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FROBENIUS SPLITTING AND DERIVED CATEGORY OF TORIC VARIETIES

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Abstract. In this paper, we will use the splitting of the Frobenius direct image of line bundles on toric varieties to explicitly construct an orthogonal basis of line bundles in the derived category $D^b(X)$ where X is a Fano toric variety with (almost) maximal Picard number.

1. Introduction

Let Y be a smooth projective variety defined over an algebraically closed field K of characteristic zero and let $D^b(Y) = D^b(\mathcal{O}_Y\text{-}mod)$ be the derived category of bounded complexes of coherent sheaves of $\mathcal{O}_Y$-modules. $D^b(Y)$ is one of the most important algebraic invariants of a smooth projective variety Y and we would like to know whether $D^b(Y)$ is freely and finitely generated or, more precisely, whether there exists a full strongly exceptional collection of coherent sheaves on Y. In spite of the increasing interest in understanding the structure of $D^b(Y)$, very little progress has been achieved. The existence of a full strongly exceptional collection of coherent sheaves on a smooth projective variety Y is very restrictive for Y, for example, the Grothendieck group $K_0(Y) = K_0(\mathcal{O}_Y\text{-}mod)$ has to be a finitely generated Abelian group. There exists a nice class of algebraic varieties, the class of smooth projective toric varieties, satisfying this condition on the Grothendieck group and King [13] conjectured the following.

Conjecture 1.1. *Every smooth complete toric variety has a full strongly exceptional collection of line bundles.*

There are a lot of contributions to the above conjecture. For instance, it turns out to be true for projective spaces [2], multiprojective spaces ([6];

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Proposition 4.16), smooth complete toric varieties with Picard number ≤ 2 ([6]; Corollary 4.13) and smooth complete toric varieties with a splitting fan ([6]; Theorem 4.12 and [7]). Nevertheless, some restrictions are required because, recently, in [11], Hille and Perling constructed an example of smooth non Fano toric surface which does not have a full strongly exceptional collection made up of line bundles. It is quite natural to conjecture the following.

CONJECTURE 1.2. *Every smooth complete Fano toric variety has a full strongly exceptional collection of line bundles.*

There are some numerical evidences towards the above conjecture (see, for instance, [8]). So far only partial results are known and we want to point out that the hypothesis Fano is not necessary. In fact, in [6]; Theorem 4.12, we constructed full strongly exceptional collections of line bundles on families of smooth complete toric varieties none of which is entirely of Fano varieties.

The goal of this paper is to investigate the structure of $D^b(X)$ where X is a smooth Fano toric variety with (almost) maximal Picard number and to prove that for such kind of varieties always exists a full strongly exceptional collection of line bundles (see Theorem 3.11). Hence, our main result provides new evidences towards Conjecture 1.2. In order to get a good candidate to be a full strongly exceptional collection of line bundles and to achieve our main result we use, as a main tool, the splitting of the Frobenius direct image of line bundles on smooth complete toric varieties. This approach will give us a full collection of line bundles on X and, in the last part of the work, we will apply Bondal's criterium (see Proposition 3.8) to conclude that such collection can be ordered in such a way that we get a full strongly exceptional collection on X.

Next, we outline the structure of this paper. In Section 2, we fix the notation and we summarize the basic facts on toric varieties needed in the sequel. In particular, we recall the classification of smooth Fano toric varieties with (almost) maximal Picard number and we explicitly describe the splitting of the Frobenius image of line bundles on toric images. Section 3 contains the main result of this work. We first briefly review the notions of exceptional sheaves, exceptional collections of sheaves and strongly exceptional collections of sheaves as well as the facts on derived categories needed later. At the end, we prove the existence of an orthogonal basis in $D^b(X)$ made up of lines bundles, where X is an n-dimensional smooth Fano toric variety with Picard number $2n-1 \leq \rho(X) \leq 2n$ if n is even; and $\rho(X) = 2n-1$ if n is odd (see Theorem 3.11).

2. Toric varieties and Frobenius splitting

In this section, we deal with d-dimensional toric varieties X with (almost) maximal Picard number. We first recall their classification (Theorem 2.2 and Proposition 2.3) and we use it to explicitly describe the splitting of the direct

image $(\pi_p)_*(\mathcal{O}_X)$ where π_p is the Frobenius morphism. To start with, we fix the notation and we recall the facts on toric varieties that we will use along this paper refereing to [10] and [17] for more details.

Let Y be a smooth complete toric variety of dimension n over an algebraically closed field K of characteristic zero characterized by a fan $\Sigma := \Sigma(Y)$ of strongly convex polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$ where N is the lattice $\mathbb{Z}^n$, i.e. N is a free Abelian group of rank n and we will denote by $e_0, \ldots, e_{n-1}$ a $\mathbb{Z}$-basis of N. Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ denote the dual lattice and $\hat{e}_0, \ldots, \hat{e}_{n-1}$ the dual basis of $e_0, \ldots, e_{n-1}$. If σ is a cone in N, the dual cone $\check{\sigma}$ is the set of vectors in M that are nonnegative in σ. This determines a commutative semigroup $\check{\sigma} \cap M$ and we set

$$U_\sigma = \mathrm{Spec}(K[S_\sigma])$$

to denote the open affine toric subvariety.

For any $0 \leq i \leq n$, we put $\Sigma(i) := \{\sigma \in \Sigma \mid \dim(\sigma) = i\}$. In particular, to any 1-dimensional cone $\sigma \in \Sigma(1)$ there is a unique generator $v \in N$, called *ray generator*, such that $\sigma \cap N = \mathbb{Z}_{\geq 0} \cdot v$. We label the set of generators in N of the 1-dimensional cones by $\{v_i \mid i \in J\}$. There is a one-to-one correspondence between such ray generators $\{v_i \mid i \in J\}$ and simple toric divisors $\{Z_i \mid i \in J\}$ on Y. The following notion is due to V. V. Batyrev (see [1]).

Definition 2.1. Let Y be a smooth toric variety. A set of toric divisors $\{Z_1, \ldots, Z_k\}$ on Y is called a *primitive set* if $Z_1 \cap \cdots \cap Z_k = \emptyset$ but $Z_1 \cap \cdots \cap \widehat{Z_j} \cap \cdots \cap Z_k \neq \emptyset$ for all j, $1 \leq j \leq k$. Equivalently, this means $\langle v_1, \ldots, v_k \rangle \notin \Sigma$ but $\langle v_1, \ldots, \widehat{v_j}, \ldots, v_k \rangle \in \Sigma$ for all j and we call $P = \{v_1, \ldots, v_k\}$ a *primitive collection*.

If $S := \{Z_1, \ldots, Z_k\}$ is a primitive set, the element $v := v_1 + \cdots + v_k$ lies in the relative interior of a unique cone of Σ, say the cone generated by $v'_1, \ldots, v'_s$ and $v_1 + \cdots + v_k = a_1 v'_1 + \cdots + a_s v'_s$ with $a_i > 0$ is the corresponding *primitive relation*.

If Y is a smooth toric variety of dimension n (hence, n is also the dimension of the lattice N) and m is the number of toric divisors of Y (and hence, the number of 1-dimensional rays in Σ) then the Picard number of Y is $\rho(Y) = m - n$ and the anticanonical divisor $-K_Y$ is given by $-K_Y = Z_1 + \cdots + Z_m$. A smooth toric Fano variety Y is a smooth toric variety with the anticanonical divisor $-K_Y$ ample.

It is well known that isomorphism classes of d-dimensional smooth Fano toric varieties correspond to isomorphism classes of smooth Fano d-polytopes, that is, fully dimensional convex lattice polytopes in $\mathbb{R}^d$ such that the origin is in the interior of the polytopes and the vertices of every facet is a basis of the integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Smooth Fano d-polytopes have been intensively studied during the last decades and completely classified up to dimension 4 ([1] and [18]). In higher dimension, they are classified under some additional

assumptions; for instance, when the polytopes have few vertices (see [14]), maximal number of vertices (see [4] and [16]) or some extra symmetries (see [5]).

