HARMONIC FUNCTIONS ON THE DISK AND REGULAR MATRIX SUMMABILITY

BY

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1. Introduction

Let $\{x_n\}$ be a sequence of points in a topological space X, and let $\mathcal{A}$ be a space of real or complex continuous functions on X. *Under what conditions is the sequence space* $\{\{f(x_n)\} : f \in \mathcal{A}\}$ *summable by a regular matrix?* This question was considered by Rudin in [4] for $X = \beta N$, the Čech compactification of the integers, and $\mathcal{A} = C^*(X)$, the space of bounded real-valued continuous functions on X. Rudin's work was extended somewhat by the present writer in [6]. Henriksen and Isbell in [2] and the present writer in [5] considered the summability of $C^*(X)$, where X is an arbitrary countable space.

Here the question is examined in the context of certain families of harmonic functions on the open unit disk D of the complex plane. Suppose $|z_n| < 1$ for $n = 1, 2, 3, \cdots$. If $\{z_n\}$ has a limit point in D or if $\{z_n\}$ approaches the boundary exponentially, then for H^∞, for example, the problem is easy. In the first case, $\{\{f(x_n)\} : f \in H^\infty\}$ is summable by a submethod of the identity. In the latter case, no regular matrix sums $\{\{f(z_n)\} : f \in H^\infty\}$.

Suppose $|z_n| < 1$ and $|z_n| \to 1$. In §3 it is proved that regular summability of $\{\{f(z_n)\} : f$ is bounded and harmonic on $D\}$ implies that the set of limit points of $\{z_n\}$ has positive Lebesgue measure on the circle. In §4 the positive regular summability of $\{\{f(z_n)\} : f \in H^1\}$ is characterized in terms of boundedness of certain convex combinations of members of the Poisson kernel. Finally, in §5 it is proved that if $0 \leqq r_n < 1$ and $\sum_{n=1}^\infty (1 - r_n) = \infty$, then there exists $\{\theta_n\}$ such that $\{\{f(r_n e^{i\theta_n})\} : f \in H^1\}$ is summable by a positive regular matrix, and that the condition $\sum_{n=1}^\infty (1 - r_n) = \infty$ is necessary.

2. Preliminaries

Let $A = (a_{nk})$ be a complex infinite matrix. The matrix A may be considered as a linear transformation of complex sequences $x = \{x_k\}$ by the formula

$$(Ax)_n = \sum_{k=1}^\infty a_{nk} x_k .$$

A is called *regular* if $\lim Ax = \lim x$ for all convergent sequences x. It is well known that A is regular if and only if $\lim_n a_{nk} = 0$ for each k, $\lim_n \sum_{k=1}^\infty a_{nk} = 1$ and $\| A \| = \sup_n \sum_{k=1}^\infty | a_{nk} | < \infty$. See [8, p. 57]. If the sequence Ax is convergent, then A is said to *sum* the sequence x. A matrix $A = (a_{nk})$ is called *positive* if $a_{nk} \geqq 0$ for all n and k.

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It is known that no regular matrix sums every sequence of zeros and ones. See [8, p. 54].

For sets S and T with $S \subset T$, let $\chi(S)$ denote the characteristic function of S; i.e. $\chi(S)(x) = 1$ if $x \in S$, $\chi(S)(x) = 0$ otherwise.

Throughout this article let D denote the open unit disk and C the unit circle in the complex plane.

The *Poisson kernel* is the family of functions P_r for $0 \leqq r < 1$ defined by

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.$$

The Poisson kernel satisfies the following:

(i) $P_r(\theta) \geqq 0$;
(ii) $1/2\pi \int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$;
(iii) if $0 < \delta < \pi$ then $\limsup_{r \to 1 |\theta| \geqq \delta} P_r(\theta) = 0$.

Let f be a Lebesgue integrable function on C. The harmonic function g on D defined by

$$g(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P_r(\theta - t)\, dt$$

is called the *Poisson integral* of f. The basic properties of the Poisson kernel and integral may be found in [3]. Note that the n^{th} Fourier coefficient of P_r is $r^{|n|}$.

