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LINEAR INDEPENDENCE OF PARSEVAL WAVELETS

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ABSTRACT. We establish several results yielding linear independence of the affine system generated by ψ in exchange for conditions on the space $V(\psi)$ of negative dilates. A typical assumption yielding linear independence is that the space $V(\psi)$ is shift-invariant. In particular, the affine system generated by a Parseval wavelet is linearly independent. As an illustration of our techniques, we give an alternative proof of the theorem of Linnell (see Proc. Amer. Math. Soc. **127** (1999), 3269–3277) on linear independence of Gabor systems.

1. Introduction

A frame for a Hilbert space $\mathcal{H}$ is a collection of vectors $\{x_i : i \in I\}$ such that there exist constants $0 < A \le B < \infty$ satisfying

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, x_i\rangle|^2 \le B\|x\|^2 \tag{1.1}$$

for all $x \in \mathcal{H}$. The optimal constants A and B above are called the *frame bounds*. An unresolved conjecture due to Feichtinger states that every frame which is norm bounded below can be decomposed into finitely many Riesz sequences. The conjecture of Feichtinger is known to be equivalent to the Kadison Singer Problem [10]. Recent progress on this problem has been made in [8], where it was proven that every frame can be decomposed into $\lceil B/C^2\rceil$ linear independent sets, where B is the upper frame bound in (1.1) and $C \le \|x_i\|$ for all $i \in I$. By considering the example of $\mathcal{H} = \mathbb{R}$ and $\{x_i = 1 : 1 \le i \le N\}$, one can see that the bound $\lceil B/C^2\rceil$ cannot be improved for general frames.

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In this paper, we consider the problem of decomposing affine systems into linearly independent sets. Let $D : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the *dilation* operator given by $(Df)(x) = \sqrt{2}f(2x)$, and let $T_y : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the *translation* operator by $y \in \mathbb{R}$, given by $T_y f(x) = f(x-y)$. Recall that a *Parseval wavelet* is a function $\psi \in L^2(\mathbb{R})$ such that the affine system generated by ψ,

$$\mathcal{W}(\psi) := \{D^j T_k \psi : j,k \in \mathbb{Z}\},$$

is a frame for $L^2(\mathbb{R})$ with frame bounds $A = B = 1$. Note that if $\|\psi\| = 1$, then $\mathcal{W}(\psi)$ is linearly independent, since $\lceil B/C^2 \rceil = 1$ in this case. In fact, it can be shown that when $\|\psi\| = 1$, $\mathcal{W}(\psi)$ is an orthonormal basis for $L^2(\mathbb{R})$. The main result of this paper is that whenever ψ is a Parseval wavelet, $\mathcal{W}(\psi)$ is linearly independent. Combining this with a recent result [9, Theorem 1.4], we obtain as a corollary that whenever $\mathcal{W}(\psi)$ is a Parseval frame, it can be partitioned into two ℓ^2-linearly independent subsets.

To achieve this goal, we establish a series of results yielding linear independence of affine systems in exchange for conditions on the shift-invariance (SI) of the space V of negative dilates. A remarkable feature of our methods is that we do not need to assume that ψ is a frame wavelet. Our main result, Theorem 1.1, guarantees linear independence of $\mathcal{W}(\psi)$ in two disjoint cases: either the space V is $\mathbb{Z}$-SI, or it lacks shift-invariance with respect to lattices sparser than $\mathbb{Z}$.

THEOREM 1.1. *Let* $V = V(\psi) = \overline{\operatorname{span}}\{D^j T_k \psi : j < 0, k \in \mathbb{Z}\}$ *be the space of negative dilates of some* $0 \neq \psi \in L^2(\mathbb{R})$. *Assume either that*:

(i) *the space* V *is* $\mathbb{Z}$*-SI, or*

(ii) *the space* V *is* not $2^j\mathbb{Z}$*-SI for any integer* $j \geq 1$.

Then, the affine system $\mathcal{W}(\psi)$ *is linearly independent.*

The linear independence of affine systems should be compared with the Gabor case. The *Gabor system* generated by $g \in L^2(\mathbb{R})$ with frequency and time shift parameters $a, b > 0$ is given by

$$\mathcal{G}(a,b,g) := \{M_{ak} T_{bl} g : k,l \in \mathbb{Z}\}.$$

Here, $M_y : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the *modulation* operator by $y \in \mathbb{R}$, given by $M_y f(x) = e^{2\pi i x y} f(x)$. Linnell [15] showed that every nonzero Gabor system is linearly independent. This solves a special case of the HRT Conjecture [14], which states that arbitrary time frequency shifts of an L^2 function are linearly independent, see [13] for details. In the affine system case, it is clear that there are affine systems which are not linearly independent (such as those generated by a compactly supported scaling function for an orthonormal wavelet), so it is perhaps more surprising that affine systems generated by Parseval wavelets are linearly independent. As an illustration of our methods, we finish the paper by giving a simple proof of Linnell's theorem [15] without using von Neumann algebra techniques. The connection between linear independence of affine and Gabor systems has also been a subject of a recent paper by Rosenblatt [18], albeit from a different perspective.