In our works [8] and [6]; we described the bounded derived category of smooth Fano d-dimensional polytopes with few vertices (see [6]; Corollary 4.13) and, in this paper, we will deal with smooth Fano d-dimensional polytopes with maximal number of vertices. It is known that $3d$ is an upper bound for the number of vertices of a Fano d-polytope and in the following theorem we recall the classification of smooth Fano d-polytopes with maximal, if d is odd, and (almost) maximal, if d is even, number of vertices. This classification turns out to be the classification of smooth Fano d-dimensional toric varieties with maximal, if d is odd, and (almost) maximal, if d is even, Picard number.

THEOREM 2.2. *Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope and $e_0, \ldots, e_{d-1}$ a basis of the lattice $\mathbb{Z}^d$. The following holds*:

(1) *The number of vertices of P is bounded by $3d$ if d is even and by $3d-1$ if d is odd.*,

(2) *If d is even and P has exactly $3d$ vertices, then P is the convex hull of the $3d$ points*

$$\begin{gathered} \pm e_0, \qquad \pm e_1, \ldots, \qquad \pm e_{d-2}, \qquad \pm e_{d-1}, \\ \pm(e_0 - e_1), \qquad \pm(e_2 - e_3), \ldots, \qquad \pm(e_{d-2} - e_{d-1}). \end{gathered}$$

(3) *If d is even and P has exactly $3d-1$ vertices, then P is the convex hull of the $3d-1$ points*

$$\begin{gathered} e_0, \qquad \pm e_1, \ldots, \qquad \pm e_{d-2}, \qquad \pm e_{d-1}, \\ \pm(e_0 - e_1), \qquad \pm(e_2 - e_3), \ldots, \qquad \pm(e_{d-2} - e_{d-1}). \end{gathered}$$

(4) *If d is odd and P has $3d-1$ vertices, then P is the convex hull of the $(3d-1)$ points*

$$\begin{gathered} e_0, \qquad \pm e_1, \ldots, \qquad \pm e_{d-1}, \\ e_1 - e_0, \qquad \pm(e_1 - e_2), \qquad \pm(e_3 - e_4), \ldots, \qquad \pm(e_{d-2} - e_{d-1}) \end{gathered}$$

or the convex hull of the $(3d-1)$ points

$$\begin{gathered} \pm e_0, \qquad \pm e_1, \ldots, \qquad \pm e_{d-1}, \\ \pm(e_1 - e_2), \qquad \pm(e_3 - e_4), \ldots, \qquad \pm(e_{d-2} - e_{d-1}). \end{gathered}$$

Proof. See [16]; Theorem 1 and [4]; Theorem 1. □

Recall that the Picard number of a d-dimensional smooth Fano toric variety is equal to the number of vertices of the associated Fano polytope minus d. So, if we denote by S_2 the blow up of $\mathbb{P}^2$ at two torus-invariant points and by S_3 the blow up of $\mathbb{P}^2$ at three torus-invariant points, the above classifying result can be read off in the following way.

PROPOSITION 2.3. *Let X be a d-dimensional smooth Fano toric variety with Picard number ρ_X. Then,*

(1) *If d is even, $\rho_X \leq 2d$ and there is up to isomorphism only one X with $\rho_X = 2d$, namely $(S_3)^{\frac{d}{2}}$, and one with $\rho_X = 2d-1$, namely $S_2 \times (S_3)^{\frac{d-2}{2}}$.*
(2) *2) If d is odd, $\rho_X \leq 2d-1$ and there are up to isomorphism precisely two X with $\rho_X = 2d-1$, namely $\mathbb{P}^1 \times (S_3)^{\frac{d-1}{2}}$ or a unique determined toric $(S_3)^{\frac{d-1}{2}}$-fiber bundle over $\mathbb{P}^1$.*

Proof. See [16] and [15]; Proposition 4.1. □

The main goal of the next section is to give an orthogonal basis made up of line bundles for the derived category $D^b(X)$ of bounded complexes of coherent sheaves on the toric varieties X described in Proposition 2.3, mainly on smooth Fano toric varieties of dimension d with (almost) maximal Picard number ρ_X. If d is even and $2d-1 \leq \rho_X \leq 2d$ or d is odd, $\rho_X = 2d-1$ and X isomorphic to $\mathbb{P}^1 \times (S_3)^{\frac{d-1}{2}}$ then, applying [6]; Theorem 4.17, we will see that there is an orthogonal basis for the derived category $D^b(X)$ of bounded complexes of coherent sheaves on X made up of line bundles. For the remaining case, namely a toric $(S_3)^{\frac{d-1}{2}}$-fiber bundle over $\mathbb{P}^1$ we will explicitly compute such basis. To this end, we need to fix some notation and to develop some technical results.

From now on, for any odd integer $d \geq 3$, we will denote by X_d the toric $(S_3)^{\frac{d-1}{2}}$-fiber bundle over $\mathbb{P}^1$ quoted in Proposition 2.3.

For any smooth projective toric variety X, we denote by $P_X(t)$ its Poincaré polynomial. It is well known that the topological Euler characteristic of X, $\chi(X)$ verifies

$$\chi(X) = P_X(-1)$$

and $\chi(X)$ coincides with the number of maximal cones of X, that is, with the rank of the Grothendieck group $K_0(X)$ of X. On the other hand, since X_d is a $(S_3)^{\frac{d-1}{2}}$-fiber bundle over $\mathbb{P}^1$ we have ([10]; pp. 92–93):

$$P_{X_d}(t) = P_{(S_3)^{\frac{d-1}{2}}}(t) P_{\mathbb{P}^1}(t).$$

Thus, putting altogether we deduce

$$\operatorname{rank}(K_0(X_d)) = P_{(S_3)^{\frac{d-1}{2}}}(-1) P_{\mathbb{P}^1}(-1) = 2 \cdot 6^{\frac{d-1}{2}}. \tag{2.1}$$

By Theorem 2.2 and Proposition 2.3, X_d is the toric variety associated to the convex hull of the $(3d-1)$ points

$$e_0, \qquad \pm e_1, \ldots, \qquad \pm e_{d-1},$$

$$e_1 - e_0, \qquad \pm(e_1 - e_2), \qquad \pm(e_3 - e_4), \ldots, \qquad \pm(e_{d-2} - e_{d-1}),$$

$e_0, \ldots, e_{d-1}$ being a basis of the lattice $\mathbb{Z}^d$. Denote by

$$v_0 = e_0, \qquad v_{2k-1} = e_k, \qquad v_{2k} = -e_k \quad \text{for } 1 \leq k \leq d-1 = 2l,$$

$$w_0 = e_1 - e_0, \qquad w_{2j-1} = e_{2j-1} - e_{2j}, \qquad w_{2j} = e_{2j} - e_{2j-1} \quad \text{for } 1 \leq j \leq l,$$

the ray generators of the fan Σ_d associated to X_d. For a later use, it is convenient to remark that the following is the list of all primitive collections on X_d, $d = 2l+1$ (see [5]; Section 2)

$$\begin{array}{ll} \{v_{2k-1}, v_{2k}\} & \text{for } 1 \leq k \leq 2l, \\ \{w_{2j-1}, w_{2j}\} & \text{for } 1 \leq j \leq l, \\ \{w_{2j-1}, v_{4j-2}\} & \text{for } 1 \leq j \leq l, \\ \{w_{2j-1}, v_{4j-1}\} & \text{for } 1 \leq j \leq l, \\ \{w_{2j}, v_{4j-3}\} & \text{for } 1 \leq j \leq l, \\ \{w_{2j}, v_{4j}\} & \text{for } 1 \leq j \leq l, \\ \{w_0, v_0\}. & \end{array} \tag{2.2}$$

For the rest of the work, we will use the following notation when we will deal with toric divisors on X_d, for $d = 2l+1 \geq 3$. We will denote by

- Z_i^+ the toric divisor associated to e_i, $0 \leq i \leq d-1$,
- Z_i^- the toric divisor associated to $-e_i$, $1 \leq i \leq d-1$,
- D_0 the toric divisor associated to $e_1 - e_0$,
- D_j^+ the toric divisor associated to $e_{2j-1} - e_{2j}$, $1 \leq j \leq l$,
- D_j^- the toric divisor associated to $-(e_{2j-1} - e_{2j})$, $1 \leq j \leq l$.

