Illinois Journal of Mathematics
Volume 55, Number 2, Summer 2011, Pages 479–494
S 0019-2082

ON THE REAL NERVE OF THE MODULI SPACE OF COMPLEX ALGEBRAIC CURVES OF EVEN GENUS

GRZEGORZ GROMADZKI AND EWA KOZŁOWSKA-WALANIA

ABSTRACT. The real locus in the moduli space of complex algebraic curves of given genus consists of curves having real forms, that is, Riemann surfaces admitting a symmetry (anticonformal involution). The real locus is covered by subsets, each formed by curves having a given topological type determined by the number of connected components and the separability type of the real models of the curves. In this paper, we study the structure of the nerve corresponding to this covering, called the real nerve of complex algebraic curves, for even genera. We find its geometrical and homological dimension, and we obtain some results concerning its global geometrical properties. In the proofs, we use the equivalent language of Riemann surfaces and their symmetries.

1. Introduction

A smooth, irreducible, real, projective algebraic curve has three important topological invariants: the number of connected components, the algebraic genus being the ordinary genus of its complexification and its separability character in its complexification. The complexification allows to map such curves of given genus g into the classical moduli space $\mathcal{M}_g$ of smooth, irreducible, complex projective algebraic curves of genus g. The image $\mathcal{M}_g^{\mathbb{R}}$, called the *real locus*, is covered by the subsets $\mathcal{M}_g^{k,\varepsilon}$ proceeding from the real algebraic curves with k connected components and given separability ε whose meaning will be explained later. Now a subset $\mathcal{M}_g^{k,\varepsilon}$ overlaps a subset $\mathcal{M}_g^{k',\varepsilon'}$ if and only if there is a complex algebraic curve of genus g having two real forms of the types (k,ε) and (k',ε'). In this paper, we study the nerve $\mathcal{N}(g)$,

Received September 30, 2009; received in final form August 20, 2012.

Both authors supported by the Research Grant N N201 366436 of the Polish Ministry of Sciences and Higher Education.

2010 *Mathematics Subject Classification.* Primary 30F. Secondary 14H.

corresponding to this covering, as in ([14], Example 3.1.6), called the *real nerve* of complex algebraic curves of given even genus g.

We shall use the equivalent language of compact Riemann surfaces and their symmetries in the proofs, which is possible due to the fact that there is a functorial equivalence between compact, connected Riemann surfaces and smooth, irreducible, complex, projective, algebraic curves. Under this equivalence, a Riemann surface X admits a symmetry σ if and only if the corresponding curve $\mathcal{C}_X$ has a real form $\mathcal{C}_X(\sigma)$. Furthermore, two such symmetries σ and τ define real forms $\mathcal{C}_X(\sigma)$ and $\mathcal{C}_X(\tau)$, birationally isomorphic over the field $\mathbb{R}$ of real numbers, if and only if they are conjugate in the group $\mathrm{Aut}^{\pm}(X)$ of all, including antiholomorphic, automorphisms of X. Finally, the set $\mathrm{Fix}(\sigma)$ is homeomorphic to a smooth projective model of the corresponding real form $\mathcal{C}_X(\sigma)$.

Recall that *a symmetry* of a compact Riemann surface X of genus $g > 1$ is an antiholomorphic involution σ of X. We call a symmetry σ *separating* if $X \setminus \mathrm{Fix}(\sigma)$ is disconnected and *nonseparating* otherwise. It is well known, by the classical results of Harnack, that the set of points fixed by σ consists of k disjoint simple closed curves which, according to the nineteenth century terminology of Hilbert, are called *ovals*. Here k varies between 0 and $g+1$ and some extra conditions, known as Harnack–Weichold conditions, are satisfied if σ is assumed to be separating. For a symmetry σ of a Riemann surface of genus g, we define (k,ε), where k denotes the number of ovals of σ and $\varepsilon = +1$ or -1 if respectively, σ is separating or not, to be the *topological type* of σ.

The above covering of the real locus $\mathcal{M}_g^{\mathbb{R}}$ gives rise to the associated nerve $\mathcal{N}(g)$, which we call *the real nerve*, being the simplicial complex whose vertices are the topological types (k,ε). The sequence of distinct types $((k_0,\varepsilon_0),\dots,(k_n,\varepsilon_n))$ is an n-simplex in $\mathcal{N}(g)$ if and only if there exists a Riemann surface X of genus g having $n+1$ symmetries of the types $(k_0,\varepsilon_0),\dots,(k_n,\varepsilon_n)$. Furthermore, the differential for the nerve $\mathcal{N}(g)$ is induced by

$$\partial_n\big((k_0,\varepsilon_0),\dots,(k_n,\varepsilon_n)\big) = \sum_{i=0}^{n}(-1)^i\big((k_0,\varepsilon_0),\dots,\widehat{(k_i,\varepsilon_i)},\dots,(k_n,\varepsilon_n)\big).$$

Some results concerning $\mathcal{N}(g)$ are known. First of all, it has $[(3g+4)/2]$ vertices, by the mentioned above results of Harnack and Weichold (c.f. [5]). By the results of Buser, Seppälä and Silhol [4], $\mathcal{N}(g)$ is connected and furthermore it was shown by Costa and Izquierdo in [6], that given g and a type (k,ε) there exists a Riemann surface X of genus g, having two symmetries σ, τ of the types (k,ε) and $(1,-1)$ respectively, which means that $(1,-1)$ is a *spine* for $\mathcal{N}(g)$ for arbitrary g. Here we find both geometrical and homological dimension of $\mathcal{N}(g)$ for even values of g and we give some results concerning its global properties.

2. Preliminaries

All the results are obtained by methods from combinatorial group theory, that is, theory of non-euclidean crystallographic groups (*NEC groups* in short), by which we mean discrete and cocompact subgroups of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$. The algebraic structure of such a group Λ is determined by the signature:

$$s(\Lambda) = \big(h; \pm; [m_1, \ldots, m_r]; \big\{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\big\}\big), \tag{1}$$

where the brackets $(n_{i1}, \ldots, n_{is_i})$ are called the *period cycles*, the integers n_{ij} are the *link periods*, m_i *proper periods* and finally h the *orbit genus* of Λ.