2. Preliminaries

The main idea of this paper is to reduce the analysis of linear independence of an affine system to the analysis of linear independence of translates of an L^2 function. Hence, we will need the following well-known lemma.

LEMMA 2.1. *If $0 \neq \psi \in L^2(\mathbb{R}^n)$, then $\{T_k\psi : k \in \mathbb{Z}^n\}$ is linearly independent.*

Lemma 2.1 is a folklore result and its standard proof uses the Fourier transform. One can also give a more complicated proof of Lemma 2.1 for functions on the line using facts about recurrence relations. This proof also holds for more general spaces such as $L^p(\mathbb{R})$, $0 < p < \infty$. The situation for $L^p(\mathbb{R}^n)$ is considerably more delicate, see [12], [17] for details.

2.1. Shift-invariant spaces. The general properties of SI spaces were studied by a number of authors, see [2], [3], [16]. Here, we only list the results that will be used later on.

DEFINITION 2.1. Suppose that Γ is a (full rank) lattice, that is, $\Gamma = P\mathbb{Z}^n$, where P is a real $n \times n$ invertible matrix. We say that a closed subspace $V \subset L^2(\mathbb{R}^n)$ is Γ-*shift-invariant* (Γ-SI), if

$$f \in V \quad \Longrightarrow \quad T_\gamma f \in V \quad \text{for all } \gamma \in \Gamma.$$

Given a countable family $\Phi \subset L^2(\mathbb{R}^n)$ and a lattice Γ, we define the Γ-SI system by

$$E^\Gamma(\Phi) = \{T_\gamma \varphi : \varphi \in \Phi, \gamma \in \Gamma\}.$$

When $\Gamma = \mathbb{Z}^n$, we often drop the superscript Γ, and we simply say that V is SI. Likewise, $E(\Phi)$ means $E^{\mathbb{Z}^n}(\Phi)$.

We will need the following two widely known lemmas regarding SI spaces. We include the proof of Lemma 2.2 for completeness. The proof of Lemma 2.3 can be found in [2], [3].

LEMMA 2.2. *Let V be a shift invariant subspace of $L^2(\mathbb{R}^n)$, and denote by P_V the orthogonal projection onto V. For every $k \in \mathbb{Z}^n$, $T_k P_V = P_V T_k$.*

Proof. Since V is a SI space, so is its orthogonal complement $V^\perp = L^2(\mathbb{R}^n) \ominus V$. Take any $f \in L^2(\mathbb{R}^n)$ and decompose it as $f = f_0 + f_1$, where $f_0 \in V$ and $f_1 \in V^\perp$. Then,

$$T_k P_V(f) = T_k f_0 = P_V(T_k f_0 + T_k f_1) = P_V T_k(f). \qquad \square$$

The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined initially for $f \in L^1(\mathbb{R}^n)$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}\, dx.$$

LEMMA 2.3. *Suppose V is a $\mathbb{Z}^n$-SI space and $\varphi \in V$. If $f \in L^2(\mathbb{R}^n)$ satisfies*

$$\hat{f}(\xi) = m_f(\xi)\hat{\varphi}(\xi) \quad a.e.\ \xi \in \mathbb{R}^n,$$

for some $\mathbb{Z}^n$-periodic measurable function m_f, then $f \in V$.

The spectral function of SI spaces was introduced by Bownik and Rzeszotnik in [5]. The following result, see [5, Lemma 2.5], can serve as its definition.

LEMMA 2.4. *If $V \subset L^2(\mathbb{R}^n)$ is SI, then there exists a countable family $\Phi \subset V$ such that $E(\Phi) = \{T_k\varphi : \varphi \in \Phi,\ k \in \mathbb{Z}^n\}$ is a Parseval frame for V. Then, the spectral function of V satisfies*

$$\sigma_V(\xi) = \sum_{\varphi\in\Phi} |\hat{\varphi}(\xi)|^2 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{2.1}$$

In particular, the above sum does not depend (except on a set of measure zero) on the choice of Φ as long as $E(\Phi)$ is a Parseval frame for V.

The *dimension function* (also called the multiplicity function) of a SI space V is a $\mathbb{Z}^n$-periodic function $\dim_V : \mathbb{R}^n \to \mathbb{N} \cup \{0, \infty\}$. It is given by

$$\dim_V(\xi) = \dim \overline{\operatorname{span}}\big\{ \big(\hat{\varphi}(\xi + k)\big)_{k\in\mathbb{Z}^n} : \varphi \in \Phi\big\},$$

where $\Phi \subset V$ is a countable set of generators of V, that is, $V = \overline{\operatorname{span}} E(\Phi)$. Alternatively, one can use the spectral function to define the dimension function of V as

$$\dim_V(\xi) = \sum_{k\in\mathbb{Z}^n} \sigma_V(\xi + k). \tag{2.2}$$

We refer the reader to [5] for the proof of this fact.