For $p \geqq 1$ let L^p be the usual Banach space of complex-valued functions on C with

$$\| f \|_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(\theta) |^p \, d\theta \right\}^{1/p}.$$

L^∞ is the space of bounded measurable functions on C with the essential supremum norm $\| f \|_\infty = \text{ess sup}_\theta | f(\theta) |$. Recall that the conjugate space of L^1 is L^∞. Let H^p denote the closed subspace of L^p consisting of those functions f such that

$$\int_{-\pi}^{\pi} f(\theta) e^{in\theta}\, d\theta = 0 \quad \text{for } n = 1, 2, 3, \ldots .$$

Then H^p consists of all functions in L^p whose Poisson integrals are analytic on D. In fact, H^p may be identified via the Poisson integral with the Banach space of analytic functions on D such that the functions $f_r(\theta) = f(re^{i\theta})$ are bounded in L^p-norm as $r \to 1$. See [3, p. 39] for details.

Note that functions on C are frequently identified for convenience with functions on the interval $[-\pi, \pi]$.

A sequence $\{z_n\}$ in D is called an *interpolating sequence* if $\{\{f(z_n)\} : f \in H^\infty\}$ is precisely the set of all bounded complex sequences. By [3, p. 203], if

$$\frac{1 - | z_n |}{1 - | z_{n-1} |} < c < 1$$

then $\{z_n\}$ is an interpolating sequence.

Let m denote normalized Lebesgue measure on $[-\pi, \pi]$. For $E \subset C$, let

$$m(E) = m(\{\theta : e^{i\theta} \in E\}).$$

3. Measure of the set of limit points

Assume that $\{z_n\} \subset D$, $|z_n| \to 1$, and that the set E of limit points of $\{z_n\}$ has Lebesgue measure zero on the circle. A certain regular matrix B corresponding to $\{z_n\}$ will now be constructed. The existence of B solves the summability question in the negative.

Using the regularity of Lebesgue measure, choose a sequence $\{F_k\}$ of disjoint closed subsets of C such that $\bigcup_{k=1}^{\infty} F_k \subset C \sim E$ and $\sum_{k=1}^{\infty} m(F_k) = 1$. Let f_k be the Poisson integral of $\chi(F_k)$. Define a matrix $B = (b_{nk})$ by $b_{nk} = f_k(z_n)$.

3.1 Lemma. *The matrix B is regular.*

Proof. For each k the closed sets F_k and E are disjoint. Let $z_n = r_n e^{i\theta_n}$. There exists $\delta > 0$ and N such that $|\theta_n - t| \geqq \delta$ for all $t \in F_k$ and $n \geqq N$. It follows from property (iii) of the Poisson kernel that

$$b_{nk} = f_k(z_n) \to 0 \quad \text{as} \quad n \to \infty$$

for each k. Also, note that for n fixed,

$$P_{r_n}(\theta_n - t) \textstyle\sum_{k=1}^{\infty} \chi(F_k)(t) = P_{r_n}(\theta_n - t)$$

almost everywhere. By the monotone convergence theorem [1, p. 112] and properties (i) and (ii) of the Poisson kernel,

$$\textstyle\sum_{k=1}^{\infty} b_{nk} = \sum_{k=1}^{\infty} f_k(z_n) = 1$$

for each n. Finally, B is positive, so $\| B \| < \infty$. The result follows.

3.2 Theorem. *Assume that $\{z_n\} \subset D$, $|z_n| \to 1$, and $m(E) = 0$ where E is the set of limit points of $\{z_n\}$. Then no regular matrix can sum all bounded harmonic functions on D restricted to $\{z_n\}$.*

Proof. Assume that the regular matrix A does sum $\{\{f(z_n)\} : f \text{ is bounded and harmonic}\}$. Construct a regular matrix B as in 3.1. Then the matrix AB is regular.

