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A COHOMOLOGICAL LOWER BOUND FOR THE TRANSVERSE LS CATEGORY OF A FOLIATED MANIFOLD

E. MACÍAS-VIRGÓS

Abstract. Let $\mathcal{F}$ be a compact Hausdorff foliation on a compact manifold. Let $E_2^{>0,\bullet} = \bigoplus\{E_2^{p,q} : p > 0, q \geq 0\}$ be the subalgebra of cohomology classes with positive transverse degree in the E_2 term of the spectral sequence of the foliation. We prove that the saturated transverse Lusternik–Schnirelmann category of $\mathcal{F}$ is bounded below by the length of the cup product in $E_2^{>0,\bullet}$. Other cohomological bounds are discussed.

1. Introduction

The *transverse* Lusternik–Schnirelmann category $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ of a foliated manifold $(M,\mathcal{F})$ was introduced in H. Colman's thesis [4], [8]. This notion (and the analogous one of saturated transverse category) has been studied by several authors in the last years [5], [6], [13], [14], [16], [25].

In [4], [8], a lower bound for $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ was given (see Theorem 3.1 below). It is related to the length of the cup product in the basic cohomology of the foliation and generalizes the corresponding classical result for the LS category of a manifold [15].

We shall prove in Theorem 4.1 that another lower bound for the transverse category is the length of the cup product in the De Rham cohomology of the ambient manifold for degrees greater than the dimension $d = \dim \mathcal{F}$ of the foliation, that is, l.c.p. $H^{>d}(M) \leq \mathrm{cat}_{\pitchfork}(M,\mathcal{F})$.

For compact Hausdorff foliations, we introduce a new lower bound for the *saturated* transverse LS category $\mathrm{cat}^s_{\pitchfork}(M,\mathcal{F})$, in terms of the spectral sequence of the foliation (Theorem 6.1). Explicitly, let $E_r^{p,q}$, $0 \leq p \leq \mathrm{codim}\,\mathcal{F}$, $0 \leq q \leq \dim \mathcal{F}$, be the de Rham spectral sequence of the foliated manifold $(M,\mathcal{F})$,

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and let $E_2^{>0,\bullet} = \bigoplus_{p>0,q\geq 0} E_2^{p,q}$ be the subalgebra of cohomology classes with positive transverse degree in the term E_2. Then the saturated transverse LS category of $(M,\mathcal{F})$ is bounded below by the length of the cup product in $E_2^{>0,\bullet}$.

The interest of this result is that the E_2 terms of the spectral sequence of a Riemannian foliation on a compact manifold are known to be finite dimensional [2], [10], [21], [22].

During the preparation of this manuscript, S. Hurder informed the author that he and H. Colman had found independently analogous results for the *tangential* LS category of a foliation (for the E_1 term) [7], thus improving known results from H. Colman and the author [9] and from W. Singhof and E. Vogt [23]. As a matter of fact, the corresponding lower bound for the tangential category of any foliation follows easily from our constructions (Theorem 8.1).

2. Transverse LS category

Let $(M,\mathcal{F})$ be a $\mathcal{C}^\infty$ foliated manifold. An open subset $U \subset M$ is said to be *transversely categorical* if the inclusion factors through some leaf L up to a foliated homotopy. That is, when we consider the induced foliation $\mathcal{F}_U$ on U, there exists a leaf L and a $\mathcal{C}^\infty$ homotopy $H : U \times \mathbb{R} \to M$ such that: each $H_t : U \to M$ sends leaves into leaves; H_0 is the inclusion $U \subset M$; and $H_1(U) \subset L$.

Definition 2.1. The *transverse LS category* $\mathrm{cat}_\pitchfork(M,\mathcal{F})$ of the foliation is the least integer $k \geq 0$ such that M can be covered by $k+1$ transversely categorical open sets.