A group Λ with signature (1) has the presentation with the following generators, called *canonical generators*:

$x_1, \ldots, x_r, e_i,\ c_{ij}, 1 \le i \le k, 0 \le j \le s_i$ and $a_1, b_1, \ldots, a_h, b_h$ if the sign is $+$ or $d_1, \ldots, d_h$ otherwise,

and relators:

$x_i^{m_i}, i = 1, \ldots, r,\ c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i,\ i = 1, \ldots, k, j = 1, \ldots, s_i$ and

$$x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1} \quad \text{or} \quad x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_h^2,$$

according to whether the sign is $+$ or $-$. The elements x_i are elliptic transformations, a_i, b_i are hyperbolic translations, d_i glide reflections and c_{ij} hyperbolic reflections. Reflections c_{ij-1}, c_{ij} are said to be *consecutive*. Every element of finite order in Λ is conjugate either to a canonical reflection or to a power of some canonical elliptic element x_i, or else to a power of the product of two consecutive canonical reflections.

Now an abstract group with such a presentation can be realized as an NEC group Λ if and only if the value

$$\eta h + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)$$

is positive, where $\eta = 1$ or 2 according to the sign being $-$ or $+$. This value turns out to be the normalized hyperbolic area $\mu(\Lambda)$ of any fundamental region for such a group and we have the following Hurwitz–Riemann formula

$$[\Lambda : \Lambda'] = \frac{\mu(\Lambda')}{\mu(\Lambda)}$$

for a subgroup Λ' of finite index in an NEC group Λ.

Now NEC groups having no orientation reversing elements are classical Fuchsian groups. They have signatures $(g; +; [m_1, \ldots, m_r]; \{-\})$, which shall be abbreviated as $(g; m_1, \ldots, m_r)$. Given an NEC group Λ, the subgroup Λ^+ of Λ consisting of the orientation preserving elements of Λ is called the *canonical Fuchsian group of* Λ. Given an NEC group with signature (1) it has, by [13], signature

$$(\eta h + k - 1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{ks_k}). \tag{2}$$

A torsion free Fuchsian group Γ is called a *surface group* and it has signature $(g; -)$. In such a case, $\mathcal{H}/\Gamma$ is a compact Riemann surface of genus g and conversely, every compact Riemann surface can be represented as such an orbit space for some surface Fuchsian group Γ. Furthermore, given a Riemann surface so represented, a finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ. The following result from [7], [8] is a principal tool in the paper.

THEOREM 2.1. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with the group G of all automorphisms of X, let $G = \Lambda/\Gamma$ for some NEC group Λ and let $\theta : \Lambda \to G$ be the canonical projection. Then the number of ovals of a symmetry σ of X equals*

$$\sum \big[\mathrm{C}\big(G, \theta(c_i)\big) : \theta\big(\mathrm{C}(\Lambda, c_i)\big)\big],$$

where C *stands for the centralizer and the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to σ.*

For a symmetry σ, we shall denote by $\|\sigma\|$ the number of its ovals. The index $w_i = [\mathrm{C}(G, \theta(c_i)) : \theta(\mathrm{C}(\Lambda, c_i))]$ will be called a *contribution* of c_i to $\|\sigma\|$.

We shall also use the result below, which follows easily from [3]. Let Λ' be a normal subgroup of an NEC group Λ. A canonical generator of Λ is *proper* (with respect to Λ') if it does not belong to Λ'. The elements of Λ expressable as a composition of proper generators of Λ are the *words* of Λ (with respect to Λ'). With these notations, we have

LEMMA 2.2 (c.f. Theorem 2.1.3 of [3]). *Let us suppose that $[\Lambda : \Lambda']$ is even and Λ has the sign $+$. Then Λ' has the sign $+$ if and only if no orientation reversing word belongs to Λ'. If $[\Lambda : \Lambda']$ is even and Λ has the sign $-$, then Λ' has the sign $-$ if and only if either a glide reflection of the canonical generators of Λ or an orientation reversing word belongs to Λ'.*

3. Geometrical dimension of $\mathcal{N}(g)$

We start with the following results of the first author and Izquierdo related to our task.

THEOREM 3.1 ([9]). *A Riemann surface of even genus g has at most four conjugacy classes of symmetries and this bound is attained for every even genus g.*

THEOREM 3.2 ([9]). *Let X be a Riemann surface of even genus and let G be a subgroup of $\mathrm{Aut}^{\pm}(X)$ generated by the nonconjugate symmetries $\sigma_1, \sigma_2, \sigma_3$ and σ_4. Then $G = \mathrm{D}_n \oplus \mathrm{Z}_2$.*

Observe that, by the Sylow theorem, we may assume that G is a 2-group. Indeed, for $\sigma_1, \sigma_2, \ldots, \sigma_k$ being the representatives of conjugacy classes of symmetries, we know that all Sylow 2-groups are conjugate and so we can assume that these symmetries generate a 2-group G. For the sake of completeness, we shall also present the two results below, concerning the maximal possible number of ovals of the set of nonconjugate symmetries on a Riemann surface of even genus.

THEOREM 3.3 ([10]). *Let X be a Riemann surface of even genus having k nonconjugate symmetries, two of which do not commute. Then the total number of ovals of these symmetries does not exceed $2g+2$ and this bound is sharp for arbitrary even g. Furthermore, let $g \geq 2$ be even and let n be arbitrary power of 2 dividing g. Then there exists a Riemann surface X of genus g with $\mathrm{Aut}^{\pm}(X) = \mathrm{D}_n \oplus \mathrm{Z}_2$, having four nonconjugate symmetries which have $2g+2$ ovals in total.*

The upper bound for the total number of ovals of three nonconjugate symmetries of a Riemann surface of even genus for $n > 2$ is also $2g+2$, however it can be proved that in such a case the bound is not attained. Now for the case of commuting symmetries, that is, $n = 2$, the following result holds.