2.2. Frame wavelets. Despite the fact that our results are motivated by the classical case of dyadic dilations in $\mathbb{R}$, we will adopt a more general setting of expansive integer-valued dilations in $\mathbb{R}^n$. More specifically, we shall assume that we are given an $n \times n$ integer-valued matrix A that is expansive, that is, all its eigenvalues have modulus greater than 1.

We say that $\psi \in L^2(\mathbb{R}^n)$ is a frame wavelet if its associated affine system

$$\mathcal{W}(\psi) = \{D^jT_k\psi : j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$$

is a frame for $L^2(\mathbb{R}^n)$. Here, the *dilation* operator $D = D_A$ is given by $Df(x) = |\det A|^{1/2} f(Ax)$ and the *translation* operator $T_kf(x) = f(x - k)$ for some $k \in \mathbb{Z}^n$. In the case when the affine system is a tight frame with constant 1, we say that ψ is a Parseval wavelet. We say that a frame wavelet ϕ

is a canonical dual to a frame wavelet ψ if $\mathcal{W}(\phi)$ is the canonical dual frame of $\mathcal{W}(\psi)$. That is,

$$D^j T_k \phi = S^{-1}(D^j T_k \psi) \quad \text{for all } j \in \mathbb{Z}, k \in \mathbb{Z}^n,$$

where S is the frame operator of $\mathcal{W}(\psi)$ given by

$$Sf = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \langle f, D^j T_k \psi\rangle D^j T_k \psi.$$

Not every frame wavelet has a canonical dual, or even a dual frame wavelet, see [7]. But, every Parseval wavelet is the canonical dual of itself, since its frame operator S is the identity on $L^2(\mathbb{R}^n)$.

For $\psi \in L^2(\mathbb{R}^n)$ we define its *space of negative dilates* by

$$V(\psi) = \overline{\operatorname{span}}\{D^j T_k \psi : j < 0, k \in \mathbb{Z}^n\}.$$

Later we will need the following result due to Weber and the first author [7].

THEOREM 2.5. *Suppose that $\psi \in L^2(\mathbb{R}^n)$ is a frame wavelet which has a canonical dual frame wavelet. That is, the canonical dual of $\mathcal{W}(\psi)$ is of the form $\mathcal{W}(\phi)$ for some frame wavelet ϕ. Then, the space $V(\psi)$ of negative dilates of ψ is $\mathbb{Z}^n$-SI.*

3. Linear independence of affine systems

In this section, we establish a series of results yielding linear independence of affine systems in exchange for conditions on the space of negative dilates. Note that we do not need to assume that ψ is a frame wavelet in any of the results except Corollaries 3.8 and 3.9. We start with the following basic result taking advantage of shift-invariance.

THEOREM 3.1. *Suppose $\psi \in L^2(\mathbb{R}^n)$ and let*

$$V = V(\psi) = \overline{\operatorname{span}}\{D^j T_k \psi : j < 0, k \in \mathbb{Z}^n\}$$

be its space of negative dilates. If V is SI and $DV \neq V$, then $\mathcal{W}(\psi)$ is linearly independent.

Proof. On the contrary, assume that there exists a nonzero finitely supported sequence $(c_{j,k})$ such that

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} c_{j,k} D^j T_k \psi = 0.$$

Applying the dilation operator, we can assume that the largest scale j such that $c_{j,k} \neq 0$ for some $k \in \mathbb{Z}^n$ is the zero scale. Hence,

$$\sum_{k\in\mathbb{Z}^n} c_{0,k} T_k \psi + \sum_{j<0} \sum_{k\in\mathbb{Z}^n} c_{j,k} D^j T_k \psi = 0. \tag{3.1}$$

Let P_V be the orthogonal projection onto V. Then, applying $I - P_V$ to (3.1), we have

$$\sum_{k\in\mathbb{Z}^n} c_{0,k}(I-P_V)T_k\psi = \sum_{k\in\mathbb{Z}^n} c_{0,k}T_k(I-P_V)\psi = 0,$$

where we used that V is SI. Hence, by Lemma 2.1, $(I-P_V)\psi = 0$. Therefore, $T_k\psi \in V$ for all $k \in \mathbb{Z}^n$, since V is SI. This implies that $DV = V$, which is a contradiction with our assumption. Hence, $\mathcal{W}(\psi)$ is linearly independent. □

Using a similar technique as in Theorem 3.1, we can prove an analogous result involving the space of positive dilates.

THEOREM 3.2. *Suppose $0 \neq \psi \in L^2(\mathbb{R}^n)$ and let*

$$Z = Z(\psi) = \overline{\operatorname{span}}\{D^jT_k\psi : j > 0, k \in \mathbb{Z}^n\}$$

be its space of positive dilates. If $DZ \neq Z$, then $\mathcal{W}(\psi)$ is linearly independent.

To prove Theorem 3.2, one needs to follow the proof of Theorem 3.1 by observing that Z is automatically SI. As a corollary of Theorem 3.2, we obtain the following result.