Given any smooth complete toric variety Y, Bondal described a method to produce a candidate collection of line bundles on Y, which for certain classes of Fano toric varieties is expected to be an orthogonal basis of the derived category $D^b(Y)$ of bounded complexes of coherent sheaves on Y. This method requires to compute the different summands appearing on the Frobenius splitting of the tautological line bundle which will be achieved applying the algorithm that we will describe now.

For any smooth complete toric variety Y of dimension n with an n-dimensional torus T acting on it and for any integer $\ell \in \mathbb{Z}$, there is a well-defined toric morphism

$$\pi_\ell : Y \longrightarrow Y$$

which restricts, on the torus T, to the map

$$\pi_\ell : T \longrightarrow T, \qquad t \mapsto t^\ell.$$

The map π_ℓ is the factorization map with respect to the action of the group of ℓ torsion of T. We fix a prime integer $p \gg 0$. By [12]; Theorem 1 and Proposition 2, $(\pi_p)_*(\mathcal{O}_Y)^\vee$ is a vector bundle of rank p^n which splits into a sum of line bundles

$$(\pi_p)_*(\mathcal{O}_Y)^\vee = \bigoplus_\chi \mathcal{O}_Y(D_\chi),$$

where the sum is taken over the group of characters of the p-torsion subgroup of T. Moreover,

$$c_1((\pi_p)_*(\mathcal{O}_Y)^\vee) = \mathcal{O}_Y\left(-\frac{p^{n-1}(p-1)}{2}K_Y\right),$$

where K_Y is the canonical divisor of Y.

For sake of completeness, we recall here the algorithm described by Thomsen in [12] that we will apply later in order to get explicitly the summands of the splitting of $(\pi_p)_*(\mathcal{O}_{X_d})$.

Given any smooth complete toric variety Y of dimension n, Picard number ρ (hence, $n+\rho$ toric divisors) and Group of Grothendieck $K_0(Y)$ of rank s (hence, s maximal cones), we consider $\{\sigma_1,\dots,\sigma_s\}$ the set of maximal cones of the fan Σ associated to Y and we denote by $v_{i_1},\dots,v_{i_n}$ the generators of σ_i. Recall that since Y is smooth, every rational cone $\sigma\in\Sigma$ is generated by a part of a $\mathbb{Z}$-basis of N. For each index i, $1\le i\le s$, we form the matrix $A_i\in GL_n(\mathbb{Z})$ having as the jth row the coordinates of v_{i_j} expressed in the basis $e_1,\dots,e_n$ of N. Let $B_i = A_i^{-1}\in GL_n(\mathbb{Z})$ and we denote by w_{ij} the jth column vector in B_i. Introducing the symbols $Y^{\hat{e}_1},\dots,Y^{\hat{e}_n}$, we form the ring

$$R = K[(Y^{\hat{e}_1})^{\pm1},\dots,(Y^{\hat{e}_n})^{\pm1}]$$

which is the coordinate ring of the torus $T\subset Y$ and for any i, $1\le i\le s$, the coordinate ring of the open affine subvariety U_{σ_i} of Y corresponding to the cone σ_i is the subring

$$R_i = K[Y^{w_{i1}},\dots,Y^{w_{in}}]\subset R,$$

where we use the notation

$$Y^w := (Y^{\hat{e}_1})^{w_1}\cdots(Y^{\hat{e}_n})^{w_n}$$

if $w=(w_1,\dots,w_n)$. For simplicity, we will also write $Y_{ij} := Y^{w_{ij}}$.

For each i and j, we denote by R_{ij} the coordinate ring of $\sigma_i\cap\sigma_j$ and we define

$$I_{ij} := \{v\in M_{n\times1}(\mathbb{Z}) \mid Y_i^v \text{ is a unit in } R_{ij}\}, \tag{2.3}$$

$$C_{ij} := B_j^{-1}B_i\in GL_n(\mathbb{Z}), \tag{2.4}$$

where we use the notation $Y_i^v := (Y_{i1})^{v_1}\cdots(Y_{in})^{v_n}$ being v a column vector with entries $v_1,\dots,v_n$.

For every $p\in\mathbb{N}$ and $w\in I_{ij}$, we define

$$P_p^n := \{v\in M_{n\times1}(\mathbb{Z}) \mid 0\le v_i<p\}$$

and the maps

$$h_{ijp}^w : P_p^n \to R_{ij},$$
$$r_{ijp}^w : P_p^n \to P_p^n$$

by means of the following equality: for any $v \in P_p$

$$C_{ij}v + w = p \cdot h^w_{ijp}(v) + r^w_{ijp}(v).$$

By [12]; Lemma 2 and Lemma 3, these maps exist and they are unique.

Recall that any toric Cartier divisor D on Y can be represented in the form $\{(U_{\sigma_i}, Y_i^{u_i})\}_{\sigma_i \in \Sigma}$, $u_i \in M_{n\times 1}(\mathbb{Z})$ (see [10]; Chapter 3.3). Once fixed the set $\{(U_{\sigma_i}, Y_i^{u_i})\}_{\sigma_i\in\Sigma}$ which represents a toric Cartier divisor D, we define

$$u_{ij} = u_j - C_{ij}u_i.$$

Notice that if $\mathcal{O}_Y(D) = \mathcal{O}_Y$ is the trivial line bundle, then for any pair i, j, we have $u_{ij} = 0$.

For any $p \in \mathbb{Z}$ and any toric Cartier divisor D on Y, $(\pi_p)_*(\mathcal{O}_Y(D))^\vee$ is defined as follows: we fix a set $\{(U_{\sigma_i}, Y_i^{u_i})\}_{\sigma_i\in\Sigma}$ representing D and we choose an index l of a cone $\sigma_l \in \Sigma$. Let D_v, $v \in P^n_p$, denote the Cartier divisor represented by the set $\{(U_{\sigma_i}, Y_i^{h_i})\}_{\sigma_i\in\Sigma}$ where, by definition

$$h_i = h^v_i := h^{u_{li}}_{lip}(v).$$

Then, we have

$$(\pi_p)_*(\mathcal{O}_Y(D))^\vee = \bigoplus_{v\in P^n_p} \mathcal{O}_Y(D_v). \tag{2.5}$$

Remark 2.4. Recall that if $h_i = (h_{i1}, \dots, h_{in})$ and $\alpha^j_{i1}, \dots, \alpha^j_{in}$ are the entries of the jth column vector of B_i, then by definition

$$Y_i^{h_i} = (Y^{\hat{e}_1{}^{\alpha^1_{i1}}} \cdots Y^{\hat{e_n}{}^{\alpha^1_{in}}})^{h_{i1}} (Y^{\hat{e}_1{}^{\alpha^2_{i1}}} \cdots Y^{\hat{e}_n{}^{\alpha^2_{in}}})^{h_{i2}} \cdots (Y^{\hat{e}_1{}^{\alpha^n_{i1}}} \cdots Y^{\hat{e}_n{}^{\alpha^n_{in}}})^{h_{in}}.$$

We denote by

$$\begin{aligned} l_{\sigma_i} = {} & (\alpha^1_{i1}h_{i1} + \alpha^2_{i1}h_{i2} + \cdots + \alpha^n_{i1}h_{in})\hat{e}_1 \\ & + (\alpha^1_{i2}h_{i1} + \alpha^2_{i2}h_{i2} + \cdots + \alpha^n_{i2}h_{in})\hat{e}_2 + \cdots \\ & + (\alpha^1_{in}h_{i1} + \alpha^2_{in}h_{i2} + \cdots + \alpha^n_{in}h_{in})\hat{e}_n \in M. \end{aligned}$$

According to this notation, if D_v is the Cartier divisor represented by the set $\{(U_{\sigma_i}, Y_i^{h_i})\}$, then

$$D_v = \beta^1_v Z_1 + \cdots + \beta^{n+\rho}_v Z_{n+\rho},$$

where

$$\beta^j_v = -l_{\sigma_k}(v_j)$$

for any maximal cone σ_k containing the ray generator v_j associated to the toric divisor Z_j. Indeed, for any pair of maximal cones σ_k and $\sigma_{k'}$ containing v_j, $l_{\sigma_k}(v_j) = l_{\sigma_{k'}}(v_j)$.