If such a covering does not exist, we put $\mathrm{cat}_\pitchfork(M,\mathcal{F}) = \infty$. Notice that since adapted charts are categorical, we have $\mathrm{cat}_\pitchfork(M,\mathcal{F}) < \infty$ when the manifold M is compact.

Remark 2.2. In the original paper [8], the definition above corresponds to cat $+1$, but presently we follow the more extended convention that contractible spaces have null LS category. Coherently, we take the length of the cup product (shortly l.c.p.) of an algebra to be the maximum $k \geq 0$ for which there exists some product $a_1 \cdots a_k \neq 0$ ($k = 0$ means that all products are null).

3. Basic cohomology

The following cohomological lower bound for $\mathrm{cat}_\pitchfork(M,\mathcal{F})$ involving the basic cohomology of the foliation was given in [8].

Recall that the *basic* cohomology $H_b = H(M,\mathcal{F})$ is defined by means of the complex $\Omega_b = \Omega(M,\mathcal{F})$ of *basic* forms, that is differential forms $\omega \in \Omega(M)$ such that $i_X\omega = 0 = i_X\, d\omega$ for any vector field X tangent to the foliation. Then the inclusion $\Omega_b \subset \Omega(M)$ induces a morphism $\pi^* : H_b \to H(M)$, which in general is not injective. Let $\pi^* H_b^{>0}$ be the image of the basic cohomology in positive degrees.

THEOREM 3.1 ([8]). *For any foliated manifold, the transverse LS category is bounded below by the length of the cup product in* $\pi^* H_b^{>0}$.

Proof. The argument is standard, but we include it for later use. Let $U \subset M$ be a transversely categorical open set. Since two homotopic maps (by a foliated homotopy) induce the same morphism in basic cohomology, the induced map $H_b^{>0}(M) \to H_b^{>0}(U)$ is null because $H_b^{>0}(L) = 0$ for any leaf L. Hence, the map $H_b^{>0}(M,U) \to H_b^{>0}(M)$ in the long exact sequence (for basic cohomology) of the pair (M,U) is surjective. Now, let $k = \mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ and let $U_0,\dots,U_k$ be $k+1$ transversely categorical open subsets covering M. If $\omega_0,\dots,\omega_k$ are basic cohomology classes of positive degrees, then each ω_i can be lifted to some $\xi_i \in H_b(M,U_i)$, and

$$\pi^*\xi_0 \cup \cdots \cup \pi^*\xi_k \in H(M, U_0 \cup \cdots \cup U_k) = H(M,M) = 0,$$

hence $\pi^*(\omega_0 \cup \cdots \cup \omega_k) = 0$. □

It should be noted that in general it is not possible to define a cup product in the relative basic cohomology, due to the lack of adequate partitions of unity. So we cannot use the length of the basic cohomology as a (better) lower bound for the transverse category. However, this can be done for a particular class of foliations (compact Hausdorff foliations), and the *saturated* transverse LS category, as we shall prove in Theorem 6.1.

4. New lower bounds

Another lower bound for $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ is almost immediate from the definitions. It was suggested by the analogous result from W. Singhof and E. Vogt for the *tangential* category of a foliation [23] and was the motivational idea for the present paper.

Let $H^{>d}(M)$ be the de Rham cohomology of the ambient manifold in degrees greater than the dimension of the foliation. Clearly, for a transversely categorical open set U the map $H^{>d}(M) \to H^{>d}(U)$ induced by the inclusion is the zero map, because it factors through $H^{>d}(L) = 0$, $d = \dim L$. Then the standard argument (this time for de Rham cohomology of M) applies, and we have proved the following theorem.

THEOREM 4.1. *The transverse LS category of a foliation is bounded below by the length of the cup product in* $H^{>d}(M)$, *for* $d = \dim \mathcal{F}$.

When comparing the latter result with Theorem 3.1, one realizes that both bounds can be explained in terms of the (de Rham) spectral sequence $E_r^{p,q} \Rightarrow H(M)$ of the foliation.