COROLLARY 3.3. *Suppose $0 \neq \psi \in L^2(\mathbb{R}^n)$ is such that the affine system $\mathcal{W}(\psi)$ is complete. If $Z \neq L^2(\mathbb{R}^n)$, then $\mathcal{W}(\psi)$ is linearly independent.*

Proof. Suppose first that $DZ = Z$. This implies that $Z = \bigcup_{j\in\mathbb{Z}} D^j(Z) = L^2(\mathbb{R}^n)$ by the completeness assumption, which is a contradiction. Hence, $DZ \neq Z$, and Theorem 3.2 can be applied. □

The following theorem plays a key role in our considerations. Theorem 3.4 shows that linear dependence of the affine system $\mathcal{W}(\psi)$ implies that the space of negative dilates $V(\psi)$ must be shift-invariant with respect to a certain sublattice of $\mathbb{Z}^n$.

THEOREM 3.4. *Suppose $0 \neq \psi \in L^2(\mathbb{R}^n)$ and $\mathcal{W}(\psi)$ is linearly dependent. Then, the space of negative dilates $V = V(\psi)$ is $A^J\mathbb{Z}^n$-SI for some $J \geq 1$. Moreover, V is finitely generated.*

Proof. Suppose that $\mathcal{W}(\psi)$ is linearly dependent. Then, there exists a nonzero finitely supported sequence $(c_{j,k})$ such that

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} c_{j,k}D^jT_k\psi = 0.$$

Applying the dilation operator, we can assume that the smallest scale j such that $c_{j,k} \neq 0$ for some $k \in \mathbb{Z}^n$ is the zero scale. Hence, for some $J \in \mathbb{N}$,

$$\varphi := \sum_{k\in\mathbb{Z}^n} c_{0,k}T_k\psi = -\sum_{j=1}^{J}\sum_{k\in\mathbb{Z}^n} c_{j,k}D^jT_k\psi. \tag{3.2}$$

For any $j_1 \le j_2 \in \mathbb{Z} \cup \{-\infty, \infty\}$, define the spaces

$$V_{j_1,j_2} = \overline{\text{span}}\{D^j T_k \psi : j_1 \le j \le j_2, k \in \mathbb{Z}^n\}. \tag{3.3}$$

Since $D^j T_k = T_{A^{-j}k} D^j$ and the dilation A preserves the lattice $\mathbb{Z}^n$, the space V_{j_1,j_2} is $A^{-j_1}\mathbb{Z}^n$-SI. Taking the Fourier transform of (3.2), we have

$$\hat{\varphi}(\xi) = m(\xi)\hat{\psi}(\xi) \in \mathcal{F}(V_{1,J}), \quad \text{where } m(\xi) = \sum_{k\in\mathbb{Z}^n} c_{0,k} e^{-2\pi i\langle k,\xi\rangle}.$$

Since the zero set of the trigonometric polynomial m has null measure, we can define a $\mathbb{Z}^n$-periodic function $m_f(\xi) = 1/m(\xi)$ for a.e. ξ. By (3.2) and (3.3), $\varphi \in V_{1,J}$. Since $V_{1,J}$ is $A^{-1}\mathbb{Z}^n$-SI, and hence $\mathbb{Z}^n$-SI, by Lemma 2.3 we have that $\psi \in V_{1,J}$. Again, using that $V_{1,J}$ is $\mathbb{Z}^n$-SI we must have

$$V_{0,J} = V_{1,J}. \tag{3.4}$$

Applying the dilation operator D^k, (3.4) yields

$$V_{k,k+J} = V_{k+1,k+J} \quad \text{for any } k \in \mathbb{Z}. \tag{3.5}$$

We claim that

$$V_{r,J} = V_{1,J} \quad \text{for all } r \le 0. \tag{3.6}$$

We shall proceed by induction. Suppose that (3.6) is true for some $r \le 0$. Then, by (3.5)

$$V_{r-1,r-1+J} = V_{r,r-1+J} \subset V_{r,J} = V_{1,J}.$$

Hence,

$$V_{r-1,J} = \overline{\text{span}}(V_{r-1,r-1+J} \cup V_{r,J}) \subset V_{1,J}.$$

Clearly, the last inclusion is the equality and (3.6) is established with r replaced by $r-1$. This proves (3.6). Consequently,

$$V_{-\infty,J} = \overline{\text{span}}\left(\bigcup_{r\le 0} V_{r,J}\right) = V_{1,J}.$$

Applying the dilation operator D^{-J-1}, we have that the space of negative dilates satisfies

$$V(\psi) = V_{-\infty,-1} = V_{-J,-1}.$$

Observe that $V_{-J,-1}$ is a $A^J\mathbb{Z}^n$-SI space. It is also finitely generated. This is because the affine system $\mathcal{W}(\psi)$ at the scale $j \ge -J$ is generated by $A^J\mathbb{Z}^n$-shifts of $|\det A|^{j+J}$ functions. Consequently, $V(\psi)$ is a finitely generated $A^J\mathbb{Z}^n$-SI space. □

As an application of Theorem 3.4, we establish our main result on the linear independence of affine systems. In particular, we will show that one can improve Theorem 3.1 by removing the hypothesis that $DV = V$. To achieve this, we need the following result about SI spaces.