THEOREM 3.4 ([10]). *Let X be a Riemann surface of even genus g having three or four nonconjugate commuting symmetries. Then the total number of ovals of the symmetries does not exceed $2g+3$ or $2g+2$ respectively and these bounds are sharp for arbitrary even g.*

Now we shall look for the geometrical dimension $\dim_{\mathrm{G}} \mathcal{N}(g)$ of $\mathcal{N}(g)$. Observe that for even g there are at most 4 nonconjugate symmetries on a Riemann surface of genus g. This means, however, that there at most four types of symmetries on this surface and, equivalently, any simplex in $\mathcal{N}(g)$ is spanned by at most 4 vertices. Hence, $\dim_{\mathrm{G}} \mathcal{N}(g) \leq 3$. The next theorem shows that in fact the equality holds for any even $g \geq 2$.

THEOREM 3.5. *For any even $g \geq 2$, the geometrical dimension of $\mathcal{N}(g)$ equals* 3.

Proof. To prove the theorem, it is enough to construct, for any even g, a Riemann surface of genus g having four symmetries of distinct topological types. Assume first that $g > 2$. Consider an NEC group Λ with signature

$$\big(0; +; [-]; \big\{(2, \overset{g+3}{\ldots}, 2)\big\}\big)$$

and an epimorphism $\theta : \Lambda \to \mathrm{Z}_2^3 = \langle x, y, z \rangle$, which maps the consecutive canonical reflections to

$$\underbrace{x, y, x, \ldots, x, y}_{g}, z, y, xyz, x.$$

Then $\Gamma=\ker\theta$ is a surface Fuchsian group of genus g and so $X=\mathcal{H}/\Gamma$ is a Riemann surface of genus g having four nonconjugate symmetries x,y,z,xyz. Now by Theorem 2.1, canonical reflection c_0 contributes with 1 oval to the symmetry x and reflections $c_2,c_4,\dots,c_{g-2}$ contribute to x with 2 ovals each. Reflections $c_1,c_3,\dots,c_{g-3}$ contribute with 2 ovals each to the symmetry y and reflections c_{g-1},c_{g+1} with 1 oval each. Reflection c_g contributes with 2 ovals to the symmetry z and reflection c_{g+2} contributes with 1 oval to the symmetry xyz. Summing up, we see that x has $(g-2)/2\cdot 2+1=g-1$ ovals, y has $(g-2)/2\cdot 2+2=g$ ovals, z has 2 ovals and xyz has 1 oval. We see that for any even $g>2$ numbers of ovals of these symmetries are distinct, hence we have a 3-simplex in $\mathcal{N}(g)$ and so $\dim_{\mathrm{G}}\mathcal{N}(g)=3$.

For $g=2$, we shall construct a Riemann surface of genus 2, having a fixed point free symmetry, separating symmetries with 1 and 3 ovals and a nonseparating symmetry with 1 oval. For, consider an NEC group Λ with signature

$$\bigl(0;+;[-];\bigl\{(2,2,2,4)\bigr\}\bigr)$$

and an epimorphism $\theta:\Lambda\to G=\mathrm{D}_4\oplus\mathrm{Z}_2=\langle x,y\rangle\oplus\langle z\rangle$ which maps the consecutive canonical reflections corresponding to the nonempty period cycle to

$$x,z,xyx,y,x.$$

Observe that, by Theorem 2.1, c_{10} contributes with 1 oval to x, c_{11} contributes with 1 oval to z and reflections c_{12} and c_{13} contribute respectively with 1 and 2 ovals to the symmetry y. Therefore, y is a separating symmetry with 3 ovals and, by Lemma 2.2, central symmetry z is separating, as there is no orientation reversing word in $\theta^{-1}(z)$. Symmetry $z(xy)^2$ is fixed point free. Recall that, by the main theorem in [2], two noncommuting symmetries on a Riemann surface of genus 2 can have at most 4 ovals in total and symmetries x and y together have exactly 4 ovals. By the results of [1] and [2] (see remarks on page 321 of [2]), in such a case symmetry x must be nonseparating. This completes the proof. □

4. Homological dimension of $\mathcal{N}(g)$

As it is usually defined, by the homological dimension of $\mathcal{N}(g)$ we understand the greatest number n such that $\mathrm{H}_n(\mathcal{N}(g),\mathbb{Z})\neq 0$. Before we state the main result about homological dimension $\dim_{\mathrm{H}}\mathcal{N}(g)$ of $\mathcal{N}(g)$, we shall give a lemma, concerning separability of the symmetries.

Lemma 4.1. *If a Riemann surface X of even genus g has 4 nonconjugate commuting symmetries with fixed points, then all the symmetries are nonseparating.*

Proof. Let $G=\langle x,y,z\rangle=\mathrm{Z}_2^3=\Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$\bigl(h;\pm;[2,\overset{r}{\ldots},2];\bigl\{(2,\overset{s_1}{\ldots},2),\dots,(2,\overset{s_k}{\ldots},2),(-),\overset{l}{\ldots},(-)\bigr\}\bigr). \tag{3}$$

Let $\theta : \Lambda \to G$ denote the canonical epimorphism. As all four of the symmetries in question have fixed points, there are canonical reflections $c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}$ which are mapped by θ respectively to x, y, z, xyz. Observe, that each of the symmetries can be represented as the product of the remaining three. Therefore, for any symmetry $\rho \in \{x, y, z, xyz\}$, the inverse image $\Gamma_\rho = \theta^{-1}(\langle \rho \rangle) \leq \Lambda$ contains an orientation reversing word. Indeed, for example, we have an orientation reversing word $c_{i_2} c_{i_3} c_{i_4} \in \Gamma_x$ and so x is nonseparating. Hence, all the symmetries are nonseparating by Lemma 2.2. □

LEMMA 4.2. *For any even $g \geq 6$, there exist five Riemann surfaces $X_1, \dots, X_5$ of genus g, having commuting, nonseparating symmetries with respectively:*

$2, 3, 4, 5$ *ovals*,
$1, 3, 4, 5$ *ovals*,
$1, 2, 4, 5$ *ovals*,
$1, 2, 3, 5$ *ovals*,
$1, 2, 3, 4$ *ovals*.