5. The spectral sequence of a foliated manifold

This is a very well-known algebraic tool which has been extensively studied; we refer the reader, for instance, to [18], [19], [21], [24] among many others. When the foliation is defined by a fibre bundle, one obtains Serre's spectral sequence, written for the de Rham cohomology as in [12].

5.1. Basic notions. Let us recall some basic notions. For a comprehensive introduction to spectral sequences, see [17]. Let $\Omega(M)$ be the de Rham complex of the ambient manifold M. We define a decreasing filtration $F^p\Omega^r(M)$, $0 \leq p \leq r$, of $\Omega^r(M)$, $0 \leq r \leq \dim M$, by the condition: $\omega \in F^p\Omega^r$ if and only if $i_{X_0} \cdots i_{X_{r-p}}\omega = 0$ whenever the vector fields $X_0, \ldots, X_{r-p}$ are tangent to the foliation.

Put $E_0^{p,q} = F^p\Omega^{p+q}/F^{p+1}\Omega^{p+q}$. Then the exterior differential d induces, for each $0 \leq p \leq \operatorname{codim}\mathcal{F}$, a differential $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$, $0 \leq q \leq \dim\mathcal{F}$, whose cohomology groups are denoted by $E_1^{p,q}$.

Often we denote $\Omega^{p,q} = E_0^{p,q}$. By taking any distribution $\mathcal{N}$ complementary to the foliation, we have $TM = T\mathcal{F} \oplus \mathcal{N}$, so $\Omega(M) = \Gamma\Lambda(T^*M)$ is isomorphic to $\Gamma\Lambda(\mathcal{N}^*) \otimes \Gamma\Lambda(T^*\mathcal{F})$, hence $\Omega^{p,q} \cong \Gamma\Lambda^p(\mathcal{N}^*) \otimes \Gamma\Lambda^q(T^*\mathcal{F})$. We shall use the well known fact that if $\omega \in \Omega^{p,q}$ then $d\omega \in \Omega^{p+q+1}(M)$ decomposes (because $d^2 = 0$) in three parts $[d\omega]^{p+1,q} + [d\omega]^{p,q+1} + [d\omega]^{p+2,q-1}$.

Now, the exterior differential induces morphisms $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ such that $(d_1)^2 = 0$, and we denote by $E_2^{p,q} = H^p(E_1^{\bullet,q}, d_1)$ its cohomology groups. The group H_b^p of basic cohomology corresponds to the term $E_2^{p,0}$.

These are the first steps in order to define morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$, $r \geq 0$, and a spectral sequence $E_{r+1} = H(E_r, d_r)$ which converges to the de Rham cohomology $H(M)$ of the ambient manifold in a finite number of steps.

We shall also need the well known fact that the exterior product of differential forms induces a well defined cup product in each term E_r of the spectral sequence, which is compatible with the bigraduation. Moreover, the morphisms d_r satisfy the usual derivation formula $d_r(\alpha \cup \beta) = d_r\alpha \cup \beta + (-1)^\alpha \alpha \cup d_r\beta$.

5.2. Foliated homotopy. The following result appeared in J. A. Álvarez López's thesis [1].

Lemma 5.1. *Let $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a foliated (i.e., sending leaves into leaves) $\mathcal{C}^\infty$ map between foliated manifolds. Then:*

(1) *The induced cohomology morphism $f^* : H(M') \to H(M)$ preserves the filtration and hence it maps $E_r^{p,q}(M')$ into $E_r^{p,q}(M)$, $0 \leq r$, for all p, q;*
(2) *If g is another map which is homotopic to f by a foliated homotopy, then $f^* = g^*$ on each $E_2^{p,q}$.*

Proof. Part (1) is clear because $X_x \in T_x\mathcal{F}$, $x \in M$, implies $f_{*x}(X_x) \in T_{f(x)}\mathcal{F}'$.