Let S be an arbitrary set of positive integers. Then

$$AB\chi(S) = A(\{\textstyle\sum_{k \in S} b_{nk}\}) = A(\{\sum_{k \in S} f_k(z_n)\}).$$

But $\sum_{k \in S} f_k$ is the Poisson integral of $\chi(\bigcup_{k \in S} F_k)$, so $\sum_{k \in S} f_k$ is a bounded harmonic function. It follows that AB sums $\chi(S)$. But this is a contradiction since no regular matrix sums every sequence of zeros and ones.

Note that the condition $m(E) > 0$ is not sufficient for regular matrix summability. In fact, there is an interpolating sequence $\{z_n\}$ such that $C = E$.

4. The principal result

4.1 THEOREM. *Assume that* $\{z_n\} \subset D$, $z_n = r_n e^{i\theta_n}$, *and* $r_n \to 1$. *Let* $P_k(t) = P_{r_k}(\theta_k - t)$ *and* $C_n =$ *convex hull of* $\{P_k : k \geqq n\}$. *Then* $\{\{f(z_n)\} : f \in H^1\}$ *is summable by a positive regular matrix if and only if there exists* $Q_n \in C_n$ *for each* n *such that* $\{\| Q_n \|_\infty\}$ *is bounded.*

Proof. Let $A = (a_{nk})$ sum $\{\{f(z_n)\} : f \in H^1\}$ with A positive regular. Now

$$\begin{aligned} A(\{f(z_k)\})_n &= \sum_{k=1}^{\infty} a_{nk} f(z_k) \\ &= \sum_{k=1}^{\infty} a_{nk} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) P_{r_k}(\theta_k - t)\, dt\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \left[\sum_{k=1}^{\infty} a_{nk} P_{r_k}(\theta_k - t)\right] dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) K_n(t)\, dt, \text{ say,} \end{aligned}$$

using the monotone convergence theorem. Let

$$\hat{K}_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) K_n(t)\, dt.$$

By the Banach-Steinhaus closure theorem [7, p. 117], each $\hat{K}_n$ is a bounded linear functional on H^1. Also, $\{\hat{K}_n(f)\}$ converges for each f in H^1. By the uniform boundedness principle [7, p. 116, Theorem 1], $\| \hat{K}_n \| = \| K_n \|_\infty \leqq M$, say, for all n. Now for each positive integer m choose p_m and q_m such that

$$\textstyle\sum_{k=1}^{m-1} a_{p_m,k} + \sum_{k=q_m+1}^{\infty} a_{p_m,k} \leqq 1/2.$$

Let $Q_m = (\sum_{k=m}^{q_m} a_{p_m,k})^{-1} \sum_{k=m}^{q_m} a_{p_m,k} P_k$. Then $\{Q_m\}$ is the required sequence of functions.

Conversely, choose $Q_n = \sum_{k=1}^{\infty} a_{nk} P_k \in C_n$ for each n such that $\{\| Q_n \|_\infty\}$ is bounded. Using a typical diagonal process, it may be assumed that

$$\hat{Q}_{p_n}(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) Q_{p_n}(t)\, dt \text{ converges as } n \to \infty$$

for each $f(t) = e^{imt}$, $m \geqq 0$. Hence, $\hat{Q}_{p_n}(P)$ converges for each polynomial P. By [7, p. 118], it follows that $\hat{Q}_{p_n}(f)$ converges for all f in H^1, since the polynomials are dense in H^1 and $\{\| \hat{Q}_n \|\}$ is bounded. But

$$\hat{Q}_{p_n}(f) = \textstyle\sum_{k=1}^{\infty} a_{p_n,k} f(z_k),$$

so the matrix $A = (a_{p_n,k})$ sums $\{\{f(z_n)\} : f \in H^1\}$.

4.2 COROLLARY. *If* $\{\{f(z_n)\} : f \in H^1\}$ *is summable by a positive regular matrix, then so is the family of restrictions to* $\{z_n\}$ *of the Poisson integrals of* L^1 *functions on* C.