Using this algorithm, we are going to prove the following proposition.

PROPOSITION 2.5. *Let X_d be the toric $(S_3)^{\frac{d-1}{2}}$-fiber bundle over $\mathbb{P}^1$, $d = 2l+1$ and $p \gg 0$ a prime integer. With the above notations the different summands of $(\pi_p)_*(\mathcal{O}_{X_d})^\vee$ are*

$$\mathcal{T}_3 \otimes \bigotimes_{k=2}^{l} \big(\mathcal{O} \oplus \mathcal{O}(Z_{2k}^- + D_k^+) \oplus \mathcal{O}(Z_{2k-1}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^-) \\ \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^+)\big),$$

where

$$\mathcal{T}_3 \cong \mathcal{O} \oplus \mathcal{O}(D_0) \oplus \mathcal{O}(Z_1^- + D_1^-) \oplus \mathcal{O}(Z_1^- + D_1^- + D_0) \\ \oplus \mathcal{O}(Z_2^- + D_1^+) \oplus \mathcal{O}(Z_2^- + D_1^+ + D_0) \\ \oplus \mathcal{O}(Z_1^- + Z_2^-) \oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^+) \\ \oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^-) \oplus \mathcal{O}(Z_1^- + Z_2^- + D_0) \\ \oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^+ + D_0) \oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^- + D_0).$$

Proof. By [12]; Theorem 1 and Proposition 2, $(\pi_p)_*(\mathcal{O}_{X_d})^\vee$ is a vector bundle of rank p^d which splits into a sum of line bundles

$$(\pi_p)_*(\mathcal{O}_{X_d})^\vee = \bigoplus_{v^d \in P_p^d} \mathcal{O}_X(D_{v^d}) \tag{2.6}$$

and using the above algorithm, we will determine all these different summands $\mathcal{O}_{X_d}(D_{v^d})$ moving $v^d \in P_p^d$. To this end, we will proceed by induction on odd d.

Assume $d = 3$. Take e_0, e_1, e_2 be a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^3$ and denote by

$$v_0 = e_0, \qquad v_1 = e_1, \qquad v_2 = -e_1, \qquad v_3 = e_2, \qquad v_4 = -e_2$$

$$w_0 = e_1 - e_0, \qquad w_1 = e_1 - e_2, \qquad w_2 = e_2 - e_1$$

the ray generators of the fan Σ_3 associated to X_3.

It follows from Remark 2.4 that in order to get all the different summands appearing in the splitting (2.6), it is enough to determine l_{σ_1}, l_{σ_2} and l_{σ_3} where σ_1, σ_2 and σ_3 are three maximal cones of Σ_3 involving all the ray generators v_i, $0 \le i \le 4$ and w_i, $0 \le i \le 2$. We choose the following three maximal cones of Σ_3:

$$\sigma_1 := \langle v_0, v_1, v_3 \rangle, \qquad \sigma_2 := \langle v_2, w_0, w_2, \rangle, \qquad \sigma_3 := \langle v_0, v_4, w_1 \rangle.$$

The matrices A_i, $1 \le i \le 3$, having as the jth row the coordinates of the j-vector of σ_i expressed in the basis e_0, e_1, e_2 are:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

and their inverses are given by

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Fix the index $l = 1$ corresponding to the cone σ_1. By (2.4),

$$C_{1i} = (B_i)^{-1} B_1 = A_i$$

and, as we pointed out before, if $\{U_{\sigma_j}, X_j^{u_j}\}_{\sigma_j \in \Sigma_3}$ represents the zero divisor then for any pair i, j,

$$u_{1i} = u_i - C_{1i} u_1 = 0.$$

Hence, for any $v \in P_p^3$, $h_i^v := h_{1ip}^{u_{1i}}(v)$ is defined by the relation

$$A_i \cdot v = p \cdot h_i^v + r_{1ip}(v)$$

for a unique $r_{1ip}(v) \in P_p^3$. For any $v = (a_0, a_1, a_2) \in P_p^3$, we define

$$\begin{aligned} d_1 &:= A_1 \cdot v = (a_0, a_1, a_2), \\ d_2 &:= A_2 \cdot v = (-a_1, -a_0 + a_1, -a_1 + a_2), \\ d_3 &:= A_3 \cdot v = (a_0, -a_2, a_1 - a_2). \end{aligned}$$

Notice that if $v \in P_p^3$, then $d_1 \in P_p^3$. Hence, we have $h_1^v = 0$ and $l_{\sigma_1} = 0$. Therefore, we obtain

$$l_{\sigma_1}(u) = 0 \quad \text{for } u \in \sigma_1 = \langle v_0, v_1, v_3 \rangle.$$

So, by Remark 2.4, we deduce that for any $v \in P_p^3$,

$$D_v = \beta_1 Z_1^- + \beta_2 Z_2^- + \beta_3 D_0 + \beta_4 D_1^+ + \beta_5 D_1^-,$$

where

$$\begin{aligned} &\beta_1 = -l_{\sigma_2}(v_2), \qquad \beta_2 = -l_{\sigma_3}(v_4), \qquad \beta_3 = -l_{\sigma_2}(w_0) \\ &\beta_4 = -l_{\sigma_3}(w_1), \quad \text{and} \quad \beta_5 = -l_{\sigma_2}(w_2). \end{aligned}$$

To determine these coefficients, we will consider different cases.

Case 1: $a_0 = a_1 = a_2 = 0$.

In that case, $D_v = 0$.

Case 2: $a_1 = a_2 = 0$ and $a_0 \neq 0$.

In that case, $d_2 = (0, -a_0, 0)$ and $d_3 = (a_0, 0, 0)$. Therefore, $h_2^v = (0, -1, 0)$, $h_3^v = 0$, $l_{\sigma_2} = \hat{e}_0$, $l_{\sigma_3} = 0$ and thus $l_{\sigma_2}(v_2) = 0$, $l_{\sigma_2}(w_0) = -1$ and $l_{\sigma_2}(w_2) = 0$ which gives us

$$D_v = D_0.$$

Case 3: $a_0 = a_2 = 0$ and $a_1 \neq 0$.

In that case, $d_2 = (-a_1, 0, -a_1)$ and $d_3 = (0, 0, a_1)$. Therefore, $h_2^v = (-1, 0, -1)$, $h_3^v = 0$, $l_{\sigma_2} = \hat{e}_0 + \hat{e}_1$, $l_{\sigma_3} = 0$ and thus $l_{\sigma_2}(v_2) = -1$, $l_{\sigma_2}(w_0) = 0$ and $l_{\sigma_2}(w_2) = -1$ which gives us

$$D_v = Z_1^- + D_1^-.$$

Case 4: $a_0 = a_1 = 0$ and $a_2 \neq 0$.

In that case, $d_2 = (0, 0, a_2)$ and $d_3 = (0, -a_2, -a_2)$. Therefore, $h_2^v = 0$, $h_3^v = (0, -1, -1)$, $l_{\sigma_3} = \hat{e}_2$, $l_{\sigma_2} = 0$ and thus $l_{\sigma_3}(v_4) = -1$, $l_{\sigma_3}(w_1) = -1$ which gives us

$$D_v = Z_2^- + D_1^+.$$

Case 5: $a_0, a_1 \neq 0$ and $a_2 = 0$.