For part (2), we can suppose $M' = M \times \mathbb{R}$, endowed with the foliation $L \times \{t\}$, $L \in \mathcal{F}$, and $f = i_0$, $g = i_1$, where $i_t(x) = (x,t)$. The maps i_0, i_1 are homotopic by the foliated homotopy i_t. Hence, the morphisms $i_0^*, i_1^* : \Omega^r(M') \to \Omega^r(M)$ are algebraically homotopic by the application $H : \Omega^{r+1}(M') \to \Omega^r(M)$ given by $H\omega = \int_0^1 i_{\partial_t}\omega\, dt$. It induces an algebraic homotopy $E_1^{p+1,q}(M') \to E_1^{p,q}(M)$, $p+q=r$, between i_0^*, i_1^* at the level E_1, hence $i_0^* = i_1^*$ at the level E_2. □

6. Compact Hausdorff foliations

A foliation $\mathcal{F}$ on the compact manifold M is said to be *compact Hausdorff* if every leaf is compact and the space of leaves is Hausdorff [11]. Several interesting results and computations of the saturated transverse LS category $\operatorname{cat}^s_\pitchfork(M,\mathcal{F})$ in this setting have been obtained by H. Colman, S. Hurder and P. G. Walczak [6], [14]. Recall that $\operatorname{cat}^s_\pitchfork(M,\mathcal{F})$ is defined [8] by considering transversely categorical open sets which are *saturated* (i.e., a union of leaves).

We have the following result.

Theorem 6.1. *Let $\mathcal{F}$ be a compact Hausdorff foliation on the manifold M. Let $E_2^{>0,\bullet} = \bigoplus_{p>0,q\geq 0} E_2^{p,q}$ be the subalgebra of classes in the E_2 term of the spectral sequence with positive transverse degree. Then* $\text{l.c.p.}\, E_2^{>0,\bullet} \leq \operatorname{cat}^s_\pitchfork(M,\mathcal{F})$.

Proof. We present here the guidelines of the proof. The details are rather technical (although not sophisticated), so we have moved them to Section 7.

First, since there are partitions of unity which are constant along the leaves, we have a Mayer–Vietoris sequence for saturated open sets, which is excessive in the E_1 level. Hence, it is possible to define a cup product in the relative E_2 terms of the spectral sequence. Finally, part (2) of Lemma 5.1 allows us to apply the standard cohomology argument cited in Theorem 3.1 because $E_2^{p,q}(L) = 0$ for $p > 0$ and any leaf L. □

Remark 6.2 (Fiber bundles). Our computation has an application to the classical LS category of a manifold. Let $F \to E \to B$ be a smooth locally trivial fiber bundle with connected fibers, $E_r^{p,q}$ the corresponding spectral sequence [12], hence $E_2^{p,q} = H^p(B;\mathcal{H}^q(F))$. The (classical) Lusternik–Schnirelmann category $\operatorname{cat} B$ of the base equals $\operatorname{cat}^s_\pitchfork(M,\mathcal{F})$ for the foliation $\mathcal{F}$ in E defined by the fibers [8], hence it is bounded below by the length of the cup product in $E_2^{p>0,\bullet} = \bigoplus_{p>0,q\geq 0} E_2^{p,q}$. (However, since we are working with real coefficients, this bound is possibly not better than the usual lower bound $\text{l.c.p.}\, H^{>0}(B) \leq \operatorname{cat} B$.)

7. Relative cup product

In this section, we develop the technical details in the proof of Theorem 6.1. Probably many of them are folk, as we follow the ideas of the book of Bott–Tu [3]. Since the proof is rather lengthly, I have tried to write it in such a way that it becomes accessible to nonspecialists.

The crucial point is the Mayer–Vietoris argument for the E_1 term in Proposition 7.1. In fact, for E_0 it is known (see Munkres's book [20]) that a cup product in the relative cohomology groups of a topological space can be defined for any excisive pair.