Proof. We shall construct the Riemann surfaces in question. Let $g \geq 6$ be even. Consider an NEC group Λ with signature

$$\big(0; +; \big[2, \overset{(g-6)/2}{\dots}, 2\big]; \big\{(2, 2, 2, 2, 2, 2, 2, 2, 2)\big\}\big).$$

Now we shall define epimorphisms $\theta_1, \dots, \theta_5 : \Lambda \to G = \mathrm{Z}_2^3$ such that $X_i = \mathcal{H}/\Gamma_i$ for $\Gamma_i = \ker \theta_i$. First of all, we define $\theta_i(x_j) = xy$ for all the canonical elliptic generators of Λ and we put $\theta_i(e) = xy$ if 4 divides g and $\theta_i(e) = 1$ otherwise. Now we only have to define θ_i on the consecutive canonical reflections.

For θ_1, we map the reflections $c_0, \dots, c_9$ respectively to

$$x, y, x, z, x, z, xyz, z, xyz, x.$$

Then, by Theorem 2.1, we see that reflections c_0, c_2 contribute to x with 1 oval each and reflection c_4 contributes to x with 2 ovals. Reflections c_3, c_7 contribute to z with 2 ovals each and reflection c_5 with 1 oval. Reflection c_1 contributes to y with 2 ovals. Reflection c_6 contributes to xyz with 2 ovals and reflection c_8 with 1 oval. The last reflection does not contribute, as it is conjugate to the first one. Summing up, we obtain a Riemann surface X_1 of genus g, having four commuting symmetries x, y, z, xyz with the following numbers of ovals: $\|x\| = 4, \|y\| = 2, \|z\| = 5, \|xyz\| = 3$.

For θ_2, consecutive canonical reflections are sent respectively to

$$x, xyz, y, x, y, z, y, z, y, x.$$

As before, we see that $\|x\| = 3, \|y\| = 5, \|z\| = 4, \|xyz\| = 1$ and we obtain a Riemann surface X_2 that we looked for.

Let θ_3 map the reflections c_i respectively to

$$x, y, z, xyz, x, xyz, z, x, z, x$$

and observe that, by Theorem 2.1 again, we have $\|x\| = 5, \|y\| = 1, \|z\| = 4, \|xyz\| = 2$. This definition gives rise to the Riemann surface X_3.

For θ_4, we map the consecutive canonical reflections c_i to

$$x, y, z, x, xyz, z, xyz, x, z, x.$$

With such a definition, $\|x\| = 3, \|y\| = 1, \|z\| = 5, \|xyz\| = 2$ and so $X_4 = \mathcal{H}/\Gamma_4$ for $\Gamma_4 = \ker\theta_4$.

Finally, let θ_5 be defined by

$$x, xyz, y, x, z, y, x, y, z, x$$

for the consecutive canonical reflections. Here $\|x\| = 4,\ \|y\| = 3,\ \|z\| = 2,$ $\|xyz\| = 1$. As a result, we obtain the Riemann surface $X_5 = \mathcal{H}/\Gamma_5$ for $\Gamma_5 = \ker\theta_5$, holding the conditions of the lemma.

The fact that all the symmetries are nonseparating follows easily from Lemma 4.1, as in each of the cases (1)–(5) we consider commuting symmetries with fixed points. $\square$

As a consequence of the above lemma, we obtain the following theorem.

THEOREM 4.3. *For any even $g \geq 6$, we have* $\dim_{\mathrm{H}} \mathcal{N}(g) = 3$.

Proof. Recall, that the boundary homomorphism is induced by

$$\partial_n\big((k_0,\varepsilon_0),\ldots,(k_n,\varepsilon_n)\big) = \sum_{i=0}^{n}(-1)^i\big((k_0,\varepsilon_0),\ldots,\widehat{(k_i,\varepsilon_i)},\ldots,(k_n,\varepsilon_n)\big).$$

Hence, the alternating sum of the five simplices

$$\begin{gathered}
\big((2,-1),(3,-1),(4,-1),(5,-1)\big),\\
\big((1,-1),(3,-1),(4,-1),(5,-1)\big),\\
\big((1,-1),(2,-1),(4,-1),(5,-1)\big),\\
\big((1,-1),(2,-1),(3,-1),(5,-1)\big),\\
\big((1,-1),(2,-1),(3,-1),(4,-1)\big),
\end{gathered}$$

given by Lemma 4.2, is a cycle which is not a homological boundary of a 4-dimensional chain. Thus, it represents a nontrivial element in the third homology group and so $\dim_{\mathrm{H}} \mathcal{N}(g) = 3$. $\square$

For the next part of the paper, we need the following result from [12].

THEOREM 4.4. *If a Riemann surface X of even genus g admits a fixed point free symmetry x and a symmetry y with nonempty fixed point set, then the order of xy is even but not divisible by* 4 *and the number of ovals of symmetry y is odd.*

Not surprisingly, for $g = 2$ the homological dimension of $\mathcal{N}(2)$ is not maximal, as the next result shows the following theorem.

THEOREM 4.5 (see also [1]). *The homological dimension of* $\mathcal{N}(2)$ *equals* 0.

Proof. Obviously the possible types of symmetries on a Riemann surface of genus 2 are $(0,-1),(1,-1),(2,-1),(1,1)$ and $(3,1)$. However, there is no Riemann surface of genus 2, which simultaneously admits symmetries of the types $(2,-1)$ and $(1,1)$ (see remarks on page 23 in [12]). Furthermore, by Theorem 1.1 in [12] there is no edge between $(2,-1)$ and $(3,1)$ as 2 and 3 have different parity. In particular, the vertex $(2,-1)$ is not a vertex of any 3-simplex. Hence, $(3,1),(1,1),(1,-1),(0,-1)$ span the unique 3-simplex in $\mathcal{N}(2)$, by the second part of Theorem 3.5, and therefore $\dim_{\mathrm{H}} \mathcal{N}(2) < 3$.