Proof. Just modify the second half of the proof of 4.1 by requiring that $\hat{Q}_{p_n}(e^{imt})$ converges for negative m as well.

4.3 *Example.* Let $\theta_k^n = 2k\pi/n$ for integers n and k satisfying $0 \leqq k < n$, and let $r_n = 1 - 1/n$. Let $\{z_n\}$ be the sequence

$$\{r_1 e^{i\theta_0{}^1}, r_2 e^{i\theta_0{}^2}, r_2 e^{i\theta_1{}^2}, r_3 e^{i\theta_0{}^3}, r_3 e^{i\theta_1{}^3}, r_3 e^{i\theta_2{}^3}, \cdots\}$$

in D. It follows from 4.1 that $\{\{f(z_n)\} : f \in H^1\}$ is summable by a positive regular matrix, for consider

$$Q_n(t) = (1/n) \textstyle\sum_{k=0}^{n-1} P_{r_n}(\theta_k^n - t).$$

Now

$$Q_n(t) = \frac{1}{n}\sum_{k=0}^{n-1} \sum_{p=-\infty}^{\infty} [r_n^{|p|} e^{ip\theta_k{}^n}] e^{-ipt}$$

$$= \sum_{p=-\infty}^{\infty} \frac{r_n^{|p|}}{n}\left[\sum_{k=0}^{n-1} e^{2\pi ipk/n}\right] e^{-ipt}.$$

Let c_p^n be the p^{th} Fourier coefficient of Q_n. Note that if p is not a multiple of n, then $c_p^n = 0$, whereas if $p = mn$, then $c_p^n = r_n^{|p|}$. Hence,

$$\| Q_n \|_\infty \leqq \sum_{m=-\infty}^{\infty} r_n^{|m|n} = \frac{2}{1 - r_n^n} - 1 \rightarrow \frac{e+1}{e-1}.$$

In particular, $\{\| Q_n \|_\infty\}$ is bounded. The boundedness of $\{\| Q_n \|_\infty\}$ will follow also from the considerations of §5.

5. Behavior of the moduli

5.1 THEOREM. *If $\{\{f(z_n)\} : f \in H^1\}$ is summable by a positive regular matrix, then $\sum_{n=1}^{\infty} (1 - | z_n |) = \infty$.*

Proof. Using 4.1 let $Q_n = \sum_{k=n}^{\infty} a_{nk} P_k \in C_n$ such that $\| Q_n \|_\infty \leqq M$, say, for all n. Now

$$a_{nk} \frac{1 + | z_k |}{1 - | z_k |} = \| a_{nk} P_k \|_\infty \leqq \| Q_n \|_\infty \leqq M,$$

so

$$1 = \sum_{k=n}^{\infty} a_{nk} \leqq M \sum_{k=n}^{\infty} \frac{1 - | z_k |}{1 + | z_k |}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1 - | z_k |}{1 + | z_k |} = \infty,$$

so

$$\sum_{k=1}^{\infty} (1 - | z_k |) = \infty.$$

By a sequence of lemmas involving estimates of the Poisson kernel, it will be shown that the requirement $\sum_{k=1}^{\infty} (1 - | z_k |) = \infty$ cannot be strengthened.

5.2 LEMMA. *Assume that* $n\theta \leqq \sqrt{6}$, $\theta = 1 - r$, *and* $r \geqq 1/2$. *Then*

$$P_r(n\theta) \leqq (12/n^2)P_r(\theta).$$

Proof.