Lemma 3.5. *Suppose that a non-zero space $V \subset L^2(\mathbb{R}^n)$ is SI and $DV \subset V$. Then, $\dim_V(\xi) = \infty$ for a.e. ξ. In particular, V is not finitely generated.*

Proof. In general, if V is SI, then DV is also SI since the dilation A preserves the lattice $\mathbb{Z}^n$. Moreover, by [5, Corollary 3.5], the dimension functions V and DV are related by

$$\dim_{DV}(\xi) = \sum_{d\in\mathcal{D}} \dim_V(B^{-1}\xi + d) \quad \text{for a.e. } \xi, \tag{3.7}$$

where $B = A^T$ and $\mathcal{D}$ is a collection of distinct coset representatives of $(B^{-1} \times \mathbb{Z}^n)/\mathbb{Z}^n$. Indeed, (3.7) is an immediate consequence of (2.2) and the identity $\sigma_{DV}(\xi) = \sigma_V(B^{-1}\xi)$, which was shown in [5].

To finish the proof, we use a standard ergodic argument. Since the linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ preserves the lattice $\mathbb{Z}^n$, it induces a measure preserving endomorphism $\tilde{B} : \mathbb{T}^n \to \mathbb{T}^n$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Moreover, $\tilde{B}$ is ergodic by [20, Corollary 1.10.1] since B is expansive. Let $E = \{\xi \in \mathbb{T}^n : \dim_V(\xi) \geq 1\}$. Since

$$\dim_V(\xi) \geq \dim_{DV}(\xi) = \sum_{d\in\mathcal{D}} \dim_V(B^{-1}\xi + d), \tag{3.8}$$

we have that $E \subset \tilde{B}^{-1}E$. Thus, $\tilde{B}^{-1}E = E$ (modulo null sets) since $\tilde{B}$ is measure preserving. Thus, by the ergodicity of $\tilde{B}$, we have either $|E| = 0$ or $|E| = 1$. Since the space V is nonzero, we must have $E = \mathbb{T}^n$. Since $\mathcal{D}$ has $|\det A|$ elements, we can apply repeatedly (3.8) to obtain that $\dim_V(\xi) \geq |\det A|^N$ a.e. for any $N \geq 1$. This implies that $\dim_V(\xi) = \infty$ a.e. □

Theorem 3.6. *Suppose $0 \neq \psi \in L^2(\mathbb{R}^n)$ and its space of negative dilates $V = V(\psi)$ is SI. Then, $\mathcal{W}(\psi)$ is linearly independent.*

Proof. We need to consider two possible scenarios. If $DV \neq V$, then $\mathcal{W}(\psi)$ is linearly independent by Theorem 3.1, and we are done.

Hence, it remains to deal with the case when $DV = V$. By Lemma 3.5, the $\mathbb{Z}^n$-SI space V is not finitely generated. Therefore, by the contrapositive of Theorem 3.4, $\mathcal{W}(\psi)$ must be linearly independent. Hence, in either case we have the linear independence of $\mathcal{W}(\psi)$. □

Combining Theorems 3.4 and 3.6 we can prove our main result on linear independence of affine systems in terms of the space of negative dilates. Roughly speaking, Theorem 3.7 guarantees linear independence if this space is either $\mathbb{Z}^n$-SI or lacks any SI with respect to lattices sparser than $\mathbb{Z}^n$.

Theorem 3.7. *Let $V(\psi)$ be the space of negative dilates of some $0 \neq \psi \in L^2(\mathbb{R}^n)$. Let*

$$J = \inf\{j \in \mathbb{Z} : V(\psi) \text{ is } A^j\mathbb{Z}^n\text{-SI}\}.$$

If $-\infty \leq J \leq 0$ or $J = \infty$, then $\mathcal{W}(\psi)$ is linearly independent.

Proof. The case $J = \infty$ means that the space V is not $A^j\mathbb{Z}^n$-SI for any $j \in \mathbb{Z}$. Hence, by the contrapositive of Theorem 3.4, $\mathcal{W}(\psi)$ must be linearly independent. On the other hand, $J \leq 0$ implies that V is $\mathbb{Z}^n$-SI, since $A^j\mathbb{Z}^n \subset A^{j-1}\mathbb{Z}^n$ for all $j \in \mathbb{Z}$. Thus, by Theorem 3.6, $\mathcal{W}(\psi)$ must be linearly independent as well. $\square$

REMARK 3.1. The idea of defining the parameter J corresponding to the shift-invariance of the space $V(\psi)$ is due to Behera [1]. If ψ is an orthogonal wavelet, then J is easily seen to be an integer ≤ 0 or $-\infty$. A result of Behera [1, Theorem 3.4] says that each of these values can be attained by some orthogonal wavelet ψ.

As a corollary of Theorems 2.5 and 3.6, we obtain the linear independence of Parseval wavelet frames.