7.1. Preliminaries. It is an exercise to prove that for a compact Hausdorff foliation there exist partitions of unity which are constant along the leaves, for any finite covering by saturated open sets.

Let W be a saturated open set. Since the inclusion $W \subset M$ is a foliated map, we have induced morphisms $(i_W)_r^* : (E_r(M), d_r) \to (E_r(W), d_r)$ between the spectral sequences. Often we denote $(i_W)_r^*\omega$ simply by $i^*\omega$ or ω_W when there is no risk of confusion.

7.2. Relative cohomology. Let us define

$$E_1^{p,q}(M, W) = E_1^{p,q}(M) \oplus E_1^{p-1,q}(W)$$

endowed with the differential $\delta = \delta_1 : E_1^{p,q}(M, W) \to E_1^{p+1,q}(M, W)$ given by

$$\delta(\mu, \omega) = \big(d_1\mu, i_1^*\mu - d_1\omega\big).$$

Since $\delta^2 = 0$, we can define $E_2^{p,q}(M, W) := H^p(E_1^{\bullet,q}(M, W), \delta)$, and we have a long exact sequence in cohomology,

$$\cdots \to E_2^{p-1,q}(W) \to E_2^{p,q}(M, W) \to E_2^{p,q}(M) \to E_2^{p,q}(W) \to \cdots,$$

induced by the (obvious) short exact sequence of complexes

$$0 \to \big(E_1^{\bullet-1,q}(W), -d_1\big) \to \big(E_1^{\bullet,q}(M, W), \delta\big) \to \big(E_1^{\bullet,q}(M), d_1\big) \to 0.$$

It is clear that this construction could already be done at the E_0 level.

7.3. Connecting morphism. Let us denote $\Omega^{p,q} = E_0^{p,q}$. Let $U, V \subset M$ be saturated open sets, and $\{\varphi_U, \varphi_V\}$ a smooth partition of unity on $U \cup V$ subordinated to the open covering $\{U, V\}$. If the functions φ_U, φ_V are constant along the leaves, then $d\varphi_U, d\varphi_V \in \Omega^{1,0}$. The *connecting morphism*

$$\Delta : \Omega^{p,q}(U \cap V) \to \Omega^{p+1,q}(U \cup V)$$

is defined by

$$\Delta(\omega) = \begin{cases} d\varphi_V \wedge \omega & \text{on } U, \\ -d\varphi_U \wedge \omega & \text{on } V. \end{cases}$$

This is well defined (we invite the reader to check it). Moreover it is a morphism of complexes $\Delta\colon (\Omega^{p,\bullet}, d_0) \to (\Omega^{p+1,\bullet}, -d_0)$ because for $\omega \in \Omega^{p,q}$ we have (for instance on U)

$$\begin{aligned} d_0\Delta(\omega) &= \big[d\Delta(\omega)\big]^{p+1,q+1} = \big[d(d\varphi_V \wedge \omega)\big]^{p+1,q+1} \\ &= [-d\varphi_V \wedge d\omega]^{p+1,q+1} = -d\varphi_V \wedge [d\omega]^{p,q+1} \\ &= -d\varphi_V \wedge d_0\omega = -\Delta d_0\omega, \end{aligned}$$

and analogously on V. Then, since $E_1^{p,q} = H^q(\Omega^{p,\bullet}, d_0)$, we have induced morphisms

$$\Delta : E_1^{p,q}(U \cap V) \to E_1^{p+1,q}(U \cup V). \tag{7.1}$$

7.4. Mayer–Vietoris sequence. Now we consider the Mayer–Vietoris sequence

$$0 \to E_1^{p,q}(U \cup V) \xrightarrow{i} E_1^{p,q}(U) \oplus E_1^{p,q}(V) \xrightarrow{\pi} E_1^{p,q}(U \cap V) \to 0 \tag{7.2}$$

defined in the usual way, that is

$$i(\xi) = \big((i_U)_1^*(\xi), (i_V)_1^*(\xi)\big)$$

and

$$\pi(\alpha, \beta) = (i_{U\cap V})_1^*(\alpha) - (i_{U\cap V})_1^*(\beta).$$

Due to the existence of basic partitions of unity we have the following lemma.