Now we shall show that $\dim_{\mathrm{H}} \mathcal{N}(2) < 1$. First of all, any 2-simplex is a face of the 3-simplex Δ spanned by $(3,1),(1,1),(1,-1),(0,-1)$. Indeed, by Theorem 4.4, there is no edge between $(2,-1)$ and $(0,-1)$. Also, we already know that there is no edge between $(2,-1)$ and $(1,1)$ or $(3,1)$. Therefore, there does not exist a nontrivial 2-cycle and so $\dim_{\mathrm{H}} \mathcal{N}(2) < 2$. Finally, the only 1-simplex not contained in the above Δ is an edge joining vertices $(2,-1)$ and $(1,-1)$. Therefore, there does not exist a nontrivial 1-cycle and hence $\dim_{\mathrm{H}} \mathcal{N}(2) < 1$. As a result, we see that $\dim_{\mathrm{H}} \mathcal{N}(2) = 0$. □

REMARK 4.6. It can be shown that $\dim_{\mathrm{H}} \mathcal{N}(4) = 1$. However, the only evidence of this fact we have, is to find explicitly the entire simplicial complex $\mathcal{N}(4)$. The last is an uphill task, which involves the Schreier coset graph technique, developed by Hoare and Singerman in [11], to determine the separability types of the symmetries. Going into the details is not the goal of this paper and the proof of this fact shall be presented separately somewhere else.

5. On global geometrical properties of $\mathcal{N}(g)$

Now we shall give some results concerning the global structure of $\mathcal{N}(g)$. As we already mentioned, $\mathcal{N}(g)$ has $[(3g+4)/2]$ vertices, by the Hurwitz–Weichold theorems (c.f. [5]). Call the vertex corresponding to a fixed point free symmetry *null*, the vertices corresponding to the remaining nonseparating symmetries *negative* and vertices corresponding to separating symmetries *positive*. Furthermore, call a simplex spanned by vertices corresponding to commuting symmetries of a Riemann surface, *a commutative simplex*.

THEOREM 5.1. *If* Δ *is a commutative* 3*-simplex in* $\mathcal{N}(g)$, *then it is also a simplex in* $\mathcal{N}(\tilde{g})$ *for even* $\tilde{g} > g$ *if*:

(a) *at least two of its vertices are negative*;
(b) *exactly one of the vertices is negative and* $\tilde{g} \equiv g(4)$;
(c) *all the non-null vertices are positive and* $\tilde{g} \equiv g(8)$.

Proof. Let Δ be a 3-simplex and let us assume that the commuting symmetries in question generate the group $G = \langle x, y, z\rangle = \mathrm{Z}_2^3 = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature (3). Let $\theta : \Lambda \to G$

denote the canonical epimorphism and so $X = \mathcal{H}/\Gamma$ for $\Gamma = \ker\theta$ is a Riemann surface of genus g having commuting symmetries x, y, z, xyz.

Observe first that if all the symmetries have fixed points, then the theorem holds. Indeed, by Lemma 4.1 the symmetries are nonseparating. Let $\tilde{g} = g + 2\alpha$ for a nonnegative integer α and consider an NEC group Λ' with signature

$$(4) \qquad \big(h; \pm; [2, \overset{r'}{\ldots}, 2]; \{(2, \overset{s_1}{\ldots}, 2), \ldots, (2, \overset{s_k}{\ldots}, 2), (-), \overset{l}{\ldots}, (-)\}\big)$$

which differs from the signature (3) only in the elliptic part, i.e. we take $r' = r + \alpha$. Observe that, with such a definition, canonical generators of the group Λ' correspond to the canonical generators of the group Λ. We have the same genus and sign, therefore we have the same number of hyperbolic generators. Also the number and lengths of period cycles are the same, which gives us the correspondence between the canonical reflections of Λ and Λ'. We define $\theta' : \Lambda' \to G$ such that θ and θ' have the same image on the corresponding generators except e_1'. We put $\theta(x_i') = xy$ for $i \geq r + 1$. If α is even, we take $\theta(e_1') = 1$ and $\theta(e_1') = xy$ otherwise. This definition gives us a Riemann surface $X' = \mathcal{H}/\Gamma'$ for $\Gamma' = \ker\theta'$, which by the Hurwitz–Riemann formula has genus $\tilde{g}$ and admits symmetries x, y, z, xyz, having the same types as in the case of surface X.

Let us assume now that one of the symmetries, say xyz, is fixed point free. Therefore, symmetries x, y, z have fixed points. Let, as in the proof of Lemma 4.1, $c_{i_1}, c_{i_2}, c_{i_3}$ be the canonical reflections in Λ which are mapped to x, y, z, respectively.

Let first at least two of the symmetries, say x, y, be nonseparating and let $\tilde{g} = g + 2\alpha$, where $\alpha \geq 0$ is an integer. Our aim is to construct a Riemann surface X' of genus $\tilde{g}$ having symmetries of the same types as x, y, z, xyz. Now it is enough to consider the definition of X' from the previous case. Observe, that in fact it gives rise to the configuration in question and it does not change the separability of the symmetry z.

Now we shall deal with the case when exactly one of the symmetries with fixed points, say x, is nonseparating. By Lemma 2.2, we know that there is a canonical glide reflection or an orientation reversing word in Γ_x but not in Γ_y and Γ_z as these are separating symmetries. It follows that there are no proper periods in the signature of Λ and that all the connecting generators are mapped by θ to 1. Indeed, an elliptic element x_i or the connecting generator e_i with nontrivial image can be mapped to xy, xz or yz. In the first case, symmetry y is nonseparating as the word $x_i c_{i_1}$ is an orientation reversing word in Γ_y. In the second case, symmetry z is nonseparating because the word $x_i c_{i_1}$ is an orientation reversing word in Γ_z. Similarly, in the third case both y and z are nonseparating. Hence, for our case it must be that $r = 0$ and for the same reason all generators e_i are mapped to 1. Furthermore, for x to be nonseparating, there must be an orientation reversing word or a canonical glide reflection in Γ_x. As there are no canonical generators mapped to xy, xz

or yz, the sign in the signature of Λ is $-$, $h>0$ and all the glide reflections are mapped to x. Indeed, if there was a canonical glide reflection d for which $\theta(d)=xyz$, then y and z would be nonseparating as $dc_{i_1}c_{i_3}$ and $dc_{i_1}c_{i_2}$ would be orientation reversing words in Γ_y and Γ_z, respectively. Now we have to construct a Riemann surface X' of genus $\tilde{g}=g+4\alpha$, where $\alpha\geq 0$ is an integer, having symmetries of the same types as for the suface X. Consider an NEC group Λ' with signature

$$\big(h';-;[-];\big\{(2,\overset{s_1}{\ldots},2),\ldots,(2,\overset{s_k}{\ldots},2),(-),\overset{l}{\ldots},(-)\big\}\big) \tag{5}$$

which differs from the signature of Λ only for the genus, i.e. we take $h'=h+\alpha$. Again, we define $\theta':\Lambda'\to G$ in the same way as θ for all the corresponding generators. In addition, we put $\theta(d_i')=x$ for $i>h$. Observe that with such a definition, for $\Gamma'=\ker\theta'$, we obtain a Riemann surface $X'=\mathcal{H}/\Gamma'$, having symmetries x,y,z,xyz of the same types as in the case of surface X.