COROLLARY 3.8. *Suppose $\psi \in L^2(\mathbb{R}^n)$ is a frame wavelet which has a canonical dual frame wavelet. In particular, ψ could be any Parseval wavelet. Then, $\mathcal{W}(\psi)$ is linearly independent.*

Proof. By Theorem 2.5, the space of negative dilates $V = V(\psi)$ is automatically SI. Hence, Theorem 3.6 applies and yields the linear independence of $\mathcal{W}(\psi)$. $\square$

REMARK 3.2. Note that the case of $DV = V$ in Corollary 3.8 is purely hypothetical, since no examples of canonical dual frame wavelets ψ with "large" space $V = V(\psi)$ of negative dilates are known. In fact, Baggett's conjecture states that the space of negative dilates V of any Parseval wavelet ψ satisfies $\bigcap_{j\in\mathbb{Z}} D^jV = \{0\}$, see [4], [6]. Consequently, if Baggett's conjecture is true, then necessarily $DV \neq V$. We also note that for general frame wavelets it might happen that V is very "large." For an example of a frame wavelet ψ such that $V = L^2(\mathbb{R}^n)$, see [4], [6]. Hence, the case of $DV = V$ in the proof of Theorem 3.6 can indeed happen. In this case, Theorem 3.6 guarantees that the affine system $\mathcal{W}(\psi)$ is necessarily linearly independent. This might seem counter-intuitive since the fact that space of negative dilates $V = L^2(\mathbb{R}^n)$ means that $\mathcal{W}(\psi)$ is highly overcomplete.

REMARK 3.3. Suppose that ϕ satisfies a refinement equation

$$\phi(x) = \sum_{k\in\mathbb{Z}^n} c_k \phi(Ax - k),$$

where only finitely many of the coefficients c_k are nonzero. In particular, ϕ could be a compactly supported scaling function of some multiresolution analysis such as a Haar-type scaling function. In the notation of Theorem 3.4, this implies that $\phi \in V_{1,1}$ and consequently, $V_{0,0} \subset V_{1,1}$. Therefore, the space of negative dilates $V(\phi) = V_{-1,-1}$ is $A\mathbb{Z}^n$-SI. Note that this is consistent with Theorem 3.7. Indeed, to conclude that the affine system $\mathcal{W}(\phi)$ is linearly independent we would need to know that $V(\phi)$ is $\mathbb{Z}^n$-SI and not merely $A\mathbb{Z}^n$-SI as in this example.

EXAMPLE 3.1. Given $\varepsilon > 0$, define the function $\psi = \psi_0 + \varepsilon\psi_1$, where

$$\hat{\psi}_0 = \mathbf{1}_{[-1/4,-1/8]\cup[1/8,1/4]}, \qquad \hat{\psi}_1 = \mathbf{1}_{[-1/2,-1/4]\cup[1/4,3/4]}.$$

By [7, Theorem 2(i)], ψ is a frame wavelet for sufficiently small $\varepsilon > 0$. Moreover, the space of negative dilates is

$$V(\psi) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-1/4, 3/8], \\ \hat{f}(\xi - 1/2) = \hat{f}(\xi) \text{ for a.e. } \xi \in [1/4, 3/8]\}.$$

Consequently, $V(\psi)$ is $2\mathbb{Z}$-SI, but not $\mathbb{Z}$-SI. Thus, Theorem 3.7 does not apply. Despite this, one can easily see that the affine system $\mathcal{W}(\psi)$ is linearly independent. Indeed, take any nontrivial finite linear combination of functions in $\mathcal{W}(\psi)$. Without loss of generality, we can assume that the largest scale j with non-zero coefficients is $j = 0$. Then, the fact that $K = \operatorname{supp} \hat{\psi}$ is bounded immediately implies that there will be no cancelations on $K \setminus (K/2)$. Thus, any such linear combination yields a nonzero function. Therefore, Theorem 3.7 gives only a sufficient, but not a necessary condition for linear independence of affine systems.

We now move to the problem of ℓ^2-linear independence of affine systems. Recall that a sequence $\{x_j : j \in \mathbb{N}\}$ in a Banach space is said to be *ℓ^2-linearly independent* if

$$\{\alpha_j\} \in \ell^2, \quad \sum_{j=1}^{\infty} \alpha_j x_j = 0 \quad \Rightarrow \quad \alpha_j = 0, \quad \forall j \in \mathbb{N}. \tag{3.9}$$

Note that the order of the indexing set matters in general; we do not assume that the convergence in the sum is unconditional. However, if $\{x_j : j \in \mathbb{N}\}$ is a Bessel sequence in a Hilbert space, then the ordering is irrelevant.

