(f_1)_r - (f_1)_s\big\|_{H^{p_1}}M_q(H,r).$$

Since $p_1<\infty$ and $q<\infty$, we know that $(f_1)_r \to f_1$ in H^{p_1} as $r\to 1$, and $H_r \to H$ in H^q as $r\to 1$ (see [2], p. 21). Thus the above quantity approaches 0 as $r,s\to 1$, which shows that the principal value exists.

For the last part of the lemma, what was already shown gives

$$\begin{aligned} \text{p.v.}\int_{\mathbb{D}} \overline{f_1}f_2f_3'\,d\sigma - \int_{\mathbb{D}} \overline{f_1}f_2(S_nf_3)'\,d\sigma &= \text{p.v.}\int_{\mathbb{D}} \overline{f_1}f_2(f_3-S_nf_3)'\,d\sigma \\ &\le C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}\big\|f_3 - S_n(f_3)\big\|_{H^{p_3}}. \end{aligned}$$

By assumption $p_3>1$. If also $p_3<\infty$, then the right-hand side approaches 0 as $n\to\infty$, which finishes the proof. □

Proof of Lemma 1.1. We know that $f^{p/2}\in H^2$ and $f^{(p/2)-1}\in H^{2p/(p-2)}$. Since h is a polynomial, we have $f^{(p/2)-1}h\in H^{2p/(p-2)}$. Also,

$$\frac{1}{2}+\frac{p-2}{2p}+\frac{1}{p}=1.$$

Thus, Lemma 1.2 with $f_1=f^{p/2}$, and $f_2=f^{(p/2)-1}h$, and $f_3=f$ gives the result. □

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