Let finally x,y,z be separating. Obviously there are no canonical glide reflections in the signature of Λ and so Λ has sign $+$. Similarly to the previous case, $r=0$ and all the generators e_i are mapped to 1. Let $\tilde{g}=g+8\alpha$ for $\alpha\geq 0$ being an integer. Consider an NEC group Λ' with signature

$$\big(h';+;[-];\big\{(2,\overset{s_1}{\ldots},2),\ldots,(2,\overset{s_k}{\ldots},2),(-),\overset{l}{\ldots},(-)\big\}\big) \tag{6}$$

with $h'=h+\alpha$ and an epimorphism $\theta':\Lambda'\to G$ which has the same image as θ on all the generators of Λ' corresponding to the generators of Λ. Moreover, we take $\theta(a_i')=\theta(b_i')=1$ for $i>h$. This definition gives rise to the Riemann surface X' of genus $\tilde{g}$ having the same symmetry type as X. $\square$

Now we shall present the necessary and sufficient condition for the existence of a 3-simplex Δ in $\mathcal{N}(g)$, such that the vertices of Δ come from nonseparating commuting symmetries with $0,t_1,t_2$ and t_3 ovals. By Theorem 4.4, we know that t_i must be odd for $i=1,2,3$.

Theorem 5.2. *Let Δ be a 3-simplex of $\mathcal{N}(g)$ spanned by the null and three negative vertices corresponding to the commuting symmetries with $0,t_1,t_2,t_3$ ovals, where $t_1<t_2<t_3$ are odd. Then $t_3<g-1$ and $t_1+t_2+t_3\leq 2g-5$.*

Proof. Let $G=\mathrm{Z}_2^3=\langle x,y,z\rangle$ be the group generated by the symmetries and let xyz be the fixed point free symmetry. Take Λ to be an NEC group with signature (3) and let $\theta:\Lambda\to G$ denote the canonical epimorphism. By the results of [10], we know that the total number of ovals of the symmetries does not exceed $2g+3$ and that at least three canonical reflections contribute with only one oval to respective symmetry. Therefore, $t\leq 2s+4l-3$ and in turn $s/4+l/2\geq(t+3)/8$, for $s=s_1+\cdots+s_k$. Observe that $\mu(\Lambda)/2\pi=(g-1)/4$ and $g-1$ is odd, therefore there are link periods in the signature of Λ and so $k\geq 1$. As xyz is fixed point free, there are no canonical reflections in Γ_{xyz}. However, x,y,z are nonseparating, which means that there are orientation reversing words or canonical glide reflections in $\Gamma_x, \Gamma_y, \Gamma_z$. If $h > 0$, then by the Hurwitz–Riemann formula

$$(g-1)/4 \geq h - 2 + 1 + l + s/4 \geq (t+3)/8,$$

which gives $t \leq 2g-5$. Assume now that $h = 0$. Then the orientation reversing words must involve the elliptic generators x_i or connecting generators e_i with nontrivial image under θ. These images can only be xy, xz or yz. Moreover, observe that there must be at least three such generators with distinct images. Indeed, if there are only two, then they have the same image for the relation $\theta(x_1 \cdots x_r e_1 \cdots e_k) = 1$ to hold. Assume without loss of generality that this image is xy. Then we get a contradiction, as there is no orientation reversing word in Γ_z and so z is separating. Obviously the relation above does not hold if there is only one x_i or e_i with nontrivial image. Therefore, we have at least three such generators and so

$$(g-1)/4 \geq -2 + 1 + 1 + l/2 + s/4,$$

giving $t \leq 2g - 5$ and the proof of the first part is finished.

Now we shall show that in fact $t_3 < g - 1$. Assume to a contrary that $t_3 = g - 1$ and let first $l = 0$. In such a case, we have only nonempty period cycles in the signature of Λ. We shall show that in a such case our symmetries would have more than $2g-2$ ovals in total, which contradicts the fact that the maximal number of ovals is $2g-5$. Let c_i be a canonical reflection such that $\theta(c_i) = z$. If $\theta(c_{i-1}) = \theta(c_{i+1})$ then c_i contributes with 2 ovals to symmetry z and reflections c_{i-1} and c_{i+1} contribute together with at least two ovals to symmetries x, y. Now if $\theta(c_{i-1}) \neq \theta(c_{i+1})$, then c_i contributes with only 1 oval to z but, as before, the reflections c_{i-1} and c_{i+1} contribute together with at least two ovals to symmetries x, y. Observe also that there is at least one reflection which contributes with 1 oval to z. Therefore, the total number $t_1 + t_2$ of ovals of symmetries x, y equals at least g and summing up we arrive to a contradiction, as $t_1 + t_2 + g - 1 \geq 2g - 1$ and on the other hand, by the first part of the proof, the total number of ovals does not exceed $2g - 5$.