In that case, $d_2 = (-a_1, -a_0 + a_1, -a_1)$ and $d_3 = (a_0, 0, a_1)$. Therefore, $h_2^v = (-1, 0, -1)$ and $h_3^v = 0$ if $-a_0 + a_1 \geq 0$ or $h_2^v = (-1, -1, -1)$ and $h_3^v = 0$ if $-a_0 + a_1 < 0$. The first case do not contribute with a new summand and in the second case $l_{\sigma_2} = 2\hat{e}_0 + \hat{e}_1$ and $l_{\sigma_3} = 0$. Thus $l_{\sigma_2}(v_2) = -1$, $l_{\sigma_2}(w_0) = -1$ and $l_{\sigma_2}(w_2) = -1$ which gives us

$$D_v = Z_1^- + D_0 + D_1^-.$$

Case 6: $a_0, a_2 \neq 0$ and $a_1 = 0$.

In that case, $d_2 = (0, -a_0, a_2)$ and $d_3 = (a_0, -a_2, -a_2)$. Therefore, $h_2^v = (0, -1, 0)$, $h_3^v = (0, -1, -1)$, $l_{\sigma_2} = \hat{e}_0$ and $l_{\sigma_3} = \hat{e}_2$. Thus, $l_{\sigma_2}(v_2) = 0$, $l_{\sigma_2}(w_0) = -1$, $l_{\sigma_2}(w_2) = 0$, $l_{\sigma_3}(w_1) = -1$ and $l_{\sigma_3}(v_4) = -1$ which gives us

$$D_v = Z_2^- + D_0 + D_1^+.$$

Case 7: $a_1, a_2 \neq 0$ and $a_0 = 0$.

In that case, $d_2 = (-a_1, a_1, -a_1 + a_2)$ and $d_3 = (0, -a_2, a_1 - a_2)$. Therefore, the possibilities that we have are

$$\begin{aligned}
&h_2^v = (-1, 0, 0), \qquad h_3^v = (0, -1, -1) \quad \text{if } a_1 - a_2 < 0;\\
&h_2^v = (-1, 0, 0), \qquad h_3^v = (0, -1, 0) \quad \text{if } a_1 - a_2 = 0;\\
&h_2^v = (-1, 0, -1), \qquad h_3^v = (0, -1, 0) \quad \text{if } a_1 - a_2 > 0.
\end{aligned}$$

Arguing as before the three news summands that we get are

$$D_v = Z_1^- + Z_2^- + D_1^+, \qquad D_v = Z_1^- + Z_2^-, \qquad D_v = Z_1^- + D_1^- + Z_2^-.$$

Case 8: $a_0, a_1, a_2 \neq 0$.

In that case, $d_2 = (-a_1, -a_0 + a_1, -a_1 + a_2)$ and $d_3 = (a_0, -a_2, a_1 - a_2)$. Arguing as before, the three news summands that we get are

$$\begin{aligned}
&D_v = Z_1^- + Z_2^- + D_0 + D_1^+, \qquad D_v = Z_1^- + Z_2^- + D_0,\\
&D_v = Z_1^- + D_1^- + Z_2^- + D_0.
\end{aligned}$$

Putting all cases together, we get that the different summands appearing in (2.6) for $d = 3$ are

$$\begin{aligned}
\mathcal{T}_3 \cong\ & \mathcal{O} \oplus \mathcal{O}(D_0) \oplus \mathcal{O}(Z_1^- + D_1^-) \oplus \mathcal{O}(Z_1^- + D_1^- + D_0)\\
&\oplus \mathcal{O}(Z_2^- + D_1^+) \oplus \mathcal{O}(Z_2^- + D_1^+ + D_0)\\
&\oplus \mathcal{O}(Z_1^- + Z_2^-) \oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^+)\\
&\oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^-) \oplus \mathcal{O}(Z_1^- + Z_2^- + D_0)\\
&\oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^+ + D_0) \oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^- + D_0)
\end{aligned}$$

and this concludes this initial case $d=3$.

For $3<d=2l+1$ take $e_0,\dots,e_{d-1}$ be a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^d$ with the convention that if $e_0,\dots,e_{d-3}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{d-2}$, we complete it to get $e_0,\dots,e_{d-3},e_{d-2},e_{d-1}$ a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. Recall that

$$v_0=e_0,\qquad v_{2k-1}=e_k,\qquad v_{2k}=-e_k\quad \text{for } 1\le k\le d-1=2l,$$

$$w_0=e_1-e_0,\qquad w_{2j-1}=e_{2j-1}-e_{2j},\qquad w_{2j}=e_{2j}-e_{2j-1}\quad \text{for } 1\le j\le l,$$

are the ray generators of the fan Σ_d associated to X_d.

As we have seen in Remark 2.4, to get all the different summands appearing in the splitting (2.6) it is enough to determine $l_{\sigma_1^d}$, $l_{\sigma_2^d}$ and $l_{\sigma_3^d}$ where σ_1^d, σ_2^d and σ_3^d are three maximal cones of Σ_d that altogether contain all the ray generators v_i, $0\le i\le 2d-2$ and w_i, $0\le i\le 2l$. We choose the following three maximal cones of Σ_d:

$$\begin{aligned}\sigma_1^d&:=\langle v_0,v_1,v_3,\dots,v_{2(d-3)-1},v_{2d-5},v_{2d-3}\rangle,\\ \sigma_2^d&:=\langle v_2,v_6,\dots,v_{2(d-4)},w_0,w_2,\dots,w_{2(l-1)},v_{2(d-2)},w_{2l}\rangle,\\ \sigma_3^d&:=\langle v_0,v_4,\dots,v_{2(d-3)},w_1,w_3,\dots,w_{2l-3},v_{2(d-1)},w_{2l-1}\rangle.\end{aligned}$$

Notice that the set of ray generators of Σ_d can be seen as the set of ray generators of Σ_{d-2} together with the ray generators $v_{2d-5},v_{2d-3},v_{2(d-2)},w_{2l},v_{2(d-1)}$, w_{2l-1} and that the following recursive relation holds:

$$\begin{aligned}\sigma_1^d&=\langle\sigma_1^{d-2},v_{2d-5},v_{2d-3}\rangle,\\ \sigma_2^d&=\langle\sigma_2^{d-2},v_{2(d-2)},w_{2l}\rangle,\\ \sigma_3^d&=\langle\sigma_3^{d-2},v_{2(d-1)},w_{2l-1}\rangle,\end{aligned}$$

where $\sigma_1^{d-2},\sigma_2^{d-2}$ and σ_3^{d-2} are the corresponding maximal cones of Σ_{d-2} that contain all its ray generators.

Thus, the matrices A_i^d, $1\le i\le 3$, having as the jth row the coordinates of the j-vector of σ_i^d expressed in the basis $e_0,\dots,e_{d-1}$ are:

$$A_1^d=\left(\begin{array}{c|c}A_1^{d-2}&0\\\hline 0&\begin{matrix}1&0\\0&1\end{matrix}\end{array}\right),\qquad A_2^d=\left(\begin{array}{c|c}A_2^{d-2}&0\\\hline 0&\begin{matrix}0&-1\\1&-1\end{matrix}\end{array}\right),$$

$$A_3^d=\left(\begin{array}{c|c}A_3^{d-2}&0\\\hline 0&\begin{matrix}-1&0\\-1&1\end{matrix}\end{array}\right),$$

where the A_i^{d-2}, $1\le i\le 3$, are the matrices having as the jth row the coordinates of the j-vector of σ_i^{d-2} expressed in the basis $e_0,\dots,e_{d-3}$. Their inverses are given by $B_1^d=A_1^d$,

$$B_2^d=\left(\begin{array}{c|c}B_2^{d-2}&0\\\hline 0&\begin{matrix}-1&0\\-1&1\end{matrix}\end{array}\right)\quad\text{and}\quad B_3^d=\left(\begin{array}{c|c}B_3^{d-2}&0\\\hline 0&\begin{matrix}-1&1\\-1&0\end{matrix}\end{array}\right),$$

where the matrix B_i^{d-2}, $1 \le i \le 3$, is the inverse of the matrix A_i^{d-2}.