COROLLARY 3.9. *Suppose $\psi \in L^2(\mathbb{R}^n)$ is a frame wavelet which has a canonical dual frame wavelet. In particular, ψ could be any Parseval wavelet. Then, $\mathcal{W}(\psi)$ can be partitioned into two ℓ^2-linearly independent sets.*

Proof. In [9], it was shown that a linearly independent Bessel sequence can be partitioned into two ℓ^2-linearly independent sequences. The result follows now from Corollary 3.8. $\square$

EXAMPLE 3.2. Let $\psi \in L^2(\mathbb{R})$ be a Parseval wavelet defined by

$$\hat{\psi} = 1_{[-1/2,-1/4]\cup[1/4,1/2]}.$$

It is easy to see that $\{T_k\psi : k \in \mathbb{Z}\}$ is not ℓ^2-linearly independent. In particular, the Parseval frame $\mathcal{W}(\psi)$ is not ℓ^2-linearly independent as well. Further information on this and related examples can be found in [19].

We close this section by giving another application of Theorem 3.4. We show how to obtain a result of Christensen and Lindner [11, Theorem 3.2] with some improvements in the constants.

COROLLARY 3.10. *Let $a \in \mathbb{N}$, $a > 1$, $b > 0$ and assume that $0 \neq \psi \in L^2(\mathbb{R})$ has compact support. Define*

$$c := \sup \operatorname{supp} \psi - \inf \operatorname{supp} \psi,$$

and suppose that there is an interval of positive length d on which $\psi \neq 0$ a.e. For any choice of integers m and n such that

$$bm > c \quad \text{and} \quad a^n(bm - c) > 2bm - c,$$

$\mathcal{W}(\psi) = \{\psi_{j,k} = D_a^j T_{kb}\psi : j, k \in \mathbb{Z}\}$ can be partitioned into mn linearly independent sets. In particular, for any $r \in \{0, \ldots, m-1\}$ and $s \in \{0, \ldots, n-1\}$, the set $\{\psi_{nj+s,mk+r} : j, k \in \mathbb{Z}\}$ is linearly independent.

Proof. We begin with the case $s = r = 0$. In the notation of the proof of Theorem 3.4, we show that $V_{-\infty,0}$ is not equal to $V_{-J,0}$ for any nonnegative integer J. Since the support of ψ contains an interval, it suffices to show that for each J, there is a set $E = E(J)$ of positive measure and a positive number $M = M(J)$ such that every interval of length M intersects E and and such that every $\psi \in V_{-J,0}$ vanishes on E. For then, there is some j sufficiently large so that $a^{nj}d > M$, which implies that the set where $D^{-nj}\psi$ is nonzero intersects the set where every function in $V_{-J,0}$ vanishes.

We proceed by induction on J, and we will show that $M(J)$ can be chosen to be $a^{nJ}(2mb - c)$. For $J = 0$, it is clear from the condition $bm > c$ that all $\psi \in V_{0,0}$ vanish on the $bm\mathbb{Z}$ periodization of a set with length at least $bm - c$. Hence, any interval of length greater than or equal to $bm + (bm - c) = M(0)$ will contain a subinterval of length $bm - c$ on which all $\psi \in V_{0,0}$ vanish.

Assume the above is true for $0, \ldots, J-1$. Note that all $D^{-nJ}T_{mbk}\psi$ vanish on an interval I of length $a^{nJ}(mb - c)$. By periodicity, all $D^{-nJ}T_{mbk}\psi$ vanish on $I_k = I + a^{nJ}mbk$ as well. Since $a^{nJ}(bm - c) = a^{n(J-1)}a^n(bm - c) > a^{n(J-1)}(2bm - c)$, it follows that each interval I_k has nontrivial intersection with $E(J-1)$. Therefore, for each $k \in \mathbb{Z}$ and $\psi \in V_{-J,0}$, ψ vanishes on a subset of I_k of positive measure. Finally, by the $a^{nJ}mb\mathbb{Z}$ periodicity of the I_k's, it follows that any interval of length $a^{nJ}(mb - c) + a^{nJ}mb = M(J)$ must contain one of the I_k's, and in particular, contain a set of positive measure on which every $\psi \in V_{-J,0}$ vanishes.

To finish the proof, replace ψ by $\tilde{\psi} = \psi_{0,r}$ and see that $\{\psi_{nj+s,mk+r} : j, k \in \mathbb{Z}\}$ is linearly independent if and only if $\{\tilde{\psi}_{nj,mk} : j, k \in \mathbb{Z}\}$ is linearly independent. □

4. Linear independence of Gabor systems

The goal of this section is to give an alternative proof of Linnell's theorem [15] on the linear independence of the Gabor system $\mathcal{G}(a,b,g)$. Linnell's proof uses von Neumann algebra techniques, and it was not previously known how to prove the same result using other techniques. We will show that the techniques of the previous section, especially Theorem 3.4, achieve this goal in the one dimensional case. The higher dimensional case is not readily accessible with these methods.

LEMMA 4.1. *Suppose that a nonzero space $V \subset L^2(\mathbb{R})$ is SI and $V \subset M_a(V)$ for some $a > 0$. Then,*

$$\int_0^1 \dim_V(\xi)\,d\xi = \infty. \tag{4.1}$$

In particular, V is not finitely generated.