$$\frac{P_r(n\theta)}{P_r(\theta)} = \frac{1 - 2r\cos\theta + r^2}{1 - 2r\cos n\theta + r^2}$$

$$\leqq \frac{1 - 2r\left(1 - \frac{\theta^2}{2} - \frac{\theta^4}{24}\right) + r^2}{1 - 2r\left(1 - \frac{n^2\theta^2}{2} + \frac{n^4\theta^4}{24}\right) + r^2}$$

$$= \frac{(1 - r)^2 + 2r\left(\frac{\theta^2}{2} + \frac{\theta^4}{24}\right)}{(1 - r)^2 + 2r\left(\frac{n^2\theta^2}{2} - \frac{n^4\theta^4}{24}\right)}$$

$$\leqq \frac{(1 - r) + 2r\theta^2}{(1 - r)^2 + \frac{rn^2\theta^2}{2}}$$

$$= \frac{1 + 2r}{1 + \frac{rn^2}{2}} \leqq \frac{12}{n^2}.$$

5.3 LEMMA. *Assume that*

$\sum_{k=1}^{n} (1 - r_k) \leqq \sqrt{6}/2$, $1/2 \leqq r_1 \leqq r_2 \leqq \cdots \leqq r_n$, *and* $1 - r_n \geqq (1 - r_1)/2$.

Let $\theta_k = k(1 - r_1)$ *for* $1 \leqq k \leqq n$. *Let*

$$R(t) = \sum_{k=1}^{n} (1 - r_k)P_{r_k}(\theta_k - t).$$

Then there exists a constant M *such that* $\| R \|_\infty \leqq M$, *independent of the choice of* $\{r_k\}$ *satisfying the above conditions.*

Proof. Let $\theta = 1 - r_1$. Note that

$$n\theta = n(1 - r_1) \leqq 2n(1 - r_n) \leqq 2\sum_{k=1}^{n} (1 - r_k) \leqq \sqrt{6}.$$

Also, note that in general $(1 - r)P_r(\theta) \leqq 2$.

Suppose first that $(n + 1)\theta \leqq t \leqq 2\pi$. Then $\cos(\theta_1 - t) \leqq \cos\theta$ and $\cos(\theta_n - t) \leqq \cos\theta$; $\cos(\theta_2 - t) \leqq \cos 2\theta$ and $\cos(\theta_{n-1} - t) \leqq \cos 2\theta$; etc. Therefore,

$$\begin{aligned} R(t) &= \sum_{k=1}^{n} (1 - r_k)P_{r_k}(\theta_k - t) \\ &\leqq (1 - r_1)P_{r_1}(\theta) + (1 - r_2)P_{r_2}(2\theta) \\ &\quad + \cdots + (1 - r_{n-1})P_{r_{n-1}}(2\theta) + (1 - r_n)P_{r_n}(\theta) \end{aligned}$$

$$\begin{aligned}
&\leqq (1 - r_1)P_{r_1}(1 - r_1) + (1 - r_2)P_{r_2}(2(1 - r_2)) \\
&\quad + \cdots + (1 - r_{n-1})P_{r_{n-1}}(2(1 - r_{n-1})) + (1 - r_n)P_{r_n}(1 - r_n) \\
&\leqq (1 - r_1)P_{r_1}(1 - r_1) + (12/2^2)(1 - r_2)P_{r_2}(1 - r_2) \\
&\quad + (12/3^2)(1 - r_3)P_{r_3}(1 - r_3) + \cdots + (1 - r_n)P_{r_n}(1 - r_n) \\
&\leqq 48 \textstyle\sum_{k=1}^{\infty} k^{-2}.
\end{aligned}$$

Now suppose $|\theta_p - t| < \theta$. Then $\cos(\theta_{p+2} - t) \leqq \cos\theta$ and $\cos(\theta_{p-2} - t) \leqq \cos\theta$; $\cos(\theta_{p+3} - t) \leqq \cos 2\theta$ and $\cos(\theta_{p-3} - t) \leqq \cos 2\theta$; etc. Therefore, as above,

$$\begin{aligned}
R(t) &= \textstyle\sum_{k=p-1}^{p+1} (1 - r_k)P_{r_k}(\theta_k - t) + \sum_{|k-p|>1} (1 - r_k)P_{r_k}(\theta_k - t) \\
&\leqq 6 + 48 \textstyle\sum_{k=1}^{\infty} k^{-2}.
\end{aligned}$$

5.4 Lemma. *Let $R(t)$ be defined as in 5.3. If*

$$n\theta + 2\sqrt{(1 - r_1^2)} \leqq t \leqq 2\pi + \theta - 2\sqrt{(1 - r_1^2)},$$

then

$$R(t) \leqq \textstyle\sum_{k=1}^{n} (1 - r_k).$$

Proof. For each k, $\cos(\theta_k - t) \leqq \cos(2\sqrt{(1 - r_k^2)}) \leqq r_k$, so

$$P_{r_k}(\theta_k - t) = \frac{1 - r_k^2}{1 - 2r_k \cos(\theta_k - t) + r_k^2} \leqq 1$$

and the result follows.