Lemma 7.1. *The Mayer–Vietoris sequence* (7.2) *is exact.*

Proof. Let

$$\begin{aligned} 0 \to \big(\Omega^{p,\bullet}(U \cup V), d_0\big) &\xrightarrow{i} \big(\Omega^{p,\bullet}(U), d_0\big) \oplus \big(\Omega^{p,\bullet}(V), d_0\big) \\ &\xrightarrow{\pi} \big(\Omega^{p,\bullet}(U \cap V), d_0\big) \to 0 \end{aligned} \tag{7.3}$$

be the usual Mayer–Vietoris short exact sequence for the ambient manifold, restricted to a fixed transverse degree p. The positive fact is that the usual section of π given by $S(\omega) = (\varphi_V\omega, -\varphi_U\omega)$ is in our setting a *morphism of complexes*, because our partitions of unity are constant along the leaves, that is, $d\varphi_U, d\varphi_V \in \Omega^{1,0}$. In fact, for $\omega \in \Omega^{p,q}(U \cap V)$ we have

$$\begin{aligned} dS(\omega) &= \big(d_0(\varphi_V\omega), -d_0(\varphi_U\omega)\big) \\ &= \big(\big[d(\varphi_V\omega)\big]^{p,q+1}, -\big[d(\varphi_U\omega)\big]^{p,q+1}\big) \\ &= \big([d\varphi_V \wedge \omega + \varphi_V \wedge d\omega]^{p,q+1}, -[d\varphi_U \wedge \omega + \varphi_U \wedge d\omega]^{p,q+1}\big) \\ &= \big(\varphi_V \wedge [d\omega]^{p,q+1}, -\varphi_U \wedge [d\omega]^{p,q+1}\big) \\ &= S\big([d\omega]^{p,q+1}\big) = Sd_0(\omega). \end{aligned}$$

Then the long exact sequence associated to (7.3) splits, and gives the short exact sequence (7.2), because $E_1^{p,q} = H^q(\Omega^{p,\bullet}, d_0)$. ∎

Notice that the morphism Δ explicitly defined in (7.1) induces in fact the connecting morphism of the long exact sequence associated to (7.2). Moreover, the section S at the E_0 level induces a section (also denoted S) such that $\pi S = \mathrm{id}$ in the E_1 sequence (7.2). However, S is not yet a morphism of complexes, and the default

$$\Delta = dS - Sd \tag{7.4}$$

is just the connecting morphism defined above, as the reader can check.

On the other hand, the section S on the right side of the sequence (7.2) induces another section $J = \mathrm{id} - S\pi$ for the *left side*, that is

$$J : E_1^{p,q}(U) \oplus E_1^{p,q}(V) \to E_1^{p,q}(U \cup V)$$

such that $Ji = \mathrm{id}$. It is an exercise to prove that

$$dJ - Jd = -\Delta\pi. \tag{7.5}$$

7.5. Cup product in relative cohomology. Now we define a product (compatible with the absolute cup product)

$$\cup : E_2^{p,q}(M,U) \otimes E_2^{r,s}(M,V) \to E_2^{p+r,q+s}(M,U \cup V) \tag{7.6}$$

in the E_2 term of the spectral sequence. It will be induced by a product in the relative E_1 level (defined in Section 7.2), that is

$$\cup : E_1^{p,q}(M,U) \otimes E_1^{r,s}(M,V) \to E_1^{p+r,q+s}(M,U \cup V).$$

This latter product is given by

$$(\mu,\alpha) \cup (\nu,\beta) = (\mu \cup \nu, \xi), \tag{7.7}$$

where $\xi \in E_1^{p+r-1,q+s}(U \cup V)$ is explicitly written as

$$\xi = J\big(\alpha \cup \nu_U, (-1)^\mu \mu_V \cup \beta\big) + (-1)^r \Delta(\alpha_{U\cap V} \cup \beta_{U\cap V}). \tag{7.8}$$