Hence, it must be that $l \geq 1$ and in particular the difference between numbers of ovals $g-1$ and t_1+t_2-1 must be covered by reflections corresponding to the empty period cycles. Let first $g - t_1 - t_2$ be divisible by 4. Then there are at least $(g-t_1-t_2)/4$ empty period cycles in the signature of Λ. Therefore,

$$\begin{aligned}
(g-1)/4 &= \mu(\Lambda)/2\pi \\
&= \eta h - 2 + k + r/2 + l + s/4 \\
&\geq \eta h - 2 + k + r/2 + (g - t_1 - t_2)/8 + (t_1 + t_2 + g - 1 + 3)/8 \\
&= \eta h - 2 + k + r/2 + g/4 + 1/4,
\end{aligned}$$

which gives $\eta h + k + r/2 \leq 3/2$. It follows that $k = 1$, $h = 0$ and $r \leq 1$. Symmetries x, y, z are nonseparating, so there are at least three generators x_i, e_i

with images xy, xz, yz. As $r < 2$, there are empty period cycles whose connecting generators have nontrivial image under θ. It follows that in fact $l \geq (g - t_1 - t_2)/4 + 1$ and $t \leq 2s + 4l - 3 - 4 = 2s + 4l - 7$, and as a result $s/4 + l/2 \geq (t + 7)/8 = (t_1 + t_2 + g + 6)/8$. This gives $(g - 1)/4 \geq -1 + (g - t_1 - t_2)/8 + 1/2 + (t_1 + t_2 + g + 6)/8$ and so $-1/4 \geq 0$, a contradiction.

Now if 4 divides $g + 2 - t_1 - t_2$, then there are at least $(g + 2 - t_1 - t_2)/4$ empty period cycles and for one of these, the connecting generator has nontrivial image under θ. Hence, $t \leq 2s + 4l - 3 - 2$ and in turn $s/4 + l/2 \geq (t + 5)/8 = (t_1 + t_2 + g + 4)/8$. Now, as before, we have

$$\begin{aligned}(g - 1)/4 &= \mu(\Lambda)/2\pi \\ &= \eta h - 2 + k + r/2 + l + s/4) \\ &\geq \eta h - 2 + k + r/2 + (g + 2 - t_1 - t_2)/8 + (t_1 + t_2 + g + 4)/8 \\ &= \eta h - 2 + k + r/2 + g/4 + 1/4,\end{aligned}$$

which gives $\eta h + k + r/2 \leq 1$. It follows that $k = 1$, $h = 0$ and $r = 0$. Symmetries x, y, z are nonseparating, so there are at least three generators e_i with images xy, xz, yz. But in our case, it means that there are at least three empty period cycles, which contribute 2 ovals to z and so it must be $l \geq (g + 2 - t_1 - t_2)/4 + 1$. It follows that $t \leq 2s + 4l - 3 - 6$ and in turn $s/4 + l/2 \geq (t + 9)/8 = (t_1 + t_2 + g + 8)/8$. We obtain a contradiction, since in such a case $(g - 1)/4 \geq -1 + (g + 2 - t_1 - t_2)/8 + (t_1 + t_2 + g + 8)/8$ and so $-1/4 \geq 1/4$. This finishes the proof, showing that in fact $t_3 < g - 1$. □

THEOREM 5.3. *Let Δ be a commutative 2-simplex of $\mathcal{N}(g)$ spanned by three vertices corresponding to symmetries with $t_1, t_2, t_3 \neq 0$ ovals, where t_1, t_2, t_3 are odd and either a vertex is positive or all the vertices are negative, $t_1 < t_2 < t_3 < g - 1$ and $t_1 + t_2 + t_3 \leq 2g - 5$. Then this simplex is the face of the 3-simplex (spanned on the null vertex).*

Proof. By Lemma 4.1, if at least one of the symmetries x, y, z is separating, then xyz is a fixed point free symmetry in question and the proof is finished.

Therefore, we only have to consider the case, when all the symmetries are nonseparating. Let X be a Riemann surface of even genus g having three nonseparating commuting symmetries x, y, z with odd and distinct numbers of ovals. Then $t_1 < t_2 < t_3 < g - 1$, $t_1 = 2t_1' + 1 < t_2 = 2t_2' + 1 < t_3 = 2t_3' + 1 < g - 1$. We also know that $t_1 + t_2 + t_3 \leq 2g - 5$ is odd and so we have two possible cases. Either $t_1 + t_2 + t_3 \equiv 2g + 3 \mod 4$ or $t_1 + t_2 + t_3 \equiv 2g + 1 \mod 4$.

Assume first that $(t_1 + t_2 - t_3 - 1)/2$ is nonnegative and that the condition $t_1 + t_2 + t_3 \equiv 2g + 3 \mod 4$ holds. In such a case, we obtain

$$2g + 3 - 4\alpha = 2t_1' + 2t_2' + 2t_3' + 3$$

for some positive integer $\alpha \geq 2$. It follows that $t_1' + t_2' + t_3'$ is an even number, which means that either t_1', t_2', t_3' are all even or exactly two of these are odd. In both cases, we obtain that $(t_a+t_b-t_c-1)/2$ are even integers, for pairwise distinct parameters $1\leq a,b,c\leq 3$. Indeed, if t_i' are even, then $t_i=4t_i''+1$ for $i=1,2,3$. It follows that $(t_a+t_b-t_c-1)/2=4(t_a''+t_b''-t_c'')/2$. Now if two of t_i' are odd, say t_a' and t_b', we see that $t_a=4t_a''+3, t_b=4t_b''+3$ and $t_c=4t_c''+1$. Now $(t_a+t_b-t_c-1)/2=4(t_a''+t_b''-t_c''+1)/2$, $(t_a+t_c-t_b-1)/2=4(t_a''+t_c''-t_b'')/2$ and $(t_b+t_c-t_a-1)/2=4(t_b''+t_c''-t_a'')/2$. Observe also, that the integers $t_1+t_3-t_2-1$ and $t_2+t_3-t_1-1$ are positive, as we assumed that $t_1<t_2<t_3$,.

Consider an NEC group Λ with signature

$$\big(0;+;[2,\overset{\alpha}{\ldots},2];\big\{(2,\overset{s}{\ldots},2)\big\}\big)$$

for which $s=(t_1+t_2+t_3+3)/2$ and let $\theta:\Lambda\to G=\mathrm{Z}_2^3=\langle x,y,z\rangle$ be the canonical epimorphism, which maps the canonical reflections corresponding to the nonempty period cycle to

$$y,\underbrace{x,y,\ldots,x,y}_{s_1},x,\underbrace{z,x,\ldots,z,x}_{s_2},z,\underbrace{y,z,\ldots,y,z}_{s_3},y,$$

where $s_1=(t_1+t_2-t_3-1)/2, s_2=(t_1+t_3-t_2-1)/2, s_3=(t_2+t_3-t_1-1)/2$. In addition, we take $\theta(x_1)=xy, \theta(x_2)=xz$ and $\theta(x_i)=yz$ for $i=3,\ldots,\alpha$ and if α is even, $\theta(e)=yz$. If α is odd, we take $\theta(e)=1$. In both cases $\mathcal{H}/\Gamma$, where $\Gamma=\ker\theta$, is a Riemann surface of genus g. By Theorem 2.1, if $\theta(c_{i-1})\neq\theta(c_{i+1})$, then reflection c_i contributes with 1 oval to symmetry $\theta(c_i)$ and with 2 ovals in the other case. Therefore, symmetries x,y,z have t_1,t_2,t_3 ovals respectively. They are all nonseparating as there are orientation reversing words in $\Gamma_x,\Gamma_y,\Gamma_z$.