5.5 Lemma. *There exists a constant N such that for any $\{r_1, r_2, \cdots, r_n\}$ satisfying*

(i) $63/64 \leqq r_1 \leqq r_2 \leqq \cdots \leqq r_n < 1$ *and*
(ii) $\sum_{k=1}^{n} (1 - r_k) \leqq 1$,

there exists $\{\theta_1, \theta_2, \cdots, \theta_n\}$ such that $\| S \|_\infty \leqq N$ where

$$S(t) = \textstyle\sum_{k=1}^{n} (1 - r_k)P_{r_k}(\theta_k - t).$$

Proof. For each positive integer p let

$$I_p = \{r_k : 2^{-p-6} < 1 - r_k \leqq 2^{-p-5}\}.$$

Renumber the sets I_p deleting those which are empty. Let R_p be the function constructed in 5.3 for the members of I_p. Then define the following:

(i) $m_p = \min\{k : r_k \in I_p\}$;
(ii) $n_p =$ cardinality of $\{k : r_k \in I_p\}$;
(iii) $\alpha_p = n_p(1 - r_{m_p}) + 2\sqrt{(1 - r_{m_p}^2)}$;
(iv) $S_p(t) = R_p(t - \sum_{k=1}^{p-1} \alpha_k)$;
(v) $S(t) = \sum_p S_p(t)$.

Note that

$$\begin{aligned}\textstyle\sum_p \alpha_p &= \textstyle\sum_p [n_p(1 - r_{m_p}) + 2\surd(1 - r_{m_p}^2)] \\ &\leqq \textstyle\sum_p [2 \sum_{r_k \in I_p} (1 - r_k) + 2\surd 2\surd(1 - r_{m_p})] \\ &\leqq 2 \textstyle\sum_{k=1}^{n} (1 - r_k) + \sum_{p=1}^{\infty} 2^{(-p-2)/2} \\ &\leqq 2 + 2^{-3/2}/(1 - 2^{-1/2}) < 2\pi.\end{aligned}$$

Therefore, by 5.3 and 5.4, $\| S \|_\infty \leqq M + \sum_{k=1}^{n} (1 - r_k) \leqq M + 1$, where M is the constant for 5.3.

5.6 THEOREM. *Assume that* $\sum_{k=1}^{\infty} (1 - r_k) = \infty$ *where* $0 \leqq r_k < 1$ *and* $r_k \to 1$. *Then there exists* $\{\theta_k\}$ *such that* $\{\{f(r_k e^{i\theta} k)\} : f \in H^1\}$ *is summable by a positive regular matrix.*

Proof. Choose an increasing sequence $\{p_n\}$ of positive integers such that

$$1/2 \leqq \textstyle\sum_{k=p_n}^{p_{n+1}} (1 - r_k) \leqq 1$$

for each n. Of course, it may be assumed that $r_1 \geqq 63/64$ and $\{r_k\}$ is increasing.

For each n let S_n be the function constructed in 5.5 for

$$\{r_{p_n}, r_{p_n+1}, \cdots, r_{p_{n+1}}\}.$$

Then $\| S_n \|_\infty \leqq N$ for each n as in 5.5. Let

$$Q_n = (\textstyle\sum_{k=p_n}^{p_{n+1}} (1 - r_k))^{-1} S_n .$$

Then $\| Q_n \|_\infty \leqq 2N$ for each n, and the result follows from 4.1.

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