Here, for the differential form μ of bidegree (p,q), the notation $(-1)^\mu$ means $(-1)^p$. On the other hand,

$$\Delta : E_1^{p-1+r-1,q+s}(U \cap V) \to E_1^{p+r-1,q+s}(U \cup V)$$

is the connecting morphism defined in (7.1), and

$$J : E_1^{p+r-1,q+s}(U) \oplus E_1^{p+r-1,q+s}(V) \to E_1^{p+r-1,q+s}(U \cup V)$$

is the left section J considered in equation (7.5). When there is no risk of confusion, we shall understand that the classes μ, β, ν, β are adequately restricted, so we should simply write

$$\xi = J\big(\alpha \cup \nu, (-1)^\mu \mu \cup \beta\big) + (-1)^r \Delta(\alpha \cup \beta).$$

Next we prove the following proposition.

Proposition 7.2. *The product* (7.8) *induces a well defined product in the* E_2 *terms.*

Proof. Although it is possible to check that by hand, we shall present a more elaborate proof, which has the advantage of being valid for any level E_r where the key ingredients Δ, J will be defined.

Since the map (7.6) has the form

$$H^p\big(E_1^{\bullet,q}, d_1\big) \otimes H^r\big(E_1^{\bullet,s}, d_1\big) \to H^{p+r}\big(E_1^{\bullet,q+s}, d_1\big),$$

what we need is a map

$$H^{p+r}\big(E_1^{\bullet,q} \otimes E_1^{\bullet,s}\big) \to H^{p+r}\big(E_1^{\bullet,q+s}\big),$$

so we must only check that the morphism

$$\cup\colon E_1^{\bullet,q}(M,U) \otimes E_1^{\bullet,s}(M,V) \to E_1^{\bullet,q+s}(M,U\cup V)$$

given in (7.7) commutes with the differentials, where as usual the left complex is endowed with the differential $d(A\otimes B) = d_1A\otimes B + (-1)^A A\otimes d_1B$.

Then we have

$$\begin{aligned}
&\delta\big((\mu,\alpha)\cup(\nu,\beta)\big)\\
&\quad= \delta\Big(\big(\mu\cup\nu, J\big(\alpha\cup\nu,(-1)^\mu\mu\cup\beta\big) + (-1)^\mu\Delta(\alpha\cup\beta)\big)\Big)\\
&\quad= \big(d_1(\mu\cup\nu), \mu\cup\nu - d_1J\big(\alpha\cup\nu,(-1)^\mu\mu\cup\beta\big) - (-1)^\mu d_1\Delta(\alpha\cup\beta)\big).
\end{aligned}$$

Now by using that $J(\mu\cup\nu,\mu\cup\nu)=\mu\cup\nu$, that $d_1J = Jd_1 - \Delta\pi$ and that $d_1\Delta = -\Delta d_1$, we have that the first coordinate is (we write $d=d_1$)

$$d\mu\cup\nu + (-1)^\mu\mu\cup d\nu \tag{7.9}$$

while the second is

$$\begin{aligned}
J(\mu\cup\nu,&\mu\cup\nu) - Jd\big(\alpha\cup\nu,(-1)^\mu\mu\cup\beta\big)\\
&+ \Delta\pi\big(\alpha\cup\nu,(-1)^\mu\mu\cup\beta\big) + (-1)^\mu\Delta d(\alpha\cup\beta)\\
&= J\big(\mu\cup\nu - d(\alpha\cup\nu), \mu\cup\nu - (-1)^\mu d(\mu\cup\beta)\big)\\
&\quad+ \Delta\pi\big(\alpha\cup\nu,(-1)^\mu\mu\cup\beta\big) + (-1)^\mu\Delta\big(d\alpha\cup\beta + (-1)^\alpha\alpha\cup d\beta\big),
\end{aligned}$$