Proof. In general, when V is SI, then $M_a(V)$ is also SI and the spectral functions are related by

$$\sigma_{M_a(V)}(\xi) = \sigma_V(\xi - a). \tag{4.2}$$

Indeed, (4.2) is an immediate consequence of Lemma 2.4, see also [5]. Thus, $V \subset M_a(V)$ implies that $\sigma_V(\xi - a) \geq \sigma_V(\xi)$ by [5, Proposition 2.6]. Let $E = \bigcup_{k\in\mathbb{Z}}(ak + \operatorname{supp}\sigma_V)$ be the $a\mathbb{Z}$-periodization of the support of σ_V. Thus,

$$\sum_{k\in\mathbb{Z}} \sigma_V(\xi + ak) = \begin{cases} \infty, & \xi \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, using (2.2),

$$\begin{aligned} \int_0^1 \dim_V(\xi)\,d\xi &= \int_0^1 \Big(\sum_{k\in\mathbb{Z}} \sigma_V(\xi + k)\Big)\,d\xi = \int_{\mathbb{R}} \sigma_V(\xi)\,d\xi \\ &= \int_0^a \Big(\sum_{k\in\mathbb{Z}} \sigma_V(\xi + ak)\Big)\,d\xi = \infty. \end{aligned}$$

□

THEOREM 4.2. *Suppose that $0 \neq g \in L^2(\mathbb{R})$ and $a, b > 0$. Then, $\mathcal{G}(a,b,g)$ is linearly independent.*

Proof. For simplicity, we shall assume that time shift parameter $b = 1$. Indeed, applying a standard dilation argument one can always reduce to the special case where $a > 0$ and $b = 1$.

On the contrary, suppose that $\mathcal{G}(a,1,g)$ is linearly dependent. Then, there exists a nonzero finitely supported sequence $(c_{j,k})$ such that

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{j,k} M_{aj} T_k g = 0.$$

Applying the modulation operator we can assume that the smallest j such that $c_{j,k} \neq 0$ for some $k \in \mathbb{Z}$ is $j = 0$. Hence, for some $J \in \mathbb{N}$,

$$\varphi := \sum_{k\in\mathbb{Z}} c_{0,k} T_k g = -\sum_{j=1}^{J} \sum_{k\in\mathbb{Z}} c_{j,k} M_{aj} T_k g. \tag{4.3}$$

For any $j_1 \leq j_2 \in \mathbb{Z} \cup \{-\infty, \infty\}$, define the SI spaces

$$V_{j_1,j_2} = \overline{\operatorname{span}}\{M_{aj} T_k g : j_1 \leq j \leq j_2,\ k \in \mathbb{Z}\}. \tag{4.4}$$

Taking the Fourier transform of (4.3), we have

$$\hat{\varphi}(\xi) = m(\xi)\hat{g}(\xi) \in \mathcal{F}(V_{1,J}), \quad \text{where } m(\xi) = \sum_{k\in\mathbb{Z}} c_{0,k} e^{-2\pi i k\xi}.$$

Since the zero set of the trigonometric polynomial m is finite, we can define a $\mathbb{Z}$-periodic function $m_f(\xi) = 1/m(\xi)$ for a.e. ξ. By (4.3) and (4.4), $\varphi \in V_{1,J}$. Since $V_{1,J}$ is SI, by Lemma 2.3, we have that $g \in V_{1,J}$. Again, using that $V_{1,J}$ is SI we must have

$$V_{0,J} = V_{1,J}. \tag{4.5}$$

Applying the modulation operator M_{ak}, (4.5) yields

$$V_{k,k+J} = V_{k+1,k+J} \quad \text{for any } k \in \mathbb{Z}. \tag{4.6}$$

We claim that

$$V_{r,J} = V_{1,J} \quad \text{for all } r \leq 0. \tag{4.7}$$

We shall induct on r. Suppose that (4.7) is true for some $r \leq 0$. Then, by (4.6)

$$V_{r-1,r-1+J} = V_{r,r-1+J} \subset V_{r,J} = V_{1,J}.$$

Hence,

$$V_{r-1,J} = \overline{\operatorname{span}}(V_{r-1,r-1+J} \cup V_{r,J}) \subset V_{1,J}.$$

Clearly, the last inclusion is the equality and (4.7) is established. Consequently,

$$V_{-\infty,J} = \overline{\operatorname{span}}\Big(\bigcup_{r\leq 0} V_{r,J}\Big) = V_{1,J}.$$

In particular, $V_{-\infty,J}$ must be finitely generated. On the other hand, applying the modulation operator M_a, we have

$$V_{-\infty,J} \subset V_{-\infty,J+1} = M_a(V_{-\infty,J}).$$

By Lemma 4.1, $V_{-\infty,J}$ is not finitely generated, which is a contradiction. Thus, $\mathcal{G}(a,1,g)$ is linearly independent. $\square$

Remark 4.1. Using metaplectic transforms, see [13, Section 5.1], one can immediately generalize Theorem 4.2 to Gabor systems corresponding to any lattice $\Lambda \subset \mathbb{R}^2$, that is,

$$\mathcal{G}(\Lambda, g) := \{M_x T_y g : (x,y) \in \Lambda\}.$$

Hence, we recover Linnell's result [15] in one dimension.

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