Assume still that $(t_1+t_2-t_3-1)/2\geq 0$ and let now the condition $t_1+t_2+t_3\equiv 2g+1 \mod 4$ be true. In such a case, we obtain

$$2g+1-4\alpha=2t_1'+2t_2'+2t_3'+3$$

for some positive integer $\alpha\geq 2$. Obviously, $t_1'+t_2'+t_3'$ is odd and so either t_1',t_2',t_3' are all odd or exactly one of these is odd. Now $(t_a+t_b-t_c-3)/2$ are even integers for pairwise distinct parameters $1\leq a,b,c\leq 3$. If t_i' are odd, then $t_i=4t_i''+3$ for $i=1,2,3$. It follows that $(t_a+t_b-t_c-3)/2=4(t_a''+t_b''-t_c'')/2$. Now if two of t_i' are even, say t_a' and t_b', we see that $t_a=4t_a''+1, t_b=4t_b''+1$ and $t_c=4t_c''+3$. Now $(t_a+t_b-t_c-3)/2=4(t_a''+t_b''-t_c''-1)/2$, $(t_a+t_c-t_b-3)/2=4(t_a''+t_c''-t_b'')/2$ and $(t_b+t_c-t_a-3)/2=4(t_b''+t_c''-t_a'')/2$. As in the previous case, the integers $t_1+t_3-t_2-3$ and $t_2+t_3-t_1-3$ are positive. Taking $p_3=t_3-2$, we obtain that $(t_1+t_2-p_3-1)/2,(t_1+p_3-t_2-1)/2$ and $(t_2+p_3-t_1-1)/2$ are even and positive. Now consider an NEC group Λ with signature

$$\big(0;+;[2,\overset{\alpha-1}{\ldots},2];\big\{(2,\overset{s}{\ldots},2),(-)\big\}\big)$$

for which $s=(t_1+t_2+t_3+1)/2$ and let $\theta:\Lambda\to G=\mathrm{Z}_2^3=\langle x,y,z\rangle$ be the canonical epimorphism, which maps the canonical reflections corresponding to the nonempty period cycle to

$$y,\underbrace{x,y,\dots,x,y}_{s_1},x,\underbrace{z,x,\dots,z,x}_{s_2},z,\underbrace{y,z,\dots,y,z}_{s_3},y,$$

where $s_1=(t_1+t_2-p_3-1)/2, s_2=(t_1+p_3-t_2-1)/2, s_3=(t_2+p_3-t_1-1)/2$. In addition, we take $\theta(c_{20})=z, \theta(x_1)=xy, \theta(e_2)=yz, \theta(x_i)=xz$ for $i=2,\dots,\alpha-1$ and if α is even, $\theta(e_1)=xz$. If α is odd, we take $\theta(e_1)=1$. In both cases, we get the configuration of symmetries in question.

Let now $t_1+t_2-t_3-1<0$ and assume that 4 divides $t_3-t_1-t_2+1$. Consider an NEC group Λ with signature

$$\big(0;+;[2,\overset{r}{\dots},2];\{(2,\overset{s}{\dots},2),(-)^l\}\big)$$

for which $s=t_1+t_2+1,\ l=(t_3-t_1-t_2+1)/4,\ r=(g+1-t_3)/2$ and let $\theta:\Lambda\to G=\mathrm{Z}_2^3=\langle x,y,z\rangle$ be the canonical epimorphism, which maps the canonical reflections corresponding to the nonempty period cycle to

$$y,x,\underbrace{z,x,\dots,z,x}_{s_1},z,\underbrace{y,z,\dots,y,z}_{s_2},y,$$

where $s_1=t_1-1, s_2=t_2-1$ and the reflections corresponding to the empty period cycles are mapped to z. As in the previous case, we take $\theta(x_1)=xy, \theta(x_2)=xz, \theta(x_i)=yz$ for $i=3,\dots,r$, $\theta(e_1)=1$ if r is odd and if r is even we take $\theta(e_1)=yz$ with $\theta(e_i)=1$ for $i\geq 2$. Again, $\mathcal{H}/\Gamma$, where $\Gamma=\ker\theta$, is a Riemann surface X' of genus g that we were looking for. Now if 4 divides $t_3-t_1-t_2+3$, we take an NEC group Λ with signature

$$\big(0;+;[2,\overset{r}{\dots},2];\{(2,\overset{s}{\dots},2),(-)^l\}\big)$$

for which $s=t_1+t_2+1,\ l=(t_3-t_1-t_2+3)/4,\ r=(g-1-t_3)/2$ and let $\theta:\Lambda\to G=\mathrm{Z}_2^3=\langle x,y,z\rangle$ be the canonical epimorphism defined as in the previous case for the canonical reflections. Also, as in the previous case, we take $\theta(x_1)=xy, \theta(x_i)=xz$ for $i=2,\dots,r$, $\theta(e_2)=yz$ with $\theta(e_i)=1$ for $i\geq 3$ and $\theta(e_1)=xz$ if r is odd and if r is even we take $\theta(e_1)=1$. This definition leads to the configuration of symmetries in question. $\square$

Acknowledgment. The authors are grateful to the referee for helpful comments and valuable suggestions concerning the final version of the paper.

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Grzegorz Gromadzki, Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland

E-mail address: `grom@mat.ug.edu.pl`

Ewa Kozłowska-Walania, Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland

E-mail address: `retrakt@mat.ug.edu.pl`