Fix the index $l = 1$ corresponding to the cone σ_1^d. By (2.4),

$$C_{1i}^d = (B_i^d)^{-1} B_1^d = A_i^d$$

and, as we pointed out before, if $\{U_{\sigma_j}, X_j^{u_j}\}_{\sigma_j \in \Sigma_d}$ represents the zero divisor then for any pair i, j,

$$u_{1i} = u_i - C_{1i} u_1 = 0.$$

Hence, for any $v^d \in P_p^d$, $h_i^{v^d} := h_{1ip}^{u_{1i}}(v^d)$ is defined by the relation

$$A_i^d \cdot v^d = p \cdot h_i^{v^d} + r_{1ip}(v^d)$$

for a unique $r_{1ip}(v^d) \in P_p^d$.

For any $v^d = (a_0, \dots, a_{d-3}, a_{d-2}, a_{d-1}) \in P_p^d$, we define

$$\begin{aligned} d_1^d &:= A_1^d \cdot v^d, \\ d_2^d &:= A_2^d \cdot v^d, \\ d_3^d &:= A_3^d \cdot v^d. \end{aligned}$$

Notice that we can see v^d as $v^d = (v^{d-2}, a_{d-2}, a_{d-1})$ and hence we have

$$\begin{aligned} d_1^d &= (d_1^{d-2}, a_{d-2}, a_{d-1}), \\ d_2^d &= (d_2^{d-2}, -a_{d-2}, -a_{d-2} + a_{d-1}), \\ d_3^d &= (d_3^{d-2}, -a_{d-1}, a_{d-2} - a_{d-1}). \end{aligned}$$

Notice that if $v^d \in P_p^d$, then $d_1^d \in P_p^d$. Hence, we have $h_1^{v^d} = 0$ and $l_{\sigma_1^d} = 0$. Therefore, it only remains to determine, for any $v^d \in P_p^d$, the functions $l_{\sigma_2^d}$ and $l_{\sigma_3^d}$ and to write down the corresponding $\mathcal{O}_{X_d}(D_{v^d})$. To this end, we will proceed by induction on odd d and we will consider four different cases.

Case 1: $a_{d-2} = a_{d-1} = 0$.

In that case, for $i = 2, 3$ we have

$$h_i^{v^d} = (h_i^{v^{d-2}}, 0, 0).$$

Therefore, $l_{\sigma_3^d} = l_{\sigma_3^{d-2}}$, $l_{\sigma_3^d} = l_{\sigma_3^{d-2}}$ and using induction the different summands that we get are

$$\begin{aligned} \mathcal{T}_3 \otimes \bigotimes_{k=2}^{l-1} \big(\mathcal{O} \oplus \mathcal{O}(Z_{2k}^- + D_k^+) \oplus \mathcal{O}(Z_{2k-1}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^-) \\ \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^+) \big) \otimes \mathcal{O}. \end{aligned}$$

Case 2: $a_{d-2} \neq 0$ and $a_{d-1} = 0$.

In that case, we have

$$\begin{aligned} h_2^{v^d} &= (h_2^{v^{d-2}}, -1, -1) \quad \text{and} \\ h_3^{v^d} &= (h_3^{v^{d-2}}, 0, 0). \end{aligned}$$

Therefore, $l_{\sigma_2^d} = l_{\sigma_2^{d-2}} + \hat{e}_{d-2}$ and $l_{\sigma_3^d} = l_{\sigma_3^{d-2}}$. Thus, using induction, the different summands that we get in this case are

$$\mathcal{T}_3 \otimes \bigotimes_{k=2}^{l-1} \big(\mathcal{O} \oplus \mathcal{O}(Z_{2k}^- + D_k^+) \oplus \mathcal{O}(Z_{2k-1}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^-) \\ \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^+)\big) \otimes \mathcal{O}(Z_{2l-1}^- + D_l^-).$$

Case 3: $a_{d-1} \neq 0$ and $a_{d-2} = 0$.
In that case, we have

$$h_2^{v^d} = (h_2^{v^{d-2}}, 0, 0) \quad \text{and} \\ h_3^{v^d} = (h_3^{v^{d-2}}, -1, -1).$$

Therefore, $l_{\sigma_2^d} = l_{\sigma_2^{d-2}}$ and $l_{\sigma_3^d} = l_{\sigma_3^{d-2}} + \hat{e}_{d-1}$. Thus, using induction, the different summands that we get in this case are

$$\mathcal{T}_3 \otimes \bigotimes_{k=2}^{l-1} \big(\mathcal{O} \oplus \mathcal{O}(Z_{2k}^- + D_k^+) \oplus \mathcal{O}(Z_{2k-1}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^-) \\ \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^+)\big) \otimes \mathcal{O}(Z_{2l}^- + D_l^+).$$

Case 4: $a_{d-1} \neq 0$ and $a_{d-2} \neq 0$.
In that case the following possibilities can occur:

(4.1) $$h_2^{v^d} = (h_2^{v^{d-2}}, -1, -1) \quad \text{and} \quad h_3^{v^d} = (h_3^{v^{d-2}}, -1, 0);$$

(4.2) $$h_2^{v^d} = (h_2^{v^{d-2}}, -1, 0) \quad \text{and} \quad h_3^{v^d} = (h_3^{v^{d-2}}, -1, 0);$$

(4.3) $$h_2^{v^d} = (h_2^{v^{d-2}}, -1, 0) \quad \text{and} \quad h_3^{v^d} = (h_3^{v^{d-2}}, -1, -1).$$

If (4.1) occurs, then $l_{\sigma_2^d} = l_{\sigma_2^{d-2}} + \hat{e}_{d-2}$ and $l_{\sigma_3^d} = l_{\sigma_3^{d-2}} + \hat{e}_{d-2} + \hat{e}_{d-1}$ and the different summands that we get in this case are

$$\mathcal{T}_3 \otimes \bigotimes_{k=2}^{l-1} \big(\mathcal{O} \oplus \mathcal{O}(Z_{2k}^- + D_k^+) \oplus \mathcal{O}(Z_{2k-1}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^-) \\ \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^+)\big) \\ \otimes \mathcal{O}(Z_{2l-1}^- + Z_{2l}^- + D_l^-).$$

If (4.2) occurs, then $l_{\sigma_2^d} = l_{\sigma_2^{d-2}} + \hat{e}_{d-2} + \hat{e}_{d-1}$ and $l_{\sigma_3^d} = l_{\sigma_3^{d-2}} + \hat{e}_{d-2} + \hat{e}_{d-1}$ and the different summands that we get in this case are

$$\mathcal{T}_3 \otimes \bigotimes_{k=2}^{l-1} \big(\mathcal{O} \oplus \mathcal{O}(Z_{2k}^- + D_k^+) \oplus \mathcal{O}(Z_{2k-1}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^-) \\ \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^+)\big) \otimes \mathcal{O}(Z_{2l}^- + Z_{2l-1}^-).$$

Finally, if 4.3 occurs, then $l_{\sigma_2^d} = l_{\sigma_2^{d-2}} + \hat{e}_{d-2} + \hat{e}_{d-1}$ and $l_{\sigma_3^d} = l_{\sigma_3^{d-2}} + \hat{e}_{d-1}$ and the different summands that we get in this case are

$$\begin{aligned}\mathcal{T}_3 \otimes \bigotimes_{k=2}^{l-1} \big(&\mathcal{O} \oplus \mathcal{O}(Z_{2k}^- + D_k^+) \oplus \mathcal{O}(Z_{2k-1}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^-) \\ &\oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^+)\big) \\ &\otimes \mathcal{O}(Z_{2l}^- + Z_{2l-1}^- + D_l^+).\end{aligned}$$

Putting together the four cases, we obtain that the different summands appearing in the splitting (2.6) are precisely

$$\begin{aligned}\mathcal{T}_3 \otimes \bigotimes_{k=2}^{l} \big(&\mathcal{O} \oplus \mathcal{O}(Z_{2k}^- + D_k^+) \oplus \mathcal{O}(Z_{2k-1}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^-) \\ &\oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^+)\big)\end{aligned}$$

which proves what we want. □

3. Orthogonal basis

This section contains the main theorem of this work and it has as a main goal to construct a full strongly exceptional collection of line bundles in the derived category $D^b(X)$, that is an orthogonal basis made up of line bundles, where X is a smooth Fano toric variety with (almost) maximal Picard number. We will start recalling the notions of exceptional sheaves, exceptional collections of sheaves, strongly exceptional collections of sheaves and full strongly exceptional collections of sheaves as well as the facts on derived categories needed in the rest of the paper.