that is

$$\begin{aligned}
&J\big(\mu\cup\nu - d\alpha\cup\nu - (-1)^\alpha\alpha\cup d\nu, \mu\cup\nu - (-1)^\mu d\mu\cup\beta - \mu\cup d\beta\big)\\
&\quad+ \Delta\pi\big(\alpha\cup\nu,(-1)^\mu\mu\cup\beta\big) + (-1)^\mu\Delta\big(d\alpha\cup\beta + (-1)^\alpha\alpha\cup d\beta\big).
\end{aligned}\tag{7.10}$$

On the other hand, $d((\mu,\alpha)\otimes(\nu,\beta))$ equals

$$\begin{aligned}
&d(\mu,\alpha)\cup(\nu,\beta) + (-1)^\mu(\mu,\alpha)\cup d(\nu,\beta)\\
&\quad= (d\mu,\mu-d\alpha)\cup(\nu,\beta) + (-1)^\mu(\mu,\alpha)\cup(d\nu,\nu-d\beta)\\
&\quad= \big(d\mu\cup\nu, J\big((\mu-d\alpha)\cup\nu,(-1)^{d\mu}d\mu\cup\beta\big) + (-1)^{d\mu}\Delta\big((\mu-d\alpha)\cup\beta\big)\big)\\
&\qquad+ (-1)^\mu\big(\mu\cup d\nu, J\big(\alpha\cup d\nu,(-1)^\mu\mu\cup(\nu-d\beta)\big) + (-1)^\mu\Delta\big(\alpha\cup(\nu-d\beta)\big)\big)
\end{aligned}$$

whose first coordinate equals (7.9), while the second is

$$J\big(\mu\cup\nu - d\alpha\cup\nu + (-1)^{\mu}\alpha\cup d\nu, (-1)^{d\mu}d\mu\cup\beta + \mu\cup\nu - \mu\cup d\beta\big)$$
$$+\Delta\big((-1)^{d\mu}\mu\cup\beta - (-1)^{d\mu}d\alpha\cup\beta + \alpha\cup\nu - \alpha\cup d\beta\big),$$

which equals (7.10), because $(-1)^{\mu} = (-1)^{\alpha+1}$ and $(-1)^{d\mu} = (-1)^{\mu+1}$, and we have done (!). ☐

REMARK. We have only used the Künneth morphism

$$H(E_U)\otimes H(E_V) \to H(E_U\otimes E_V),$$

but in fact it is an isomorphism because the relative groups $E_2(M,W)$ are finite dimensional for compact Hausdorff foliations on a compact manifold.

8. Tangential LS category

The following result is an easy consequence of our computation in Section 7. It has been found independently by S. Hurder and H. Colman [7].

THEOREM 8.1. *Let $(M,\mathcal{F})$ be any foliated manifold. Then the tangential LS category is bounded below by the length of the cup product in $E_1^{\bullet,>0} = \bigoplus_{p\geq 0,q>0} E_1^{p,q}$, the subalgebra of E_1 of cohomology classes with positive tangential degree.*

Proof. Roughly speaking, the tangential LS category [9] is defined by means of open sets which deform to a transversal *along the leaves*. As it is well known [1], this kind of *integrable* homotopy is an invariant of the E_1 term of the spectral sequence (compare with our Lemma 5.1), hence the inclusion of a tangentially categorical open set vanishes in cohomology for positive tangential degrees.

But on the other hand, the Mayer–Vietoris sequence is always exact in the E_0 term, so it is possible to define the relative E_1 cohomology, and the standard argument applies. ☐

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E. Macías-Virgós, Departamento de Xeometria e Topoloxia, Facultade de Matemáticas, Universidade de Santiago de Compostela, 15782-Spain

E-mail address: quique.macias@usc.es