Definition 3.1. Let Y be a smooth projective variety.

(i) A coherent sheaf F on Y is *exceptional* if $\mathrm{Hom}(F,F) = K$ and $\mathrm{Ext}^i_{\mathcal{O}_Y}(F, F) = 0$ for $i > 0$,

(ii) An ordered collection $(F_0, F_1, \ldots, F_m)$ of coherent sheaves on Y is an *exceptional collection* if each sheaf F_i is exceptional and $\mathrm{Ext}^i_{\mathcal{O}_Y}(F_k, F_j) = 0$ for $j < k$ and $i \geq 0$.

(iii) An exceptional collection $(F_0, F_1, \ldots, F_m)$ is a *strongly exceptional collection* if in addition $\mathrm{Ext}^i_{\mathcal{O}_Y}(F_j, F_k) = 0$ for $i \geq 1$ and $j \leq k$.

(iv) An ordered collection $(F_0, F_1, \ldots, F_m)$ of coherent sheaves on Y is a *full (strongly) exceptional collection* if it is a (strongly) exceptional collection and F_0, $F_1, \ldots, F_m$ generate the bounded derived category $D^b(Y)$.

Remark 3.2. The existence of a full strongly exceptional collection $(F_0, F_1, \ldots, F_m)$ of coherent sheaves on a smooth projective variety Y imposes a rather strong restriction on Y, namely that the Grothendieck group $K_0(Y) = K_0(\mathcal{O}_Y\text{-}mod)$ is isomorphic to $\mathbb{Z}^{m+1}$.

It is natural to ask whether $D^b(Y)$ is freely and finitely generated. More precisely, we are lead to consider the following problem.

Problem 3.3. To characterize smooth projective varieties Y which have a full strongly exceptional collection of coherent sheaves and, even more, if there is one made up of line bundles.

This problem is far from being solved and in this paper we will restrict our attention to the particular case of toric varieties. Toric varieties admit a combinatorial description which allows many invariants to be expressed in terms of combinatorial data. We will use this fact to describe the derived category of smooth Fano toric varieties with (almost) maximal Picard number and, in particular, we will give positive contributions to the above problem and to the following conjecture.

CONJECTURE 3.4. *Every smooth complete Fano toric variety X has a full strongly exceptional collection of line bundles.*

So far, only partial results are known but there are some numerical evidences towards Conjecture 1.2. (For detailed information about Conjecture 1.2, the reader can consult [9], [6], [8] and the references quoted there).

Let us start dealing with smooth Fano d-dimensional toric varieties with Picard number $2d$ or $2d-1$ which are products of toric varieties of smaller dimension. In that case, we will use the following result.

PROPOSITION 3.5. *Let X_1 and X_2 be two smooth projective varieties and let $(F_0^i, F_1^i, \ldots, F_{n_i}^i)$ be a full strongly exceptional collection of locally free sheaves on X_i, $i = 1, 2$. Then*

$$(F_0^1 \boxtimes F_0^2, F_1^1 \boxtimes F_0^2, \ldots, F_{n_1}^1 \boxtimes F_0^2, \ldots, F_0^1 \boxtimes F_{n_2}^2, F_1^1 \boxtimes F_{n_2}^2, \ldots, F_{n_1}^1 \boxtimes F_{n_2}^2)$$

is a full strongly exceptional collection of locally free sheaves on $X_1 \times X_2$.

Proof. See [6]; Proposition 4.16. □

Applying this result we get the following proposition.

PROPOSITION 3.6. *Let X be a d-dimensional smooth Fano toric variety which is isomorphic to either $(S_3)^{\frac{d}{2}}$ or $S_2 \times (S_3)^{\frac{d-2}{2}}$ if d is even, and $\mathbb{P}^1 \times (S_3)^{\frac{d-1}{2}}$ if d is odd. Then, X has a full strongly exceptional collection made up of line bundles.*

Proof. It is well known that $\mathbb{P}^1$ has a full strongly exceptional collection made up of line bundles. On the other hand, by [6]; Proposition 4.19, S_2 and S_3 both have a full strongly exceptional collection of line bundles. Thus, we can conclude by applying reiteratively Proposition 3.5. □

Now we will deal with the remaining case of a d-dimensional smooth Fano toric variety X with Picard number $2d-1 \leq \rho_X \leq 2d$, namely d will be odd

and X isomorphic to X_d: a toric $(S_3)^{\frac{d-1}{2}}$-fiber bundle over $\mathbb{P}^1$. In this case, we will apply Bondal's criterium. Roughly speaking, this criterium asserts that, under certain restrictions, the different summands of the splitting of the Frobenius direct image $(\pi_p)_*(\mathcal{O}_{X_d})$ of the tautological line bundle can be ordered in such a way that they form a full strongly exceptional collection of line bundles. We are going to recall it after fixing some notation.

Notation 3.7. For any irreducible toric curve C in an n-dimensional toric variety X, we denote by $D_1^C, \dots, D_{n-1}^C$ the irreducible toric divisors containing C and we denote by $(a_1^C, \dots, a_{n-1}^C)$ the corresponding intersections numbers of the divisors D_i^C with C.

PROPOSITION 3.8 (Bondal's criterium). *Let X be a smooth n-dimensional toric variety. Assume that for any irreducible toric curve C on X, the coefficients a_i^C verify $a_i^C \geq -1$ for $1 \leq i \leq n-1$ and that no more than one is equal to -1. Then, for $p \gg 0$, a suitable order of the different summands of $(\pi_p)_*(\mathcal{O}_X)^\vee$ form a full strongly exceptional collection of line bundles on X.*

Proof. See [3]. □

Remark 3.9. Let X be a smooth toric variety of dimension d and let C be an irreducible toric curve. Let us compute the numerical invariants a_i^C. To this end, we consider $u_1, \dots, u_{n-1}$ the generators of the cone corresponding to C and let u_+, u_- be the additional generators of the two maximal cones adjacent to it. Then, there is a relation

$$u_+ + u_- + \sum_{i=1}^{n-1} a_i^C u_i = 0$$

in which the coefficients a_i^C are the required intersection numbers.

THEOREM 3.10. *Let X_d be a toric $(S_3)^{\frac{d-1}{2}}$-fiber bundle over $\mathbb{P}^1$. Then, a suitable order of the summands of*

$$\begin{aligned}\mathcal{T}_3 \otimes \bigotimes_{k=2}^{l} \big(&\mathcal{O} \oplus \mathcal{O}(Z_{2k}^- + D_k^+) \oplus \mathcal{O}(Z_{2k-1}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^-) \\ &\oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^-) \oplus \mathcal{O}(Z_{2k-1}^- + Z_{2k}^- + D_k^+)\big),\end{aligned}$$

where

$$\begin{aligned}\mathcal{T}_3 \cong\ &\mathcal{O} \oplus \mathcal{O}(D_0) \oplus \mathcal{O}(Z_1^- + D_1^-) \oplus \mathcal{O}(Z_1^- + D_1^- + D_0) \\ &\oplus \mathcal{O}(Z_2^- + D_1^+) \oplus \mathcal{O}(Z_2^- + D_1^+ + D_0) \\ &\oplus \mathcal{O}(Z_1^- + Z_2^-) \oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^+) \\ &\oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^-) \oplus \mathcal{O}(Z_1^- + Z_2^- + D_0) \\ &\oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^+ + D_0) \oplus \mathcal{O}(Z_1^- + Z_2^- + D_1^- + D_0)\end{aligned}$$

form a full strongly exceptional collection of line bundles on X_d.

TABLE 1. Two dimensional cones σ_C associated to any irreducible toric curve C and the additional generators u_+, u_- of the two 3-dimensional cones adjacent to it

	σ_C	u_+	u_-		σ_C	u_+	u_-
1	$\langle v_1, v_3\rangle$	v_0	w_0	10	$\langle v_3, w_0\rangle$	v_1	w_2
2	$\langle v_1, v_0\rangle$	v_3	w_1	11	$\langle v_3, w_2\rangle$	v_0	w_0
3	$\langle v_1, w_0\rangle$	v_3	w_1	12	$\langle v_4, v_0\rangle$	v_2	w_1
4	$\langle v_1, w_1\rangle$	v_0	w_0	13	$\langle v_4, w_0\rangle$	v_2	w_1
5	$\langle v_2, v_4\rangle$	v_0	w_0	14	$\langle v_4, w_1\rangle$	v_0	w_0
6	$\langle v_2, v_0\rangle$	v_4	w_2	15	$\langle v_0, w_1\rangle$	v_1	v_4
7	$\langle v_2, w_0\rangle$	v_4	w_2	16	$\langle v_0, w_2\rangle$	v_2	v_3
8	$\langle v_2, w_2\rangle$	v_0	w_0	17	$\langle w_1, w_0\rangle$	v_1	v_4
9	$\langle v_3, w_0\rangle$	v_1	w_2	18	$\langle w_2, w_0\rangle$	v_2	v_3

Proof. First of all, notice that we have exactly $2\cdot 6^{\frac{d-1}{2}}$ summands which by (2.1) is the rank of the Grothendieck group $K_0(X_d)$. Hence, the cardinality of any full strongly exceptional collection on X_d is $2\cdot 6^{\frac{d-1}{2}}$ (see Remark 3.2).

By Theorem 2.5 and Proposition 3.8, we will conclude if we prove that X_d verifies Bondal's criterium. To this end, let $e_0,\dots,e_{d-1}$ be a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^d$ and denote by

$$v_0 = e_0, \qquad v_{2k-1} = e_k, \qquad v_{2k} = -e_k \quad \text{for } 1 \le k \le d-1 = 2l, \tag{3.1}$$

$$\begin{aligned} w_0 &= e_1 - e_0, \qquad w_{2j-1} = e_{2j-1} - e_{2j}, \\ w_{2j} &= e_{2j} - e_{2j-1} \quad \text{for } 1 \le j \le l, \end{aligned} \tag{3.2}$$

the ray generators of the fan Σ_d associated to X_d. We will proceed by induction on odd d.

Let $d = 3$. In Table 1, we write down the two-dimensional cones σ_C associated to any irreducible toric curve $C \subset X_d$ and the additional generators u_+, u_- of the two 3-dimensional cones adjacent to it.

By Remark 3.9, we have to check that for any irreducible toric curve C in the above table the coefficients of the relation

$$u_+ + u_- + a_1^C u_1 + a_2^C u_2 = 0 \tag{3.3}$$

being u_1, u_2 the ray generators of σ_C, are greater or equal to -1 and at most there is one equal to -1.

Consider the first case. If

$$v_0 + w_0 + a_1^C v_1 + a_2^C v_3 = 0$$

then by (3.1) and (3.2), we must have $a_1^C = -1$ and $a_2^C = 0$. Hence, the condition is verified. The remaining cases can also be checked by direct computation, and we left the details to the reader.

Let $d>3$ be an odd integer. Let C be any irreducible toric curve, denote by $u_1,\dots,u_{n-1}$ the generators of the cone σ_C corresponding to C and let u_+, u_- be the additional generators of the two maximal cones adjacent to it. Then, there is a unique relation

$$u_+ + u_- + \sum_{i=1}^{n-1} a_i^C u_i = 0 \tag{3.4}$$

and by Remark 3.9 we have to prove that all the coefficients a_i^C of this relation are greater or equal to -1 and at most there is one equal to -1.

First of all, notice that $\sigma_+ := \langle u_+, u_1, \dots, u_{d-1}\rangle$ is a maximal cone in Σ_d. Thus, it contains at least two vectors, z_1, z_2 belonging to the set

$$S := \{\pm e_{d-2}, \pm e_{d-1}, \pm(e_{d-2}-e_{d-1})\}. \tag{3.5}$$

But it follows from (2.2) that this set does not contain three vectors defining a 3-dimensional cone in Σ_d. Thus, σ_+ contains exactly two vectors z_1, z_2 belonging to the set S and moreover, the only possibilities for the pair (z_1, z_2) are

$$\begin{array}{ll} (e_{d-2}, e_{d-1}), & (-e_{d-2}, -e_{d-1}),\\ (e_{d-2}, e_{d-2}-e_{d-1}), & (-e_{d-2}, e_{d-1}-e_{d-2}),\\ (e_{d-1}, -e_{d-2}+e_{d-1}), & (-e_{d-1}, e_{d-2}-e_{d-1}) \end{array} \tag{3.6}$$

because, by (2.2), any other pair is a primitive collection. The same argument shows that $\sigma_- := \langle u_-, u_1, \dots, u_{d-1}\rangle$ contains exactly two vectors z_1', z_2' belonging to the set S and the only possibilities for the pair (z_1', z_2') are the ones in (3.6).

From the definition is clear that the sets $\{z_1, z_2\}$ and $\{z_1', z_2'\}$ coincide or they have at least one vector in common. Keeping in mind this remark, we will distinguish two cases.

Case 1: $\{z_1, z_2\} = \{z_1', z_2'\}$.

In that case, necessarily $z_1, z_2 \in \{u_1, \dots, u_{d-1}\}$. Renumbering if necessary, we can assume that $z_1 = u_{d-2}$ and $z_2 = u_{d-1}$. Since the relation (3.4) must be verified, z_1 has to be canceled against z_2. But (z_1, z_2) is one of the pairs (3.6). Therefore, the only possibility is $a_{d-2}^C = a_{d-1}^C = 0$ and the relation (3.4) turns to be

$$u_+ + u_- + \sum_{i=1}^{d-3} a_i^C u_i = 0.$$

Hence, by hypothesis of induction, the coefficients a_i^C are greater or equal to -1 and at most one is equal to -1.

Case 2: $\{z_1, z_2\}$ and $\{z_1', z_2'\}$ have one vector in common.

In that case, renumbering, if necessary, the only possibility is $z_1 = u_{d-1} = z_1'$, $z_2 = u_+$ and $z_2' = u_-$. Thus, we must have $z_2 + z_2' + a_{d-1}^C z_1 = u_+ + u_- +$

$a^C_{d-1}u_{d-1} = 0$. Therefore, the only possibility is $a^C_{d-1} = 0$, 1 or -1 and we get the relation

$$\sum_{i=1}^{d-2} a^C_i u_i = 0.$$

If in this relation there is one u_i of type $u_i = e_j - e_{j-1}$, then it should be canceled with $-e_j$ and e_{j-1}; or e_j and $-e_{j-1}$; or e_j and e_{j-1}; or $-e_j$ and $-e_{j-1}$. But according to the list of primitive collections (2.2), none of this possibilities can occur since none of them define a 3-dimensional cone. So, the relation only contains vectors u_i of type $\pm e_i$ and thus, the only possibility is $a^C_i = 0$ for all $1 \leq i \leq d-2$.

Therefore, for any irreducible toric curve $C \subset X_d$, Bondal's condition is verified and therefore, the collection can be ordered in such a way that we get a full strongly exceptional collection of line bundles. □

Summing up we get our main result.

THEOREM 3.11. *Let X be a smooth Fano d-dimensional toric variety with Picard number ρ_X with $2d - 1 \leq \rho_X \leq 2d$. Then, X has a full strongly exceptional collection of line bundles.*

Proof. It follows from Proposition 2.3, Proposition 3.6 and Proposition 3.10. □

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